CONSTRUCTION OF OUTCOME FUNCTIONS GUARANTEEING EXISTENCE AND PARETO OPTIMALITY OF NASH EQUILIBRIA

by

Leonid Hurwicz and David Schmeidler

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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Introduction

Recent formulations of social decisions problems focus on rules by which a group of individuals (the society) arrives at a choice among available alternatives. Such a rule, which we shall call the outcome function (Gibbard's "game form," see [3]), specifies the permissible strategies of each individual and associates with every combination of permissible strategies a particular element (outcome) of the set of alternatives. The desirability of choices made is evaluated in terms of preferences each individual is assumed to have among the alternatives.

Within this framework, Gibbard and Satterthwaite [3, 8] have shown that where there are more than two alternatives, there does not exist a non-dictatorial outcome function with dominant strategies for all preference profiles. This result encourages the search for non-dictatorial outcome functions retaining some of the main advantages of having dominant strategies even when such strategies do not exist, in particular the Pareto optimality of outcomes and the Nash equilibrium property of solutions.

Along these lines, the present paper deals with the question whether there are non-dictatorial outcome functions with the following two properties: for every preference profile there exists a Nash equili-

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brium, and, for every profile, every Nash equilibrium outcome is Pareto optimal. (We call such outcome functions acceptable.)

The latter property is not necessarily satisfied by outcome functions possessing dominant strategies, but we regard it as highly desirable given the possibility of multiple Nash equilibria.

We find that for two person societies there are no non-dictatorial acceptable outcome functions regardless of the number of alternatives. (In the Gibbard-Satterthwaite framework, which does not require the Pareto-optimality of all Nash equilibrium outcomes, there is no impossibility result when there are only two alternatives.)

When the society consists of more than two persons, the results depend on the class of admissible profiles. If indifferences are ruled out, non-dictatorial acceptable outcome functions are shown to exist. The construction and properties of such functions constitute the main subject of our paper. On the other hand, if all profiles with indifferences are also permitted, it is shown that no acceptable outcome functions exist. However, when the requirement of Pareto optimality is replaced by that of weak Pareto optimality (with a corresponding notion of weak acceptability) it turns out that every function acceptable for strict preference profiles is weakly acceptable even where indifferences are present.

The problem studied in this paper also arises in the context of recent work in the theory of optimal resource allocation. Thus, in models of economies with public goods, Groves and Ledyard [4] have devised outcome functions guaranteeing the Pareto optimality of Nash equilibrium allocations over a certain range of preference profiles and initial endowments. Analogous outcome functions have been constructed for pure exchange economies (Hurwicz [6]). However, even for certain "classical" preference profiles, the Groves-Ledyard outcome functions either lack Nash-equilibria or generate non-optimal Nash equilibrium outcomes (Ledyard [7]), and similar phenomena are likely to appear in pure exchange models. These difficulties point to the importance of requiring the existence of Nash equilibria for all profiles of a given class and not merely the optimality of the outcomes whenever the Nash equilibria happen to exist.

The Nash equilibria obtained in our paper, like those in the work of Groves and Ledyard [4] and Hurwicz [6], are of the naive ("non-manipulative" in the terminology of [5]) type, and may not be immune to manipulation or misrepresentation of preferences. (See Farquharson [2], Gibbard [3], p. 591, and Hurwicz [5].)

Notations and Definitions

An outcome function f maps, by definition, a non-empty Cartesian product $S_1 \times \cdots \times S_n$ of finite sets into a non-empty finite set A. We refer to A as the set of outcomes and S_i as the set of strategies of person i, with $N = \{1,\ldots,n\}$ denoting the set of persons. The set of transitive total (i.e., complete and reflexive) binary relations on A is denoted by Σ'. Denoting by R a generic element of Σ', an n-list $(R_1,\ldots,R_n)$ of such elements is called a profile. The set of all profiles is denoted by Σ'^n. The subset of antisymmetric relations in Σ' is denoted by Σ. The elements of Σ^n will be called strict profiles.

An outcome function f and a profile $\underline{R} = (R_1,\ldots,R_n)$ in Σ'^n define an n-person ordinal game is strategic form. An n-list of strategies $\underline{x} = (x_1,\ldots,x_n)$ in $S_1 \times \ldots \times S_n \overset{\equiv}{\text{def}} \underline{S}$ is said to be a Nash equilibrium (NE for short) of this game if, for every i in N and for every y in S_i : $f(\underline{x})\ R_i\ f(x_1,\ldots,x_{i-1}, y, x_{i+1},\ldots,x_n)$. The outcome $f(\underline{x})$ is then called a Nash outcome. Given this game, an outcome a is said to be Pareto optimal (PO for short) if there is no other outcome b in A such that: for all i in N , bR_ia and, for some j in A , not aR_jb .

An outcome function is said to be acceptable for a set of profiles if, for every profile in this set, there is a NE and every NE is PO.

Some writers (e.g., Arrow and Hahn [1], p. 91) use a concept of what we shall call weak Pareto optimality. An outcome $a \in A$ is said to be weakly Pareto optimal if there is no outcome $b \in A$ such that $b\ P_ia$ for all $i \in N$. (bP_ia iff bR_ia and not aR_ib .)

An outcome function is said to be weakly acceptable for a set of profiles if, for each profile in the set, there is a NE and every NE is weakly Pareto optimal.

An outcome function f is said to be weakly (or β-) dictatorial if there is a person, say, j , such that for every a in A and every $\underline{x}$ in $\underline{S}$ there is a y in S_j with $f(x_1,\ldots,x_{j-1},y, x_{j+1},\ldots,x_n) = a$. It is said to be strongly (or α-) dictatorial if there is a person, say, j , such that for every a in A there is a y in S_j with $f(x_1,\ldots,x_{j-1}, y, x_{j+1},\ldots,x_n) = a$ regardless of the strategies x_k used by others.

An outcome function will be called non-dictatorial if there is no weak dictator.

Using this terminology, the problem dealt with in our paper is whether there exist non-dictatorial outcome function acceptable (or weakly acceptable) for all profiles, or all strict profiles. The question is non-trivial only if there are at least two outcomes in A, two persons in N, and each person's strategy set S_i contains at least two elements.

Summary of Results*

Theorem 1 states that for two persons a function that is acceptable for all strict profiles is dictatorial. Corollary 1 extends this result to weak profiles.

In Section 2 we provide a complete analysis of the case involving three persons, two outcomes, and two-element strategies for strict profiles. The "kingmaker" outcome function is shown to be acceptable for strict profiles, but it lacks symmetry across persons. It is shown that there exists only one type of outcome function possessing such symmetry, viz. one in which split votes yield one outcome and unanimous votes another. It should be noted that the simple majority outcome function, although symmetric, is not acceptable.

In Section 3 we deal with weak profiles (where indifferences are allowed). Theorem 2 states that for the class of all such profiles no acceptable outcome function exists. It follows, however, from Lemmas 1 and 2 of Section 3 that functions acceptable for strict profiles are weakly acceptable for weak profiles.

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*Some of the results presented in this paper were obtained independently by Eric Maskin.

Theorem 3a of Section 4 states that there is a non-dictatorial outcome function ("kingmaker") acceptable for all strict profiles regardless of the number of persons and alternatives. Theorem 3b deals with a broader class of non-dictatorial functions acceptable for all strict profiles; in those functions the person making decisions is chosen by an elective process. Examples of alternative (non-elective) hierachial structures are also given.

Since the outcome functions of Theorems 3a and 3b lack symmetry across persons, we devote Section 5 to the problem of constructing functions having this symmetry property. It is shown (Lemmas 1 and 2, and Theorem 4) that functions symmetric across persons and acceptable for an arbitrary number of persons over strict profiles can be constructed when the number of alternatives does not exceed three. The case of an arbitrary number of alternatives is not treated.

In Section 6 we consider performance functions and games in characteristic function form as means of evaluating the relative power of players.

Section 1: The Case of Two Persons

The simplest non-trivial case is one in which $A = \{a,b\}$, $N = \{1,2\}$, and $\#S_1 = \#S_2 = 2$. Here we can easily exhibit all possible outcome functions. Figure 1 contains this list, omitting outcome functions obtainable from those shown by permutations of persons, outcomes or strategies. In Figure 1 an outcome function is represented by a matrix, with the first person controlling the rows, the second person controlling the columns, and the entries representing the corresponding outcomes.

$$\underset{f_1}{\begin{matrix} a & b \\ a & b \end{matrix}} \qquad \underset{f_2}{\begin{matrix} a & a \\ a & a \end{matrix}} \qquad \underset{f_3}{\begin{matrix} a & a \\ a & b \end{matrix}} \qquad \underset{f_4}{\begin{matrix} a & b \\ b & a \end{matrix}}$$

FIGURE 1

It is clear that the outcome function f_1 is strongly dictatorial. Outcome functions f_2 and f_3 yield non-optimal Nash equilibria when both persons strictly prefer b to a . For this profile every Nash equilibrium is obviously non-optimal with f_2 . For f_3 , there is a non-optimal NE for this profile at the encircled entry (top row, left column). With f_4 outcome function, there is no NE when the two persons have opposite strict preferences. Thus only a strongly dictatorial outcome function has the two properties of existence and optimality of Nash equilibria.

Remark: Note that under f_3, which is non-dictatorial, every player always has a dominant strategy relative to his preferences; hence f_3 is non-manipulable in the Gibbard-Satterthwaite sense on Σ'^2. This shows that non-manipulability does not imply acceptability.

When there are no more than two persons the following generalization is obtained

Theorem 1 Let $n \leq 2$, and, let f be acceptable for Σ^n.

Then f is strongly dictatorial.

Remark

A special case of this result is obtained when Σ' is substituted for Σ in the statement of Theorem 1; this will be referred to as Corollary 1.

Proof Clearly, f is strongly dictatorial for $n = 1$ by definition. For $n = 2$, we shall think of outcome functions as matrices with persons I and II respectively controlling rows and columns. We first note that, given an arbitrary non-trivial partition of the set of outcomes A into B and $\bar{B}$, $(B \neq \emptyset,\ \bar{B} \equiv A \setminus B,\ \bar{B} \neq \emptyset)$ it is not the case that every row and every column contains elements of both B and $\bar{B}$. Otherwise, NE cannot exist either for profiles of the form

I	II
B	$\bar{B}$
$\bar{B}$	B .

(We represent preferences of, say, the first person by

$$\begin{array}{c} \text{I} \\ \hline B \\ \bar{B} \end{array}$$

when this person prefers every element of B to every element of $\bar{B}$.)

Therefore, there is either a B-row (i.e., a row consisting of elements of B only) or a $\bar{B}$-row or a B-column or a $\bar{B}$-column. Consider the first case, i.e., suppose that there is a B-row. We then show that there must also be a $\bar{B}$-row.

Note first that in this case there can be no B-column. Otherwise, the intersection of this column with the B-row is an element b^* of B which would constitute a non-PO NE for profiles of the form

$$\begin{array}{c|c} \text{I} & \text{II} \\ \hline \bar{B} & \bar{B} \\ b^* & b^* \\ B\setminus\{b^*\} & B\setminus\{b^*\} \end{array}.$$

Now suppose that there is no $\bar{B}$-row. Then there is no NE for profiles of the form

$$\begin{array}{c|c} \text{I} & \text{II} \\ \hline \bar{B} & B \\ B & \bar{B} \end{array}$$

or for profiles of the form

I	II
$\overline{B}$	B
B	$\overline{B}$.

For person I will always want and be able to prevent a NE with a B-outcome (since there is no B-column), while person II will always want and be able to prevent a NE with a $\overline{B}$-outcome (since, by hypothesis, there is no $\overline{B}$-row). This contradiction of existence of Nash equilibria for all profiles in Σ^2 implies that there must be a $\overline{B}$-row.

Clearly, by symmetry, it follows that, for any non-trivial partition $\{B, \overline{B}\}$ of A, there are either two rows, a B-row and a $\overline{B}$-row, or two columns, a B-column and a $\overline{B}$-column.

We now apply the last observation to partitions of the form $\{ \{a\}, A \setminus \{a\} \}$ where a is a single element of A. It follows that, for every a in A there is either a row of a's or a column of a's. This, in turn, implies that either there is a row of a's for every a in A, or there is a column of a's for every a in A.

In the first case, person I is a strong dictator; in the second case person II is a strong dictator. Q.E.D.

Remark

Although we have used the language of matrices, the proof was not limited to finite A or S_i. Hence the theorem holds for infinite outcome and strategy sets, provided there are no more than two persons.

Section 2: The Case of Three Persons with Strict Preferences

It is helpful to start by considering the simplest non-trivial subcase, that of two alternatives, $A = \{a, b\}$, with two strategies for each player. In contrast to the case of two persons, $n = 2$, we find that there are several types of outcome functions guaranteeing the Pareto optimality of all Nash equilibria and the existence of Nash equilibria for all (strict) profiles.

One such type of outcome function will be referred to as "kingmaker." An example is given in Figure 2, with person III as the kingmaker.

III

z_1

I \ II	y_1	y_2
x_1	a	a
x_2	b	b

z_2

I \ II	y_1	y_2
x_1	a	b
x_2	a	b

Figure 2

Here person I controls the rows (strategies x_1 , x_2), II controls columns (strategies y_1 , y_2) , while III has the choice between the left and right matrices (strategies z_1 , z_2).

It is clear that if III chooses strategy z_1 (the left matrix), player I acquires dictatorial powers (becomes a "king for a day"). Similarly, if III chooses z_2 , player II acquires dictatorial powers. However, it is clear that none of the three players is a dictator in the sense of our definitions. It will be shown below (Theorem 3a) that a "kingmaker" outcome function guarantees the Pareto-optimality of all Nash equilibria and their existence for any number of persons and outcomes provided all preferences are strict (no indifference), i.e., all profiles are in Σ^n .

The kingmaker outcome function may be criticized for its lack of symmetry across persons. It is therefore interesting that at least for $n = 3$, and $\#A=2=\#S_i$ for all i in N, there do exist outcome functions possessing such symmetry property. We call an outcome function f <u>symmetric across persons</u> if $f(x_i, y_j, z_k)$, i, j, k ε {1,2} is invariant with respect to permutations of i, j, k. Thus, for instance

$$f(x_1, y_2, z_1) = f(x_2, y_1, z_1) = f(x_1, y_1, z_2) ,$$

etc. (See also Section 5 below.)

In Figures 3^1 and 3^2 we show two such outcome functions, omitting the labels for strategies

b	a		a	a
a	a		a	b

Figure 3^1

a	b		b	b
b	b		b	a

Figure 3^2

We now show that, for the class of cases here considered, there are no outcome functions symmetric across persons other than those in Figures 3^1 and 3^2. First, observe than an outcome function symmetric across persons must be of the form shown in Figure 4^1. (Conventions concerning strategies are the same as in Figure 2, but strategy labels are omitted.)

x	y		y	v
y	v		v	w

Figure 4^1

Now we must require $x \neq y$, since otherwise there would arise a non-Pareto optimal NE at x when the profile is

I	II	III
$\bar{x}$	$\bar{x}$	$\bar{x}$
x	x	x

,

with $\bar{x} \neq x$, $\bar{x} \in A$. Similarly, we must have $v \neq w$. This leaves four possible assignments of the values a, b to x, y, v, w . Two of these assignments, viz. $x = a = w, y = b = v$ and $x = b = w$, $y = a = v$ yield the functions of Figures $3^{\underline{2}}$ and $3^{\underline{1}}$ respectively. The other two result in the non-existence of NE for certain profiles.

<u>Remark</u>: Note, in particular, that simple majority voting is not acceptable because it can yield non-PO Nash Equilibria. The inspection of this outcome function (see Fig. $4^{\underline{2}}$) shows that

a	a
a	b

a	b
b	b

Fig. $4^{\underline{2}}$

there is a NE at the encircled outcome, corresponding to a unanimous vote for a, regardless of the prevailing profile. Such NE is non-PO when everyone prefers b to a. There also is a NE corresponding to a unanimous vote for b (lower right-hand corner of the second matrix), which is non-PO when everyone prefers a to b.

We notice immediately that the outcome function in Figure $3\frac{2}{}$ is obtained from that in Figure $3\frac{1}{}$ by replacing a with b and *vice versa*. However, neither function by itself is symmetric with respect to outcomes, i.e., as between a and b . On the other hand it is clear that the kingmaker function is symmetric with respect to outcomes, though the precise meaning of this symmetry must still be supplied. It, therefore, becomes natural to ask whether there are outcome functions that are symmetric both across persons and across outcomes.

An outcome function is said to be *symmetric across outcomes* if, for every permutation $\pi : A \to A$ of the outcome set, there are n permutations σ_i , $\sigma_i : S_i \to S$, $i \in N$, such that

$$\pi(f(\sigma_1(x_1),\ldots,\sigma_n(x_n))) = f(x_1,,,.x_n)$$

for all $(x_1,\ldots,x_n)$ in $\underline{S}$.

Clearly, the kingmaker function is symmetric across outcomes by this definition, while the functions in Figures $3\frac{1}{}$ and $3\frac{2}{}$ are not. But we demonstrated above that the latter were the only functions (for $n = 3$, $\#A = 2$, $\#S_i = 2$ for $i \in N$) possessing the property of symmetry across persons. It follows that for this case there are no outcome functions symmetric both across outcomes and persons.

Are there other outcome functions that cannot be derived from either kingmaker or Figure 3 functions by suitable permutations ?

It turns out that there are. One is shown in Figure 5^{1}.

a	a
a	b

b	b
b	a

Figure 5^{1}

a	b
a	b

a	b
b	a

Figure 5^{2}

We note that interchanging a and b together with the permutation of the two strategies of player III leaves this outcome function unchanged. Hence we have here, as with the kingmaker function, symmetry across outcomes.

By interchanging persons II and III we obtain an outcome function (see Figure 5^{2}) that is somewhat different in appearance from that of Figure 5^{1}. It is seen that for the function of Figure 5^{2} player III, by selecting his first strategy, can give dictatorial powers to player II; however, unlike with the kingmaker function, he cannot do this for player I.

Must an outcome function by symmetric across either persons or outcomes? That the answer is in the negative is shown by yet another pair of outcome functions (equivalent by interchange of players) presented in Figures 6^{1} and 6^{2}.

a	a
a	b

b	a
a	b

Figure 6^{1}

a	b
a	a

a	a
b	b

Figure 6^{2}

Clearly, these functions are symmetric neither across persons nor across outcomes.

A systematic inspection reveals that for $n = 3$, $\#A = 2$, $\#S_i = 2$, $i \in N$, every outcome function acceptable for Σ^n

is reducible to one of the four types (kingmaker, Figure 3, Figure 5, Figure 6) by suitable permutations of persons, strategies, or outcomes.

Section 3: The Case of Unrestricted Domain Σ'^n

An Impossibility Result

Theorem 2 Let $f : \underline{S} \to A$ be an outcome function with $n \geqq 2$, $\#A \geqq 2$. If for every $\underline{R}$ in Σ'^n there is a NE, then there is a profile $\underline{R}^*$ such that a NE outcome for $\underline{R}^*$ is non-PO. I.e., for $n \geq 2$, $\#A \geq 2$, there is no f acceptable for Σ'^n.

Proof Let $\underline{R}$ be a profile of the form

I	II	III	...	n
a	b	b	...	b
b	a	a	...	a
C	C	C	...	C

where $C = A \setminus \{a, b\}$. Then only a and b are Pareto optimal. Suppose a is a NE outcome. Then consider the profile $\underline{R}^*$ of the form

I	II	III	...	n
ab	b	b	...	b
C	a	a	...	a
	C	C		C

where I is indifferent between a and b. Clearly a is still a NE outcome, but it is no longer PO.

On the other hand if b is a NE outcome then the following profile $\underline{R}^{**}$ yields a non-PO NE

I	II	III	...	n
a	ab	ab	...	ab
b	c	c	...	c
c				

If neither a nor b is a NE outcome for $\underline{R}$, then any NE outcome for $\underline{R}$ is non-PO.

Q.E.D.

Remark

Note that even a strong dictator does not guarantee the optimality of all NE's when indifferences is allowed.

Weak Acceptability

We now establish two lemmas which imply that outcome functions acceptable for Σ^n (strict preferences) are weakly acceptable for Σ'^n (allowing indifferences).

Lemma 1 Let $f : \underline{S} \rightarrow A$ be an outcome function such that for every $\underline{R}$ in Σ^n each NE outcome is PO. Then for every R' in Σ'^n each NE outcome is weakly PO.

Lemma 2 Let $f : \underline{S} \rightarrow A$ be an outcome function such that for every $\underline{R}$ in Σ^n there is an NE. Then there is a NE for every $\underline{R}'$ in Σ'^n.

Remark

In virtue of Theorems 3a and 3b below, these lemmas imply that there exist outcome functions weakly acceptable for Σ'^n when $n > 2$.

Proof of Lemma 1 Let $\underline{x}$ be a NE for a profile $\underline{R}'$ in Σ'^n, with $f(\underline{x}) = a$. Then define the profile $\underline{R}$ in Σ^n corresponding to $\underline{R}'$ as follows. For each $i \in N$, R_i is any ordering in Σ such that, for all b in A, $b \; R_i a$ if and only if $b \; P'_i a$, and $a \; R_i b$ if $a \; R'_i b$. ($b \; P'_i a$ means, as usual, $b \; R'_i a$ but not $a \; R'_i b$.)

Clearly $\underline{x}$ is still a NE for $\underline{R}$. This is so because for every person any outcome preferable to a under the ordering R_i is also preferable under the ordering R'_i.

Furthermore, by the assumption on f, there is no b in A such that $b \; R_i a$ for all $i \in N$. But by definition of $\underline{R}$ it follows that there is no b in A such that $b \; P'_i a$ for all $i \in N$. Hence a is weakly Pareto optimal.

Q.E.D.

<u>Proof of Lemma 2</u> Let Q be a fixed (arbitrarily chosen) ordering in Σ. (I.e., Q is reflexive antisymmetric.) Given a profile $\underline{R}'$ in Σ'^n, we define the corresponding profile $\underline{R}$ in Σ^n as follows. For each $i \in N$, $a\,R_i\,b$ if and only if either $a\,P_i'\,b$ or $a\,I_i'\,b$ and aQb. (Here, as usual, I_i is the symbol for indifference. I.e., $a\,I_i'\,b$ means $a\,R_i'\,b$ and $b\,R_i'\,a$.)

By our assumption on f, there is a NE, say $\underline{x}$, for $\underline{R}$. But clearly $\underline{x}$ is also a NE for $\underline{R}'$. Otherwise there would be some person $\underline{i}$ who by changing strategy could obtain an outcome strictly preferred under R_i' to $f(\underline{x})$. But the same strategy change would have been feasible and preferable under R_i, contradicting the NE property of $\underline{x}$ under $\underline{R}$.

Q.E.D.

Section 4: Existence of Outcome Functions Acceptable for Σ^n When $n > 2$

In this section we shall first show that the kingmaker outcome function introduced earlier for three persons (see Figure 2) can be defined for arbitrary $n > 2$ and turns out to be acceptable. We also construct other types of outcome functions acceptable for arbitrary large n. One category involves what we shall call indirect kingmakers; another category provides for the election of a "king for a day," i.e., someone with dictatorial powers.

Theorem 3a The following ("kingmaker") outcome function f is acceptable for Σ^n and nondictatorial for $n > 2$:

$$S_i = A \qquad \text{for all } i \text{ in } \{1,\ldots,n-1\}$$

$$S_n = \{1,\ldots,n-1\},$$

and

$$f(\underline{x}) = x_{x_n} \qquad \text{for every } \underline{x} \text{ in } \underline{S} .$$

(Note that the last equation is equivalent to

$$f(x_1,\ldots,x_{n-1}, j) = x_j ,$$

where $j = x_n$. j is the "king for a day.")

Remark

Theorem 3a makes the n-th person a kingmaker. Obviously, any person can be selected for this role. Clearly f is non-dictatorial for $n > 2$.

Proof Let $\underline{x}$ be a NE for the profile $\underline{R}$ in Σ^n, with $x_n = j$, i.e., with j-th person as "king for a day," Then $f(\underline{x}) = x_j$. Hence, by the definition of NE, $x_j R_j a$ for all $a \in A$. Since $\underline{R}$ is in Σ^n, this implies Pareto optimality of x_j. Hence every NE is PO.

Given a profile $\underline{R}$, define x_i to be the first (most preferred) outcome under the ranking R_i, $i = \{1,\ldots,n-1\}$. Then set $x_n = j$ where $x_j R_n x_i$ for $i=1,2,\ldots,n-1$. The n-list $\underline{x}$ so defined is a NE. Since j in N is the "king for a day", the outcome $f(\underline{x})$ is j's first choice under the ranking R_j. If player n (the kingmaker) were to choose someone other than j as "king for a day," the outcome would still be in the set $\{x_i : i = 1,\ldots,n-1\}$ in which x_j is the kingmaker's first choice under R_n. As for players other than j or n, their strategy change would not affect the outcome. Thus no person has an incentive to depart from his component of $\underline{x}$, and so $\underline{x}$ is a NE.

Q.E.D.

We now proceed to give an example of an <u>indirect kingmaker</u> outcome function.

Let $n = 7$, $S_1 = \{2, 3\}$, $S_2 = \{4, 5, 7\}$, $S_3 = \{5, 6, 7\}$, and $S_i = A$ for $i \in \{4, 5, 6, 7\}$. Here person 1 is the indirect kingmaker, while 2 and 3 are direct kingmakers subordinated to him. A "king for a day" will be selected from $\{4, 5, 6, 7\}$. If 1 chooses 2 as the direct kingmaker, the latter can pick a king for a day from among $\{4, 5, 7,\}$; if 1 chooses 3 as the direct kingmaker, the latter picks a king for a day from among $\{5, 6, 7\}$. Finally, the chosen king for a day selects an outcome from A. This type of outcome function is represented in Figure $7^{\underline{1}}$ as directed graph.

The proof of acceptability given for Theorem 3a carries over to this type of outcome function, which is also clearly non-dictatorial.

It is not necessary that each player appear only at one point of the graph. This is shown by a modified form of the previous function in which 1 is substituted for 7 as a potential king for a day, while retaining his indirect kingmaker powers (see Figure $7^{\underline{2}}$).

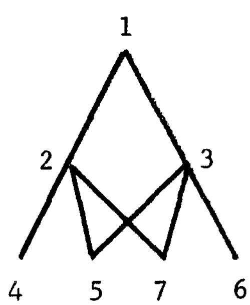


Figure $7^{\underline{1}}$

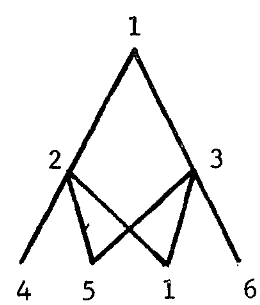


Figure $7^{\underline{2}}$

Formally, we have here $n = 6$, $S_1 = A \times \{2, 3\}$, $S_2 = \{4, 5, 1\}$, $S_3 = \{5, 6, 1\}$, and $S_i = A$ for $i \in \{4, 5, 6\}$.

We shall not try to provide a characterization of the class graphs of this type yielding acceptable non-dictatorial outcome functions.

Another class of variants of the kingmaker function yielding acceptable nondictatorial outcome functions, is given by systems in which the king for a day is chosen by an elective process. Thus, for $n \geqq 3$, let $N = N_1 \cup N_2$ where $\#N_1 \geqq 1$ and $\#N_2 \geqq 2$, and the two sets may overlap. Each member of N_1 votes for his choice of king for a day from among members of N_2. Different voting systems may be used. In one example of such a system the median of the frequency distribution of the voters' choices (under any fixed order of N_2) is elected and, in turn, chooses an element of A. (This system will be applied only when $\#N_1$ is odd.)

As another example of an admissible type of election procedure we consider the plurality principle together with the following tie-breaking rule: if the maximal number of votes is obtained by more than one member of N_2, the one designated by the smallest integer is elected king for a day.

It is clear that under either voting system any Nash equilibrium outcome will be Pareto optimal because it is the first choice of the king for a day. On the other hand an N-list of strategies in which all the members of N_1 vote for the same member of N_2 and every member of N_2 votes for his first choice in A is a NE for $\#N_1 \geqq 3$. (When the sets N_1 and N_2 overlap, those belonging to both sets simultaneously vote for a king for a day and also choose an element of A.)

Another NE, valid for $\#N_1 \geqq 2$ is obtained when all members of N_1 vote for the member of N_2 designated by the smallest (in N_2) integer.

When $\#N_1 = 1$, these voting systems are equivalent to the kingmaker outcome function of Theorem 3a. (But then the kingmaker cannot belong to N_2; otherwise he becomes a dictator.)

To give a more general formulation, we introduce the following class of outcome functions to be used in the election process.

An outcome function g with M as the set of persons, T_i as the strategy set of each i in M and B as the set of outcomes, is said to be <u>fixed at</u> an outcome b if there is an M-list $\underline{z}$ in $\underline{T}$ such that for every i in M and every v_i in T_i,

$$g(\underline{z}) = g(z_1,\ldots,z_{i-1}, v_i, z_{i+1},\ldots,z_m) = b .$$

An outcome function g is said to be <u>fixed</u> if it is fixed at some outcome.

Now consider as such an outcome function, with $N_1 = M$ and $N_2 = B = T_i$ for all i in N_1, the plurality voting system with the tie-breaking rule described above. Let j^* be the smallest integer in N_2. Then this system constitutes an outcome function fixed at j^* for $\#N_1 \geq 2$.

Furthermore, for $\#N_1 \geq 3$, both the plurality and the median voting systems are fixed at every member of N_2. A "fixing" strategy is a unanimous vote for that member.

To state the condition needed to ensure non-dictatorship we introduce the following definitions.

Given an outcome function g with domain and range as above, a person $\underline{k}$ in M is said to <u>α-dictate</u> via g an outcome b in B if there is a strategy y_k in T_k such that $g(\underline{y}) = b$ for any list of strategies $(y_j)_{j\in M\setminus\{k\}}$.

Person k is said to <u>β-dictate</u> via g an outcome b in B if for any list of strategies $(y_j)_{j \in M \setminus \{k\}}$ there is a strategy $y_k \in T_k$ yielding $g(\underline{y}) = b$.

If a person k α-dictates [resp. β-dictates] every outcome, then such a person is an α-[resp. β-dictator] as defined in the Introduction.

<u>Theorem 3b</u> Let $N = N_1 \cup N_2$, $N_1 \neq \emptyset \neq N_2$. Let g ("the voting system") be an outcome function with the set of persons (voters) N_1 , the set of outcomes (potential kings for a day) N_2 , and for each $i \in N_1$, the set of strategies denoted by T_i. Define an outcome function f with the set of persons N , the set of outcomes A, and the following strategy sets:

T_i for $i \in N_1 \setminus N_2$,

A for $i \in N_2 \setminus N_1$,

$T_i \times A$ for $i \in N_1 \cap N_2$.

For an arbitrary list of strategies

$$\underline{x} = ((y_i)_{i \in N_1 \setminus N_2} , (z_i)_{i \in N_2 \setminus N_1} , ((y_i, z_i))_{i \in N_1 \cap N_2})$$

we define the corresponding value of the function f by

$$f(\underline{x}) = z_j \quad \text{in } A ,$$

where $j = g((y_i)_{i \in N_1})$ is the elected king for a day.

Then:

I. The function f is acceptable for Σ^n if g is fixed.

II. The function f has no α-dictator (resp. β-dictator) if and only if

(i) $\#\{\text{range of } g\} \geqq 2$ and $\#A \geqq 2$,

and

(ii) $\ell \neq k$ whenever k in N_1 α-dictates (resp. β-dictates) via g the outcome ℓ in N_2.

Proof

I. Let $\underline{R}$ in Σ^n be given. Let $\underline{x}$ be a NE, say $f(\underline{x}) = z_j$, for $j = g(y)$. Clearly z_j is the first choice of j, hence $\underline{x}$ is PO.

To prove the existence of NE for $\underline{R}$, we use the assumption that the voting system g is fixed at some potential king k in N_2 via a strategy $\underline{y}$. Set z_k to be the first choice of k in A. Then any strategy list $\underline{x}$ compatible with $\underline{y}$ and z_k is a NE. Indeed k is the only person who can affect the outcome and he has no incentive to do so.

Thus f is acceptable.

II. We first show that if conditions (i) and (ii) with α [resp. (ii) with β] are satisfied then there is no α-dictator [resp. β-dictator].

Given an arbitrary i in N and an arbitrary a in A, we show that i cannot α-dictate [resp. β-dictate] a. We consider in turn three categories of persons.

1. Let $i \in N_1 \setminus N_2$. Consider any strategy list $\underline{x}$ such that

$$z_j = b \neq a \quad \text{for all} \quad j \in N_2 .$$

Then $f(\underline{x}) = b$ regardless of $y_i \equiv x_i$. Hence i does not β-dictate a. Hence he is neither a β-dictator nor an α-dictator.

2. Let $i \in N_2 \setminus N_1$. By (i), there is $\underline{y}$ such that $g(\underline{y}) \neq i$. Now consider any strategy list $\underline{x}$ such that

$$z_j = b \neq a \quad \text{for all} \quad j \in N_2 \setminus \{i\}$$

and

$$\underline{y} \text{ such that } g(\underline{y}) \neq i .$$

Then $f(\underline{x}) = b$ regardless of x_i. Hence again i is neither a β-dictator nor an α-dictator.

3. Let $i \in N_1 \cap N_2$ and let $y_i \in T_i$ be given. Consider $\underline{x}$ such that

$$z_j = b \neq a \quad \text{for all} \quad j \in N_2 \setminus \{i\}$$

and

$$(y_k)_{k\in N_1\setminus\{i\}} \quad \text{such that} \quad g(\underline{y}) \neq i \ .$$

(The existence of such $(y_k)_{k\in N_1\setminus\{i\}}$ follows from (ii) with α .) Then $f(\underline{x}) = b$ and i is not an α-dictator. On the other hand, if (ii) with β is assumed, there exists a strategy list

$$(y_k)_{k\in N_1\setminus\{i\}}$$

such that

$$g(\underline{y}) \neq i \quad \text{regardless of} \quad y_i \ .$$

Let $\underline{x}$ be such that $z_j = b \neq a$ for all $j \in N_2 \setminus \{i\}$ and $(y_k)_{k\in N_1\setminus\{i\}}$ as above. Then $f(\underline{x}) = b$ regardless of of x_i and so i is not a β-dictator.

To complete the proof we show that the violation of either of condition (i) or (ii) yields the corresponding type dictator.

If the range of g is a one element set, say ℓ in N_2 , then clearly ℓ is an α-dictator (and hence a β-dictator).

Now let k in $N_1 \cap N_2$ dictate himself. Then he can dictate any outcome a by choosing a strategy (y_k, z_k) such that

$$z_k = a$$

and y_k yielding $g(\underline{y}) = k$ regardless of $(y_r)_{r\in N_1\setminus\{k\}}$.

On the other hand, suppose k in $N_1 \cap N_2$ β-dictates himself. Then given any $(y_r)_{r\in N_1\setminus\{k\}}$ and any $(z_r)_{r\in N_2\setminus\{k\}}$, person k has a strategy y_k such that $g(y) = k$. By combining y_k with $z_k = a$, person k can β-dictate any outcome a. Hence he is a β-dictator.

Q.E.D.

To illustrate the conditions required to ensure non-dictatorship, we consider two examples.

Example 1. Let $N_2 = \{1, 2\}$ and $\#N_1 = 2$, say $N_1 = \{i, j\}$, $i \neq j$. Let the plurality voting system prevail, with candidate designated by the lower number winning in case of a tie. For such g, person 1 is a fixed outcome of the voting system that can be dictated by either i or j, while 2 cannot be dictated by anybody. It then follows from condition (ii) that $1 \notin N_1$ and any membership of N_1 excluding 1 would ensure non-dictatorship. E.g., we could have $N_1 = \{2, 3\}$ or $N_1 = \{3, 4\}$.

Example 2. Let $N = N_1 = N_2 = \{1, 2, 3\}$ and let g be the plurality voting system with the above tie-breaking rule. Here every participant is both a voter and a potential king for a day, but the tie-breaking rule introduces an asymmetry across persons. It can be verified that both conditions (i) and (ii) above are satisfied and hence f is acceptable and non-dictatorial.

Remark

It is possible to combine the elective elements of Theorem 3b with the indirect appointive elements of the type represented by the graphs in Figures 7 above to yield non-dictatorial acceptable outcome functions.

Section 5: Outcome Functions Symmetric Across Persons

In Section 2 above, we discussed the problem of existence of symmetric outcome functions. It was seen that, in general, acceptable non-dictatorial functions that are symmetric both across outcomes and across persons do not exist. It was shown that when $n=3$, $\#A = 2$, $\#S_i = 2$ for all $i \in N$, outcome functions symmetric across persons do exist. We shall now show that this result can be generalized.

An outcome function $f : S_1 \times \ldots \times S_n \to A$ is said to be <u>symmetric across persons</u> if and only if:

(1) $S_1 = S_2 = \ldots = S_n$

(2) for any permutation π of the set N, we have
$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

It follows that, for $S_1 = S_2 = \ldots S_n$, f is symmetric across persons if and only if: for every $s \in S_o \equiv S_1 = \ldots = S_1$,

$$\#\{i \in N: \; x_i = s\} = \#\{i \in N: \; y_i = s\} ,$$

implies

$$f(\underline{x}) = f(\underline{y})$$

where $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_n)$

We shall now restate this in terms of what may be called the distribution of votes. Let N and S_o be fixed (so that n is also fixed), and write $\underline{S} = S_o \times \ldots \times S_o$ (n-fold Cartersian product). Then, for each $\underline{x} \in \underline{S}$, there exists a unique distribution $p_{\underline{x}}$ on S_o given by

$$p_{\underline{x}}(s) = \#\{i \in N : \; x_i = s\} \qquad \forall\, s \in S_o \,.$$

Denoting by J the set of nonnegative integers, we shall write

$$D_n(S_o) = \{p \in J^{S_o} : \sum_{s \in S_o} p(s) = n\} \,.$$

We shall denote by d the mapping, $d : \underline{S} \to D_n(S_o)$ associating with an element $\underline{x}$ of $\underline{S}$ the corresponding distribution $p_{\underline{x}}$.

We see that f is symmetric across persons if and only if: $\forall \underline{x}\,,\, \underline{y} \in \underline{S}$,

$$d(\underline{x}) = d(\underline{y}) \text{ implies } f(\underline{x}) = f(\underline{y}) \,.$$

Therefore, an outcome function f is symmetric across persons if and only if there exists a function $g : D_n(S_o) \to A$ such that $f = g \circ d$.

Note that for outcome functions symmetric across persons $\#S_1 = \#S_2 = \ldots = \#S_n$. It is clear that one cannot hope to find acceptable outcome functions with $\#S_i < \#A$. On the other hand if acceptable functions are found with $\#S_i = \#A$, one can trivially enlarge the strategy spaces to any desired extent. It is therefore natural to seek outcome functions with $\#S_i = \#A$. Since in cases we have investigated such functions do exists, there is no loss in assuming $\#S_i = \#A$ for all $i \in N$. In fact, we may take $S_o = A$.

We do not at this time have a result for arbitrary $\#A$. In what follows, we shall confine ourselves to separate treatments of two special cases: $\#A = 2$, and $\#A = 3$.

The Case $\#A = 2, n \geq 3$

As indicated above, we shall in this subsection require $S_o = S_i = A$ for all i, so that $\#S_i = \#A = 2$ for all $i \in N$. Therefore, a symmetric across persons outcome function f will be obtained if we find a function $g : D_n(A) \to A$.

Write $A = \{a, b\} = S_o$. An element of $D_n(A)$ is uniquely characterized by the pair of integers $(p(a) , p(b))$, or, since n is given, by $p(a)$ alone. Hence, for a given n , a function $g : D_n(A) \to A$ is represented by the sequence

$$< g(0) , g(1),\ldots,g(n) >$$

where $g(i)$ is the outcome for $p(a) = i$.

In this notation, the symmetric outcome function of Figure $3\frac{1}{}$ is represented by the sequence $< b, a, a, b >$.

We shall now characterize the functions g , for $n \geqq 3$, yielding acceptable outcome functions $f = g \circ d$, so that f is symmetric across persons.

Lemma 1 Let $n \geqq 3$. Every NE is PO for all profiles in Σ^n if and only if the function g has the following property:

$$(*) \begin{cases} g(0) \neq g(1) \text{ ; } g(n-1) \neq g(n) \text{ ; and, for all} \\ 0 < i < n \text{ , either } g(i+1) \neq g(i) \text{ or } g(i-1) \neq g(i) \text{ .} \end{cases}$$

Proof (i) Sufficiency. Let $\underline{x}$ be a NE strategy list inducing the distribution $p_{\underline{x}}(a) = i$ and $p_{\underline{x}}(b) = n - i$, and suppose that every person in N prefers the other outcome to $g(i)$. Then by condition (*) of the Lemma there is always some person who can swing the outcome to either $g(i-1)$ or $g(i+1)$ which he would prefer.

(ii) Necessity. Suppose that for some $0 \leqq i \leqq n$, $g(i)$ violates condition (*) of the Lemma. Consider the profile where everyone prefers the other outcome to $g(i)$. Then a strategy list where i persons vote a and $n - i$ vote b yields the non-optimal outcome $g(i)$. However, this strategy list is a NE, since a change in one person's vote cannot swing the outcome to anything different from $g(i)$.

Lemma 2 Let $n \geqq 3$. Under condition (*) of Lemma 1, there exists a NE for every profile in Σ^n if and only if

either

$$(**) \qquad g(1) = g(2) \quad \text{or} \quad g(n-2) = g(n-1)$$

or

$$(***) \qquad \begin{cases} \exists j \text{ such that } g(j) = g(j+1) = a \\ \text{and} \\ \exists k \text{ such that } g(k) = g(k+1) = b . \end{cases}$$

Proof (i) Sufficiency. For the unanimous profile case, a NE always exists, since by (*) the range of g is the whole set A .

When the profile is not unanimous, we consider in turn the cases where (**) and (***) hold.

Suppose (**) holds with $g(1) = g(2)$. Since the profile is not unanimous there is at least one person preferring $g(1)$ to $g(0)$. The strategy list where such a person votes a and everyone else votes b is a NE with the outcome $g(1)$. Indeed this person has no reason to change his vote, and a vote change by anyone else would yield $p(a) = 2$ and so not affect the outcome.

By (**), if $g(1) \neq g(2)$, we have $g(n-1) = g(n)$, and the proof is analogous.

Suppose now the profile is not unanimous and (***) holds. Denote by m the number of persons preferring a to b .

Case 1 There exist j and k satisfying (***) such that $j < k$.

Subcase 1.1: $m \geqq j$. The following strategy list yields a NE here: j persons preferring a to b vote a , while everyone else votes b . This is a NE because no a-voter has an incentive to switch his vote to b and so to swing the outcome to $g(j-1)$, which by (*) equals b , and nobody else can affect the outcome since $g(j + 1) = g(j)$.

Subcase 1.2: $m < j$. Here $n - m > n - k$, since $k > j$. The following strategy list yields a NE: $n - k$ persons preferring b to a vote b , while everyone else votes a . The argument is like that in the preceeding subcase.

Case 2 $j > k$. This can be reduced to Case 1 by exchanging the names a and b .

(ii) Necessity. Suppose that (*) is satisfied but neither (**) nor (***) hold. This means that in the sequence $< g(0) , g(1),\dots,g(n) >$ either $g(i) = g(i + 1) = a$ is absent or $g(i) = g(i + 1) = b$ is absent, and if, for some i , $g(i) = g(i + 1)$, then $i \notin \{0, 1, n - 2, n - 1\}$.

Suppose first that $g(i) = g(i + 1) = b$ is absent. Then there is no NE for the profile where person 1 prefers a to b and everyone else prefers b to a . This is seen as follows. Suppose first that those preferring b to a split their votes between a and b . If the outcome is a , there is someone in this group who by changing his vote can swing the outcome to an outcome b adjacent in the g-sequence $< g(0),\dots,g(n) >$. If the outcome is b , then person 1 can swing the outcome to a , because on either side of an interior b in the g-sequence there is an a , and we cannot be at an endpoint of the g-sequence when the votes of those preferring b are split.

Now suppose that all those preferring b to a (i.e., persons $2,\ldots,n$) vote the same way. Here the outcome must be one of the following: $g(0)$, $g(1)$, $g(n-1)$, or $g(n)$. When (*) holds but neither (**) nor (***) do, the g-sequence begins either with $\langle g(0), g(1), g(2) \rangle = \langle a, b, a \rangle$ or with $\langle g(0), g(1), g(2) \rangle = \langle b, a, b \rangle$, and ends either with $\langle g(n-2), g(n-1), g(n) \rangle = \langle a, b, a \rangle$ or with $\langle g(n-2), g(n-1), g(n) \rangle = \langle b, a, b \rangle$.

A direct check shows that in none of these cases is there a NE yielding any of the outcomes $g(0)$, $g(1)$, $g(n-1)$, or $g(n)$ for the above profile.

The proof for the case where $g(i) = g(i+1) = a$ is absent is obtained by exchanging the names a and b. Q.E.D.

Lemmas 1 and 2 provide a complete characterization of outcome functions that are acceptable and symmetric across persons for the case $\#A = \#S_i = 2$ for all i in N, $n \geqq 3$.

For $n = 3$, the two Lemmas imply that the outcome functions in Figures $3\frac{1}{}$ and $3\frac{2}{}$ are the only ones possible. It is obvious that for every $n \geqq 3$ there exist functions satisfying the conditions of the two Lemmas. The following is one example (with (**) satisfied):

$$g(0) = g(2r+1) = a \qquad \text{for all } r = 1,2,\ldots,[\tfrac{n-1}{2}],$$

and

$$g(i) = b \qquad \text{otherwise,}$$

where $[x]$ is the largest integer not exceeding x.

The smallest number of persons where (*) and (***) can be satisfied with (**) violated is $n = 7$. An example is given by the following function:

$$g(0) = g(2) = g(3) = g(6) = a \ ,$$

$$g(1) = g(4) = g(5) = g(7) = b \ .$$

Some General Considerations When $\#A \geqq 2$, and $n \geqq 3$

The method of analysis used above for $\#A = 2$ can be generalized for bigger sets of outcomes, with $S_o = A$.

We start by defining the set of distributions accessible from a given distribution by (at most) a single person's vote switch.

Definition For every $\bar{q}$ in $D_n(A)$, we shall write

$$\mathcal{N}(\bar{g}) = \{q \in D_n(A) : \forall a \in A \ , \ |q(a) - \bar{q}(a)| \leqq 1 \ , \text{ and}$$

$$\#\{a \in A : q(a) \neq q(\bar{a})\} \leqq 2\} \ .$$

Clearly, $\forall q$, $\bar{q} \in D_n(A)$, if $q \neq \bar{q}$ and $q \in \mathcal{N}(\bar{g})$, then q can be reached from $\bar{q}$ by one person switching his vote. Since, for any $q \in D_n(A)$, $\sum_{a \in A} q(a) = n$, it follows that $|q(a) - \bar{q}(a)| = 2$ for $q \neq \bar{q}$.

We now state a generalization of Lemma 1 above.

Lemma 3 For every $n \geqq 3$, $\#A \geqq 2$, and for every profile in Σ^n, each NE is PO if and only if

$$(\#) \qquad \forall q \in D_n(A) \ , \ q \ (\mathcal{N}(g)) = A \ .$$

Remark

The "accessibility condition" (#) is an extension of Condition (*) in Lemma 1.

Proof (i) Sufficiency. Suppose that $\bar{q}$ is a distribution induced by a NE strategy list for some profile in Σ^n. If $g(\bar{q})$ were not PO for this profile, there would be another outcome preferred by everyone, and this outcome is accessible from $g(\bar{q})$ by a single vote switch in virtue of Condition (#). This contradicts the supposition that $\bar{q}$ is NE.

(ii) Necessity. Suppose that the Condition (#) is violated so that there is a $\bar{q}$ in $D_n(A)$ and $a \in A$ such that $a \notin g(\eta(\bar{q}))$.

Consider a profile in which everyone prefers a as his first choice and $g(\bar{q})$ as his second choice. Then the strategy list inducing $\bar{q}$ in a non-PO Nash Equilibrium.

Q.E.D.

The Case $\#A = 3$, $n \geq 3$

Here, again, we take $S_o = S_i = A$ for all i, so that $\#S_i = \#A = 3$. In this case, it is possible to provide a simple representation of the function g by means of the following two-way table where the entry in the i-th row, j-th column represents the value of g when a receives i votes, b receives j votes and c receives $n-i-j$ votes, $i,j \geqq 0$, $i + j \leqq n$, and we write $A = \{a,b,c\}$.

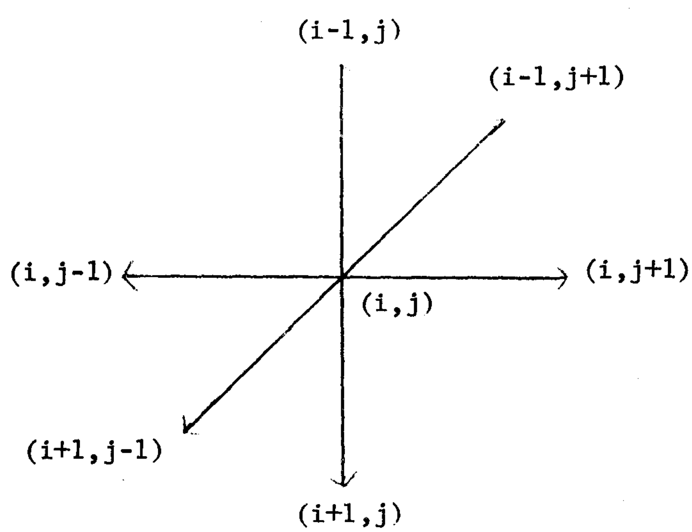
no. of votes for a \ no. of votes for b	0	1	n-1	n
0				
1				X
n-1				X
n		X	X	X

(X indicates a voting pattern that is impossible because $i + j > n$.)

The accessible set $\mathcal{N}(q)$ can here be written as $\mathcal{N}(i,j)$ and is given by

$$\mathcal{N}(i,j) = \{(k,\ell) : |k - i| \leqq 1 , |\ell - j| \leqq 1 , |k + \ell - i - j| \leqq 1 ,$$
$$k \geqq 0 , \ell \geqq 0 , k + \ell \leqq n\}$$

and can be represented graphically as follows



(It is understood that all points indicated by arrows are accessible provided their indices are non-negative and do not add up to more than n .)

The construction for $n \geqq 3$, $\#A = 3$, is accomplished by first obtaining acceptable g's for $n = 3, 4$ and then providing an algorithm of extension for arbitrary $n > 4$.

The following table gives a function g acceptable for $n = 3$, $\#A = 3$. (It may be noted that other g functions, not obtainable from considerations of symmetry, have also been found.)

votes for a \ votes for b		0	1	2	3
0		a	b	b	c
1		c	a	a	X
2		c	a	X	X
3		b	X	X	X

It may be verified that for every entry (i,j) in this table the accessible set $\eta(i,j) = \{a,b,c\}$. Hence, every NE is PO for all profiles in Σ^3. It remains to show that this g possesses a NE for every profile in Σ^3.

Since the range of g is A , NE's exist for all profiles with unanimity as to the first choice.

If at least one person has a as first choice, the following voting pattern yields a NE: one person, whose first choice is a , votes for a , while the other two vote for b . The a-voter has no incentive to switch his vote since the outcome at $(1,2)$ is a, which is his first choice; neither b-voter can affect the outcome by switching his vote either to a or to c . Hence there is a NE at $(i,j) = (1,2)$, with a as outcome.

Suppose a is nobody's first choice and there is no unanimity as to first choice.

Consider first the case where a is undominated (hence PO). Then there exist two (distinct) persons, say i and j , such that i prefers a to c and j prefers a to b . Here the following voting patterns yields a NE at the (1,1) entry in the above table: let i vote for b , j for a , while the third person votes for c .

Second, let a be dominated, say by b . Then, since we are at present assuming the absence of unanimity as to first choice, there must be a person whose first choice is b . A NE at (0,1) is therefore obtained when this person votes b while the others vote c . If, on the other hand, a is dominated by c , there is an analogous NE at (1,0) with a person whose first choice is c voting for a while others vote for c .

Thus, we have shown that the function g in the above table is acceptable for all profiles in Σ^3 .

When there are four persons $(n = 4)$, with #A remaining at 3, the following g function will be shown to be acceptable.

	0	1	2	3	4
0	b	c	b^{II}	b	c
1	a^{I}	a	b	a	X
2	a	c^{III}	c	X	X
3	b	c	X	X	X
4	a	X	X	X	X

The Pareto optimality of all NE's is proved here as it was for $n = 3$. As for existence, we distinguish 3 types of profiles. First, let a be at least one person's first choice. Then the entry $(i,j) = (1,0)$, marked I, yields a NE when such a person votes a and others vote c. If a is nobody's first choice, suppose b is the first choice of at least two persons. For such profiles, a NE exists at the entry marked II (i.e., at $(0,2)$) when such two persons vote b and the others vote c.

For the remaining profiles at least three have c as their first choice. Then NE are attained at the entry marked III (i.e., $(2,1)$) when two of such three persons vote a, one votes b, and the fourth person votes c. Hence the function g of the above table is acceptable for all profiles in Σ^4.

Note that all of the locations of the NE's mentioned above (I, II, III) satisfy the condition $i + j < 4$. In general, for any n, we shall say that a function g has the "<u>interiority property</u>" if, for every profile in Σ^n, there is a NE at a location (i,j) with $i + j < n$. (Recall that i and j denote respectively the numbers of votes cast for a and b.) So, for instance, the above g for $n = 4$ has the "interiority property," while that for $n = 3$ does not.

Although the preceding argument only shows the existence of acceptable functions g for $n = 3, 4$, $\#A = 3$, it turns out that these results can be used to construct acceptable g functions for all $n \geqq 3$ as long as $\#A = 3$. Formally, we then obtain the following

<u>Theorem 4</u>: If $\#A = 3$, then for every $n \geq 3$, there exists a function g acceptable for all profiles in Σ^n.

Proof For $n = 3,4$ the Theorem has already been proved by construction. For $n > 4$ the proof is carried out by induction whose validity is established in the Induction Step Lemma below.

Before stating the Lemma, we show for illustrative purposes how the induction step is carried out for $n = 4$.

The problem is to fill the blank spaces (forming what we shall call the "new diagonal"(*)) in the table below in such a way that both the existence and Pareto-optimality properties present for $n = 4$ are preserved.

	0	1	2	3	4	5
0	b	c	b*	b	c^{Δ}	
1	a*	a	b	a^{Δ}		X
2	a	c*	c^{Δ}		X	X
3	b	c^{Δ}		X	X	X
4	a^{Δ}		X	X	X	X
5		X	X	X	X	X

In this table, the letter entries define the function g for $n = 4$ which is acceptable and with "interior" NE's for all profiles in Σ^4 marked by asterisks.

It can be verified that any extension $\tilde{g}$ of this g (i.e., any way of filling of the new diagonal) will possess "interior" NE's for all profiles in Σ^5, since the NE's for $n = 4$ retain their "interior" NE status for $n = 5$, and there are enough of them for

(*) To be distinguished from the "old (boundary) diagonal," marked by triangles in the above table.

all the new profiles. So the problem reduces to that of assuring the Pareto-optimality of all the NE's that appear in the extension $\tilde{g}$. It may be verified that the extension presented in the following table satisfies this requirement, and is therefore acceptable for $n = 5$; also it has "interior" NE's (marked by asterisks)

a \ b	0	1	2	3	4	5
0	b	c	b*	b	c	a
1	a*	a	b	a	b	X
2	a	c*	c	a	X	X
3	b	c	b	X	X	X
4	a	b	X	X	X	X
5	c	X	X	X	X	X

for all profiles in Σ^5 .

Induction Step Lemma For $\#A = 3$ and $n \geqq 4$, let $g : D_n(A) \to A$ be acceptable for all profiles in Σ^n and such that for every profile in Σ^n there is an "interior" NE. Then there exists an extension $\tilde{g} : D_{n+1}(A) \to A$ of g having the corresponding properties for all profiles in Σ^{n+1} .

Proof First we show that any extension $\tilde{g}$ of g to $D_{n+1}(A)$ possesses a NE for each profile in Σ^{n+1} and has the interiority property.

Let a profile in Σ^{n+1} be given. Consider the profile in Σ^n obtained by deleting (the preferences of) one of the $n + 1$ persons. By hypothesis the function g has a NE at some (i,j), with $i + j < n$. We then obtain a NE of $\tilde{g}$ for the original profile at the same location (i,j) when the "deleted" person votes c and the others retain the strategies used in the n-person game. These strategy choices yield a NE at (i,j) because the "deleted" person cannot affect the outcome (since $g(i,j + 1) = g(i + 1, j) = g(i,j))$, and the others either cannot or have no incentive to destroy the NE status of (i,j) in the n-person game. Also, for the n+1-game the interiority property is preserved since $i + j < n < n + 1$.

It remains to show that there exists an extension $\tilde{g}$ for which all NE are Pareto-optimal. By Lemma 3, we must require that, for any q in $D_{n+1}(A)$, $\tilde{g}(\tilde{\eta}(q)) = A$. (Here $\tilde{\eta}(q)$ denotes the "neighborhood" of q in the $(n + 1)$ by $(n + 1)$ table, i.e., in the domain of $\tilde{g}$; we shall write $\eta(q)$ to denote the "neighborhood" of q in the $n \times n$ table, i.e., in the domain of g.)

Note that for q represented by (i,j) such that $i + j \leqq n$, we have $g(\eta(i,j)) = A$ by the assumed acceptability of g. Since $\eta(i,j) \subseteq \tilde{\eta}(i,j)$, it follows that $\tilde{g}(\tilde{\eta}(i,j)) = A$. Therefore, we need only verify that there exists an extension $\tilde{g}$ such that, for every i,j with $i + j = n + 1$ (i.e., on the new diagonal), $\tilde{g}(\tilde{\eta}(i,j)) = A$. We shall establish this by induction on i.

For $i = 0,1$, the extension $\tilde{g}$ is defined by

$$\tilde{g}(0,n + 1) = \tilde{g}(1,n - 1) \equiv g(1,n - 1)$$

$$\tilde{g}(1,n) = \tilde{g}(0,n - 1) \equiv g(0,n - 1) .$$

First note that $\eta(0,n) = \{(0,n), (0,n-1), (1,n-1)\}$ and hence, by Lemma 3, $\{g(0,n), g(0,n-1), g(1,n-1)\} = A$. Therefore, $\tilde{g}(\tilde{\eta}(0,n+1)) = A$. Furthermore, $\tilde{g}(\tilde{\eta}(1,n) \setminus \{(2,n-1)\}) = A$, and also $\tilde{g}(1,n) \neq \tilde{g}(1,n-1)$. Proceeding by induction, suppose that we have defined the values for $\tilde{g}(i,n+1-i)$ up to and including $i = k \leqq n$, in such a manner that $\tilde{g}(\tilde{\eta}(i,n+1-i)) \setminus \{(i+1, n-i)\}) = A$ and $\tilde{g}(k,n+1-k) \neq \tilde{g}(k,n-k)$. Then we define $\tilde{g}(k+1, n-k)$ as follows.

If

$$g(k+1, n-k-1) = g(k, n-k)$$

or

$$g(k+1, n-k-1) = \tilde{g}(k, n+1-k)$$

then set $\tilde{g}(k+1, n-k)$ to be the unique element in A different from both $g(k, n-k)$ and $\tilde{g}(k, n+1-k)$.

On the other hand, if

$$g(k+1, n-k-1) \neq g(k, n-k)$$

and

$$g(k+1, n-k-1) \neq \tilde{g}(k, n+1-k)$$

then set

$$\tilde{g}(k+1, n-k) = g(k, n-k) .$$

Finally, for $k = n$, apply the first alternative (i.e., set $\tilde{g}(n+1,0)$ equal to the unique element in A different from both $g(n,0)$ and $\tilde{g}(n,1)$).

Clearly, the two properties, $\tilde{g}(k + 1, n - k) \neq \tilde{g}(k + 1, n - k - 1)$ and $\tilde{g}(\tilde{\eta}(k + 1, n - k)) \setminus \{(k + 2, n - k - 1)\} = A$, are satisfied for $k < n$; and, when $k = n$, $\tilde{g}(\tilde{\eta}(k + 1, n - k)) = A'$.

Q.E.D.

Section 6: Performance Correspondences and Characteristic Functions.

In general, a choice correspondence associates a subset of the set of outcomes with a given profile.

Given an outcome function f, its performance correspondence φ is defined as the choice correspondence which associates with a profile $\underline{R}$ the set of NE of f for $\underline{R}$. Thus $\varphi(\underline{R})$ is the set of NE of the ordinal game $(f, \underline{R})$.

Clearly f is acceptable on a class of profiles containing $\underline{R}$ if and only if $\varphi(\underline{R})$ is nonempty and is contained in the $\underline{R}$-Pareto-optimal subset of A for every $\underline{R}$ in the class. A choice correspondence ψ is said to be dictatorial if there is exactly one person in N for whom, given any profile $\underline{R}$ in the domain of ψ, the set $\psi(\underline{R})$ contains an element a in A such that $aR_i b$ for all b in A.

Clearly, if an outcome function is dictatorial, so is its performance correspondence. It is also obvious that the performance correspondence associated with an outcome function that is symmetric across persons is not dictatorial. We have not determined whether, in general, a non-dictatorial outcome function acceptable on Σ^n generates a non-dictatorial performance correspondence. It can easily be seen, however, that the outcome function in Theorem 3a generates non-dictatorial performance correspondences. In the case of the kingmaker outcome function (Theorem 3a), for each person there is a profile such that the only NE outcome is his least preferred alternative: (An example; the given person's last choice is the first choice of all others). In the case of Theorem 3b, we restrict our attention to voting systems g that are fixed at all points of the set N_2 of potential kings. Then for every profile $\underline{R}$, the value $\psi(\underline{R})$ of the performance correspondence contains the first choices of all potential kings for a day. Hence the uniqueness requirement in the definition of a dictatorial choice correspondence is violated.

We may note that had the uniqueness requirement been deleted from the definition of a dictatorial choice correspondence (thus making the class of non-dictatorial correspondences smaller), the performance correspondence associated with the kingmaker outcome function would have remained non-dictatorial.

Furthermore, no person is a "dummy" in the kingmaker outcome function. (The i-th person is called a performance correspondence <u>dummy</u> if on the domain of φ, the set $\varphi(\underline{R})$ is dependent of R_i.)

Note however, that for outcome function of Theorem 3b, with N_1 and N_2 disjoint, every person in N_1 is a dummy of the associated performance correspondence - without necessarily being a dummy of the outcome function. For example, if the fixed function g is that of plurality voting with some tie-breaking rule (see Section 4 above), then no person is a dummy of the outcome function. Yet $\varphi(\underline{R})$ consists of first choices of potential kings for a day (members of N_2), and hence is independent of the preferences of the members of N_1.

Dictatorship and dummy status constitute the extremes of power positions of game participants. An alternative approach to the analysis of power relationships is to investigate games in von Neumann and Morgenstern characteristic function form corresponding to a given outcome function.

First, for an outcome function $f:S \to A$, we define the characteristic function $v_\alpha(\cdot)$ as follows: for a set of persons $M \subset N$, $v_\alpha(M) = 1$ if and only if the coalition M α-dictates every outcome a in A; otherwise we set $v_\alpha(M) = 0$. The characteristic function $v_\beta(\cdot)$ is defined analogously. (A coalition M is said to α-dictate the outcome v if there is an M-list of strategies yielding the outcome v regardless of the strategies chosen by those outside the coalition M. A coalition M is said to β-dictate the outcome a if for any choices of strategies by persons outside M there are strategies for members of M yielding a.)

Characteristic function $v(\cdot)$ is said to have a dictator if there is a player i such that, for every coalition M, $v(M) = 1$ if and only if i belongs to M. It follows from our definitions in the Introduction that an outcome function f has an α-dictator if and only if v_α has a dictator.

It is also evident that an outcome function dummy is necessarily a characteristic function dummy (in the sense that he does not affect the value of a coalition by joining it). It is not known, however, whether the converse is true.

An advantage of the characteristic functions approach is that it reveals certain symmetries that are not otherwise apparent. Thus, for instance, in the three-person kingmaker outcome function of Section 4, the characteristic function has the values $v_\alpha(M) = v_\beta(M) = 1$ for $\#M \geq 2$ and zero otherwise. Hence there is symmetry of the characteristic function with respect to players even though the outcome function is not symmetric across persons.

Another example of a symmetric characteristic function associated with an asymmetric outcome function is a special case covered by Theorem 3b where $N_1 = N_2$, $\#N_1$ is odd, and voting is by plurality with some tie-breaking rule.

Clearly, the tie-breaking rule makes f asymmetric. However, both v_α and v_β are those of a simple majority game and hence are symmetric.

A more general notion of a characteristic function can be introduced as follows. We define $V_\alpha : M \to B$ where $M \subseteq N$ and $B \subseteq A$, where $b \in V(M)$ if and only if M α-dictates b. V_β is defined analogously. It is seen that $V_\alpha (V_\beta)$ is more informative than $v_\alpha (v_\beta)$ and thus its symmetry properties may be regarded as more plausible indicators of equal power of the players.

So far we have been discussing the construction of characteristic functions given the outcome functions. It is of considerable interest to consider the reverse process of finding outcome functions that will "implement" given characteristic functions.

For instance, a simple majority characteristic function game for n persons (n odd) can be "implemented" by an outcome function of the type described in Theorem 3b, with $N_1 = N_2 = N$, $\#N = n$, and with plurality voting system and some tie-breaking rule. The use of such an outcome function provides a well-defined procedure for realizing the given characteristic function, with a guarantee that the outcome will necessarily be Pareto-optimal. It would be of interest to develop a method of specifying outcome functions for broader classes of characteristic function.

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