EXPECTATIONS EQUILIBRIUM AND INFORMATIONAL EFFICIENCY FOR STOCHASTIC ENVIRONMENTS

by

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1. Introduction

This paper explores the relationship between two distinct approaches to the theory of information in general competitive equilibrium. One approach, initiated by Radner, [13] and [14], is through the theory of competitive equilibrium under uncertainty. In this framework, the exogenous state of the world is random, and agents have state dependent objectives. Each agent has an exogenously given partition of the set of states. An equilibrium price system, which associates an equilibrium price with each state, also generates a partition. In [14], Radner discussed the case in which agents use this partition to augment their individual information, and emphasized the efficiency of equilibrium when the price partition is at least as fine as every agent's private partition.

More recently, the information communicated by prices was studied in a different context by Green [2]. The model investigated by Green was one in which some agents were exogenously informed of the state and others conditioned their expectations on the equilibrium price. Conditions were established under which any equilibrium price system communicates complete information. Further results of this nature were obtained in different contexts by Grossman [4] and Kihlstrom and Mirman [9].

However, attempts to establish the general existence of price conditional expectations equilibria produced counterexamples (Green [3] and Kreps [10]). These examples rest on an informational discontinuity which was mentioned by Radner [14, p. 51]. In [8], I showed that difficulties were liable to occur whenever expectations were conditioned on endogenous variables. A striking exception to the general negative result of that paper is the case in which each agent's expectations are conditioned on the information generated by his allocation and the equilibrium price. Any equilibrium under complete information is sustained under this information structure [8, p. 11-12].

This result suggests that the expectations equilibrium problem is related to the theory of informationally decentralized allocation mechanisms, initiated by Hurwicz [7]. An allocation mechanism includes a message process by which agents communicate information about the environment. A major goal of this theory is the characterization of the minimal amount of information which must be conveyed by a decentralized message process in order for the resulting allocations to meet certain requirements. The amount of information conveyed by a message process has been studied in terms of various concepts of the "size" of the space of equilibrium messages (Hurwicz [5] and [6], Mount and Reiter [11], Osana [12], and Walker [17]), and in terms of the fineness of the partition generated by the map from environments to equilibrium messages (Hurwicz [7] and Reiter [16]).

The competitive message process, as formalized by Mount and Reiter [11], communicates competitive equilibrium prices and individual net trades. Thus the above-mentioned existence of complete information equilibria is obtained if each agent uses the information generated by his own endowment and the competitive message. This observation suggests that the concept of expectations equilibrium can be extended to general message processes by stipulating that the relevant utility function for each agent is his expected utility function conditioned on private information and the information generated by the equilibrium message.

The object of this paper is the study of message processes for which expectations equilibria generally exist. Such processes are termed "admissible". In section 3, we obtain a necessary condition for admissibility which is close to the decentralization condition known as "privacy". Sections 4 and 5 provide two results on the informational minimality of the competitive message process. In section 4, it is shown that any admissible message process which supports nonwasteful allocations uses a message space which is locally at least as large as the competitive message space. Section 5 is devoted to the case in which expectations are based on partial data computed from the competitive message. The main result of this paper is that admissibility requires that if the competitive message is condensed at all, it must be condensed to a constant. This result implies that the partition generated by the competitive message process is minimal among partitions generated by continuous, nonconstant, privacy-preserving message processes. Section 6 contains a discussion of the amount of information agents need in order to associate the elements of the message partition with the respective messages.

Throughout the paper, the analysis is confined to the class of Cobb-Douglas exchange environments with two states of the world. With the exception of section 6, the main results are directly inherited by any set of stochastic environments which contains this set. This inheritance property is discussed more specifically below, and the strength of the two event case is also discussed in [8]. However, the model used here will not be acceptably general until it is expanded to include production decisions and a careful treatment of financial commodities.

2. Definitions

2.1 Definitions: There are K agents, indexed by the superscript i, and L commodities, indexed by the subscript j, with $2 \leq K < \infty$ and $2 \leq L < \infty$. The $i^{\underline{th}}$ agent has a consumption space $X^i = \operatorname{int} R^L_+$, an endowment space $\Omega^i = \operatorname{int} R^L_+$, and a space of net trades $Y^i = X^i - \Omega^i$. Let $X = \Pi^K_{i=1} X^i$, $\Omega = \Pi^K_{i=1} \Omega^i$, and $Y = \{y \in \Pi^K_{i=1} Y^i : \Sigma^K_{i=1} y^i \leq 0\}$.

The space of static environments is defined as in [11]. Let $U^i = \{\alpha^i \in \operatorname{int} R^L_+ : \Sigma^L_{j=1} \alpha^i_j = 1\}$, where each α^i in U^i represents a utility function $u^i : X^i \to R$ defined by $u^i(x^i) = \Sigma^L_{j=1} \alpha^i_j \ln x^i_j$; let U^i be topoligized as a subspace of R^L, and let $U = \Pi^K_{i=1} U^i$. For each i, let $E^i = \Omega^i \times U^i$. The space of static environments is the space $E = \Pi^K_{i=1} E^i$.

2.2. Remarks: A "state of the world", as this term is commonly used, is a complete description of everything that is relevant to every agent. In the present context, a state of the world is simply an endowment and a K-tuple of utilities $u \in U$. Although time will not be formally represented in the model, we will suppose that the endowment is realized in the present, whereas the state dependent utilities may not be realized until the future, after exchange takes place. Suppose there are only two possible states, a, represented by $(\omega_a, u_a) \in E$, and b, represented by $(\omega_b, u_b) \in E$. If $\omega^i_a \neq \omega^i_b$, the $i^{\underline{th}}$ agent can infer the state directly from his endowment, and can thus predict his utility function with certainty before trading takes place. Even if $\omega^i_a = \omega^i_b$, the $i^{\underline{th}}$

agent may have exogenous information which enables him to identify the state before trade. Thus the description of a two-event stochastic exchange environment requires, in addition to the specification of the two states and their probabilities, the description of the exogenous information available to agents before trading.

2.3 Definitions: Let $I = \Pi_{i=1}^{K} \{0,1\}$, with generic element $\eta = (\eta^i)_{i=1}^{K}$. An element of I is called an exogenous information structure. The i^{th} agent is said to be informed if $\eta^i = 1$, and uninformed otherwise. Define the space of stochastic environments by $S = \{(\lambda,\eta,e_a,e_b) \in (0,1) \times I \times E \times E$: for each i, if $\omega_a^i \neq \omega_b^i$ then $\eta^i = 1\}$, with generic element s. The first component, λ, of a stochastic environment is defined to be the probability of state a.

2.4 Remarks: Since any element in $E \times E$ can be a pair of states, there is no imposed relationship between the state dependent characteristics of different agents. The strength of this "decomposability" assumption (Hurwicz [7])can be seen by considering a special case of the model introduced by Green in [2]. Suppose that the commodities traded are contingent claims, and agents are uncertain about the true probability distribution on the events $1 \leq j \leq L$. In the present model, two states (ω_a, u_a) and (ω_b, u_b) could be associated with two such probability distributions only if for each j, $\alpha_{aj}^i / \alpha_{bj}^i = \alpha_{aj}^{i'} / \alpha_{bj}^{i'}$ for all i and i'. Thus the results obtained here will not be directly applicable to this case.

2.5 Definitions: Let $r_a: S \to E$ and $r_b: S \to E$ be given by:

$$r_a(\lambda,\eta,e_a,e_b)^i = \begin{cases} e_a^i & \text{if } \eta^i = 1, \text{ and} \\ (\omega_a^i,\ \lambda u_a^i + (1-\lambda)u_b^i) & \text{otherwise;} \end{cases}$$

$$r_b(\lambda,\eta,e_a,e_b)^i = \begin{cases} e_b^i & \text{if } \eta^i = 1, \text{ and} \\ (\omega_b^i,\ \lambda u_a^i + (1-\lambda)u_b^i) & \text{otherwise.} \end{cases}$$

2.6 Remarks: The function r_a associates with each stochastic environment the endowment realized in state a, and for each i, the i^{th} agent's conditional expected utility function given the exogenous information he receives in state a. The function r_b associates with each stochastic environment the analogous realization for state b. Suppose we associate with each s the competitive equilibrium for $r_a(s)$, and the competitive equilibrium for $r_b(s)$. This is a highly simplified version of the equilibrium concept studied in [13], in which agents choose their excess demands by maximizing expected utility conditional on exogenous information. In general, any allocation mechanism for static environments can be composed with r_a and r_b to determine an allocation mechanism on S.

An unsatisfactory feature of this type of mechanism is that agents make no use of the information generated by the endogenous variables of the mechanism itself. In the theory developed by Hurwicz [7], an allocation mechanism includes as part of its description a message process by which agents exchange information about the environment through the exchange of messages. Uninformed agents would naturally use this process as a source of information about the state. Suppose that a message process associates with each static environment an equilibrium message. If uninformed agents condition their expected utility on the equilibrium message, the resulting static environment is influenced by the message. This feedback is the focus of the expectations equilibrium concept defined below.

<u>2.7 Allocation Processes</u>: A <u>message process</u> is a pair (μ,M), where M is a Hausdorff space and μ is a continuous function from E onto M. The space M is called a <u>message space</u>. An <u>allocation process</u> is a triple (μ,M,g), where (μ,M) is a message process, and $g: M \to Y$. An allocation process (μ,M,g) is said to be <u>nonwasteful</u> if for each $e \in E$, the trade $g \cdot \mu(e)$ is Pareto efficient for e.

<u>2.8 Expectations Equilibrium</u>: An <u>expectations equilibrium</u> for a stochastic environment $s = (\lambda,\eta,e_a,e_b)$ and a message process (μ,M) is a set $\mathcal{E} \subset M$ such that either:

i) $\mathcal{E} = \{\mu(e_a), \mu(e_b)\}$, and $\mu(e_a) \neq \mu(e_b)$; or

ii) $\mathcal{E} = \{\mu \cdot r_a(s)\}$, and $\mu \cdot r_a(s) = \mu \cdot r_b(s)$.

A message process (μ,M) is said to be <u>admissible</u> if for each $s \in S$, an expectations equilibrium exists for s and (μ,M).

<u>2.9 Remarks</u>: The above definition of an allocation process differs from that of Mount and Reiter [11, pp. 169-170] by requiring M to be Hausdorff and μ to be single-valued. If μ is set-valued, the ability of an uninformed agent to distinguish between states according to their equilibrium messages may depend on the particular messages selected, and thus may be undetermined. The Hausdorff requirement insures that the topological structure

of a message process preserves observable distinctions between events.

A message process assigns a message to each static environment, so an equilibrium for a two event stochastic environment will be a pair of messages, one for each state. If the messages are distinct, uninformed agents can infer the state from the message, so the message in each state should agree with each agent's known state dependent characteristics. If the two messages are the same, the common message should agree with the state dependent characteristics of the informed agents in each state, and the common endowment and expected utility of the uninformed agents. These two cases are described by 2.8 (i) and (ii) respectively. An expectations equilibrium will fail to exist if $\mu(e_a) = \mu(e_b)$ and $\mu \cdot r_a(s) \neq \mu \cdot r_b(s)$.

An uninformed agent is, by interpretation, an agent for whom the relevant distinction between the two states is not realized until after an allocation is made. Put somewhat differently, an uninformed agent's allocation is an investment with an uncertain return. However, the return may be predictable given information which is directly available to other agents. An expectations equilibrium allows an uninformed agent to obtain this information if it is distinguished by the equilibrium messages.

3. A Necessary Condition for Admissibility

3.1 Lemma: Suppose that (μ,M) is an admissible message process. Then for each $e_a, e_b \in E$ at least one of the following two conditions is satisfied:

i) $\mu(e_a) \neq \mu(e_b)$; or

ii) for each $\lambda \in (0,1)$ and each i^o such that $\omega_a^{i^o} = \omega_b^{i^o}$, $\mu(e_{a_{\lambda_{i^o}}}) = \mu(e_{b_{\lambda_{i^o}}})$, where

$$e^i_{a_{\lambda_{i^o}}} = \begin{cases} (\omega_a^i,\ u_a^i) & \text{if } i \neq i^o, \\ (\omega_a^i,\ \lambda u_a^i + (1-\lambda)u_b^i) & \text{if } i = i^o; \end{cases}$$

and $$e^i_{b_{\lambda_{i^o}}} = \begin{cases} (\omega_b^i,\ u_b^i) & \text{if } i \neq i^o \\ (\omega_b^i,\ \lambda u_a^i + (1-\lambda)u_b^i) & \text{if } i = i^o. \end{cases}$$

Proof: Let $e_a, e_b \in E$ and suppose that $\mu(e_a) = \mu(e_b)$. Let $1 \leq i^o \leq K$ such that $\omega_a^{i^o} = \omega_b^{i^o}$; and let $\eta \in I$ be given by $\eta^i = 1$ for all $i \neq i^o$, and $\eta^{i^o} = 0$. Then for each $\lambda \in (0,1)$, $r_a(\lambda,\eta,e_a,e_b) = e_{a_{\lambda_{i^o}}}$ and $r_b(\lambda,\eta,e_a,e_b) = e_{b_{\lambda_{i^o}}}$ so the Lemma follows from 2.8 (ii).

3.2 Definition: For each i and each $e_a, e_b \in E$, define:

$$e_a *^i e_b = \begin{cases} (e_b^1, \ldots, e_b^{i-1}, e_a^i, e_b^{i+1}, \ldots, e_b^K) & \text{if } \omega_a^i = \omega_b^i, \text{ and} \\ e_b & \text{otherwise.} \end{cases}$$

Given $e_a, e_b \in E$, the notation $\mu(e_a) \cap \mu(e_b)$ denotes the intersection of one point sets.

3.3 Proposition: Suppose that (μ,M) is an admissible message process. Then for each i and each $e_a, e_b \in E$

$$(*) \qquad \mu(e_a) \cap \mu(e_b) = \mu(e_b *^i e_a) \cap \mu(e_a *^i e_b).$$

Proof: Let $e_a, e_b \in E$ and let $1 \leq i \leq K$. If $\omega_a^i \neq \omega_b^i$, the assertion is trivial, so suppose that $\omega_a^i = \omega_b^i$. First suppose that $\mu(e_a) = \mu(e_b)$. Then by 3.1, $\mu(e_{a_{\lambda_i}}) = \mu(e_{b_{\lambda_i}})$ for each $\lambda \in (0,1)$. If $\lambda \to 1$,

$e_{a_{\lambda_i}} \to e_a$ and $e_{b_{\lambda_i}} \to e_a *^i e_b$. Therefore, by the continuity of μ,

$\mu(e_a) = \mu(e_a *^i e_b)$. Similarly, letting $\lambda \to 0$ yields $\mu(e_b *^i e_a) = \mu(e_b)$,

so $\mu(e_a) = \mu(e_b) = \mu(e_a *^i e_b) = \mu(e_b *^i e_a)$. Second, suppose that

$\mu(e_a *^i e_b) = \mu(e_b *^i e_a)$. The result follows by the above argument and the observation that $(e_a *^i e_b) *^i (e_b *^i e_a) = e_a$ and $(e_b *^i e_a) *^i (e_a *^i e_b) = e_b$.

3.4 Remarks: Since $(*)$ restricts the behavior of μ only at the endpoints of intervals in E, it is not in general sufficient for admissibility. For example, suppose that μ is 1 - 1 on E. Let $e_a, e_b \in E$ with $\omega_a \neq \omega_b$, and let μ' be obtained by identifying the values of μ at the (at most 2^{K-1}) static environments generated by repeated applications of the $*^i$ operations to e_a and e_b. Then μ' satisfies $(*)$ but may not be admissible. However, Lemma 3.8 (ii) below indicates that for most economically interesting message processes on E, (*) is sufficient for admissibility.

Condition (*) was suggested by the "crossing condition" used by Mount and Reiter [11] to characterize a decentralization property introduced by Hurwicz [7, Definition 4, p. 32]. This characterization is stated formally below.

3.5 Definitions ([11], pp. 170-171): A correspondence μ on E to a set M is said to be privacy preserving if for each i there exists a correspondence $\mu^i: E^i \twoheadrightarrow M$ such that $\mu(e) = \bigcap_{i=1}^{K} \mu^i(e^i)$ for each $e \in E$. For each $e_a, e_b \in E$, let $e_a \boxtimes^i e_b = (e_b^1, \ldots, e_b^{i-1}, e_a^i, e_b^{i+1}, \ldots, e_b^K)$.

3.6 Proposition ([11], Lemma 5, p. 171): A correspondence μ on E to a set M is privacy preserving if and only if for each i and each $e_a, e_b \in E$

$$(\times) \quad \mu(e_a) \cap \mu(e_b) = \mu(e_b \boxtimes^i e_a) \cap \mu(e_a \boxtimes^i e_b).$$

3.7 Remarks: Since a message process associates with each static environment an equilibrium message, a privacy preserving message process is one for which the static equilibrium conditions can be verified for each agent independently. This means that in equilibrium, all information which an agent receives about the rest of the environment is embodied in the equilibrium message. In an expectations equilibrium, this information condition is imposed by definition for uninformed agents. The static equilibrium message is an uninformed agent's only source of information about whether the environment is e_a or e_b. Thus (*) arises as a crossing condition for uninformed agents.

3.8 Lemma: Suppose that (μ, M, g) is a nonwasteful allocation process and that (μ, M) satisfies (*). Then

i) if $e_a, e_b \in E$ and $\omega_a^i = \omega_b^i$ for some i, then $\mu(e_a) = \mu(e_b)$ only if $\alpha_a^i = \alpha_b^i$; and

ii) for each $(\lambda, \eta, e_a, e_b) \in S$, $\{\mu(e_a), \mu(e_b)\}$ is an expectations equilibrium.

Proof: Let $e_a, e_b \in E$ with $\omega_a^{i^o} = \omega_b^{i^o}$ and suppose that $\mu(e_a) = \mu(e_b)$. Let $y = g \cdot \mu(e_a) = g \cdot \mu(e_b)$. Since y is a Pareto efficient trade for e_a and e_b, let p_a be the common normalized gradient of u_a^i at $\omega_a^i + y^i$ for each i, and let p_b be the common normalized gradient of u_b^i at $\omega_b^i + y^i$ for each i. If $p_a = p_b$, then $\alpha_a^{i^o} = \alpha_b^{i^o}$ since $\omega_a^{i^o} = \omega_b^{i^o}$. If $p_a \neq p_b$, then y is not a Pareto efficient trade for $e_b *^{i^o} e_a$. But (*) implies that $\mu(e_b *^{i^o} e_a) = \mu(e_a)$, so we must have $p_a = p_b$, which proves (i).

To prove (ii), it suffices to show that for any $s = (\lambda, \eta, e_a, e_b) \in S$ such that $\mu(e_a) = \mu(e_b)$, $\mu \cdot r_a(s) = \mu \cdot r_b(s) = \mu(e_a)$. It follows from (i) that if $\mu(e_a) = \mu(e_b)$, then $e_a^i = e_b^i$ for all i such that $\eta^i = 0$. Thus if $\mu(e_a) = \mu(e_b)$, then $r_a(s) = e_a$ and $r_b(s) = e_b$, which completes the proof.

3.9 Remarks: The argument used to prove 3.8 (i) is standard (see [6, pp. 1 - 7] and [11, Lemma 29, p. 187] for other applications and further references). Statement 3.8 (ii) should not be given too much importance, since it depends heavily on the present definition of E. In fact, if one merely relaxes the requirement that $\Sigma_j \alpha_j^i = 1$, one can construct examples, similar to the one in 3.4, showing that condition (*) and nonwastefulness do not imply admissibility. For an example of an admissible message process for which $\{\mu(e_a), \mu(e_b)\}$ is not always an expectations equilibrium, let $\mu: E \to R^K$ be given by $\mu(e) = (||\alpha^1||, \ldots, ||\alpha^K||)$, where $||\cdot||$ denotes the euclidean norm.

4. The Competitive Message Process

4.1 Definitions: Let Δ denote the relative interior of the unit simplex in R^L_+, and let $M_c = \{(p,y) \in \Delta \times Y\colon\ \Sigma_i y^i = 0$ and for each i, $py^i = 0\}$. For each i, let $\mu^i_c\colon\ E^i \twoheadrightarrow M$ be defined by $\mu^i_c(e^i) = \{(p,y)\colon\ y^i$ maximizes $u^i(\omega^i + y^i)$ subject to $py^i \leq 0\}$. Let $\mu_c\colon E \to M_c$ be given by $\mu_c(e) = \bigcap^K_{i=1} \mu^i_c(e^i)$. Then (μ_c, M_c) is the competitive message process [11, Definition 25, p. 185].

4.2 Remarks: Since Cobb-Douglas exchange economies satisfy the gross substitutes condition [1, p. 225], μ_c is a C^1 function from the KL(L-1) dimensional manifold E to the K(L-1) dimensional manifold M_c.

The admissibility of (μ_c, M_c) follows from Lemma 3.8 (ii), but we will give a simpler proof which does not use the Cobb-Douglas assumption. As is indicated in [8, Proposition 3.6 and Remarks 3.7], the admissibility of the competitive message process also extends to the case in which there are many possible states, and the function μ_c is a selection from the static equilibrium correspondence.

4.3 Proposition: The competitive message process is admissible. Moreover, for each $(\lambda, \eta, e_a, e_b) \in S$, $\{\mu_c(e_a), \mu_c(e_b)\}$ is an expectations equilibrium.

Proof: Let $(\lambda, \eta, e_a, e_b) \in S$ such that $\mu_c(e_a) = \mu_c(e_b)$. Let $(p,y) = \mu_c(e_a)$ and, if $\eta^i = 0$, let $\omega^i = \omega^i_a = \omega^i_b$ and let $x^i = y^i + \omega^i$. Since $(p,y) \in \mu^i_c(e^i_a)$ and $(p,y) \in \mu^i_c(e^i_b)$, it follows by definition that x^i maximizes u^i_a subject to $px^i \leq p\omega^i$, and x^i also maximizes u^i_b subject to the same constraint. Then x^i maximizes any convex combination of u^i_a and u^i_b subject to the same constraint, so $(p,y) \in \mu^i_c(\omega^i, \lambda u^i_a + (1-\lambda)u^i_b)$ for all λ. Thus whenever $\mu_c(e_a) = \mu_c(e_b)$, we have $\mu_c \cdot r_a(s) = \mu_c \cdot r_b(s) = \mu_c(e_a)$. This completes the proof.

4.4 Remarks: There are stochastic environments $(\lambda, \eta, e_a, e_b) \in S$ for which $\mu_c(e_a) \neq \mu_c(e_b)$ and $\mu_c \cdot r_a(s) = \mu_c \cdot r_b(s)$, in which case $\{\mu_c \cdot r_a(s)\}$ is also an expectations equilibrium.

The following result, which is adapted directly from [6] and [11, Theorem 35, p. 190], shows that the competitive message space is locally of minimal "size" subject to the requirements of nonwastefulness and admissibility.

4.5 Theorem: Suppose that (μ, M, g) is a nonwasteful allocation process and that (μ, M) satisfies (*). Then M has a subset which is locally homeomorphic to M_c.

Proof: Let $e = (\omega, u) \in E$, and let $E_\omega = \{e' \in E\colon\ \omega' = \omega\}$. Then E_ω, identified with U, is a $K(L-1)$ dimensional manifold. By Lemma 3.8 (i), μ is $1-1$ on E. Let V be a neighborhood of $\mu(e)$ in M. Then $\mu^{-1}(V)$ contains a compact neighborhood V' of e in E_ω. Since M is Hausdorff, it follows that $\mu\colon V' \to \mu(V')$ is a homeomorphism. Since int V' and M_c are manifolds of the same dimension, the result follows easily.

4.6 Remark: The strength of Theorem 4.5 lies in its applicability to any allocation process defined on a larger class of environments, whose restriction to E satisfies the hypothesis.

5. Admissible Data Processes

In applying the competitive message process to stochastic environments, we have thus far required uninformed agents to use the entire competitive message to forecast their future utility. Proposition 4.3 shows that this guarantees not only the existence of an expectations equilibrium, but the existence of the equilibrium which would obtain if all agents were informed.

In this section we consider the case in which uninformed agents base their forecasts on some condensation of the competitive message, for example price, or the volume of trade.

5.1 Notation: In this section only, the competitive message process will be denoted (μ, M), without the previous subscripts.

5.2 Definitions: A data process is a pair (f,Z), where Z is a Hausdorff space and f is a continuous function from M onto Z. A data process (f,Z) is said to be admissible if the message process $(f \cdot \mu, Z)$ is admissible. If f is one to one, (f,Z) is said to be discrete; and (f,Z) is said to be trivial if Z contains only one element. Given a data process (f,Z), a subset $F \subset M$ is said to be observable if $F = f^{-1}(z)$ for some $z \in Z$.

5.3 Remarks: If (f,Z) is a data process, the message process $(f \cdot \mu, Z)$ would be a condensation of μ, in the terminology of Reiter [16, Definition 3, p. 329], if f were "locally sectioned" [11, Definition 6, p. 173]. The definition of admissibility embodies the requirement that all uninformed agents use the same data process. The main result of this paper is that all admissible data processes are either discrete or trivial.

The following lemma is a corollary of Proposition 3.3.

5.4 Lemma: Suppose that (f,Z) is an admissible data process. Then for each i and each $e_a, e_b \in E$,

$$(*') \qquad f\cdot\mu(e_a) \cap f\cdot\mu(e_b) = f\cdot\mu(e_b *^i e_a) \cap f\cdot\mu(e_a *^i e_b).$$

5.5 Definitions: Given an ordered K-tuple of static exchange environments $\{e_i\}_{i=1}^K \subset E$, let $E^{ii} = (\mu^i)^{-1}\cdot\mu(e_i)$ for each i. Then for each i, E^{ii} is an L dimensional manifold. That is, if $(p^o, y^o) = \mu(e_i)$, and $e^i = (\omega^i, \alpha^i) \in E^{ii}$, then for each j, $\alpha_j^i = p_j^o(\omega_j^i + y_j^{oi})/p^o\omega^i$; so E^{ii} is locally diffeomorphic to Ω^i. Let $E^* = \Pi_{i=1}^K E^{ii}$ and let μ^* denote the restriction of μ to E^*. Let e^* be the environment $(e_i^i)_{i=1}^K \in E^*$ and let $m = \mu^*(e^*)$. The K-tuple $\{e_i\}_{i=1}^K$ is a local generator at m if the differential $D\mu^*(e^*)$ has full rank.

5.6 Lemma: Let (f,Z) be a data process and let F be an observable subset of M. Suppose that (f,Z) satisfies $(*')$, and that for each $m \in F$, $\mu^{-1}(F)$ contains a local generator at m. Then $F = M$.

Proof: The lemma will be proved by showing that F is open and closed. Since M is connected, the lemma will follow. That F is closed follows from the continuity of f and the fact that Z is Hausdorff.

To show that F is open, let $m \in F$ and let $\{e_i\}_{i=1}^K \subset \mu^{-1}(F)$ be a local generator at m. Using the notation in 5.5, we have that for each i, $E^{ii} \subset (\mu^i)^{-1}(F)$. We now show that there is an open neighborhood V of e^* in E^* such that $V \subset \mu^{-1}(F)$. Let $(p,y) = m$, and for each i, let $(p_i,y_i) = \mu(e_i)$ and let $(\omega^i,\alpha^i) = e_i^i$. Let V^i be an open neighborhood of ω^i in Ω^i such that for each $\omega'^i \in V^i$, $\omega'^i + y^i \in X^i$ and $\omega'^i + y_i^i \in X^i$. As noted in 5.5, V^i can be viewed as an open neighborhood of e_i^i in E^{ii}. For each $e'^i \in V^i$, let e_i' be the environment obtained by replacing e_i^i with e'^i in e_i, and note that $\mu(e_i') = \mu(e_i)$ by the definition of E^{ii}. Let $V = \Pi_{i=1}^K V^i$. Then for each environment $(e'^i)_{i=1}^K \in V$, there is an environment $e'' \in \mu^{-1}(m)$ with $\omega''^i = \omega'^i$ for each i. This implies that $(e'^i)_{i=1}^K = e_K' *^K(\cdots(e_2' *^2(e_1' *^1 e''))\cdots)$. Since each such e'' and e_i' is in $\mu^{-1}(F)$, it follows from $(*')$ that $V \subset \mu^{-1}(F)$. Finally, since $D\mu^*(e^*)$ is surjective, $\mu(V)$ contains an open neighborhood of m.

5.7 Remarks: The second paragraph in the proof of 5.6 indicates the role of a local generator. Suppose that a subset of M contains a collection of messages $\{m_i\}$ associated with a local generator at m. Then by repeated applications of the $*^i$ operations to environments in $\mu^{-1}(m)$ and $\mu^{-1}(\{m_i\})$, one generates environments whose equilibrium messages compose a neighborhood of m. In principle, this method of analysis seems applicable to any message process for which the required differential exists. However, any application requires the construction of local generators, which is accomplished here for the competitive message process.

Lemma 5.8 establishes a sufficient condition for a collection of environments to be a local generator. Lemmas 5.9 and 5.10 show that if (f,Z) is admissible and F is an observable set containing more than one message, such a collection can be constructed in $\mu^{-1}(F)$ at any message in F. The length of these steps is due largely to the Cobb-Douglas assumption. If the set of environments is enlarged in such a way that μ remains a C^1 function, the enlargement of E^* may make it easier for $D\mu^*$ to be surjective. In an earlier version of this paper, the set of environments was enlarged to include C^2 perturbations of Cobb-Douglas utility functions which preserve the gross substitutes condition. In that case the construction of local generators is considerably less tedious. The advantage of working with a smaller class of environments is that the hereditary property of the result is stronger. This is discussed further in 5.12 below.

<u>5.8 Lemma</u>: Let $\{e_i\}_{i=1}^{K} \subset E$, let $(p^*,y^*) = \mu(e^*)$, and for each i, let $(p_i,y_i) = \mu(e_i)$. Suppose that for each agent i and each commodity j, $y^{*i}_j \neq y^i_{ij}$. Then $\{e_i\}_{i=1}^{K}$ is a local generator at (p^*,y^*).

<u>Proof</u>: For each i, let $M^i \subset \Delta \times R^L$ be defined by $M^i = \{(p,y^i):$ $py^i = 0\}$. Let the function $h^i\colon \Delta \times E^{ii} \to M^i$ be given by $h^i(p,e^i) = (p,y^i)$, where y^i is the excess demand at p of the agent described by e^i. To show that $D\mu^*(e^*)$ is surjective, it suffices to show that $Dh^i(p^*,e^{*i})$ is surjective for each i. Let $M^i_{p^*} = \{y^i \in R^L : p^*y^i = 0\}$, and let $d^i : E^{ii} \to M^i_{p^*}$ be given by $d^i(e^i) = h^i(p^*,e^i)$. Then it suffices to show that $Dd^i(e^{*i})$ is

surjective. As in the proof of Lemma 5.6, let V^i be an open neighborhood of ω^{*i} such that for each $\omega^i \in V^i$, $\omega^i + y^{*i} \in X^i$ and $\omega^i + y^i_i \in X^i$, and view V^i as a neighborhood of $e^{*i} \in E^{ii}$. For each commodity j, and each $\omega^i \in V^i$, let $r_j(\omega^i) = (p^*\omega^i/p_i\omega^i)(p_{ij}/p^*_j)$. Then, on V^i, the function d^i is given by $d^i_j(\omega^i) = r_j(\omega^i)y^i_{ij} + \omega^i_j(r_j(\omega^i) - 1)$. By hypothesis, $y^i_{ij} \neq y^{*i}_j = d^i_j(\omega^{*i})$ for all j, so $r_j(\omega^{*i}) \neq 1$ for all j. It follows easily that $Dd^i(\omega^{*i})$ is surjective.

<u>5.9 Lemma</u>: Suppose that (f,Z) is a data process which satisfies $(*')$. Let F be an observable subset of M and let $(p,y) \in F$. Suppose that F contains more than one message. Then there is an ordered K-tuple of environments $\{e_i\}_{i=1}^K \in \mu^{-1}(F)$ such that

i) $\mu(e^*) = (p,y)$;

ii) for each i, the price component of $\mu(e_i)$ differs from p; and

iii) $\omega^{i'}_{ij} = \omega^{i'''}_{i''j'}$ for all agents i, i', i'', i''' and all commodities j, j'.

<u>Proof</u>: Let $(p,y) \neq (p',y') \in F$; and let $e \in \mu^{-1}(p,y)$ and $e' \in \mu^{-1}(p',y')$ with $\omega^i_j = \omega'^i_j = k$ for all i, j, for some $k > 0$. If $p \neq p'$, choose k large enough so that for each i, $\alpha^i \neq \alpha'^i$. For each i, let $(p_i,y_i) = \mu(e^{*i}e')$. By $(*')$, $(p_i,y_i) \in F$, so if $p_i \neq p$, let $e_i = e^{*i}e'$. If $p_i = p$, there are two possibilities to consider. First suppose that $p' = p$. Since $p_i = p$ and $\omega = \omega'$, we

must have $y^i = y'^i$. However, since $(p,y) \neq (p',y')$, we must have $y^\ell \neq y'^\ell$ for some $\ell \neq i$, so let $e_i = e^{*\ell}(e^{*i}e')$. Second, suppose that $p' \neq p$. Then by the choice of e and e', $\alpha^\ell \neq \alpha'^\ell$ for all ℓ. Since $p_i = p$ and $\omega = \omega'$, we must have $y^\ell \neq y_i^\ell$ for each $\ell \neq i$. Therefore, for any $\ell \neq i$, let $e_i = e^{*\ell}(e^{*i}e')$. Then $\{e_i\}_{i=1}^K$ satisfies the Lemma, since $e^* = e$.

5.10 Lemma: Suppose that (f,Z) is a data process which satisfies $(*')$. Let F be an observable subset of M and let $(p^*,y^*) \in F$. Suppose that F contains more than one message. Then $\mu^{-1}(F)$ contains a local generator at (p^*,y^*).

Proof: Applying Lemma 5.9, let $\{e_i\}_{i=1}^K$ satisfy 5.9(i) - (iii), and let $(p_i,y_i) = \mu(e_i)$ for each i. If $y_{ij}^i \neq y_j^{*i}$ for all i, j, then $\{e_i\}_{i=1}^K$ is a local generator by Lemma 5.8. Therefore suppose that $y_{ij^o}^i = y_{j^o}^{*i}$ for some i, j^o. However, since $p_i \neq p^*$, there are commodities ℓ and ℓ' for which $y_{i\ell}^i \neq y_\ell^{*i}$ and $y_{i\ell'}^i \neq y_{\ell'}^{*i}$. Using 5.9 (iii), let k be the common endowment of each commodity held by each agent, and let G be the one dimensional submanifold of $(\mu^i)^{-1}(p^*,y^*)$ defined by $G = \{e^i: \omega^i + y^{*i} \in X^i, \omega^i + y_i^i \in X^i;$ $\omega_j^i = k$ for all $j \neq \ell, \ell'$; and $p^*\omega^i = k\}$. Let $e^i \mapsto \omega_\ell^i$ be the coordinate system for G. Condition $(*')$ implies that for each $e^i \in G$, the environment $(e_i^1, \ldots, e^i, \ldots, e_i^K) \in \mu^{-1}(F)$. Accordingly, let $\rho: G^i \to \Delta$ be given by setting $\rho(e^i)$ equal to the price coordinate of $\mu(e_i^1, \ldots, e^i, \ldots, e_i^K)$, and consider the function

$e^i \mapsto \rho(e^i)\omega^i$, which associates with each $e^i \in G$ the value of ω^i at the price $\rho(e^i)$. Then $\left.\frac{d[\rho(e^i)\omega^i]}{de^i}\right|_{e^i_i} = p_{i\ell} - p_{i\ell'}(p^*_\ell/p^{*'}_\ell)$, since $\omega^i_{ij} = k$ for all j. Also $y^i_{ij} \neq y^{*i}_j$ if and only if $p_{ij} \neq p^*_j$, so we can choose ℓ and ℓ' so that $p_{i\ell} - p_{i\ell'}(p^*_\ell/p^*_{\ell'}) \neq 0$.

Thus the function $e^i \to \rho(e^i)\omega^i$ is locally injective. Also, the budget shares α^i_j for all commodities other than ℓ and ℓ' are constant on G. Since $\rho(\cdot)\omega^{i'}_i = k$ for all $i' \neq i$, it follows that for each $j \neq \ell,\ell'$, the function on G obtained by projecting $\mu(e^1_i, \ldots,\cdot,\ldots e^K_i)$ to the $i^{\underline{th}}$ agent's excess demand for commodity j is locally injective. This implies that for some $e^i \in G$, if $(p'_i, y'_i) = \mu(e^1_i, \ldots e^i, \ldots e^N_i)$ then $y'^i_{ij} \neq y^{*i}_j$ for all j. The Lemma follows directly.

<u>5.11 Theorem</u>: A data process (f,Z) satisfies (*') if and only if (f,Z) is either discrete or trivial.

<u>Proof</u>: Since (μ,M) is admissible, any discrete data process is admissible. Condition (*') is obvious for trivial data processes. Necessity follows directly from Lemma 5.10.

<u>5.12 Remarks</u>: Theorem 5.11 indicates that if the expectations of uninformed agents are based on any nontrivial condensation of the competitive message process, expectations equilibria can fail to exist, even in the class of

two-event Cobb-Douglas exchange environments. It follows that if (f,Z) is a data process which admits the existence of expectations equilibria on any class of stochastic exchange environments containing S, (f,Z) must be discrete or trivial.

This result depends on the requirement that all uninformed agents use the same data process. If this assumption is relaxed, the proof of 4.3 indicates that complete information expectations equilibria exist when the $i^{\underline{th}}$ agent's expectations are conditioned on (p,y^i), for each i.

Theorem 5.11 also establishes an interesting informational efficiency property of the competitive message process. If (f,Z) is a data process $(f\cdot\mu,Z)$ is a message process,and the partition of E induced by μ is as fine as that induced by $f\cdot\mu$. In the language of [16], μ reveals as much as $f\cdot\mu$. Then 5.11 implies that any privacy preserving message process which reveals less than the competitive process must map E to a constant. The crossing condition $(\times)$ implies that the partition induced by a privacy preserving process is a "product partition", having the characteristic property that each element is of the form $\Pi_{i=1}^{K}A^i$, with $A^i \subset E^i$ for each i. Thus, among partitions induced by data processes, the competitive partition is a minimal nontrivial product partition.

This property appears to be quite different from the type of informational efficiency established in [6] and [11], of which Theorem 4.5 is an application. Both properties seem sufficiently fundamental to the structure of the competitive process that their formal relationship should be explored. Another question for future research is whether the continuity assumption, which is used in Lemmas 5.4 and 5.6, can be dropped.

6. The Knowledge Required to Form Expectations[2]

The expectations equilibrium concept presumes that uninformed agents can predict their future utility from the current message whenever the two states generate distinct messages. However, the fact that $m \neq m'$ does not by itself enable an agent to predict state a when m is observed, and state b when m' is observed. This section is addressed to the question of how much information is used in making such predictions. Unlike the main results of the previous sections, the results obtained here are not directly inherited by stochastic environments with many states, and should be regarded as merely suggestive.

It is perhaps most natural to suppose that expectations are formed on the basis of empirical observation. If an uninformed agent observes that the message m is generally followed by the return he obtains in state a, he will predict state a given m. An expectations equilibrium is sustained under this type of expectation formation. Whether or not it can be dynamically achieved in this way is a question which will not be treated here.

The observed relationship between messages and states for a particular stochastic environment can be represented by an ordered pair (m_a, m_b), such that the set $\{m_a, m_b\}$ is an expectations equlibrium. Knowledge of the ordered pair (m_a, m_b) clearly suffices for associating states with elements of the expectations equilibrium set $\{m_a, m_b\}$. Of course if $m_a = m_b$, the message is associated with the given probability distribution of the two

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[2] The topic of this section was motivated by some comments of Professor Radner.

states.

This leads to the question of whether the entire ordered pair (m_a, m_b) is needed. Any condensation of ordered pairs can be represented by a function ν on $M \times M$, so the question becomes: What functions ν have the property that for each stochastic environment, $\nu(m_a, m_b)$ contains sufficient information for associating conditional probabilities of states with elements of an equilibrium set $\{m_a, m_b\}$. Consider a message process $\{\mu, M\}$, such as the competitive process, with the property that for each stochastic environment $(\lambda, \eta, e_a, e_b)$, $\{\mu(e_a), \mu(e_b)\}$ is an expectations equilibrium. Let $m_a = \mu(e_a)$, $m_b = \mu(e_b)$, and let $\nu(m_a, m_b) = m_a$. Knowledge of m_a enables agents to predict states according to the following rule: if $m = m_a$, expect state a, otherwise expect state b.

The difference between the expectations formed with the information (m_a, m_b), and those formed with only the information m_a, is that the latter are never probabalistic. If $m_a = m_b$, uninformed agents will always expect state a with certainty.

However, if $\{\mu(e_a), \mu(e_b)\}$ is always an expectations equilibrium, it must be the case that whenever $\mu(e_a) = \mu(e_b)$, $\mu \cdot r_b(s) = \mu(e_a)$. Thus agents will be mistakenly certain of state a only when the resulting equilibrium message is the same as if they had formed the statistically correct expectations given the message. In the particular case of the competitive process, an uninformed agent's excess demand would be unaffected if his expectations were corrected. (Under the Cobb-Douglas assumption, the force of this observation is somewhat diminished by the implication of 3.8(i) that $\mu_c(e_a) = \mu_c(e_b)$ only if $u_a^i = u_b^i$ for each uninformed agent i. However, the proof of 4.3 indicates that the Cobb-Douglas assumption can be relaxed for this argument.) These remarks indicate that in determining the amount of information used to form expectations, it may be useful to require only correct expectations equilibria, rather than correct expectations. We now show that this requirement can be formalized using a privacy condition for uninformed agents.

<u>6.1 Definitions</u>: An <u>inference process</u> is a pair (ν, N), where $\nu\colon M \times M \to N$. An <u>information process</u> is a pair of processes $(\mu, M; \nu, N)$, where (μ, M) is a message process and (ν, N) is an inference process.

Given a message process (μ, M), let $\mu^o\colon S \to M \times M$ be given by $\mu^o(\lambda, \eta, e_a, e_b) = (\mu(e_a), \mu(e_b))$. An <u>information equilibrium</u> for a stochastic environment s and an information process $(\mu, M; \nu, N)$ is a set $\mathcal{I} \subset M \times N$ such that $\mathcal{I} = \mathcal{E} \times \{\nu \cdot \mu^o(s)\}$, where $\mathcal{E}$ is an expectations equilibrium for s and (μ, M). An information process $(\mu, M; \nu, N)$ is said to be <u>admissible</u> if (μ, M) is admissible. An <u>equilibrium correspondence</u> for an admissible information process is a correspondence $\phi\colon S \twoheadrightarrow M \times N$ such that for each $s \in S$, $\phi(s)$ is an information equilibrium.

For each pair of stochastic environments $s = (\lambda, \eta, e_a, e_b)$ and $s' = (\lambda', \eta', e_a', e_b')$, and each i, let

$$s' *^i s = \begin{cases} (\lambda, \eta, e_a' \boxtimes^i e_a, e_b' \boxtimes^i e_b) & \text{if } \lambda = \lambda' \text{ and } \eta^i = \eta'^i = 0; \text{ and} \\ s & \text{otherwise.} \end{cases}$$

An information process is said to be <u>adequate</u> if it admits an equilibrium correspondence ϕ such that for each $s, s' \in S$ and each i,

$$(*'') \qquad \phi(s) \cap \phi(s') = \phi(s' *^i s) \cap \phi(s *^i s').$$

6.2 Proposition: An equilibrium correspondence ϕ satisfies (*") if and only if there exist correspondences $\phi^i : (0,1) \times \{(e_a^i, e_b^i): \omega_a^i = \omega_b^i\} \twoheadrightarrow M \times N$ for each i, and $\phi^A: (0,1) \times \Pi_{i\epsilon A} (E^i \times E^i) \twoheadrightarrow M \times N$ for each $A \subset \{1,\ldots K\}$ such that for each stochastic environment $s = (\lambda,\eta,e_a,e_b)$,

$\phi(s) = \{\phi^A[\lambda, (e_a^i, e_b^i)_{i\epsilon A}]\} \cap \{\cap_{i\epsilon A}\phi^i(\lambda,e_a^i,e_b^i)\}$, where $A = \{i: \eta^i = 1\}$.

Proof: This follows directly from Proposition 3.6.

6.3 Remarks: Given a message m, the expectation which the $i^{\underline{th}}$ agent associates with m determines whether he responds to m according to the utility function u_a^i, u_b^i, or $\lambda u_a^i + (1 - \lambda)u_b^i$. If $(\mu,M;\nu,N)$ is an adequate information process then $\nu\cdot\mu^o(s)$ contains all the information an uninformed agent needs to make the appropriate response to any message he observes in an expectations equilibrium. For example, let $e_1,e_2 \epsilon E$ with $\omega_1^i = \omega_2^i$ and suppose that $m_1 = \mu(e_1)$ and $m_2 = \mu(e_2)$, with $m_1 \neq m_2$. Let $s = (\lambda,\eta,e_1,e_2)$ and $s' = (\lambda,\eta,e_2,e_1)$ with $\eta^i = 0$ and suppose that $\mu\cdot r_a(s) \neq \mu\cdot r_b(s)$ and $\mu\cdot r_a(s') \neq \mu\cdot r_b(s')$. Then $\{m_1,m_2\}$ is the unique expectations equilibrium for s and s', but the relation between messages and states differs between the two cases. If $\nu(m_1,m_2) = \nu(m_2,m_1)$, then to satisfy (*") we must have $\{m_1,m_2\} = \{\mu(e_1 \boxtimes^i e_2), \mu(e_2 \boxtimes^i e_1)\}$. Since μ must be privacy preserving (Proposition 6.4 below), this may require that $\{m_1,m_2\} \subset \mu^i(e_1^i)$ and $\{m_1,m_2\} \subset \mu^i(e_2^i)$; or in other words, that the $i^{\underline{th}}$ agent's response to the two messages is independent of the expectations he associates with them. If the $i^{\underline{th}}$ agent's response is not independent of his expectations, ν must distinguish between the ordered pairs (m_1,m_2) and (m_2,m_1) so that the $i^{\underline{th}}$

agent can form the correct expectations. For the competitive message process, these remarks are formalized in Lemma 6.9 below.

6.4 Proposition: Suppose that $(\mu,M;\nu,N)$ is an adequate information process. Then the message process (μ,M) is privacy preserving.

Proof: Let $e,e' \in E$, let $1 \leq i \leq K$, and let $s = (\lambda,\eta,e,e)$ and $s' = (\lambda,\eta,e',e')$, with $\eta^i = 0$. If ϕ is an equilibrium correspondence for $(\mu,M;\nu,N)$, then $\phi(s) = \{\mu(e)\} \times \{\nu(\mu(e),\mu(e))\}$ and $\phi(s') = \{\mu(e')\} \times \{\nu(\mu(e'),\mu(e'))\}$. Therefore

$\phi(s) \cap \phi(s') = \phi(s' *^i s) \cap \phi(s *^i s')$ only if

$\mu(e) \cap \mu(e') = \mu(e' \boxtimes^i e) \cap \mu(e \boxtimes^i e')$, which completes the proof.

6.5 Proposition: Suppose that (μ,M) is a privacy preserving admissible message process. Let $N = M \times M$ and let ν be the identity map on $M \times M$. Then $(\mu,M;\nu,N)$ is an adequate information process.

Proof: Let the equilibrium correspondence ϕ be given by

$$\phi(s) = \begin{cases} \{\mu(e_a),\mu(e_b)\} \times \{\nu \cdot \mu^o(s)\} & \text{if } \mu(e_a) \neq \mu(e_b)\text{; and} \\ \{\mu \cdot r_a(s)\} \times \{\nu \cdot \mu^o(s)\} & \text{otherwise.} \end{cases}$$

Then condition (*") can be verified directly, using the fact that if $s = (\lambda,\eta,e_a,e_b)$ and $s' = (\lambda',\eta',e_a',e_b')$ with $\lambda' = \lambda$ and $\eta^i = \eta'^i = 0$, then $r_a(s' *^i s) = r_a(s') \boxtimes^i r_a(s)$.

6.6 Definition: An information process $(\mu,M;\nu,N)$ is said to be complete if the correspondence $\phi\colon S \twoheadrightarrow M \times N$, given by $\phi(\lambda,\eta,e_a,e_b) = \{\mu(e_a),\mu(e_b)\} \times \{\nu \cdot \mu^o(s)\}$, is an equilibrium correspondence and satisfies (*").

<u>6.7 Proposition</u>: Suppose that (μ,M) is a privacy preserving message process such that for each stochastic environment $(\lambda,\eta,e_a e_b)$, $\{\mu(e_a),\mu(e_b)\}$ is an expectations equilibrium. Let $N = M$ and let ν be the projection $(m_a,m_b) \mapsto m_a$. Then $(\mu,M;\nu,N)$ is a complete information process.

<u>Proof</u>: The proof is direct.

<u>6.8 Remarks</u>: Proposition 6.7 formalizes the remarks made at the beginning of this section. The message process $(\omega,\alpha) \mapsto (||\alpha^1||,\ldots,||\alpha^K||)$, where $||\cdot||$ is the euclidean norm, (also discussed in 3.9) is an admissible privacy preserving process but does not satisfy the hypothesis of 6.7. If ν is the projection $(m_a,m_b) \mapsto m_a$, then the information process $(\mu,M;\nu,N)$ is not adequate.

Proposition 6.10 below indicates that the projections $(m_a,m_b) \mapsto m_a$ and $(m_a,m_b) \mapsto m_b$ are minimal inference processes for the competitive message process, subject to the adequacy requirement.

<u>6.9 Lemma</u>: Suppose that $(\mu_c,M_c;\nu,N)$ is an adequate information process. If m and m' are distinct messages in M_c then $\nu(m,m') \neq \nu(m',m)$.

<u>Proof</u>: Let m and m' be distinct messages in M_c. It is straightforward to construct a stochastic environment $s = (1/2,\eta,e,e')$ such that

i) $\mu_c(e) = m$ and $\mu_c(e') = m'$;

ii) $\mu_c \cdot r_a(s) \neq \mu_c \cdot r_b(s)$; and

iii) for some i, $\eta^i = 0$, $\mu_c(e *^i e') \notin \{m,m'\}$, and $\mu_c(e' *^i e) \notin \{m,m'\}$.

Let $s' = (1/2,\eta,e',e)$. Then (i) and (ii) imply that $\{m,m'\}$ is the unique expectations equilibrium for s and s'. If $\nu(m,m') = \nu(m',m)$, then (iii) is in conflict with (*"), so we must have $\nu(m,m') \neq \nu(m',m)$.

6.10 Proposition: Suppose that $(\mu_c, M_c; \nu, N)$ is an adequate information process, that N is a Hausdorff space, and that ν is continuous. Then for each $m \in M_c$, at least one of the functions $\nu(m,\cdot)$ and $\nu(\cdot,m)$ is one to one.

Proof: Let $m \in M_c$. We will show that $\nu(m,\mu_c(\cdot))$ and $\nu(\mu_c(\cdot),m)$ are privacy preserving. Let $1 \leq i \leq K$ and let $e_1, e_2 \in E$. It is straightforward to construct stochastic environments $s_1 = (\lambda, \eta_1, e_{a1}, e_1)$ and $s_2 = (\lambda, \eta_2, e_{a2}, e_2)$ such that $\eta_1^i = \eta_2^i = 0$, $\{m, \mu_c(e_1)\}$ is the unique expectations equilibrium for s_1, and $\{m, \mu_c(e_2)\}$ is the unique expectations equilibrium for s_2. Let ϕ be an equilibrium correspondence for $(\mu_c, M_c; \nu, N)$. Then $\{m\} \times \{\nu(m,\mu_c(e_1)) \cap \nu(m,\mu_c(e_2))\} \subset \phi(s_1) \cap \phi(s_2)$. If $\nu(m,\mu_c(e_1)) = \nu(m,\mu_c(e_2))$, then (*") requires that $\nu(m,\mu_c(e_1)) = \nu(m,\mu_c(e_2 \boxtimes^i e_1)) = \nu(m,\mu_c(e_1 \boxtimes^i e_2))$. Similarly, $\nu(\mu_c(\cdot),m)$ is privacy preserving.

It follows from Theorem 5.11 that the functions $\nu(m,\cdot)$ and $\nu(\cdot,m)$ are either constant or one to one, and Lemma 6.9 implies that at least one of the two functions must be nonconstant. This completes the proof.

6.11 Remarks: Proposition 6.10 does not extend to all message processes which realize a nonwasteful allocation process. For example, if μ is the identity, an inference process which yields a complete information process can be constructed as follows. Let ν be the identity on $E \times E$ and let $e \in E$.

Let $e_1, e_2, e_3, e_4 \in E$, let E' be the set of static environments generated by repeated applications of the $\boxtimes^i$ operations to e_1 and e_2, and let E'' be the set of environments similarly generated by e_3 and e_4. Then if $E' \cap E'' = \phi$ and $e \in E' \cup E''$, identify the values of ν on $\{e\} \times E'$, and identify (with a different value) the values of ν on $E'' \times \{e\}$. If the continuity assumption can be dropped from Lemma 5.6, it can also be dropped from the proposition.

The discussion of this section has been limited to inference processes which are empirical in the sense that, if an information process is complete, the inference process is a function of the messages which are observed in an expectations equilibrium. It would also seem possible to use the adequacy condition (*") to analyse inference processes which convey information about the stochastic environment which is not reflected in the observed messages.

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