SHADOW PRICING, INTERNATIONAL TRADE AND THE THEORY OF THE SECOND-BEST

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1. Introduction

Consider a small open economy in which domestic market prices are distorted by fixed, non-optimal taxes. What are the optimal shadow prices for use in guiding public sector production? The existence of distortionary taxes is clearly crucial to this question and places it immediately in the domain of the economics of the second-best. It is by now widely accepted that, given the usual "small country" assumption, the appropriate shadow prices for traded commodities are their international prices. This result has been found to hold despite the existence of non-optimal taxes in the domestic markets for traded and non-traded commodities.[1] On the other hand, there is still considerable disagreement over the appropriate "second-best" shadow prices for non-traded commodities in the presence of distortionary taxes.

The influential contributions of Little and Mirrlees [6], [7] and [8] have recommended connecting the public production of non-traded commodities to international trade by valuing them at their "foreign exchange equivalent," and this proposition has been supported in later work by Dasgupta and Stiglitz [3]. Unfortunately, there has been substantial ambiguity about the precise meaning of the concept of a foreign exchange equivalent, both in the above writings and in those of their critics; in particular, it has been unclear precisely what rates of transformation are to be used in converting non-traded commodities into traded commodities in the measurement of their foreign exchange equivalent.[2] Boadway [1] has claimed that the optimal shadow prices cannot be interpreted as "foreign exchange equivalents," although a proof is not provided, or a clear definition of the term. Instead, the correct shadow price of a non-traded commodity is said to be its producer price plus the change in total tax revenue when public production is changed by one unit, a proposition that seems to have its origin in Harberger [4], [5]. Still another approach is taken by Dasgupta, Marglin and Sen [2] who attempt to value non-traded commodities and foreign exchange in terms of the impact of their generation in the public sector on the value of final consumption at consumer prices.

This paper derives optimal shadow prices in the presence of distortionary taxes from a simple, explicit optimization model in which the tax distortions are incorporated directly as constraints. This exercise is performed under three different sets of assumptions about the tradability of consumer goods on the one hand and of factors on the other. In each case, we then perform a comparative statics exercise on the equilibrium solution to study the validity and precise meaning of the various shadow pricing rules that have been advocated, as well as to indicate the relationships among them.

2. Some Preliminaries

Consider an economy with four commodities. Commodities 1 and 2 are final consumption goods and commodities 3 and 4 are factors available in fixed total supply. These fixed supplies are denoted K_3 and K_4, respectively. There are two firms, one "private" and the other "public." The private firm's implicit production function is denoted $f(y) = 0$ where $y = (y_1, \ldots, y_4)$ is a vector of the private firm's outputs of commodities 1 and 2 and inputs of commodities 3 and 4, respectively, where $y_i \geqq 0$, $i = 1, \ldots, 4$. The public firm's implicit production function is denoted $g(x) = 0$ where $x = (x_1, \ldots, x_4)$ and is defined similarly. The functions f and g are each assumed to be strictly concave and twice continuously diffentiable, but not necessarily identical. There is a single consumer whose utility function (strictly quasi-concave, strictly increasing and twice continuously differentiable) is denoted $U(c)$ where $c = (c_1, c_2)$ and $c_i \geqq 0$ is his consumption of commodity i.

The consumer is assumed to maximize his utility subject to the budget constraint

$$M - q_1 c_1 - q_2 c_2 \geqq 0 \,, \tag{1}$$

where M is his money income and q_1 and q_2 are the market prices of commodities 1 and 2 faced by the consumer. We normalize on commodity 1 by setting $q_1 = 1$. The components of M are: total payments to factors, M^F; the net profits of the public and private firms, M^{Π}; and total tax revenues, M^T. Hence

$$M = M^F + M^{\Pi} + M^T \,. \tag{2}$$

The consumer treats the market prices q_1 and q_2 parametrically and ignores the effect of any tax component in those prices on his budget constraint. Assuming a strictly interior solution as we will, for simplicity, throughout this paper, this implies that in equilibrium

$$U_2/U_1 = q_2 \ .$$

The manager of the private firm is assumed to maximize profits, given by

$$\Pi^y = \sum_{i=1}^{2} p_i y_i - \sum_{i=3}^{4} p_i y_i \ ,$$

subject to the implicit production function $f(y) = 0$, treating the market prices $p_1, \ldots, p_4$ parametrically. We assume throughout that commodity 1 is untaxed, hence $p_1 = q_1 = 1$. For an interior solution we have, in equilibrium,

$$f_2/f_1 = p_2 \ ,$$

$$- f_3/f_1 = p_3 \ ,$$

and

$$- f_4/f_1 = p_4 \ .$$

The manager of the public firm is assumed to maximize shadow profit, given by

$$\Pi^x = \sum_{i=1}^{2} s_i x_i - \sum_{i=3}^{4} s_i x_i \ ,$$

subject to $g(x) = 0$, treating the shadow prices $s_1, \ldots, s_4$ parametrically. For a strictly interior solution this implies, normalizing by setting $s_1 = 1$, that

$$g_2/g_1 = s_2 \ ,$$

$$- g_3/g_1 = s_3 ,$$

and

$$- g_4/g_1 = s_4 .$$

The determination of the shadow prices s_2, s_3 and s_4 is the responsibility of a fourth agent, a "project planner," whose task is to set these shadow prices so as to maximize the consumer's utility. However, this planner is assumed to have no control over any taxes which may be present, and hence must regard the existence of such taxes as constraints on his planning exercise. His problem is thus an exercise in the economics of the second-best. Our concern is with the properties of the "second-best" utility-maximizing shadow prices.

Three different versions of the basic model are considered, differing in their assumptions as to which commodities are traded and which non-traded. In each case, two commodities are assumed to be traded internationally at fixed prices (normalized at unity in each case), and two commodities are non-traded. In Model I, both consumer goods are traded and both factors are non-traded. This is, of course, the familiar "2 x 2" model of international trade theory. In Model II, one consumer good (commodity 1) is traded and the other (commodity 2) is non-traded; one factor (commodity 3) is a traded raw material and the other (commodity 4) is non-traded. In Model III both consumer goods are non-traded and both factors are traded raw materials. In each case, we assume the existence of a fixed tax, denoted t, on the consumption of commodity 2 and a fixed tariff, denoted τ, on the importation of one of the traded commodities. The market prices of non-traded factors and consumer goods are in each case assumed to adjust automatically such that the corresponding resource balance constraints and commodity balance constraints specified below are satisfied as strict equalities.

3. Model I: All Consumer Goods Traded; All Factors Non-traded

Let commodities 1 and 2 be traded and commodities 3 and 4 be non-traded. We assume, for simplicity, that commodity 1 is a net export and commodity 2 is a net import. The trade balance constraint is

$$x_1 + y_1 - c_1 + x_2 + y_2 - c_2 \geqq 0 \tag{3}$$

and the resource balance constraints are

$$K_3 - x_3 - y_3 \geqq 0 \tag{4}$$

and

$$K_4 - x_4 - y_4 \geqq 0 \ . \tag{5}$$

Let there be a tax at the rate t on the consumption of commodity 2 and a tariff at the rate τ on imports of commodity 2. From the profit maximizing behavior of the private firm we have (recalling that we are assuming strictly interior solutions throughout and assuming that commodity 2 continues to be a net import despite the tariff)

$$f_2/f_1 = p_2 = 1 + \tau \tag{6}$$

and from the utility maximization of the consumer

$$U_2/U_1 = q_2 = 1 + \tau + t \ . \tag{7}$$

3.1 Optimal Shadow Prices

The second-best optimization problem is now

Problem I: max $U(c_1, c_2)$ subject to $f(y)=0$, $g(x)=0$ and (3) to (7) .

We form the Lagrangian function

$$L^I \equiv U(c_1, c_2) + \lambda(x_1 + y_1 - c_1 + x_2 + y_2 - c_2)$$

$$+ \rho_3(K_3 - x_3 - y_3) + \rho_4(K_4 - x_4 - y_4)$$

$$+ \mu g(x) + \gamma f(y) + \delta_1(f_2/f_1 - (1+\tau))$$

$$+ \delta_2(U_2/U_1 - (1+\tau+t)) \ . \qquad (8)$$

The first-order conditions for an interior solution are, taking optimal public production first,

$$\lambda^* + \mu^* g_i = 0, \quad i = 1, 2 , \qquad (9)$$

and $$- \rho_j^* + \mu^* g_j = 0, \quad j = 3, 4 , \qquad (10)$$

where an asterisk (*) denotes that the Lagrangian multiplier concerned is evaluated at the optimum.

From these equations we have

$$s_2 = g_2/g_1 = 1 , \qquad (11)$$

$$s_3 = - g_3/g_1 = \rho_3^*/\lambda^* , \qquad (12)$$

and $$s_4 = - g_4/g_1 = \rho_4^*/\lambda^* . \qquad (13)$$

From (11) we see that the traded commodity should be shadow priced at its international price (relative to the numeraire good, commodity 1) while the shadow prices of the non-traded commodities depend on the values of the Lagrangian multipliers λ^*, ρ_3^* and ρ_4^*. Note that λ^* is the shadow price of foreign exchange (in utility numeraire).

We now proceed similarly for the control vectors c and y to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -e_1 \\ 1 & 0 & 0 & 0 & 0 & -e_2 \\ 1 & 0 & 0 & f_1 & b_1 & 0 \\ 1 & 0 & 0 & f_2 & b_2 & 0 \\ 0 & 1 & 0 & f_3 & b_3 & 0 \\ 0 & 0 & 1 & f_4 & b_4 & 0 \end{bmatrix} \begin{bmatrix} \lambda^* \\ \rho_3^* \\ \rho_4^* \\ \gamma^* \\ \delta_1^* \\ \delta_2^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (14)$$

where $b_i \equiv \partial(f_2/f_1)/\partial y_i$ and $e_i \equiv \partial(U_2/U_1)/\partial c_i$. In the above system we now have six linear equations in six unknowns which can be solved for the values of the Lagrangian multipliers at the optimum, giving

$$s_3 = \rho_3^*/\lambda^* = p_3 + \tau\frac{(b_3 + p_3b_1)}{(b_2 - p_2b_1)}, \qquad (15)$$

$$s_4 = \rho_4^*/\lambda^* = p_4 + \tau\frac{(b_4 + p_4b_1)}{(b_2 - p_2b_1)}. \qquad (16)$$

3.2 Final Consumption Interpretation

Definition 1. The final consumption effect of a commodity is the rate of change of the value of final consumption, at consumer prices, with respect to the public sector's net output of that commodity.[3]

Proposition 1. In the case of Model I, the optimal shadow price of a non-traded factor is its final consumption effect, relative to that of the numeraire commodity, valued at the optimum.

Proof: Writing H_k for the final consumption effect of commodity k we have, for commodity 3 (a net input),

$$\frac{H_3}{H_1} = - \frac{dc_1/dx_3 + q_2 dc_2/dx_3}{dc_1/dx_1 + q_2 dc_2/dx_1} \ . \qquad (17)$$

Now, totally differentiating the private firm's production function and equations (3), (6) and (7) with respect to x_1 and x_3, and noting that (from equation (4)) $dy_3/dx_3 = -1$ and that (from equation (5)) $dx_4 = 0$ implies $dy_4 = 0$, we obtain the system

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & p_2 & 0 & 0 \\ b_1 & b_2 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \end{bmatrix} \begin{bmatrix} dy_1/dx_1 & dy_1/dx_3 \\ dy_2/dx_1 & dy_2/dx_3 \\ dc_1/dx_1 & dc_1/dx_3 \\ dc_2/dx_1 & dc_2/dx_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -p_3 \\ 0 & b_3 \\ 0 & 0 \end{bmatrix} . \qquad (18)$$

This system can be solved, using Cramer's rule, for the desired derivatives dc_i/dx_j, where $i = 1, 2$ and $j = 1, 3$. Substituting into (17) gives

$$\frac{H_3}{H_1} = p_3 + \tau \frac{(b_3 + p_3 b_1)}{(b_2 - p_2 b_1)} = s_3 \ .$$

The proof proceeds identically for commodity 4.

The intuitive explanation for this result is straightforward. Clearly

$$dU/dx_k = U_1 dc_1/dx_k + U_2 dc_2/dx_k = U_1(dc_1/dx_k + q_2 dc_2/dx_k) \ .$$

Hence,

$$\frac{dc_1/dx_k + q_2 dc_2/dx_k}{dc_1/dx_1 + q_2 dc_2/dx_1} = \frac{dU/dx_k}{dU/dx_1} \ .$$

Consider the addition of a constant α_1 to the trade balance constraint and a constant α_k to the resource constraint for factor k, giving

$$x_1 + y_1 - c_1 + x_2 \; y_2 - c_2 + \alpha_1 \geqq 0 \;, \tag{19}$$

and

$$K_k - x_k - y_k + \alpha_k \geqq 0 \,, \quad k = 3,\, 4 \;. \tag{20}$$

Now, from the duality relationships of non-linear programming, it is well known that the values of the Lagrangian multipliers at the optimum are given by

$$\frac{\rho_k^*}{\lambda^*} = \left.\frac{dU/d\alpha_k}{dU/d\alpha_1}\right|_{\substack{\alpha_1 = 0 \\ \alpha_k = 0}} \qquad k = 3,\, 4 \;.$$

But it is easily seen that

$$\frac{dU/d\alpha_k}{dU/d\alpha_1} = - \frac{dU/dx_k}{dU/dx_1} \;, \qquad k = 3,\, 4 \;.$$

Consequently, at the optimum,

$$\frac{\rho_k^*}{\lambda^*} = - \frac{dU/dx_k}{dU/dx_1} = - \frac{dc_1/dx_k + q_2 dc_2/dx_k}{dc_1/dx_1 + q_2 dc_2/dx_1} \;. \tag{21}$$

3.3 Government Revenue Interpretation

Definition 2. The government revenue effect of a commodity is its producer price plus the rate of change of total tax revenues with respect to the public sector's net output of that commodity.[4]

Proposition 2. In the case of Model I, the optimal shadow price of a non-traded factor is its government revenue effect, relative to that of the numeraire commodity, valued at the optimum.

Proof: Writing R^T for total government tax revenues, and R_k for the government revenue effect of commodity k we have

$$R^T = tc_2 + \tau(c_2 - x_2 - y_2) \ , \tag{22}$$

$$\frac{R_3}{R_1} = \frac{p_3 - dR^T/dx_3}{1 + dR^T/dx_1} = \frac{p_3 - (\tau+t)dc_2/dx_3 + \tau dy_2/dx_3}{1 + (\tau+t)dc_2/dx_1 - \tau dy_2/dx_1} \ . \tag{23}$$

Now returning to equation system (18) and solving for the above derivatives as before we again obtain

$$\frac{R_3}{R_1} = p_3 + \tau\frac{(b_3 + p_3b_1)}{(b_2 - p_2b_1)} = s_3 \ . \tag{24}$$

The proof is isomorphic for commodity 4.

It is of some interest to see how (23) can be derived directly from (17). By differentiating the trade balance constraint totally with respect to x_1 and x_3 we obtain

$$-\frac{dc_1/dx_3 + q_2dc_2/dx_3}{dc_1/dx_1 + q_2dc_2/dx_1} = \frac{- dy_1/dx_3 - dy_2/dx_3 + (\tau+t)dc_2/dx_3}{1 + dy_1/dx_1 + dy_2/dx_1 + (\tau+t)dc_2/dx_1} \ . \tag{25}$$

Now, differentiating $f(y) = 0$ totally with respect to x_1 and x_3 ,

$$dy_1/dx_1 + dy_2/dx_1 = - \tau dy_2/dx_1 \ , \tag{26}$$

$$dy_1/dx_3 + dy_2/dx_3 = - p_3 - \tau dy_2/dx_3 \ . \tag{27}$$

Equation (25) now becomes

$$\frac{p_3 + \tau dy_2/dx_3 - (\tau+t)dc_2/dx_3}{1 - \tau dy_2/dx_1 + (\tau+t)dc_2/dx_1} = \frac{p_3 - dR^T/dx_3}{1 + dR^T/dx_1} \ . \tag{28}$$

The intuitive interpretation for this result is as follows. Consider the effect of public production (use) of a commodity on the consumer's budget constraint. Since all government revenue is handed over to him, a change in government revenue directly affects his budget constraint. Writing R for total government revenues, $R = R^{\Pi} + R^{T}$, where R^{Π} is the net profit of the public firm, so that

$$- \frac{dR/dx_3}{dR/dx_1} = \frac{- dR^{\Pi}/dx_3 - dR^{T}/dx_3}{dR^{\Pi}/dx_1 + dR^{T}/dx_1} = \frac{p_3 - dR^{T}/dx_3}{1 + dR^{T}/dx_1} .$$

Hence, the "government revenue effect" of a commodity gives the effect of a change in public production or use of that commodity on the consumer's budget constraint via its effect on total government revenues.

In Model I, the shadow price of foreign exchange is given by

$$\lambda^* = U_1 \left[1 + \frac{(\tau+t)e_1}{e_1 - e_2} \right] . \tag{29}$$

In Boadway [1], Proposition 2 is stated correctly except for an error concerning the shadow price of foreign exchange. Boadway states that the shadow price of a traded commodity is its international price and the shadow price of a non-traded commodity is its government revenue effect. This assumes implicitly that the government revenue effect of the numeraire commodity in equation (23) is simply its producer price (unity) or, alternatively, that the shadow price of foreign exchange is given by $\lambda^* = U_1$. Boadway's analytical procedure is to <u>define</u> the shadow price of the non-traded commodity n to be $1/U_1 dU/dx_n$. But the shadow price of the traded commodity k is not set equal to

$$\frac{1}{U_1}\frac{dU}{dx_k} = \frac{1}{U_1}\lambda^* r_k = \left[1 + \frac{(\tau+t)e_1}{e_1 - e_2}\right] r_k \ , \tag{30}$$

where r_k is the international price of commodity k (unity in our analysis), but simply r_k . Consequently, Boadway implicitly assumes that $\lambda^*=U_1$. [1, p. 426]

3.4 Foreign Exchange Interpretation

Definition 3. The foreign exchange effect of a commodity is the rate of change of the total foreign exchange earnings generated by the private sector with respect to the public sector's net output of that commodity.

Proposition 3. In the case of Model I, the optimal shadow price of a non-traded factor is its foreign exchange effect, as given by Definition 3, defined at the optimum.

Proof: The private firm's contribution to total foreign exchange earnings is given by $E^y = y_1 + y_2$. Hence,

$$dE^y/dx_3 = dy_1/dx_3 + dy_2/dx_3 \ .$$

By solving for these derivatives from equation system (18) we obtain, on substituting,

$$\frac{dE^y}{dx_3} = p_3 + \tau\frac{(b_3 + p_3 b_1)}{(b_2 - p_2 b_1)} \ . \tag{31}$$

Definition 3'. The foreign exchange effect of a commodity is the rate of change of the value of final consumption at international prices with respect to the public sector's net output of that commodity.

Proposition 3'. In the case of Model I the optimal shadow price of a non-traded factor is its foreign exchange effect, as given by Definition 3', defined at the optimum.

Proof: Given Proposition 3 it is only necessary to show that, for Model I, Definitions 3 and 3' are equivalent. Writing E^c for the value of final consumption at international prices

$$dE^c/dx_3 = dc_1/dx_3 + dc_2/dx_3 \ .$$

But, differentiating the trade balance constraint totally with respect to x_3

$$dc_1/dx_3 + dc_2/dx_3 = dy_1/dx_3 + dy_3/dx_3 \ .$$

Hence, $dE^c/dx_3 = dE^y/dx_3$.

The intuitive explanation for these results is again quite straightforward. When all final consumption goods are traded, public and private production can affect the consumer's utility only via the trade balance constraint--by providing foreign exchange. The consumer's utility is maximized by maximizing foreign exchange earnings; this is achieved, even given the tax distortions, by allocating non-traded factors between public and private use such that their marginal contribution to foreign exchange earnings is equated between the two sectors.

3.5 Market Behavior Interpretation

The shadow price expressions of equations (15) and (16) can be interpreted in terms of observable market behavioral quantities as follows. The strict concavity of f implies that the first-order conditions for profit maximization of the private firm generate single valued supply functions (i=1, 2) and input demand functions (i=3, 4) of the form $y_i = Y_i(p_2, p_3, p_4)$ which, by the implicit function theorem, are differentiable. Substituting these functions into the equations $f(y) = 0$ and $b(y) = p_2$ and differentiating with respect to p_2, p_3 and p_4 we obtain

$$\begin{bmatrix} Y_{22} & Y_{23} & Y_{24} \\ Y_{23} & Y_{33} & Y_{34} \\ Y_{24} & Y_{34} & Y_{44} \end{bmatrix} \begin{bmatrix} (b_2 - p_2 b_1) \\ (b_3 + p_3 b_1) \\ (b_4 + p_4 b_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

where $Y_{ij} = \partial y_i / \partial p_j$. Now, substituting into equations (15) and (16), we have

$$s_3 = p_3 - \tau \frac{\begin{vmatrix} Y_{23} & Y_{34} \\ Y_{24} & Y_{44} \end{vmatrix}}{\begin{vmatrix} Y_{33} & Y_{34} \\ Y_{34} & Y_{44} \end{vmatrix}} \quad \text{and} \quad s_4 = p_4 + \tau \frac{\begin{vmatrix} Y_{23} & Y_{33} \\ Y_{24} & Y_{34} \end{vmatrix}}{\begin{vmatrix} Y_{33} & Y_{34} \\ Y_{34} & Y_{44} \end{vmatrix}} .$$

A special case of some interest is the partial equilibrium case of zero cross-price effects; that is, $Y_{ij} = 0$, $i \neq j$. The above expressions then reduce simply to $s_3 = p_3$ and $s_4 = p_4$.

4. Model II: Some Consumer Goods Traded; Some Factors Traded

Now let commodities 1 and 3 be traded and commodities 2 and 4 be non-traded. We thus have a non-traded final consumption good (commodity 2) and a traded raw material (commodity 3). We assume, for simplicity, that commodity 3 is a net import and commodity 1 is a net export. The trade balance constraint is now

$$x_1 + y_1 - c_1 + K_3 - x_3 - y_3 \geqq 0 \ , \tag{32}$$

and the commodity balance constraint for commodity 2 is

$$x_2 + y_2 - c_2 \geqq 0 \ , \tag{33}$$

while the resource constraint for commodity 4 is

$$K_4 - x_4 - y_4 \geqq 0 \ . \tag{34}$$

Let there again be a tax at the rate t on the consumption of commodity 2 and a tariff at the rate τ on imports of commodity 3. From the profit-maximizing behavior of the private firm we have (assuming strictly interior solutions as before, and that commodity 3 continues to be a net import despite the tariff)

$$- f_3/f_1 = p_3 = 1 + \tau \ , \tag{35}$$

and from utility maximization of the consumer

$$U_2/U_1 = q_2 = p_2 + t = f_2/f_1 + t \ . \tag{36}$$

4.1 Optimal Shadow Prices

The second-best optimization problem is clearly

<u>Problem II</u>: max $U(c_1, c_2)$ subject to $f(y)=0$, $g(x)=0$ and (32) to (36).

We form the Lagrangian function

$$L^{II} \equiv U(c_1, c_2) + \lambda(x_1 + y_1 - c_1 + K_3 - x_3 - y_3) + \rho_2(x_2 + y_2 - c_2) + \rho_4(K_4 - x_4 - y_4) + \mu g(x) + \gamma f(y) + \eta_1(f_3/f_1 + 1 + \tau) + \eta_2(U_2/U_1 - f_2/f_1 - t). \quad (37)$$

Deriving first-order conditions for optimal public production as before, we have, for an interior solution

$$s_2 = g_2/g_1 = \rho_2^*/\lambda^* , \quad (38)$$

$$s_3 = - g_3/g_1 = 1 , \quad (39)$$

and

$$s_4 = - g_4/g_1 = \rho_4^*/\lambda^* . \quad (40)$$

We now see that the optimal shadow price of the traded input is its international price while the shadow prices of the non-traded commodities depend on the Lagrangian multipliers ρ_2^*, ρ_4^* and λ^* (the shadow price of foreign exchange).

Proceeding similarly for the control vectors c and y we obtain the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -e_1 \\ 0 & 1 & 0 & 0 & 0 & -e_2 \\ 1 & 0 & 0 & f_1 & a_1 & -b_1 \\ 0 & 1 & 0 & f_2 & a_2 & -b_2 \\ -1 & 0 & 0 & f_3 & a_3 & -b_3 \\ 0 & 0 & -1 & f_4 & a_4 & -b_4 \end{bmatrix} \begin{bmatrix} \lambda^* \\ \rho_2^* \\ \rho_4^* \\ \gamma^* \\ \eta_1^* \\ \eta_2^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \quad (41)$$

where $a_i \equiv \partial(f_3/f_1)/\partial y_i$, $b_i \equiv \partial(f_2/f_1)/\partial y_i$ and $e_i \equiv \partial(U_2/U_1)/\partial c_i$. Solving by Cramer's rule we obtain

$$s_2 = \frac{\rho_2^*}{\lambda^*} = p_2 + \frac{tB_{(4)} + \tau(a_2 - p_2a_1)(e_2 - q_2e_1)}{B_{(4)} + (a_3 + p_3a_1)(e_2 - q_2e_1)} , \tag{42}$$

and

$$s_4 = \frac{\rho_4^*}{\lambda^*} = p_4 + \frac{t\, B_{(2)} - \tau[B_{(3)} + (a_4 + p_4a_1)(e_2 - q_2e_1)]}{B_{(4)} + (a_3 + p_3a_1)(e_2 - q_2e_1)} , \tag{43}$$

where $B_{(j)}$ is the determinant of the matrix

$$\begin{bmatrix} 1 & p_2 & -p_3 & -p_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

with the jth column deleted.

4.2 Final Consumption Interpretation

Proposition 4: In the case of Model II, the optimal shadow prices of the non-traded consumption good and the non-traded factor are their final consumption effects, relative to that of the numeraire commodity, defined at the optimum.

Proof: Writing H_k for the final consumption effect of commodity k as before we have

$$\frac{H_2}{H_1} = \frac{dc_1/dx_2 + q_2dc_2/dx_2}{dc_1/dx_1 + q_2dc_2/dx_1} , \tag{44}$$

and

$$\frac{H_4}{H_1} = - \frac{dc_1/dx_4 + q_2dc_2/dx_4}{dc_1/dx_1 + q_2dc_2/dx_1} . \tag{45}$$

Now, totally differentiating the private firm's implicit production function and equations (32), (33), (35) and (36) with respect to x_1, x_2 and x_4 we obtain, noting that $dy_4/dx_4 = -1$,

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & p_2 & -p_3 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 \\ b_1 & b_2 & b_3 & -e_1 & -e_2 \end{bmatrix} \begin{bmatrix} \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_4} \\ \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_4} \\ \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_4} \\ \frac{dc_1}{dx_1} & \frac{dc_1}{dx_2} & \frac{dc_1}{dx_4} \\ \frac{dc_2}{dx_1} & \frac{dc_2}{dx_2} & \frac{dc_2}{dx_4} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -p_4 \\ 0 & 0 & a_4 \\ 0 & 0 & b_4 \end{bmatrix} . \quad (46)$$

We can now solve for the derivatives dc_i/dx_j (i=1, 2 and j=1, 2 and 4) and substitute into equations (44) and (45). This gives

$$\frac{H_2}{H_1} = p_2 + \frac{tB_{(4)} + \tau(a_2 - p_2a_1)(e_2 - q_2e_1)}{B_{(4)} + (a_3 + p_3a_1)(e_2 - q_2e_1)}$$

and

$$\frac{H_4}{H_1} = p_4 + \frac{tB_{(2)} - \tau[B_{(3)} + (a_4 + p_4a_1)(e_2 - q_2e_1)]}{B_{(4)} + (a_3 + p_3a_1)(e_2 - q_2e_1)} .$$

Proposition 4 is expected, given that, by argument similar to that of section 3.2,

$$\frac{\rho_2^*}{\lambda^*} = \frac{dU/dx_2}{dU/dx_1} \quad \text{and} \quad \frac{\rho_4^*}{\lambda^*} = - \frac{dU/dx_4}{dU/dx_1} .$$

4.3 Government Revenue Interpretation

Proposition 5: In the case of Model II the optimal shadow prices of the non-traded consumer good and the non-traded factor are their government revenue effects, relative to that of the numeraire commodity, defined at the optimum.

Proof: In light of Proposition 4, it is sufficient to show that $R_2/R_1 = H_2/H_1$ and $R_4/R_1 = H_4/H_1$. Total tax revenues are given by

$$R^T = tc_2 + \tau(y_3 + x_3 - R_3) \ .$$

Hence

$$dR^T/dx_k = tdc_2/dx_k + \tau dy_3/dx_k \qquad k=1,\ 2,\ 4\ . \tag{47}$$

Differentiating the private firm's production function and equations (32) and (33) totally with respect to x_1, x_2 and x_3 we obtain, on substituting,

$$\tau dy_3/dx_1 = dc_1/dx_1 + p_2dc_2/dx_1 - 1\ ,$$

$$\tau dy_3/dx_2 = dc_1/dx_2 + p_2(dc_2/dx_2 - 1)$$

and

$$\tau dy_3/dx_4 = dc_1/dx_4 + p_2dc_2/dx_4 + p_4\ .$$

Substituting into equation (47) gives

$$\frac{p_2 + dR^T/dx_2}{1 + dR^T/dx_1} = \frac{dc_1/dx_2 + q_2dc_2/dx_2}{dc_1/dx_1 + q_2dc_2/dx_1} = \frac{H_2}{H_1}\ ,$$

$$\frac{p_4 - dR^T/dx_4}{1 + dR^T/dx_4} = -\frac{dc_1/dx_4 + q_2dc_2/dx_4}{dc_1/dx_1 + q_2dc_2/dx_1} = \frac{H_4}{H_1}\ .$$

Finally we note that, in the case of Model II,

$$\lambda^*/U_1 = 1 + \frac{e_1\{\tau(a_2 - p_2a_1) - t(a_3 + p_3a_1)\}}{B_{(4)} + (a_3 + p_3a_1)(e_2 - p_2e_1) - \tau e_1(a_2 - p_2a_1)} . \qquad (48)$$

The version of Proposition 5 stated in [1] must be modified accordingly.

4.4 Foreign Exchange Interpretation

The existence of a non-traded final consumption good poses an immediate problem for the definition of the foreign exchange effect of either the non-traded consumption good or the non-traded factor in Model II. Public production (use) of these commodities will clearly affect the private firm's foreign exchange earnings and hence the amount of the traded consumer good (commodity 1) available for consumption. But there will also be an effect on private production of the non-traded consumer good (commodity 2) and the amount of that good available for consumption. The problem is to define the foreign exchange effect of public production in such a way as to eliminate this effect on the amount of the non-traded consumer good available for consumption and to channel the effect of public production entirely into extra foreign exchange earnings, that is, extra potential consumption of traded commodities.

Definition 4. The foreign exchange effect of a commodity is the rate of change of the foreign exchange earnings generated by the private sector (foreign exchange value of final consumption of traded commodities) with respect to the public sector's net output of that commodity, when the consumption of non-traded commodities is held constant.

To see what this means, differentiate the commodity balance constraint for good 2 with respect to x_2, to obtain

$$1 + dy_2/dx_2 = dc_2/dx_2 \ .$$

If dc_2/dx_2 is to be held to zero, then $dy_2/dx_2 = -1$. That is, for there to be no net increase in consumption, private production of good 2 must be made to decline at the same rate as public production increases. This amounts to an additional constraint on the behavior of the private firm, over-determining equation system (46). Hence, to perform the conceptual experiment described by Definition 4 we must, in general, violate the market equilibrium condition for commodity 2, namely that $f_2/f_1 = U_2/U_1 - t$.

Differentiating the private firm's implicit production function and (35) totally with respect to x_2 and x_4 we obtain, setting $dc_2/dx_2 = 1 + dy_2/dx_2 = 0$ and $dc_2/dx_4 = dy_2/dx_4 = 0$,

$$\begin{bmatrix} 1 & -p_3 \\ a_1 & a_3 \end{bmatrix} \begin{bmatrix} dy_1/dx_2 & dy_1/dx_4 \\ dy_3/dx_2 & dy_3/dx_4 \end{bmatrix} = \begin{bmatrix} p_2 & -p_4 \\ a_2 & a_4 \end{bmatrix} . \tag{49}$$

Solving by Cramer's rule

$$\frac{dc_1}{dx_2} = \frac{dy_1}{dx_2} - \frac{dy_3}{dx_2} = p_2 + \tau\frac{(a_2 - p_2a_1)}{(a_3 + p_3a_1)} \tag{50}$$

and

$$-\frac{dc_1}{dx_4} = -\frac{dy_1}{dx_4} + \frac{dy_3}{dx_4} = p_4 - \tau\frac{(a_4 + p_4a_1)}{(a_3 + p_3a_1)} . \tag{51}$$

To see the meaning of these expressions, consider the following problem.

<u>Problem II'</u>: max c_1 subject to $f(y)=0$, $g(x)=0$, equations (32), (34), (35) and $x_2 + y_2 - c_2^o \geqq 0$, where c_2^o is a pre-specified positive constant.

We form the Lagrangian function

$$L^{II'} \equiv c_1 + \lambda'(x_1 + y_1 - c_1 + K_3 - x_3 - y_3) + \rho_2'(x_2 + y_2 - c_2^o)$$

$$+ \rho_4'(K_4 - x_4 - y_4) + \mu' g(x) + \gamma' f(y) + \eta_1'(f_3/f_1 + 1 + \tau) \ . \qquad (52)$$

Deriving first-order conditions as before we obtain

$$s_2' = g_2/g_1 = p_2 + \tau\frac{(a_2 - p_2 a_1)}{(a_3 = p_3 a_1)} \ , \qquad (53)$$

$$s_3' = - g_3/g_1 = 1 \ , \qquad (54)$$

and

$$s_4' = - g_4/g_1 = p_4 - \tau\frac{(a_4 + p_4 a_1)}{(a_3 + p_3 a_1)} \ . \qquad (55)$$

Clearly, Problem II' is isomorphic to the valuation of the traded factor at its international price and the valuation of the non-traded consumer good and factor at their foreign exchange effects as given by Definition 4. It is now of interest to see what must hold for Problems II and II' to be equivalent. We will focus on the case of the non-traded consumer good.

Proposition 6. In the case of Model II, the optimal shadow price of the non-traded consumer good is not necessarily equal to its foreign exchange effect as given by Definition 4, defined at the optimum.

Proof: Suppose that Problems II and II' are equivalent. Then in Problem II', $c_2^o = c_2^*$, the optimal value of c_2 in Problem II. Now, setting equations (42) and (53) equal to one another and rearranging, we obtain

$$B_{(4)}[t(a_3 + p_3 a_1) - \tau(a_2 - p_2 a_1)] = 0 \ . \qquad (56)$$

That this is a necessary condition for equations (42) and (53) to be identical follows directly from the fact that their identity implies (56). That it is also a sufficient condition can be seen by substituting either (i) $B_{(4)} = 0$ or (ii) $t(a_3 + p_3 a_1) = \tau(a_2 - p_2 a_1)$ into equation (42). In case (i), equation (42) reduces immediately to equation (53). In case (ii), both (42) and (53) reduce to $s_2 = s_2' = p_2 + t = q_2$. Hence, the optimal shadow price of the non-traded consumer good is its foreign exchange effect if and only if (56) holds. But (56) does not necessarily hold. For example, a sufficient condition to exclude either case (i) or case (ii) is that sign $(a_1, a_2, a_3) = (+, 0, +)$ and sign $(b_1, b_2, b_3) = (-, +, 0)$. It is easily checked by differentiating $a = f_3/f_1$ and $b = f_2/f_1$, that these sign assumptions are fully consistent with the strict concavity of f.

Hence, while it is a mistake to claim that the optimal shadow price of the non-traded consumer good is never interpretable as a foreign exchange equivalent, it is clear that, in Model II, this interpretation is not generally valid. It is easily shown that this result also holds for the non-traded factor in Model II. Clearly, it is the existence of a non-traded final consumption good whose consumption is in general affected by public production that causes the difficulty. For the final consumption of good 2 to be unaffected by public production we must have $dy_2/dx_2 = -1$. The necessary condition for this to occur automatically is, solving from equation system (46),

$$B_{(4)} = e_1[p_2(a_3 + p_3 a_1) + \tau(a_2 - p_2 a_1)] \ . \qquad (57)$$

The above sign assumptions, together with the assumption that $e_1 \equiv (U_{12} - q_2 U_{11})/U_1 > 0$, are also sufficient to preclude (57) from holding.

4.5 Market Behavior Interpretation

The shadow price expressions of equations (42) and (43) can be interpreted in terms of market behavioral quantities by substituting the output supply and input demand functions $y_i = Y_i(p_2, p_3, p_4)$ into the equations $f(y) = 0$, $a(y) = -p_3$ and $b(y) = p_2$. Differentiating with respect to p_2, p_3 and p_4 gives

$$\begin{bmatrix} Y_{22} & Y_{23} & Y_{24} \\ Y_{23} & Y_{33} & Y_{34} \\ Y_{24} & Y_{34} & Y_{44} \end{bmatrix} \begin{bmatrix} (a_2 - p_2 a_1) & (b_2 - p_2 b_1) \\ (a_3 + p_3 a_1) & (b_3 + p_3 b_1) \\ (a_4 + p_4 a_1) & (b_4 + p_4 b_1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{58}$$

Now, differentiating the equation $e(c) = q_2$ with respect to q_2 gives, writing C_{i2} for $\partial c_i/\partial q_2$,

$$e_1 C_{12} + e_2 C_{22} = 1 . \tag{59}$$

Assuming (a) zero cross-price effects on the production side $(Y_{ij}=0,\ i\neq j)$, and (b) either (i) $e_1 = 0$ or (ii) $C_{12}/C_{22} = -q_2$ on the demand side, we obtain

$$s_2 = p_2 + \frac{t\partial c_2/\partial q_2}{\partial c_2/\partial q_2 - \partial y_2/\partial p_2} = \frac{p_2 \partial y_2/\partial p_2 - q_2 \partial c_2/\partial q_2}{\partial y_2/\partial p_2 - \partial c_2/\partial q_2} , \tag{60}$$

and

$$s_4 = p_4 . \tag{61}$$

The first of these is the well-known partial equilibrium Harberger "weighted average" formula [4].[5]

5. Model III: All Consumer Goods Non-traded; All Factors Traded

We now consider briefly the case where both consumer goods are non-traded and both factors are traded raw materials. Let commodity 3 be a net import and commodity 4 be a net export. The trade balance constraint is

$$K_3 - x_3 - y_3 + K_4 - x_4 - y_4 \geqq 0 \ , \tag{62}$$

and the commodity balance constraints are

$$x_1 + y_1 - c_1 \geqq 0 \ , \tag{63}$$

and

$$x_2 = y_2 - c_2 \geqq 0 \ . \tag{64}$$

Let there be a tax at the rate t on the consumption of good 2 and a tariff at the rate τ on imports of commodity 3. This implies that

$$U_2/U_1 = q_2 = p_2 + t \tag{65}$$

and

$$- f_3/f_1 = p_3 = 1 + \tau \ . \tag{66}$$

The second-best optimization problem is now

Problem III: max $U(c_1, c_2)$ subject to $f(y)=0$, $g(x)=0$ and (62) to (66)

Given the techniques developed in the previous sections the reader may easily derive the optimal shadow prices for the non-traded consumption goods and verify that the optimal relative shadow prices for the traded factors are their international prices. Given our earlier results it is easily verified that[6]

$$\frac{s_2}{s_1} = \frac{dU/dx_2}{dU/dx_1} = \frac{dc_1/dx_2 + q_2 dc_2/dx_2}{dc_1/dx_1 + q_2 dc_2/dx_1} \ . \tag{67}$$

Our concern here is to show that the final consumption effect and the government revenue effect of the non-traded consumption goods are equivalent in Model III, but that the notion of the foreign exchange effect of a non-traded commodity has no meaning in Model III.

Total tax revenues are given by

$$R^T = tc_2 + \tau(x_3 + y_3 - K_3)$$

and the government revenue effect of commodity 2 is given by

$$p_2 + dR^T/dx_2 = p_2 + tdc_2/dx_2 + \tau dy_3/dx_2 \ . \tag{68}$$

Differentiating the private firm's production function totally with respect to x_2

$$dy_1/dx_2 + p_2dy_2/dx_2 - (1+\tau)dy_3/dx_2 - dy_4/dx_2 = 0 \ . \tag{69}$$

Differentiating the trade balance constraint we obtain

$$dy_3/dx_2 + dy_4/dx_2 = 0 \ . \tag{70}$$

Hence (69) becomes

$$\tau dy_3/dx_2 = dy_1/dx_2 + p_2dy_2/dx_2 \ . \tag{71}$$

Now, differentiating the commodity balance constraints,

$$dy_1/dx_2 = dc_1/dx_2 \quad \text{and} \quad 1 + dy_2/dx_2 = dc_2/dx_2 \tag{72}$$

and equation (68) becomes

$$p_2 + dR^T/dx_2 = dc_1/dx_2 + q_2dc_2/dx_2 \tag{73}$$

and we see that the government revenue effect of commodity 2 is equivalent to its final consumption effect. The same obviously applies to commodity 1.

Now consider the effect of public production of non-traded commodities on the private sector's foreign exchange earnings. Totally differentiating the trade balance constraint,

$$dy_3/dx_1 + dy_4/dx_1 = 0 \tag{74}$$

and

$$dy_3/dx_2 + dy_4/dx_2 = 0 \, . \tag{75}$$

Since there is no consumption of traded commodities in Model III, the trade balance constraint implies that the foreign exchange effects of non-traded commodities are by definition zero. The concept of a foreign exchange effect therefore has no meaning in such a model.

6. Conclusions

This paper has analysed the properties of "second-best" optimal shadow prices for guiding public production in a small open economy in the presence of fixed distortionary taxes. This exercise has shown that, using a traded commodity as numeraire:

1. The second-best optimal shadow price of a traded commodity is its international price.

2. The second-best optimal shadow price of a non-traded commodity is its "final consumption effect," relative to that of the numeraire commodity, defined at the optimum.

3. The second-best optimal shadow price of a non-traded commodity is also given by its "government revenue effect," relative to that of the numeraire commodity, defined at the optimum. In these models this concept is logically equivalent to that of a "final consumption effect."

4. The second-best optimal shadow price of a non-traded commodity is given by its "foreign exchange effect," defined at the optimum, when all final consumption goods are traded. When only some are traded, and some are non-traded, this result does not hold generally, and when all final consumption goods are non-traded, the concept of a "foreign exchange effect" is meaningless.

Some limitations of the analysis should be made explicit. First, we have considered only distortions due to non-optimal taxes, and some of the results, especially those on the equivalence of final consumption effects and government revenue effects, are dependent on this.[7] More broadly, however, the analytical exercise pursued here of deriving shadow prices from the first-order conditions from an optimization model provides information about the properties of the optimal solution, but not necessarily a mechanism for reaching that solution.[8] This point is overlooked throughout the literature on shadow pricing. Shadow pricing formulae derived from first-order conditions are spoken of as "rules" for benefit-cost analysis as if it were known that iterative application of those rules, starting from some non-optimal position, would lead to the attainment of the optimum that the first-order conditions describe. This implicit assumption is not always correct, and it obfuscates the informational and convergence problems involved in such an adjustment process.

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FOOTNOTES

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1 See, for example, [7], [9], [3], [1] and [10].

2 Dasgupta and Stiglitz [3, pp. 16, 17, 23 and 30] actually seem to suggest that the rate of transformation that is meant to be used is the rate of transformation between the non-traded commodity concerned and the traded numeraire commodity applying in the public firm at the optimum. This is tautologically correct because the object of the shadow pricing exercise is precisely to find this rate of transformation; it could hardly be an input into the calculation.

3 Note that "net output" is negative in the case of a net input and that all other inputs and outputs of the public sector are being held constant in Definition 1.

4 See footnote 3 above.

5 It is worth noting that, despite its numeraire error referred to above, the Boadway [1] rule also reduces to equation (60) in the case of the above assumptions.

6 Note that the numeraire commodity in (67) is non-traded.

7 In [1, p. 429] it is claimed without proof that results similar to the "government revenue effect" results derived here generalize to distortions due to "monopoly, externalities, risk, etc." This claim is unjustified.

8 The author has tried to demonstrate this point and suggest some solutions in a somewhat simpler model, in [11].