SHADOW PRICING, INFORMATION AND STABILITY IN A SIMPLE OPEN ECONOMY

by

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1. Introduction

Given the existence of distortions in market prices which cannot, for the time being, be corrected, the question arises whether it is possible to increase welfare by basing production decisions in the public sector on some alternative set of prices. This is the essential question of shadow pricing. Since market distortions are assumed to exist, and are to be taken as given, shadow pricing is essentially a problem in the economics of the second-best. The above question can be thought of as consisting, first, of whether welfare-increasing shadow prices exist, and second, of whether it is possible to find such prices in practice. Given the usual convexity assumptions there is little controversy over the first question, but there is substantial disagreement over the second.

The literature on shadow pricing falls largely into two categories. Writers in the first group emphasize the informational problems of finding the correct second-best shadow prices, although the precise nature of the informational problems involved is not usually made explicit. A familiar proposition, however, is that when there are non-traded commodities the informational problems of finding the dual solution to the second-best optimal public sector production program (shadow prices) are equivalent to the informational problems of finding the corresponding primal solution.[1] Thus it is suggested that unless the planner concerned with determining shadow prices is endowed with information on such crucial matters as the public sector's production possibility set, he could not possibly hope to determine the second-best optimal set of shadow prices. These writers recommend that, in practice, public sector production decisions be based on domestic market prices, both for traded and non-traded commodities, even though these prices are known to be distorted.[2]

Writers in the second group attempt to derive rules for public sector shadow pricing from the first-order conditions from an optimization model. Sometimes this exercise is conducted relatively formally, sometimes more casually, but the most common recommendation to emerge is that traded commodities should be valued at their international prices, irrespective of the existence of tariffs, and that non-traded commodities should be valued at their "foreign exchange equivalent."[3] There has been considerable ambiguity over the precise meaning of the latter concept, but a simplistic interpretation is that it is the rate of transformation between the non-traded commodity concerned and a traded commodity in the private sector.[4] In general, writers in this category do not consider explicitly how the economy moves from some initial non-optimal position to the second-best optimum described by the first-order conditions, although the shadow pricing rules are clearly meant to facilitate the transition. Some writers have suggested that an iterative adjustment mechanism is entailed, but its precise nature is not made explicit.[5] This has tended to obfuscate the informational and stability issues involved in such adjustment processes.

This paper attempts to clarify some of the informational and stability problems involved in shadow pricing in the context of a simple economic model. This model is described in section 2. In section 3 a fixed distortionary tariff is introduced and the necessary conditions for a second-best optimum are derived. We then interpret the meaning of these first-order conditions in terms of shadow prices. Section 4 then examines the informational problems involved in moving from an initial non-optimal position to this second-best optimum by means of shadow pricing. On the assumption that iterative determination of shadow prices is informationally feasible, section 5 examines the convergence of the most obvious adjustment process and shows that, for an important class of examples, this process is non-convergent. Finally, section 6 considers the implications of the determination of second-best shadow prices being informationally infeasible.

2. Notation and Assumptions

Consider an economy with three commodities:

1--a traded, domestically produced consumption good (say, wheat), whose fixed international price is normalized at unity;

2--a traded input, not domestically produced (say, fertilizer), whose fixed international price is normalized at unity; and

3--a non-traded input (say, land), whose fixed total supply is normalized at unity.

There is a single consumer whose utility function is simply $U = c_1$, where c_1 is his consumption of commodity 1; $c_1 \geqq 0$. There are two producers, a private firm, and a public firm. The private firm's production function is given by $y_1 = f(y_2, y_3)$, where y_1 is the firm's output of commodity 1 and y_2 and y_3 are its inputs of commodities 2 and 3; $y_1, y_2, y_3 \geqq 0$. The public firm's production function is $x_1 = g(x_2, x_3)$, where $x_1, x_2, x_3 \geqq 0$ and are defined similarly. The functions f and g are assumed to be strictly concave and twice continuously differentiable, but not necessarily identical. Commodity 1 is used as numeraire, so both market and shadow prices of commodities 2 and 3 are expressed in terms of commodity 1. The market prices are denoted p_2 and p_3 and the shadow prices are denoted s_2 and s_3.

Commodity 1 can be exported to purchase commodity 2, but such transactions must satisfy the trade balance constraint:

$$g(x_2, x_3) + f(y_2, y_3) - x_2 - y_2 - c_1 \geqq 0. \tag{1}$$

Furthermore, total use of commodity 3 must not exceed the available supply.

Thus:

$$1 - x_3 - y_3 \geqq 0. \tag{2}$$

There are three agents in the economy whose knowledge and behavior concern us: the private firm's manager, the public firm's manager and a planner. The private firm's manager is assumed to know the precise form of f and the market prices p_2 and p_3 obtaining at any instant, but nothing else. He is assumed to behave so as to maximize profits treating market prices parametrically. His problem is thus

$$\max_{y_2, y_3} \quad \Pi^y = f(y_2, y_3) - p_2 y_2 - p_3 y_3 .$$

The necessary conditions are

$$f_2 \leqq p_2, \qquad y_2(f_2 - p_2) = 0$$

and

$$f_3 \leqq p_3, \qquad y_3(f_3 - p_3) = 0 .$$

Similarly, the public firm's manager is assumed to know the precise form of g and the shadow prices s_2 and s_3 given him by the planner at any instant, but nothing else. He maximizes shadow profit, treating shadow prices parametrically. His problem is thus

$$\max_{x_2, x_3} \quad \Pi^x = g(x_2, x_3) - s_2 x_2 - s_3 x_3 .$$

The necessary conditions are

$$g_2 - s_2 \leqq 0, \qquad x_2(g_2 - s_2) = 0 ,$$

and

$$g_3 - s_3 \leqq 0, \qquad x_3(g_3 - s_3) = 0 .$$

The planner is assumed to know the domestic market prices p_2 and p_3 obtaining at any instant, the international price of commodity 2 (normalized here at unity) and the magnitude of any tariff on commodity 2. He is also capable of observing the private firm's input and output levels at any instant. However, unless otherwise stated, the planner is assumed not to know any details of the functions f and g, except that they are both concave. Furthermore, he is assumed not to know the magnitude of the constraint on the availability of commodity 3 (here normalized at unity). The planner is assumed to control only the shadow prices s_2 and s_3, which he transmits to the public project. In particular, the planner has no control over the government's tax policy or any price controls, etc. These are set by other agencies of the government and the planner must take them as given.

Finally, we consider the determination of p_3. The strict concavity of f implies, ignoring some irregular cases,[6] that the Hessian matrix $[f_{ij}]_{i,j=2,3}$ of second derivatives of f is negative definite. This implies, by the implicit function theorem, that the first-order conditions for the private firm generate differentiable demand functions of the form

$$y_2 = Y_2(p_2, p_3)$$

$$y_3 = Y_3(p_2, p_3) \ .$$

Given x_3 and p_2 predetermined at x_3^o and p_2^o, respectively, p_3 is assumed to adjust instantaneously such that

$$Y_3(p_2^o, p_3) = 1 - x_3^o \ .$$

3. Necessary Conditions for a Second-Best Optimum

In this section we derive the necessary conditions for optimal public sector production in the presence of a fixed market distortion and interpret their meaning in terms of shadow prices. Consider first the "first-best" optimization problem.

Problem 1: max c_1 subject to (1) and (2).

We form the Lagrangian

$$L^* \equiv c_1 + \lambda_1\left(g(x_2, x_3) + f(y_2, Y_3) - c_1 - x_2 - y_2\right) + \lambda_2(1 - x_3 - y_3) \; . \quad (3)$$

The necessary conditions for optimal private sector production are

$$f_2^* \leqq 1, \qquad y_2^*(f_2^* - 1) = 0$$

and

$$f_3^* \leqq \lambda_2^* \qquad y_3^*(f_3^* - \lambda_2^*) = 0 \; ,$$

where $(^*)$ denotes evaluation of the quantity concerned at this "first-best" optimum. Assuming $y_3^* > 0$ the necessary conditions for optimal public sector production become

$$g_2^* \leqq 1, \qquad x_2^*(g_2^* - 1) = 0 \; ,$$

and

$$g_3^* \leqq f_3^* \qquad x_3^*(g_3^* - f_3^*) = 0 \; .$$

Hence, comparing these expressions with the above conditions for profit and shadow profit maximization, the "first-best" optimal shadow prices are

$$s_2^* = p_2^* = 1 \quad \text{and} \quad s_3^* = p_3^* \; .$$

Both the shadow price and the market price of the traded input should be its international price, while the non-traded input should be valued identically in the two sectors ($s_3^* = p_3^*$).

We now introduce a distortion into the market for the traded input in the form of a fixed tariff, τ, where $-1 < \tau < \infty$. Assuming $y_2 > 0$, profit maximization of the private firm now requires

$$f_2 = 1 + \tau , \tag{4}$$

which violates one of the necessary conditions for a "first-best" optimum. The "second-best" optimization problem now becomes

<u>Problem 2</u>: max c_1 subject to (1), (2) and (4).

We form the Lagrangian

$$L^{**} \equiv c_1 + \lambda_1\big(g(x_2, x_3) + f(y_2, y_3) - c_1 - x_2 - y_2\big) + \lambda_2(1 - x_3 - y_3) + \lambda_3\big(f_2 - (1 + \tau)\big). \tag{5}$$

Assuming $(y_2^{**}, y_3^{**}) > 0$, the necessary conditions for optimal public sector production become (assuming $f_{22}^{**} \neq 0$)

$$g_2^{**} \leqq 1, \quad x_2^{**}(g_2^{**} - 1) = 0 \quad \text{and}$$

$$g_3^{**} \leqq p_3^{**} - \tau\, f_{23}^{**}/f_{22}^{**},\ x_3^{**}(p_3^{**} - \tau\, f_{23}^{**}/f_{22}^{**}) = 0 ,$$

where (**) denotes evalaution at this "second-best" optimum.

The "second-best" optimal shadow prices are thus

$$s_2^{**} = 1 \quad \text{and} \tag{6}$$

$$s_3^{**} = p_3^{**} - \tau\, f_{23}^{**}/f_{22}^{**} , \qquad (f_{22}^{**} \neq 0) \tag{7}$$

Note that the presence of a fixed tariff on the traded input implies that the shadow prices of both inputs should differ from their domestic market prices--provided, in the case of the non-traded input, that $f_{23}^{**} \neq 0$. The second-best shadow price of the traded input is still its international price, regardless of the existence of a tariff, while the second-best shadow price of the non-traded input depends both on the tariff and the second derivatives of the private firm's production function. We now consider whether this latter shadow price can be interpreted as a "foreign exchange equivalent."

The net contribution of the private firm to aggregate foreign exchange earnings is clearly $y_1 - y_2$. The net effect of an extra unit of commodity 3 on the firm's contribution to foreign exchange earnings is therefore $dy_1/dy_3 - dy_2/dy_3$. Now, totally differentiating the private firm's production function,

$$dy_1 = f_2 dy_2 + f_3 dy_3 ,$$

and

$$\frac{dy_1}{dy_3} = f_2 \frac{dy_2}{dy_3} + f_3 = (1 + \tau) \frac{dy_2}{dy_3} + f_3 \tag{8}$$

Totally differentiating equation (4)

$$f_{22} dy_2 + f_{23} dy_3 = 0 ,$$

and hence

$$\frac{dy_2}{dy_3} = - \frac{f_{23}}{f_{22}} . \tag{9}$$

From (4), (8) and (9) we now have, at the second-best optimum,

$$\frac{dy_1^{**}}{dy_3^{**}} - \frac{dy_2^{**}}{dy_3^{**}} = p_3^{**} - \tau \frac{f_{23}^{**}}{f_{22}^{**}} , \tag{10}$$

which is the same as equation (7).

Thus the shadow price of the non-traded commodity is the net effect that releasing a unit of that commodity to the private firm has on aggregate foreign exchange earnings, at the optimum.[7] There seems little point in debating whether this is what previous authors "really" meant by a "foreign exchange equivalent", since that term has been used rather ambiguously; the point is that the above argument makes precise the sense in which that concept is a correct interpretation of equation (7). The above argument also demonstrates that the logical basis for the "foreign exchange equivalent" shadow pricing procedure derives from the first-order conditions from an optimization problem. Note especially that for the optimal second-best shadow price of the non-traded input to be viewed correctly as a "foreign exchange equivalent", the latter concept should not be interpreted as simply the private firm's rate of transformation between that input and a traded output, as a more simplistic interpretation would suggest; this is simply p_3 in equation (7). Of course, the formal correspondence between equation (7) and the concept of a "foreign exchange equivalent" says nothing about the informational problems of determining s_3^{**}. We now turn to this question.

4. Informational Requirements

We now suppose the economy to be producing initially at some point removed from the second-best optimum, meaning that equations (6) and (7) are not satisfied, and consider the planner's informational problems in finding shadow prices that will move the economy to this optimum. Clearly, there is no problem in the case of the traded input, since we have assumed its international price to be known. Our attention will focus on the case of the non-traded input.

Consider, first, the planner's informational requirements for finding the numerical value of s_3^{**} in equation (7). Clearly, this value will, in general, depend on the forms of the functions f and g. This has led some writers to suggest that, in economies where not all commodities are traded, the informational requirements of finding the set of shadow prices associated with an optimal production program (i.e. the dual solution) are equivalent to those of finding the physical quantities involved in that production program (i.e. the primal solution). If this were so, there would be a logical contradiction in supposing that a planner could determine the value of s_3^{**} in a single step without full information about g, f and the resource constraint. But this is untrue; very partial information may be sufficient.

Suppose, for example, that the planner knows only that f is additively separable and linear in y_3. Then it has the form

$$f(y_2, y_3) = h(y_2) + by_3 , \tag{11}$$

where $b > 0$ is an unknown constant, and the functions h and g are unknown except that g is concave and $h'' < 0$. Then (7) reduces to

$$s_3^{**} = b \ . \tag{12}$$

From the necessary conditions for profit maximization it is clear that whenever $y_3 > 0$, $f_3 = b = p_3$, the observed market price.

Thus without any a priori knowledge of b, or of the functions g or h, the planner can infer the optimal shadow prices immediately; namely, $s_2^{**} = 1$ and $s_3^{**} = p_3$. But he is still unable to determine the physical quantities involved in the optimal public sector production program without knowledge of g, and this is possessed only by the public firm's manager.

More generally, of course, f cannot be assumed to have this convenient form and the planner's informational requirements for determining the numerical value of s_3^{**} in a single step will be more severe. Let us suppose these requirements to be prohibitive, for the sake of argument, and consider the informational problems in applying the "rule" given by (7) iteratively. This is what a number of writers seem to intend. At time $t = 0$, let the public sector be producing non-optimally. Let $s_2^o = 1$, but

$$s_3^o \neq p_3^o - \tau \ f_{23}^o/f_{22}^o \ , \tag{13}$$

where numerical superscripts denote points in time. Now let observations be made on the right hand side of (13) and s_3^1 be set equal to this. Thus

$$s_3^1 = p_3^o - \tau \ f_{23}^o/f_{22}^o \ . \tag{14}$$

Now let the market price of commodity 3 adjust fully to this change in public production, and let the private firm adjust fully to this price change. Then let the planner revise s_3 by again applying the "rule" given by (7), so

$$s_3^2 = p_3^1 - \tau\, f_{23}^1/f_{22}^1 \,, \tag{15}$$

etc. That is, $s_2^t = 1$ for all t and

$$s_3^{t+1} = p_3^t - \tau\, f_{23}^t/f_{22}^t \,, \qquad t=0,\ 1,\ \ldots \tag{16}$$

What are the planner's informational requirements in this adjustment process?

Consider the term f_{23}^t/f_{22}^t. Differentiating equation (4) with respect to p_3 we have[8]

$$f_{22} \frac{\partial y_2}{\partial p_3} + f_{23} \frac{\partial y_3}{\partial p_3} = 0 \,.$$

Hence
$$\frac{f_{23}}{f_{22}} = - \frac{\partial y_2/\partial p_3}{\partial y_3/\partial p_3} = - \frac{\xi_{23} y_3}{\xi_{33} y_2} \tag{17}$$

where $\xi_{23} = \dfrac{p_3 \partial y_2}{y_2 \partial p_3}$ is the price elasticity of demand for commodity 2 with respect to p_3, etc.

Equation (16) now reduces to

$$s_3^{t+1} = p_3^t + \tau\, \xi_{23}^t\ y_3^t/\xi_{33}^t\ y_2^t \,, \qquad t=0,\ 1,\ \ldots \tag{18}$$

From the observability of p_3^t, τ, y_2^t and y_3^t, it follows that if the ratio of elasticities ξ_{23}^t/ξ_{33}^t were known, the above iterative process would be informationally feasible.

Alternatively, suppose that the planner knows only that f is a Cobb-Douglas production function with the general form

$$f(y_2,\ y_3) = a\ y_2^{\alpha}\ y_3^{\beta} \,, \tag{19}$$

where a, α and β are unknown parameters except that $a,\ \alpha,\ \beta > 0$

and $\alpha + \beta < 1$. Nothing is known of g except that it is concave. Differentiating twice we obtain

$$\frac{f_{23}}{f_{22}} = \frac{\beta y_2}{(\alpha-1)y_3} .$$

Now, writing

$$\frac{f_3}{f_2} = \frac{\beta}{\alpha}\frac{y_2}{y_3} = \frac{p_3}{p_2} ,$$

we obtain

$$\frac{f_{23}}{f_{22}} = \frac{p_3\alpha}{p_2(\alpha-1)} .$$

Now using the result that $\alpha = p_2y_2/y_1$, we have, substituting into (16)

$$s_3^{t+1} = p_3^t \ \frac{y_1^t - y_2^t}{y_1^t - y_2^t(1+\tau)} . \tag{20}$$

This expression involves only observable market variables. Hence, while the informational problems of shadow pricing are very real, no logical contradiction is involved in supposing them to be solvable for some classes of environments (namely where f is known to take some specific forms), especially where shadow pricing is to be conducted iteratively. This does not mean, of course, that these problems are solvable for all classes of environments; but it is false to claim that the informational requirements for shadow pricing are equivalent to the informational requirements for determination of the physical characteristics of the optimal public sector production program. It is a mistake to dismiss the possibility of finding welfare-increasing shadow pricing procedures on these grounds.

5. Stability

Strictly speaking, the first-order conditions from an optimization problem describe some relationships that must hold at the optimum concerned, but say nothing about how to reach that optimum, starting from an initial non-optimal position. Nevertheless, the frequent use of such first-order conditions to provide "rules" for public sector shadow pricing, optimal taxation, etc., suggests that the formulae so obtained are meant to be used as a means of moving from some initial position to the optimum concerned. This is particularly so of the shadow pricing literature. The adjustment mechanisms involved are ordinarily not made explicit, however, and, in addition to the informational issues involved, this leaves unanswered questions about the convergence of those mechanisms. We now consider the stability of the iterative process most naturally suggested by the analysis of section 3 and discussed in the preceding section. We show that, for an important class of examples, this process is non-convergent. This exercise suggests that questions of stability are of more than simply theoretical interest for shadow pricing.

Assuming that g_{22} is bounded away from zero,[9] the equation

$$g_2(x_2, x_3) = 1$$

can be solved for x_2 as a function of x_3. Using this, we can now express $g_3(x_2, x_3)$ as a function of x_3 alone, which we will denote $G(x_3)$. Likewise, assuming that f_{22} is always non-zero the equation

$$f_2(y_2, y_3) = 1 + \tau$$

can be solved for y_2 as a function of y_3 and τ. We now use this and the resource balance equation $y_3 = 1 - x_3$ to write

$$f_3(y_2, y_3) - \tau f_{23}(y_2, y_3)/f_{22}(y_2, y_3) = F(x_3; \tau),$$

where τ is always a fixed parameter.

Our iterative adjustment process is now

$$G(x_3^{t+1}) = F(x_3^t; \tau) \qquad t=0, 1, 2, \ldots \tag{21}$$

This process has the second-best optimal solution as a stationary point; but it converges to that solution if and only if

$$\lim_{t \to \infty} G(x_3^{t+1}) = G(x_3^{**}).$$

Now taking a Taylor's series expansion about the point x_3^{**} we obtain a linear approximation to the left member of equation (21),

$$G(x_3^{t+1}) \simeq G(x_3^{**}) + G'(x_3^{**})(x_3^{t+1} - x_3^{**}) \ .$$

Proceeding similarly with the right member of (21) we obtain the following linear approximation to equation (21).

$$G(x_3^{**}) + G'(x_3^{**})(x_3^{t+1} - x_3^{**}) = F(x_3^{**}; \tau) + F'(x_3^{**})(x_3^t - x_3^{**}) \tag{22}$$

We then lag this equation by one period and subtract the lagged equation from (22) to obtain a first-order difference equation approximation to equation (21):

$$G'(x_3^{**})(x_3^{t+1} - x_3^t) = F'(x_3^{**}; \tau)(x_3^t - x_3^{t-1}) \ . \tag{23}$$

This difference equation is convergent if and only if

$$|F'(x_3^{**};\ \tau)/G'(x_3^{**})| < 1\ . \tag{24}$$

Note that $F'(x_3^{**};\ \tau)$ and $G'(x_3^{**})$ will ordinarily have opposite signs, so their quotient will be negative.

Proposition 1: If f and g are any pair of identical Cobb-Douglas production functions having the form

$$f(y_2,\ y_3) = a\ y_2^{\alpha}\ y_3^{\beta}$$

$$g(x_2,\ x_3) = a\ x_2^{\alpha}\ x_3^{\beta}$$

where $a, \alpha, \beta > 0$, $\alpha + \beta < 1$ and $\tau > 0$, then the iterative adjustment process

$$G(x_3^{t+1}) = F(x_3^t;\ \tau), \qquad t=0,\ 1,\ \ldots$$

is non-convergent.[10]

Proof: see Appendix 1.

A numerical example of such a divergent process is presented in Table 1, where $a = 1$, $\alpha = \beta = 1/3$, and $\tau = 1$. The initial point (t=0) is an equilibrium solution where $f_2^o = 1 + \tau$, $g_2^o = 1$, and $g_3^o = f_3^o = p_3^o$. The second-best optimal solution is presented in the final row. Obviously, this iterative process does not converge to the optimum. Although the optimal solution exists and is a stationary point of this iterative process (i.e. an equilibrium), it is unstable. In the case of this example, $F'(x_3^{**};\ \tau)/G'(x_3^{**}) = -\ 1.28$. The value of consumption achieved after each iteration (c_1^t) is given in the final column. Note that c_1^t declines steadily as the iterative process continues. This example is illustrated in Figure 1.[11] It is easily verified that the non-convergence result does not depend on the initial value of x_3, provided it is not x_3^{**}. In such cases, no matter how close the economy is to the optimum initially, the iterative process given by equation (21) will not produce convergence to the optimum.

Proposition 2: For every non-convergent shadow pricing process where $F'(x_3^{**};\ \tau)/G'(x_3^{**}) \leqq -1$, there exists a simple revised process of the form

$$G(x_3^{t+1}) = \rho\ F(x_3^t;\ \tau) + (1-\rho)G(x_3^t), \qquad t=0,\ 1,\ 2,\ \ldots, \tag{24}$$

where $0 < \rho < 1$, which is convergent.

What this amounts to is that there always exists a value of ρ small enough such that this damped adjustment process will converge. Taking a Taylor's series expansion about x_3^{**} as before and proceeding similarly we find that this process is convergent if

$$\left|\frac{\rho F'(x_3^{**};\ \tau) + (1-\rho)G'(x_3^{**})}{G'(x_3^{**})}\right| < 1\ . \tag{25}$$

Denoting $F'(x_3^{**};\ \tau)/G'(x_3^{**})$ by K, we now suppose $-\infty < K \leqq -1$. Now choose ρ such that $0 < \rho < \frac{2}{1-K} \leqq 1$. The condition for convergence is now

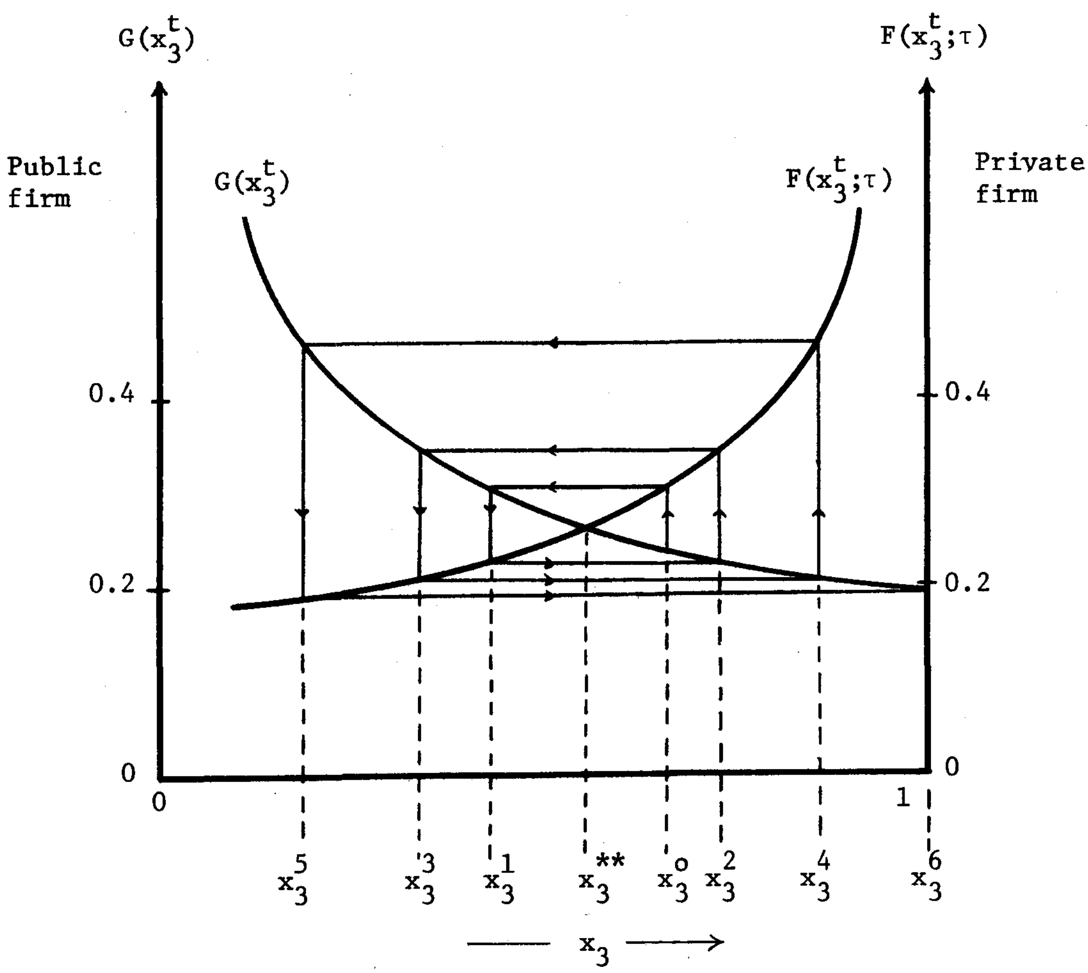
$$|\rho K + 1 - \rho| = |1 - \rho(1-K)| = |1 - m| < 1,$$

where $\rho = m/1 - K$, $0 < m < 2$. But $-1 < 1 - m < 1$, so $|1 - m| < 1$. So any ρ such that $0 < \rho < 2/1 - K$ will give convergence. Therefore, by choosing successively smaller values of ρ, a non-convergent shadow pricing process could always be converted into a convergent one.

Table 1: Non-Convergent Shadow Pricing Adjustment Process

	x_3^t	$G(x_3^t)$	$F(x_3^t;\tau)$	p_3^t	c_1^t
t = 0	0.6667	0.2357	0.2946	0.2357	0.5107
t = 1	0.4267	0.2946	0.2247	0.1797	0.5090
t = 2	0.7339	0.2247	0.3297	0.2638	0.5052
t = 3	0.3407	0.3297	0.2095	0.1676	0.5009
t = 4	0.8440	0.2095	0.4306	0.3445	0.4880
t = 5	0.1997	0.4306	0.1901	0.1521	0.4764
t = 6	1.000	0.1925	-	-	0.3849
Optimal Solution	0.5614	0.2569	0.2569	0.2055	0.5137

Figure 1: A Non-Convergent Shadow Pricing Adjustment Process



6. Some Alternative Solutions

Let us now suppose that the second-best solution described in section 3 is unattainable, in that the planner is unable to shadow price the non-traded input correctly. This might be so, for example, because the informational problems involved are unmanageable or because of a bureaucratic constraint on the shadow pricing methods he can employ. We will suppose the planner to be constrained to shadow price the non-traded input at its domestic market price. We now ask whether, given this, it is better to shadow price the traded input at its international price or at its (tariff-distorted) domestic market price. Clearly, both these procedures are informationally feasible. The first is what a simplistic interpretation of the Little-Mirrlees and Dasgupta-Stiglitz recommendations would consist of (valuing the non-traded input at its producer price and the traded input at its international price), while the second is what Weckstein and Rudra recommend (valuing all commodities at their domestic market prices). We will ignore questions of stability and consider only the values of consumption achieved at the respective equilibrium solutions.

First we consider whether, given that the non-traded input is to be valued at its equilibrium market price in <u>both</u> the private and the public firms, the <u>optimal</u> shadow price for the traded input is still its international price.

<u>Problem 3</u>: max c_1 subject to (1), (2), (4) and $g_3 = f_3$.

We formulate the Lagrangian

$$L^{***} = c_1 + \lambda_1\left(g(x_2, x_3) + f(y_2, y_3) - c_1 - x_2 - y_2\right) + \lambda_2(1 - x_3 - y_3)$$
$$+ \lambda_3\left(f_2 - (1 + \tau)\right) + \lambda_4(f_3 - g_3). \quad (26)$$

Proceeding as before we obtain, for $y_2, y_3, x_3 > 0$,

$$s_2^{***} = 1 + \tau\, f_{23}^{***} g_{23}^{***} \left[f_{22}^{***} (g_{33}^{***} + f_{33}^{***}) - (f_{23}^{***})^2 \right]^{-1} \tag{27}$$

where (***) denotes evaluation of the derivative concerned at this "third-best" optimum.

From the strict concavity of f we know that the Hessian matrix of second derivatives $\left[f_{ij} \right]_{i,j=2,3}$ is negative definite, with the possible exception of a countable number of isolated points,[12] so $f_{22}, f_{23} < 0$ and $f_{22}f_{33} - f_{23}^2 > 0$. Likewise $g_{33} < 0$, so the term in square brackets is positive. Ordinarily, we expect $f_{23}, g_{23} \geqq 0$; but note that if <u>either</u> of these terms is vanishing (for example, if either f or g is additively separable), $s_2^{***} = 1$ as before. If both are non-zero, $s_2^{***} > 1$ and the international price is no longer the optimal shadow price for the traded input. Obviously, if the non-traded input cannot be valued correctly for informational reasons, presumably equation (27) cannot be evaluated either. But given that $s_2^{***} > 1$, it is not obvious whether the international price (unity) or the domestic market price $(1 + \tau)$ is superior. We now define

<u>Problem 4</u>: Let $s_2^{(4)} = 1$, $s_3^{(4)} = p_3^{(4)}$, and

<u>Problem 5</u>: Let $s_2^{(5)} = 1 + \tau$, $s_3^{(5)} = p_3^{(5)}$,

where $p_3^{(4)}$ and $p_3^{(5)}$ denote the equilibrium market prices of the non-traded input resulting from these two shadow pricing strategies, respectively. We now ignore questions of convergence and compare the equilibrium levels of consumption of commodity 1 resulting from these strategies, denoted $c_1^{(4)}$ and $c_1^{(5)}$, respectively.

Proposition 3:

Let f and g be any pair of identical Cobb-Douglas production functions of the form

$$f(y_2, y_3) = a\, y_2^{\alpha}\, y_2^{\beta} ,$$

$$g(x_2, x_3) = a\, x_2^{\alpha}\, x_3^{\beta} ,$$

where $a, \alpha, \beta > 0$ and $\alpha + \beta < 1$, then $c_1^{(4)} \geqq c_1^{(5)}$ for all τ such that $-1 < \tau < \infty$.

Proof: see Appendix 2.

For this limited class of production functions, then, it is always better to value the traded input at its international price than at its tariff-distorted domestic market price when the tariff is the only distortion present. It is very difficult to extend the proof of Proposition 2 beyond the class of identical Cobb-Douglas production functions, but the author has been unable to construct a counter-example when the public and private sectors are allowed to have different Cobb-Douglas production functions. Nevertheless, such examples may exist. In the first 5 columns of Table 2 we illustrate the complete numerical solutions to Problems 1 through 5 for the following specific case.

Numerical Example I: Let

$$f(y_2, y_3) = y_2^{1/3}\, y_3^{1/3} \quad \text{and}$$

$$g(x_2, x_3) = x_2^{1/3}\, x_3^{1/3} .$$

For Problems 2 through 5 we have set $\tau = 1$.

Note that c_1 decreases moving across the table, and that $c_1^{(4)}$ greatly exceeds $c_1^{(5)}$. Figure 2 illustrates the relationship between $c_1^{(4)}$ and $c_1^{(5)}$ in this case when τ is allowed to vary.

Once a distortion is introduced directly into the market for the non-traded input, however, this result no longer holds. It is then no longer true that when the shadow price of the non-traded input is set at its (distorted) domestic market price, it is always better to shadow price the traded input at its international price than at its domestic market price--even for the restricted class of production functions considered in Proposition 3. This is illustrated by Problem 6 in the final column of Table 2.

Problem 6: Let $s_2^{(6)} = 1$ and $s_3^{(6)} = p_3^{(5)}$.

Think of the economy initially being at the solution to Problem 5 and the market price of the non-traded input thereafter being held fixed at this initial value. Now let the planner change the shadow price of the traded input from its domestic market price to its international price, but continue to shadow price the non-traded input at its (fixed) market price. In this case the public firm expands to eliminate the private firm completely and c_1 falls dramatically. The solutions for Problems 1 through 6 are summarized in Figure 3 for the above Cobb-Douglas example. The solutions are denoted (1), (2), ..., (6), respectively.

Finally, we present an example to show that when the non-traded input is shadow priced at its domestic market price, shadow pricing the traded input at its international price can be worse than shadow pricing it at its tariff-distorted domestic market price, even when there are no distortions present in the market for the non-traded input.

Numerical Example II: Let

$$f(y_2,\ y_3) = 99y_2y_3 - 50y_2^2 - 50y_3^2 + 3y_3\ ,$$

$$g(x_2,\ x_3) = 99x_2x_3 - 50x_2^2 - 50x_3^2 + 3x_3\ ,$$

and let $\tau = 1$. It is easily verified that f and g are strictly concave,[13] and that $f(0,\ 0) = g(0,\ 0) = 0$. The complete equilibrium solutions for Problems 1 through 6 are presented in Table 3. This example has a number of interesting features,[14] but note especially that in this case s_2^{***} is close to $1 + \tau$ (Problem 3). Given that s_3 is set equal to p_3, it is better in this case to value the traded input at its domestic market price than at its international price $(c_1^{(5)} > c_1^{(4)})$, even though p_3 is a perfectly competitive equilibrium price and the tariff is the only distortion present.

This is a disturbing result. If the informational problems of finding the correct second-best shadow prices of non-traded commodities are prohibitive, requiring that their market prices be used instead, then we cannot, in general, be sure that it is better to shadow price traded commodities at their international prices than at their domestic market prices. This is so even when the only distortions present in the economy are tariffs on the traded commodities and the markets for the non-traded commodities are perfectly competitive. Examples of the above kind are admittedly difficult to construct; but this problem should be appreciated.

Table 2: Numerical Example I

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
c_1	0.5443	0.5137	0.5110	0.5107	0.4811	0.3849
x_1	0.4082	0.4326	0.4550	0.4714	0.2887	0.5774
x_2	0.1361	0.1442	0.1438	0.1571	0.0481	0.1925
x_3	0.5	0.5614	0.6548	0.6667	0.5	1
s_2	1	1	1.0544	1	2	1
s_3	0.2722	0.2569	0.2316	0.2357	0.1925	0.1925
y_1	0.4082	0.2704	0.2399	0.2357	0.2887	0
y_2	0.1361	0.0451	0.0400	0.0393	0.0481	0
y_3	0.5	0.4386	0.3452	0.3333	0.5	0
p_2	1	2	2	2	2	-
p_3	0.2722	0.2055	0.2316	0.2357	0.1925	-

Table 3: Numerical Example II

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
c_1	1.5225	1.5176	1.5127	1.3944	1.5125	1.0200
x_1	1.2463	1.2463	1.2514	1.6834	1.2313	2.0000
x_2	0.4850	0.4850	0.4850	0.7313	0.4750	0.9800
x_3	0.5000	0.5000	0.5097	0.7487	0.5000	0.9975
s_2	1	1	1.9610	1	2	1
s_3	1.0150	1.0150	0.0443	0.5200	0.0250	0.0250
y_1	1.2463	1.2313	1.2117	0.6710	1.2313	0
y_2	0.4850	0.4750	0.4654	0.2287	0.4750	0
y_3	0.5000	0.5000	0.4903	0.2513	0.5000	0
p_2	1	2	2	2	2	-
p_3	1.0150	0.0250	0.0443	0.5200	0.0250	-

Figure 2: Equilibrium Consumption Level with Varying Tariff (Numerical Example I)

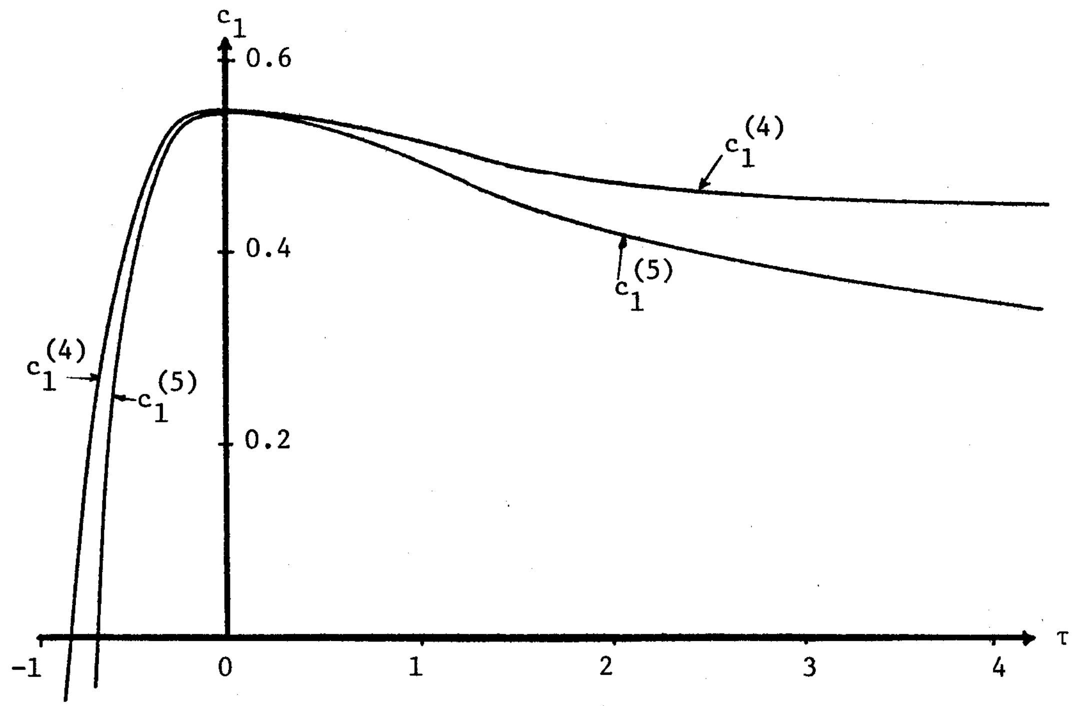
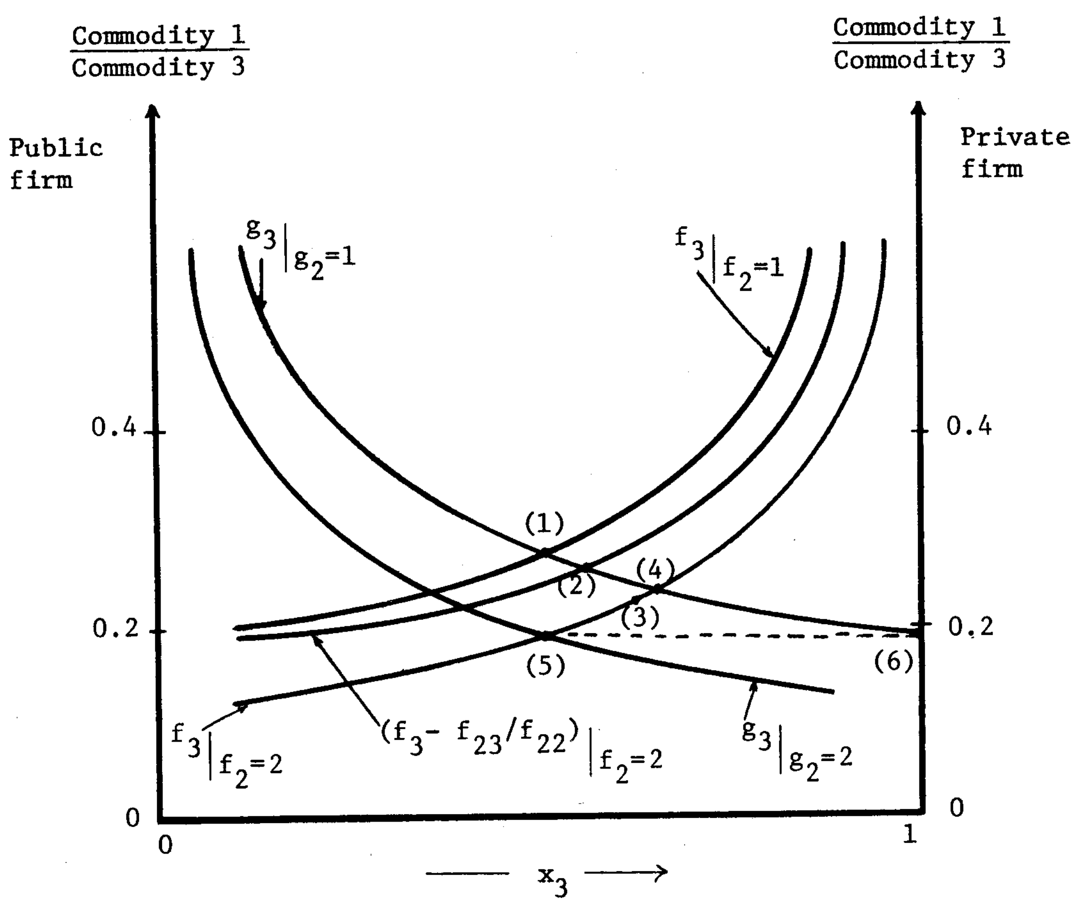


Figure 3: Solutions to Numerical Example I



7. Summary and Conclusions

Shadow pricing is essentially a topic in the economics of the second-best. The technical literature on shadow pricing derives shadow pricing "rules" from the first-order conditions from an optimization model and this typically results in rules which link public production to international trade. In the case of non-traded commodities, however, this exercise merely provides a description of some relationships that must hold at the second-best optimum concerned, and says nothing about the mechanism by which the economy can move to that optimum, starting from some initial non-optimal position. This point has been overlooked in the literature, and the informational and convergence properties of possible adjustment processes to achieve this transition have not been investigated for a mixed, open economy. The present paper attempts to study these issues with a highly simplified but still fruitful model.

There are no informational problems involved in the correct second-best shadow pricing of traded commodities; given the usual "small country" assumption, these are simply their international prices. But in the case of non-traded commodities the informational problems are significant. It is important to distinguish the informational requirements of moving <u>directly</u> from an initial non-optimal allocation to the second-best optimum (determining the numerical values of the second-best optimal shadow prices of non-traded commodities in a single step) from those of moving to that optimum iteratively. The informational requirements of the latter are more reasonable. This raises the question of the stability of the iterative adjustment process concerned, and we have shown that for an important and realistic class of examples, the most obvious adjustment process, where shadow prices are revised over time according to the "rule" given by the first-order conditions, is non-convergent. In such cases a revised adjustment process is needed, and we have shown how this can be done.

Finally, we have considered what should be done when the informational requirements of determining the second-best optimal shadow prices of non-traded commodities are judged to be prohibitive. If the shadow prices of these commodities are set at their domestic market prices, the question arises whether it is still desirable to shadow price traded commodities at their international prices, rather than their tariff-distorted domestic prices. For an important class of examples, we have shown that this is so, provided that the tariffs concerned are the only distortions present. But we have also shown that there exist reasonable cases where even this result does not hold. In such cases, when non-traded commodities are shadow priced at their domestic producer prices, shadow pricing traded commodities at their international prices is actually worse than shadow pricing them at their tariff-distorted domestic prices, even though these tariffs are the only distortions present in the economy!

Appendix 1: Proof of Proposition 1

Proceeding as in the text for

$$g(x_2, x_3) = a\, x_2^{\alpha}\, x_3^{\beta}\ , \text{ and}$$

$$f(y_2, y_3) = a\, y_2^{\alpha}\, y_3^{\beta}$$

where $a, \alpha, \beta > 0$, $\alpha + \beta < 1$ and $\tau > 0$, we obtain

$$G(x_3^{t+1}) = a^{1/1-\alpha}\ \beta\ \alpha^{\alpha/1-\alpha}\ (x_3^{t+1})^{\frac{\alpha+\beta-1}{1-\alpha}}\ , \text{ and}$$

$$F(x_3^t;\ \tau) = a^{1/1-\alpha}\left[\beta\left(\frac{\alpha}{1+\tau}\right)^{\alpha/1-\alpha} + \tau\,\frac{\beta}{1-\alpha}\left(\frac{\alpha}{1+\tau}\right)^{1/1-\alpha}\right](1-x_3^t)^{\frac{\alpha+\beta-1}{1-\alpha}}$$

Writing K for $F'(x_3^{**};\ \tau)/G'(x_3^{**})$ we obtain

$$K = -\left[\left(\frac{1}{1+\tau}\right)^{\alpha/1-\alpha} + \tau\,\frac{\alpha}{1-\alpha}\left(\frac{1}{1+\tau}\right)^{1/1-\alpha}\right]^{\frac{1-\alpha}{\alpha+\beta-1}}\ . \tag{28}$$

Clearly $1-\alpha/\alpha+\beta-1 < -1$, so $|K| \gtreqless 1$ as $|R| \lesseqgtr 1$, where R is the term in square brackets in (28). We now show that $|R| < 1$.

Suppose $|R| \geqq 1$. Then

$$\left(\frac{1}{1+\tau}\right)^{\alpha/1-\alpha}\left[1 + \left(\frac{\tau}{1+\tau}\right)\left(\frac{\alpha}{1-\alpha}\right)\right] \geqq 1\ ,$$

or
$$1 + \left(\frac{\tau}{1+\tau}\right) r \geqq (1+\tau)^{r}$$

and
$$1 + (r+1)\tau \geqq (1+\tau)^{r+1}$$

where $r = \alpha/1-\alpha > 0$, so $r + 1 > 1$. Let $r + 1$ be a rational number (if not, we can approximate it by a rational arbitrarily closely so as to preserve the inequality) equal to n/m, where n and m are positive integers such that $n > m$. Then

$$\left(1 + \frac{n\tau}{m}\right)^m \geqq (1 + \tau)^n . \tag{29}$$

We now take a binomial expansion of each side

$$\left(1 + \frac{n\tau}{m}\right)^m = 1 + \frac{n}{m}\tau + \frac{m(m-1)}{2}\left(\frac{n\tau}{m}\right)^2 + \ldots + m\left(\frac{\tau n}{m}\right)^{m-1} + \left(\frac{n}{m}\tau\right)^m . \tag{30}$$

$$(1 + \tau)^n = 1 + n\tau + \frac{n(n-1)\tau^2}{2} + \ldots + n\tau^{n-1} + \tau^n . \tag{31}$$

It is now easily verified that each of the $m+1$ terms of (30) is smaller than or equal to the corresponding term of (31). But the following $n-m$ terms of (31) are all positive. This contradicts (29). Hence $|R| < 1$, and thus $|K| > 1$, and the proof is complete. A simple extension of the proposition is that if

$$g(x_2, x_3) = a\, x_2^\alpha\, x_3^\beta , \text{ and}$$

$$f(y_2, y_3) = b\, y_2^\alpha\, y_3^\beta ,$$

where $a, b, \alpha, \beta > 0$, $\alpha + \beta < 1$ and $\tau > 0$, then $|K| > 1$ whenever $a \geqq b$.

Appendix 2: <u>Proof of Proposition 3</u>

For brevity, we will outline the argument. The equilibrium levels of consumption resulting from Problems 4 and 5 are, respectively,

$$c_1^{(4)} = \left\{ \left(\frac{\alpha}{1+\tau}\right)^{\frac{\alpha}{1-\alpha}} \left[1 + (1+\tau)^{\frac{\alpha}{1-\alpha-\beta}}\right]^{\frac{1-\alpha-\beta}{1-\alpha}} \right.$$

$$\left. - \left(\frac{\alpha}{1+\tau}\right)^{\frac{1}{1-\alpha}} \left[1 + (1+\tau)^{\frac{1-\beta}{1-\alpha-\beta}}\right] \left[1 + (1+\tau)^{\frac{\alpha}{1-\alpha-\beta}}\right]^{\frac{-\beta}{1-\alpha}} \right\} a^{\frac{1}{1-\alpha}}$$

and
$$c_1^{(5)} = 2^{\frac{1-\alpha-\beta}{1-\alpha}} \left[\left(\frac{\alpha}{1+\tau}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{1+\tau}\right)^{\frac{1}{1-\alpha}}\right] a^{\frac{1}{1-\alpha}}$$

Now consider τ as a variable and a, α and β as parameters. We now write $c_1^{(5)} - c_1^{(4)} \equiv D(\tau;a,\alpha,\beta)$. Proposition 3 states that $D(\tau;a,\alpha,\beta) \leqq 0$ for all $a, \alpha, \beta > 0$ such that $\alpha + \beta < 1$ and all τ such that $-1 < \tau < \infty$. It is clear that $D(0;a,\alpha,\beta) = 0$. It can also be shown that $D_\tau(0;a,\alpha,\beta) = 0$ and that $D_\tau(\tau;a,\alpha,\beta) \leqq 0$ for $\tau \,\varepsilon(-1,0)$ and $D_\tau(\tau;a,\alpha,\beta) \leqq 0$ for $\tau \,\varepsilon(0,\infty)$. These relationships hold for all a, α, β satisfying the above conditions. Hence, since $D(\cdot)$ is continuous in τ, it is impossible that $D(\tau;a,\alpha,\beta) > 0$; thus $D(\tau;a,\alpha,\beta) \leqq 0$.

FOOTNOTES

* This paper owes its origin to a suggestion from Leonid Hurwicz. The author is responsible for all views and any errors it contains.

1 The proposition is trivially false in the case where all commodities are traded.

2 Examples are R. S. Weckstein, "Shadow Prices and Project Evaluation in Less-Developed Countries," Economic Development and Cultural Change XX (April 1972), 474-494, and A. Rudra, "Use of Shadow Prices in Project Evaluation," Indian Economic Review VII (N.S.)(April 1972), 1-15.

3 Examples are P. Dasgupta and J. E. Stiglitz, "Benefit-Cost Analysis and Trade Policies," Journal of Political Economy LXXXII (January/February 1974), 1-33, I. M. D. Little and J. A. Mirrlees, Manual of Industrial Project Analysis in Developing Countries (Paris: Organization for Economic Cooperation and Development, 1969), and I. M. D. Little and J. A. Mirrlees, Project Appraisal and Planning for Developing Countries (New York: Basic Books, 1974).

4 For a critique of the Dasgupta-Stiglitz argument on this point see P. G. Warr, "A Note on Shadow Pricing with Fixed Taxes," Discussion Paper No. 74-52 (December 1974), Department of Economics, University of Minnesota.

5 See, for example, P. Dasgupta, S. A. Marglin and A. K. Sen, Guidelines for Project Evaluation (New York: United Nations Industrial Development Organization, 1972), Ch. 17.

6 There are irregular cases in which the derivatives f_{ii} may vanish, even though f is strictly concave, but only at a countable number of isolated points. These points form a "negligible" subset of the domain of f.

7 For a contrary claim, see R. Boadway, "Benefit-Cost Shadow Pricing in Open Economies: An Alternative Approach," Journal of Political Economy LXXXIII (April 1975), 419-430.

8 Since equation (4) must hold for all p_3 and, from the implicit function theorem, the demand functions $y_2 = Y_2(p_2, p_3)$ and $y_3 = Y_3(p_2, p_3)$ are differentiable, this differentiation is legitimate.

9 This follows from the negative definiteness of $[g_{ij}]_{i,j=2,3}$.

10 Obviously, the requirement that $\alpha + \beta < 1$ is necessary to ensure the strict concavity of f and g. Readers attached to constant returns to scale may think of a third factor being specific to the two firms in equal quantities, whose exponent in the production functions is $1 - \alpha - \beta$.

11 Note the resemblance of Figure 1 to the "cob-web" diagrams of intermediate price theory.

12 See footnote 6.

13 $[f_{ij}] = [g_{ij}] = \begin{bmatrix} -100 & 99 \\ 99 & -100 \end{bmatrix}$, which is negative definite. Hence f and g are strictly concave.

14 For example, note that the first-best and second-best optimal shadow prices are identical, even though f and g are not additively separable.