PURE STRATEGY NASH EQUILIBRIUM POINTS AND THE LEFSCHETZ FIXED POINT THEOREM

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Discussion Paper No. 75-60, September 1975

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ABSTRACT

A pure strategy Nash equilibrium point existence theorem is established for a class of n-person games with possibly nonacyclic (e.g., disconnected) strategy spaces. The principal tool used in the proof is a Lefschetz fixed point theorem for multivalued maps which extends the well known "Eilenberg-Montgomery fixed point theorem" to nonacyclic spaces. Special cases of the existence theorem are also discussed.

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1. INTRODUCTION

A pure strategy Nash equilibrium point existence theorem is established for a class of n-person games with possibly nonacyclic (e.g., disconnected) strategy sets. The principal tool used in the proof is a Lefschetz fixed point theorem for multivalued maps, due to Eilenberg and Montgomery, which extends their better known "Eilenberg-Montgomery fixed point theorem" (EMT) [9, Theorem 1, page 215] to nonacyclic spaces. Special cases of the existence theorem are also discussed.

A number of economists and game theorists investigating pure strategy solution concepts have used the EMT [see 1, 6, 7, 17]. On the other hand, Lefschetz fixed point theorems do not seem to have appeared in either the economic or game theory literature. The Lefschetz approach to fixed point theorems may ultimately prove to be of particular importance in economic and game theory for two reasons: generality of spaces which can be considered; interesting related questions which can be investigated.

Concerning the first reason, the objective of many fixed point theorems (e.g., the EMT) has been to establish that certain spaces have the fixed point property with respect to a class of maps which includes all continuous maps. Since disconnected spaces do not have the fixed point property with respect to all continuous maps, the hypotheses in

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*Research underlying this paper was supported by National Science Foundation Grant GS-31276X.

these theorems generally include an acyclicity restriction (e.g., convexity) which implies connectedness. In contrast, the Lefschetz fixed point theorem is basically a statement about homotopy classes of maps rather than the class of all continuous maps; hence more general spaced (e.g., disconnected) can be considered.

Concerning the second reason, the Lefschetz approach to fixed point theorems leads naturally to the concept of the "Nielsen number" of a map $f: Y \to Y$, a homotopy-invariant lower bound for the "number" of fixed points of f [see 11, section 3; and 5, Chapters VI - VII]. The Nielsen number provides a lower bound for the number of Nash equilibrium points in certain n-person games (see the proof of 2.8, below, where a one-to-one correspondence is established between the pure strategy Nash equilibrium points of an n-person game and the fixed points of a map). Secondly, for sufficiently restricted spaces and maps, a converse to the Lefschetz fixed point theorem can be obtained [see 11, section 3; and 5, Chapter VIII].

2. THE EXISTENCE THEOREM

A number of needed intermediary results will be established prior to the existence theorem (2.8). The following definitions and conventions will be used.

2.1 DEFINITIONS AND CONVENTIONS. A standard n-person game in normal form is given by a vector

$$(\prod_{i \in n^*} \Theta_i,\ \prod_{i \in n^*} U_i : \prod_{i \in n^*} \Theta_i \to R^n)$$

where $n^* \equiv \{1,,,n\}$ is the <u>player set</u>, and, for each $i \in n^*$, $\Theta_i = \{\theta_i, \ldots\}$ is the ith player's <u>strategy set</u> and $U_i : \prod_{n^*} \Theta_i \to R$ is the ith player's <u>objective function</u>. A joint strategy $(\theta_1', \ldots, \theta_n') \in \prod_n \Theta_i$ is a <u>pure strategy Nash equilibrium point</u> if for each $i \in n^*$,

$$U_i(\theta_1', \ldots, \theta_{i-1}', \theta_i, \theta_{i+1}', \ldots, \theta_n') \le U_i(\theta_1', \ldots, \theta_i', \ldots, \theta_n') ,$$

for all $\theta_i \in \Theta_i$.

Čech homology based on all open coverings with coefficients in a fixed field F [see 10, Chapter IX] will be used throughout this paper, except where otherwise specified. A nonempty compact Hausdorff space Y is <u>acyclic with respect to Čech homology over F</u> (C_F - acyclic) provided the Čech homology F-vector spaces $\{\check{H}_i(Y; F)\}$ satisfy $\check{H}_0(Y ; F) \cong F$ and $\check{H}_i(Y ; F) \cong 0$ for $i \neq 0$.

Let Y be a compact Hausdorff space and let $f : Y \to Y$ be a continuous map. For each integer n, let $(f_*)_n : \check{H}_n(Y ; F) \to \check{H}_n(Y ; F)$ denote the F-homomorphism induced by f [see 10, Chapter IX, Section 4]. If $\check{H}_n(Y ; F)$ is finitely generated, define <u>Trace $(f_*)_n$</u> to be the trace of the matrix associated with $(f_*)_n$ with respect to any basis of $\check{H}_n(Y ; F)$.

Let $T : Y \to Y$ be a multivalued map on a compact Hausdorff space Y . Let $B(T) \equiv \{(x, y) \mid x \in Y , y \in T(x)\}$ denote the graph of T . Define continuous maps $r : B(T) \to Y$ and $t : B(T) \to Y$ by

$$r(x, y) = y \quad , \quad t(x, y) = x \quad .$$

Then the <u>Lefschetz number</u> $L(T;F)$ of the map T with respect to the field F is defined to be

$$\sum_i (-1)^i \operatorname{Trace}(r_* \circ t_*^{-1})_i \quad ,$$

whenever the sum exists and is finite.

Let X and Y be compact Hausdorff spaces. A multivalued map $T: X \to Y$ will be called <u>closed</u> if its graph $B(T) \equiv \{(x, y) \mid x \in X, y \in T(x)\}$ is a closed subset of $X \times Y$.

Let X be a normal topological space. If for every normal space Y, every closed subset A of Y, and every map $f: A \to X$ there exists an extension of f to a neighborhood of A in Y, then X is called an <u>absolute neighborhood retract</u> (ANR).

<u>Remark</u>. This definition of ANR is not completely standard. For example, Hu [14] uses "ANR" for spaces having the (above) extension property with respect to metrizable rather than normal spaces Y. Nevertheless, for compact metrizable (hence normal) spaces X, Hu has shown [14, 5.2 - 5.3, pages 93 - 94] that his definition and the above definition are equivalent.

<u>Conventions</u>. The word "group" will always mean "abelian group." The term "compactum" (plural "compacta") will be used for "compact metrizable space." All products of topological spaces are assumed to carry the product topology. References to the coefficient field F will hereafter be suppressed.

Many of the sources referred to below do not use Čech homology; for example, Eilenberg and Montgomery [9] use Vietoris homology, Begle [2, 3] uses generalized Vietoris homology, and Dold [8] uses singular homology. The following proposition clarifies the relationship among these homology theories and justifies references to these various sources in later parts of the paper.

2.2 PROPOSITION.

(i) Vietoris and Čech homology groups are isomorphic over compacta, and generalized Vietoris and Čech homology groups are isomorphic over compact Hausdorff spaces, with respect to any coefficient group;

(ii) Čech homology over compact Hausdorff spaces satisfies the seven axioms of the Eilenberg-Steenrod system [see 10] with respect to any coefficient field (considered as a vector space over itself);

(iii) Čech and singular homology groups over ANR compacta are isomorphic with respect to any coefficient field.

Proof. The first part of statement (i) follows from Theorem 26.1 [15, page 273] and the second part is proved in Ref. 3 [pages 536 - 537]. Statement (ii) is proved by Eilenberg and Steenrod [10, Chapter IX].

For any ANR compactum M, if H' and H'' are two homology theories satisfying the seven Eilenberg and Steenrod axioms with isomorphic coefficient groups G' and G'', then $H'_n(M; G')$ is isomorphic to $H''_n(M; G'')$ for all integers n [24, Theorem 7.3, page 141].

Thus statement (iii) follows from statement (ii) and Chapter VII in Ref. [10], where it is proved that singular homology satisfies the seven Eilenberg and Steenrod axioms for arbitrary coefficient group.

Q.E.D.

2.3 LEMMA. Let Y be a compact Hausdorff space such that the Čech homology vector spaces $\check{H}_i(Y)$ are finitely generated for each integer i and vanish for sufficiently large i. Let $T : Y \to Y$ be a closed multivalued map such that $T(y)$ is C-acyclic for each $y \in Y$. Then $L(T)$ exists and is finite.

<u>Proof</u>. Let $B \equiv \{(x, y) | x \in Y, y \in T(x)\}$ denote the graph of T and let continuous maps $r : B \to Y$ and $t : B \to Y$ be defined as above by

$$r(x, y) = y \quad ; \quad t(x, y) = x \quad .$$

Since by assumption $T(x)$ is C-acyclic for each $x \in Y$, $t^{-1}(x) \equiv (\{x\} \times (T(x)) \approx T(x)$ is C-acyclic for each $x \in Y$. By the Vietoris-Begle Mapping Theorem [3], t_* maps $\check{H}(B)$ isomorphically onto $\check{H}(Y)$; hence $(t_*^{-1})_i$ is a well defined single-valued map and $(r_* \circ t_*^{-1})_i$ is a homomorphism of $\check{H}_i(Y)$ into itself for each integer i.

Since by assumption $\check{H}_i(Y)$ has a finite basis for each integer i and $\check{H}_i(Y) = 0$ for sufficiently large i, Trace $(r_* \circ t_*^{-1})_i$ is well defined for each i and vanishes for sufficiently large i. Thus $L(T) \equiv \sum_i (-1)^i$ Trace $(r_* \circ t_*^{-1})_i$ exists and is finite.

Q.E.D.

2.4 COROLLARY. Let M be an ANR compactum and let $T : M \to M$ be as in 2.3. Then $L(T)$ exists and is finite.

Proof. By Corollary 7.2 [14, page 141] and 2.2 (ii), these exists an integer m such that $\check{H}_n(M)$ is finitely generated for every $n \leq m$ and $\check{H}_n(M) = 0$ for $n > m$. The claim now follows from 2.3.

Q.E.D.

2.5 COROLLARY. Let Y be a C-acyclic compact Hausdorff space, and let $T : Y \to Y$ be as in 2.3. Then $L(T) = 1$.

Proof. By C-acyclicity of Y, $\check{H}_n(Y) = 0$ for $n > 0$ and $\check{H}_o(Y)$ has only one generator. By 2.3, $L(T) = \sum_i (-1)^i \operatorname{Trace} (r_* \circ t_*^{-1})$ is well defined. In particular, $(r_* \circ t_*^{-1})_n : \check{H}_n(Y) \to \check{H}_n(Y)$ is the zero map for $n > 0$ and $(r_* \circ t_*^{-1})_o : \check{H}_o(Y) \to \check{H}_o(Y)$ is the identity map. Hence

$$L(T) = \operatorname{Trace} (r_* \circ t_*^{-1})_o = 1 \quad .$$

Q.E.D.

2.6 THEOREM [9, Theorem 5, page 217]. Let M be an ANR compactum and let $T : M \to M$ be a closed multivalued map such that $T(x)$ is C-acyclic for each $x \in M$. If $L(T) \neq 0$, then there exists $x \in M$ such that $x \in T(x)$.

Remarks. Begle [2] has generalized 2.6 to compact homologically locally connected spaces; R. B. Thompson [see 11] has generalized 2.6

(for single-valued maps T) to compact spaces which admit a "weak semicomplex structure;" and A. Granas and L. Gorniewicz [12] have generalized 2.6 to topologically complete ANR's. See also [16] and [18] for additional generalizations.

2.7 COROLLARY (EMT) [9, Theorem 1, page 215]. Let M be a C-acyclic ANR compactum and let $T : M \to M$ be a closed multivalued map such that $T(x)$ is C-acyclic for each $x \in M$. Then there exists $x \in M$ such that $x \in T(x)$.

Remark. A C-acyclic, compact Hausdorff space is connected [15, 11.18, page 261].

Proof. By 2.5, $L(T) = 1$. Thus 2.7 follows from 2.6.

Q.E.D.

2.8 THEOREM. Let $\Gamma \equiv (\prod_{n^*} \Theta_i, \prod_{n^*} U_i : \prod_{n^*} \Theta_i \to R^n)$ be an n-person game. If

1) $\prod_{n^*} \Theta_i$ is an ANR compactum;
2) $U_i : \prod_{n^*} \Theta_i \to R$ is continuous for each $i \in n^*$;
3) $T(\theta)$ is C-acyclic for each $\theta \in \prod_{n^*} \Theta_i$, where $T \equiv \prod_{n^*} T_i : \prod_{n^*} \Theta_i \to \prod_{n^*} \Theta_i$ is given in terms of its coordinate maps by

$$T_i(\theta') = \{\theta_i^* \in \Theta_i \mid U_i(\theta_1', \ldots, \theta_{i-1}', \theta_i^*, \theta_{i+1}', \ldots, \theta_n') = \max_{\theta_i \in \Theta_i} U_i(\theta_1', \ldots, \theta_{i-1}', \theta_i, \theta_{i+1}', \ldots, \theta_n')\},$$

for each $i \in n^*$, for each $\theta' \equiv (\theta_1', \ldots, \theta_n') \in \prod_{n^*} \Theta_i$;

4) $L(T) \neq 0$, where T is as in condition 3),

then Γ has at least one pure strategy Nash equilibrium point.

Proof. It will first be shown that T is a well defined closed map. Let $Y \equiv \Theta$ and $X \equiv \Theta_2 \times \dots \times \Theta_n$. Then Y and X are compact Hausdorff spaces; in particular, $Y \times \{x\}$ is a compact subset of $Y \times X$ for each $x \in X$. Let $T^*: X \to Y$ be given by

$$T^*(x) = \{y \in Y | U_1(y, x) = \sup_{Y \times \{x\}} U_1(\cdot)\} \quad .$$

Since by assumption U_1 is continuous over $Y \times X \equiv \prod_{n^*} \Theta_i$, T^* is well defined. By the Maximum Theorem [4, page 116], T^* is closed. Since graph $(T_1) = Y \times (\text{graph } (T^*))$, T_1 is well defined and closed. Similarly, $T_2, \dots, T_n$ are well defined and closed. Hence the product $T \equiv \prod_{n^*} T_i$ is well defined and closed.

Combining conditions 1), 3), 4), and the above paragraph, it follows from 2.6 that there exists $\theta' \in \prod_{n^*} \Theta_i$ such that $\theta' \in T(\theta')$. By definition of T, this implies that Γ has at least one pure strategy Nash equilibrium point.

Q.E.D.

2.9 COROLLARY. Let $\Gamma \equiv (\prod_{n^*} \Theta_i, \prod_{n^*} U_i : \prod_{n^*} \Theta_i \to R^n)$ be an n-person game. If

1′) $\prod_{n^*} \Theta_i$ is a C-acyclic ANR compactum;

2′) Conditions 2) and 3) in 2.8 are satisfied,

then Γ has at least one pure strategy Nash equilibrium point.

Proof. By 2.5, $L(T) = 1$. The claim follows from 2.8.

Q.E.D.

3. SPECIAL CASES

In this section special cases of the existence theorem 2.8 will be discussed. The following definitions will be used.

3.1 DEFINITIONS. If $i : A \to X$ is the inclusion map for a pair of spaces $A \subseteq X$, then A is a <u>retract of X</u> if there exists a continuous map $r : X \to A$ such that $r \circ i$ is the identity map on A. The space A is called a <u>neighborhood retract</u> (in X) if A has a neighborhood in X of which it is a retract. A space Y is called a <u>euclidean neighborhood retract</u> (ENR) if there exists a neighborhood retract $X \subseteq R^n$ (some n) which is homeomorphic to Y. A space X is <u>contractible</u> if the identity map $i : X \to X$ is homotopic to a constant map on X. A space X is <u>locally contractible at a point $p \in X$</u> if every open set U containing p contains an open set V containing p such that V is contractible over U to a point in U. The space X is <u>locally contractible</u> if it is locally contractible at every point.

ANR compacta appear in the hypotheses of both the existence theorem 2.8 and its corollary 2.9. Most of the spaces commonly used in economic and game theory are ANR compacta. For example, compact convex subsets of Banach spaces are ANR compacta [14, 6.4 and 6.5, page 96]; finite dimensional locally contractible compacta (e.g., finite discrete spaces) are ANR compacta [14, 7.1, page 168]; and locally euclidean compacta (e.g., compact n-manifolds) are ANR compacta [14, 8.3, page 98]. By a theorem of Haver [13, page 281], locally contractible compacta that are a countable union of finite dimensional compacta are ANR compacta. As these examples demonstrate, the hypotheses of 2.9 are significantly more restrictive than those of 2.8; for the C-acyclicity restriction on the product of strategy spaces in 2.9 implies a strong global type of connectedness, whereas the property of being an ANR compactum is a local property.

The following proposition clarifies the relationship between the Lefschetz number, C-acyclicity, and contractibility. This latter restriction is used by Debreu [7].

3.2 PROPOSITION. Let Y be a compact Hausdorff space. Then

$$Y \text{ contractible } \underset{\not\Leftarrow}{\Rightarrow} Y \text{ C-acyclic } \underset{\not\Leftarrow}{\Rightarrow} L(f) \neq 0 \text{ for all continuous maps } f : Y \to Y .$$

<u>Remark</u>. Corollary 2.5 implies that the right-hand side forward implication can be strengthened to: Y C-acyclic $\Rightarrow L(T) \neq 0$ for all closed maps $T : Y \to Y$ such that $T(y)$ is C-acyclic for each $y \in Y$. Whether or not the converse of this stronger implication holds is apparently not known.

<u>Proof</u>. By Theorem 3.4 [10, page 238] and Theorem 5.1 [10, page 240], one point spaces are C-acyclic and homotopic maps on a compact Hausdorff space induce the same homomorphism on the Čech homology groups of that space. Thus contractible compact Hausdorff spaces are C-acyclic. By 2.5, C-acyclicity of Y implies $L(f) = 1$ for every continuous map $f : Y \to Y$.

As for the converse implications, for every positive integer n, real projective $2n$-space RP^{2n} is a finite polyhedron which is acyclic with respect to singular homology over the rational field Q, but not contractible; and complex projective $2n$-space CP^{2n} is a finite polyhedron which is not acyclic with respect to singular homology over the field Z_2 of integers mod 2, yet $L(f; Z_2) \neq 0$ for every continuous map $f : CP^{2n} \to CP^{2n}$ [5, pages 31 - 32]. Since finite polyhedra are ANR compacta, it follows by 2.2 (iii) that the proof of 3.2 is complete.

Q.E.D.

The following proposition demonstrates that conditions 1) and 1') in 2.8 and 2.9 may be replaced by restrictions on the individual strategy sets.

3.3 PROPOSITION. Let M and N be ANR compacta. Then $M \times N$ is an ANR compactum; and if in addition M and N are C-acyclic, then so is $M \times N$.

Proof. The topological product of a finite number of metrizable ANRs is a metrizable ANR [14, 7.6, page 97]; compactness of $M \times N$ follows by Tychonoff's theorem.

Suppose M and N are C-acyclic. By 2.12 [8, page 181] and 2.2 (iii), for each integer n,

$$\check{H}_n (M \times N) \cong \oplus_{p+q=n} \check{H}_p (M) \otimes \check{H}_q (N) , \tag{1}$$

where $\otimes$ denotes tensorproduct (with respect to F) . By definition of tensorproduct, $0 \otimes F \cong 0$ and $F \otimes F \cong F$. Hence by (1) and C-acyclicity of M and N ,

$$\check{H}_n (M \times N) \cong \begin{cases} 0 & \text{if} \quad n \neq 0 \\ F & \text{if} \quad n = 0 \, ; \end{cases}$$

i.e., $M \times N$ is C-acyclic.

Q.E.D.

3.4 COROLLARY. Condition 1) in 2.8 may be replaced by

1^{o}) For each $i \in n^*$, Θ_i is an ANR compactum;

and condition 1') in 2.9 may be replaced by

1*) For each $i \in n^*$, Θ_i is a C-acyclic ANR compactum.

The restriction "$T(\theta)$ is C-acyclic for each $\theta \in \prod_{n^*} \Theta_i$" in condition 3) of 2.8 may be replaced by "$T_i(\theta)$ is a C-acyclic ANR for each $\theta \in \prod_{n^*} \Theta_i$, $i \in n^*$."

<u>Proof</u>. The first statement is immediate from 3.3. As established in the proof of 2.8, the coordinate maps T_i , $i \in n^*$, are closed. In particular, the image sets $T_i(\theta)$, $\theta \in \prod_{n^*} \Theta_i$, are closed hence compact subsets of Θ_i , for each $i \in n^*$. Metrizability of the image sets is clear, since each Θ_i , $i \in n^*$, is metrizable. The second statement in 3.4 now follows from 3.3.

Q.E.D.

In conclusion, we discuss an interesting class of "generalized convex" spaces which satisfy the hypotheses of 2.7.

Definitions. A subset D of a real topological vector space is starconvex if there exists at least one point $p \in D$ such that

$$d \in D,\ t \in [0, 1] \Rightarrow td + [1 - t]\,p \in D \quad . \tag{2}$$

The collection D^* of all points $p \in D$ which satisfy (2) will be called the starkernel of D. (Clearly a convex set is a starconvex set which coincides with its starkernel.)

3.6 PROPOSITION. Every starconvex set is contractible.

Proof. Let D be a starconvex subset of a real topological vector space, and let $id : D \to D$ be the identity map on D. Let $g : D \to \{p\}$ be the constant map taking D into a point p in the starkernel of D. Define $\Phi : D \times [0, 1] \to D$ by

$$\Phi\,(d, t) = [1 - t]\, d + t\, p\,, \quad d \in D\,,\ t \in [0, 1] \quad .$$

Since p is in the starkernel of D, Φ is well defined. Continuity of Φ is obvious. Finally, $\Phi\,(\cdot\,, 0) \equiv id(\cdot)$ and $\Phi\,(\cdot\,, 1) \equiv g(\cdot)$. Hence Φ is a deformation of id into g; i.e., D is contractible (to p) .

Q.E.D.

3.7 LEMMA. Every ENR is an ANR.

Proof. Without loss of generality, assume D is a neighborhood retract in R^n, for some n. By Theorem 7.1 [14, page 168], every metrizable, finite dimensional, locally contractible space is an ANR. To prove the lemma, it thus remains to show that D is locally contractible.

Every open n-ball in R^n is contractible (3.6). Since every open set in R^n contains an open n-ball, R^n (and hence each open subset of R^n) is locally contractible. By assumption, D is a retract of an open set in R^n. Since retracts preserve local contractibility [14, 9.2, page 26], D is locally contractible.

Q.E.D.

3.8 PROPOSITION. Let $D \subseteq R^n$ be a compact starconvex set whose starkernel D^* contains an n-ball B. Then D is a C-acyclic ANR compactum.

Proof. After parallel translation and multiplication by some positive constant, it can be assumed that the n-ball $B \subseteq D^*$ is the standard ball $B^n \equiv \{x \in R^n \mid \|x\| \leq 1\}$, where $\|\cdot\|$ denotes euclidean distance. Then for every $d \in D$ and $0 \leq r < 1$, the point rd lies in the interior D^o of D; for rd lies in the open cone obtained by projecting the interior $(B^n)^o$ of B^n from d. Since by assumption $B^n \subseteq D^*$, this cone lies in D. By compactness of D, every ray from 0 must therefore contain exactly one point on the boundary ∂D of D.

Since $0 \in B^n \subseteq D^* \subseteq D$, ∂D is bounded away from 0; and ∂D is compact as a closed subset of D. The map $f : \partial D \to S^{n-1} \equiv \{x \in R^n \mid \|x\| = 1\}$ given by

$$F(d) = d/\|d\|$$

is therefore a well defined homeomorphism. By radial extension, one then obtains a homeomorphism

$$F : D \approx B^n \tag{3}$$

given by

$$F(rd) = rd/\|d\|, \quad d \in \partial D, \quad 0 \leq r \leq 1 .$$

Every space homeomorphic with B^n is a neighborhood retract of R^n, hence an ENR [8, 8.5, page 81]. It thus follows from (3) and 3.7 that D is an ANR. By 3.6 and 3.2, D is C-acyclic.

Q.E.D.

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