AXIOMATIZATION FOR AN EXPECTED UTILITY MODEL

WITH ENDOGENOUSLY DETERMINED GOALS*

by

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ABSTRACT

In [7] a "goal-control expected utility model" was formulated which allows the decision maker to specify his acts in the form of "controls" (partial contingency plans) and to simultaneously choose goals and controls in end-mean pairs. It was shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of this model.

In this paper the goal-control expected utility representation for the goal-control model primitives is axiomatized.

1. INTRODUCTION

In [7] a "goal-control expected utility model" was formulated which allows the decision maker to specify his acts in the form of "controls" (partial contingency plans) and to simultaneously choose goals and controls in end-mean pairs. It was shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of this model.

In this paper the expected utility representation for the goal-control model primitives is axiomatized. The primitives are reviewed in order to make this paper reasonably self-contained. However, for a detailed discussion of the goal-control model together with examples illustrating the expected utility representation, the reader is referred to [7].

2. PRIMITIVES FOR THE GOAL-CONTROL MODEL

Let $G = \{g, \ldots\}$ be a set of candidate goals, and for each $g \in G$ let $\Lambda_g = \{\lambda_g, \ldots\}$ be a set of controls. The primitives for the goal-control model ("gc-model") are then characterized by a vector

$$(\langle \Theta, \succ \rangle, \{\langle \Omega_\theta, \succ_\theta \rangle \mid \theta \in \Theta\}, \{\langle \mathcal{E}_\theta, \geq_\theta \rangle \mid \theta \in \Theta\})$$

where

$\Theta = \{\theta, \ldots\} = \bigcup_{g \in G} \{(g, \lambda_g) \mid \lambda_g \in \Lambda_g\}$ is the policy choice set consisting of candidate goal-control pairs (policies);

$\succ$ (policy preference order) is a weak order[1] on Θ;

and for each policy $\theta \in \Theta$,

$\Omega_\theta = \{\omega_\theta, \ldots\}$ is a nonempty set of state flows associated with the policy θ;

$\succ_\theta$ (θ-conditioned preference order) is a weak order on Ω_θ;

$\mathcal{E}_\theta = \{E_\theta, \ldots\}$ is an algebra[2] of subsets of Ω_θ whose elements E_θ will be called event flows associated with the policy θ;

$\geq_\theta$ (θ-conditioned probability order) is a weak order on $\mathcal{E}_\theta$.

The controls may be operationally interpreted as possibly conditioned sequences of actions (i.e., partial contingency plans) entirely under the control of the decision maker at the time of his choice. The candidate goals $g \in G$ may be operationally interpreted as potential objectives (e.g., production targets) whose realization the decision maker can attempt to achieve through appropriate choice of a control. The grouping of the controls into sets $\{\Lambda_g \mid g \in G\}$ reflects the possibility that different sets of controls may be relevant for different goals; e.g., for a decision maker in San Francisco, the control "travel by bus" is suitable for the goal "vacation in Los Angeles" but not for the goal "vacation in Hawaii." A control $\lambda_g \in \Lambda_g$ may or may not provide for the communication of the goal g to other persons in the decision maker's problem environment.

The weak order $\succcurlyeq$ on Θ can be operationally interpreted as a preference order as follows. For all $\theta', \theta'' \in \Theta$,

$\theta' \succcurlyeq \theta'' \Leftrightarrow$ the choice of policy θ' is at least as desirable to the decision maker as the choice of policy θ''.

The decision maker is assumed to choose a policy (candidate goal-control pair) $\theta' \in \Theta$ which is optimal in the sense that $\theta' \succcurlyeq \theta$ for all $\theta \in \Theta$. Throughout this paper we use "choose policy $\theta = (g, \lambda_g)$" and "implement control λ_g with g as the objective" interchangeably.

For each $\theta \in \Theta$, the set Ω_θ of state flows ω_θ can be interpreted as the decision maker's answer to the following question: "If I choose policy θ, what distinct situations (i.e., state flows ω_θ) might obtain?" The state flows may include references to past, present, and future happenings. In order for subsequent probability assessments to be realistically feasible, the state flow sets should include the decision maker's background information concerning the problem at hand.

The θ-conditioned preference orders $\succ_\theta$ can be interpreted as follows. For all $\omega, \omega' \in \Omega_\theta$, $\theta \in \Theta$,

$\omega \succ_\theta \omega' \Leftrightarrow$ the realization of ω is at least as desirable to the decision maker as the realization of ω', given the event "decision maker chooses θ."

Similarly, the θ-conditioned probability orders $\geq_\theta$ can be interpreted as follows. For all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$,

$E \geq_\theta E' \Leftrightarrow$ in the judgment of the decision maker, the realization of E is as likely as the realization of E', given the event "decision maker chooses θ."

A state flow ω may be relevant for the decision maker's problem under distinct potential policy choices; e.g., $\omega \in \Omega_\theta \cap \Omega_{\theta'}$ for some $\theta, \theta' \in \Theta$. Similarly, the algebras

$\{\mathcal{E}_\theta\}$ may overlap. Given state flows $\omega, \omega' \in \Omega_\theta \cap \Omega_{\theta'}$ for some $\theta, \theta' \in \Theta$, it may hold that $\omega \succ_\theta \omega'$ whereas $\omega' \succ_{\theta'} \omega$. Verbally, the relative utility of the state flows ω and ω' may depend on which conditioning event the decision maker is considering, "decision maker chooses θ" or "decision maker chooses θ'." Similarly for the relative likelihood of event flows $E, E' \in \mathcal{E}_\theta \cap \mathcal{E}_{\theta'}$, $\theta, \theta' \in \Theta$.

3. AXIOMATIZATION: INTRODUCTION

In sections 4 and 5 axioms will be given which ensure that the gc-model has an expected utility representation in the following sense: To each policy $\theta \in \Theta$ there corresponds a finitely additive probability measure $\sigma(\cdot \mid \theta) : \mathcal{E}_\theta \to [0, 1]$ satisfying

$$\sigma(E \mid \theta) \geq \sigma(E' \mid \theta) \Leftrightarrow E \geq_\theta E' , \qquad (1)$$

for all $E, E' \in \mathcal{E}_\theta$, and a utility function $u(\cdot \mid \theta) : \Omega_\theta \to R$ satisfying

$$u(\omega \mid \theta \geq u(\omega' \mid \theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega' , \qquad (2)$$

for all $\omega, \omega' \in \Omega_\theta$, such that

$$\int_{\Omega_\theta} u(\omega \mid \theta)\ \sigma(d\omega \mid \theta) \geq \int_{\Omega_{\theta'}} u(\omega \mid \theta')\ \sigma(d\omega \mid \theta') \Leftrightarrow \theta \succcurlyeq \theta' , \qquad (3)$$

for all $\theta, \theta' \in \Theta$.

This expected utility representation for the policy preference order $\succcurlyeq$ can be interpreted as follows. To each state flow $\omega \in \Omega_\theta$, $\theta \in \Theta$, the decision maker assigns a utility number $u(\omega | \theta)$ representing the desirability of $\{\omega\}$ obtaining, conditioned on the event "decision maker chooses θ," and a probability number $\sigma(\{\omega\} | \theta)$ representing the likelihood of $\{\omega\}$ obtaining, conditioned on the event "decision maker chooses θ." He then calculates the expected utility

$$\int_{\Omega_\theta} u(\omega \mid \theta)\ \sigma(d\omega \mid \theta)$$

corresponding to each choice of policy $\theta \in \Theta$, and chooses a policy which yields maximum expected utility.

Before beginning the statement of axioms, it might be helpful to briefly discuss the relationship of the gc-expected utility model axiomatization to previously established axiomatizations.

Ideally, an expected utility axiomatization should be calculationally feasible and all the primitives should be relevant for the decision maker's problem. In actuality, most expected utility axiomatizations extend the "basic primitives" (i.e., the primitives essential for the decision maker's problem) for mathematical reasons, and this extension often implies an impossible calculational ability on the part of the decision maker. (See Fishburn [1] and Krantz, Luce _et.al._ [3] for reviews of the expected utility literature.)

In certain axiomatizations the basic primitive sets are explicitly extended by introducing over these sets a collection of extraneous gambles, usually infinite in number, which the decision maker is required to order in preference. In other axiomatizations the basic primitive sets are implicitly extended. For example, in the expected utility model of L. Savage [4] the primitive sets consist of a set S of "states of the world," a set C of "consequences," and a set F containing _all_ "acts" (functions) taking S into C. For most decision problems the presence of constant functions in F represents an extension of the basic primitive set of acts available to the decision maker. In addition, Savage's axioms require S to be uncountably infinite. Since the decision maker is assumed to order in preference all functions in F, the uncountability of S introduces a calculational infeasibility.

Reliance on a single primitive preference order seems to be the principal reason why extraneous elements are introduced into the primitives of most individual choice models. In order for a preference order over consequences to be derived from a primitive preference order over a set B of acts or gambles, the consequence set must somehow be imbedded into B; e.g., through constant acts or degenerate gambles. Similarly, in order for probability judgments to be assessed from a primitive preference order over B, the acts or gambles in B must be suitably varied.

In contrast to most individual choice models, the gc-model primitives include three different types of orders whose existence is implied by the desired expected utility representation: a policy preference order $\succ$, θ-conditioned preference orders $\succ_\theta$, and θ-conditioned probability orders $\geq_\theta$. Consequently, the expected utility representation is obtained under minimal restrictions on the basic primitives.

Specifically, in section 4 the representations (2) and (3) are established under three assumptions (Axioms I - III) which include the temporary assumption (Axiom I) that probability representations satisfying (1) have been obtained. Axiom II is a finiteness restriction on the state flow sets. Axiom III requires the decision maker's primitive preference and probability orders to be compatible with the existence of a certain weak order over a mixture set constructed from primitive elements. As will be discussed in 4.2 below, this weak order can (but need not) be interpreted as a preference order over extraneous gambles. Given Axioms I and II, Axiom III will be shown (4.4) to be necessary and sufficient for the existence of the desired representations (2) and (3).

In section 5 two different axiomatizations for probability representations satisfying (1) are presented. The first axiomatization, due essentially to C. Kraft, J. Pratt, and A. Seidenberg, establishes necessary and sufficient conditions for the existence of the desired probability representations. The second axiomatization, due to Krantz, Luce <u>et. al</u>., establishes only sufficient conditions for the existence of the desired representations, but uniqueness is guaranteed. Uniqueness is of interest in relation to the preference order interpretation offered for Axiom III in section 4 (see 4.2). On the other hand, when the state flow sets Ω_θ are assumed to be finite as in Axiom II, the Krantz <u>et. al</u>. nonnecessary condition which ensures uniqueness is strong. These points will be further discussed in section 5.

Under both axiomatizations the resulting probability representations are finitely rather than countably additive. In view of Axiom II, this is all that is needed for the expected utility representation. If Axiom II were to be eventually weakened to allow for σ-algebras, the extension to countably additive probability representations would present no problems. A simple necessary and sufficient condition for a finitely additive probability representation on a σ-algebra to be countably additive has been obtained by C. Villegas (see [3, pages 215 - 216]).

4. AXIOMATIZATION: UTILITY

Let the primitives $(\langle\Theta, \succ\rangle, \{\langle\Omega_\theta, \succ_\theta\rangle \mid \theta \in \Theta\}, \{\langle\mathcal{E}_\theta, \geq_\theta\rangle \mid \theta \in \Theta\})$ for a gc-model be given (see section 2). The first axiom presented below will be replaced in section 5 by conditions on the primitives.

AXIOM I (TEMPORARY). To each policy $\theta \in \Theta$ there corresponds a finitely additive probability measure $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]$ satisfying

$$\sigma(E|\theta \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E' \quad ,$$

for all $E, E' \in \mathcal{E}_\theta$.

In the next axiom finiteness of the state flow sets $\{\Omega_\theta\}$ will be assumed in order to use 4.3 below. Although finiteness of the state flow sets is realistic, it is often convenient to work with connected sets, e.g., intervals of the real line. Moreover, as will be seen in section 5, this finiteness restriction is not essential for establishing the existence of the desired probability representations as in Axiom I. Thus it would be desirable to weaken Axiom II to allow for infinite state flow sets.

AXIOM II. For every policy $\theta \in \Theta$, the associated set Ω_θ of state flows is a finite set $\{\omega'_\theta, \ldots, \omega_\theta^{n_\theta}\}$, and the associated algebra $\mathcal{E}_\theta$ of event flows is given by $\mathcal{E}_\theta = 2^{\Omega_\theta}$ (i.e., the set of all subsets of Ω_θ) .

In the next axiom the decision maker's primitive preference and probability orders will be required to be compatible with the existence of a certain extraneous weak order. A preference order interpretation for the axiom will be discussed after it is stated. The following definitions and notation will be used.

4.1 DEFINITIONS AND NOTATION. A set M is the <u>mixture set</u> for a set K if

1) $K \subseteq M$;
2) For all $t \in [0, 1]$ and $B, D \in M$, there exists an element $tB + [1-t]D \in M$;
3) For all $t, r \in [0, 1]$ and $B, D \in M$,
   (a) $1B + 0D = B$;
   (b) $tB + [1-t]D = [1-t]D + tB$;
   (c) $t[rB + [1-r]D] + [1-t]D = trB + [1-tr]D$;
4) M is the minimal set with properties 1), 2), and 3).

For each policy $\theta \in \Theta$ let $M\Omega_\theta = \{\psi_\theta, \ldots\}$ denote the mixture set for $\Omega_\theta = \{\omega_\theta^1, \ldots, \omega_\theta^{n_\theta}\}$ (see Axiom II); and let $T(\theta) \equiv \sum_{i=1}^{n_\theta} \omega_\theta^i \, \sigma(\{\omega_\theta^i\} \mid \theta) \in M\Omega_\theta$, where $\sigma(\cdot \mid \theta) : \mathcal{E}_\theta \to [0, 1]$ is the finitely additive probability representation for the θ-conditioned probability order $\langle \mathcal{E}_\theta , \geq_\theta \rangle$ whose existence is guaranteed by Axiom I.

Let $Q \equiv \{(\psi_\theta \mid \theta) \mid \psi_\theta \in M\Omega_\theta, \theta \in \Theta\}$, and let $MQ = \{b, c, d, \ldots\}$ denote the mixture set for Q .

AXIOM III. There exists a weak order $\succsim^*$ over MQ which satisfies the following five conditions: For all $\theta, \theta' \in \Theta$, $\omega, \omega' \in \Omega_\theta$, $\psi, \psi' \in M\Omega_\theta$, and $b, c, d \in MQ$,

1) $(\omega|\theta) \succsim^* (\omega'|\theta) \Leftrightarrow \omega \succsim_\theta \omega'$;
2) $(T(\theta)|\theta) \succsim^* (T(\theta')|\theta') \Leftrightarrow \theta \succsim \theta'$;
3) $c >^* b,\ 0 < t < 1 \Rightarrow tc + [1 - t]d >^* tb + [1 - t]d$;
4) $d >^* c >^* b \Rightarrow$ there exist $t, s \in (0, 1)$ such that $tb + [1 - t]d >^* c >^* sb + [1 - s]d$;
5) $t(\psi|\theta) + [1 - t](\psi'|\theta) \sim^* (t\psi + [1 - t]\psi'|\theta)$ for all $t \in [0, 1]$,

where $>^*$ is defined on MQ by $[b >^* d] \equiv [b \succsim^* d$ and not $(d \succsim^* b)]$; and $\sim^*$ is defined on MQ by $[b \sim^* d] \equiv [b \succsim^* d$ and $d \succsim^* b]$.

<u>Remarks</u>. Axiom III - 2) is well defined only if Axiom I holds. Assuming conditions 1) and 2) are compatible, a weak order on MQ satisfying conditions 1) and 2) always exists. (By assumed connectedness and transitivity of the orders $\langle\Theta, \succsim\rangle$ and $\{\langle\Omega_\theta, \succsim_\theta\rangle | \theta \in \Theta\}$, the partial order $\succsim^o$ induced on MQ by the compatible conditions 1) and 2) is transitive and reflexive. Hence $\succsim^o$ can be extended to a weak order over MQ (see [6]).)

4.2 EXTRANEOUS GAMBLE - PREFERENCE ORDER INTERPRETATION FOR $\langle MQ, \succsim^*\rangle$. The mixture sets $M\Omega_\theta$, $\theta \in \Theta$, may be interpreted as sets of extraneous gambles as follows. For each set $\{t_1, \ldots t_{n_\theta}\}$ of nonnegative coefficients satisfying $\sum_i t_i = 1$, let the corresponding element $\psi = \Sigma_i \, \omega_\theta^i \, t_i \in M\Omega_\theta$ be interpreted as the gamble which awards "prize" ω_θ^i with "probability" t_i. Under this interpretation, if the probability representation $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]$ for the probability order $\langle \mathcal{E}_\theta, \geq_\theta \rangle$ guaranteed by Axiom I is unique, then

$$T(\theta) \equiv \Sigma_{i=1}^{n_\theta} \omega_\theta^i \, \sigma(\{\omega_\theta^i\} \mid \theta) \quad \in M\Omega_\theta$$

is the gamble which the decision maker will participate in if he chooses policy θ, according to his own judgments. If $\sigma(\cdot|\theta)$ is not unique, then $T(\theta)$ approximates this gamble.

Similarly, the mixture set MQ for $Q \equiv \{(\psi_\theta|\theta) | \psi_\theta \in M\Omega_\theta , \theta \in \Theta\}$ may be interpreted as a set of extraneous gambles as follows. Let each element $(\psi|\theta) \in Q$ be interpreted as the event "decision maker participates in gamble ψ" conditioned on the event "decision maker chooses policy θ." Then for each set $\{r_1, \ldots, r_m\}$ of nonnegative coefficients with $\Sigma_j \, r_j = 1$, and each set of elements $\{(\psi_{\theta_j}|\theta_j) \in Q | j = 1, \ldots, m\}$, the element $b \equiv \Sigma_j \, (\psi_{\theta_j}| \, \theta_j) \, r_j \in MQ$ can be interpreted as the gamble which awards "prize" $(\psi_{\theta_j}|\theta_j)$ with "probability" r_j. To "participate" in the gamble b, the decision maker imagines that with probability r_j he must participate in the gamble ψ_{θ_j}, with θ_j as his policy choice.

The weak order $\succ^*$ can then be interpreted as a preference order over the gambles in MQ as follows.

$b \succ^* c \Leftrightarrow$ Participation in the gamble b is at least as desirable to the decision maker as participation in the gamble c .

Under this gamble-preference order interpretation for $\langle MQ, \succ^* \rangle$, conditions 1) - 5) in Axiom III can be given straightforward interpretations. Condition 1) is tautological, and condition 2) is essentially tautological if the probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ are unique. Verbally, condition 2) reads: The desirability of participating in the gamble $T(\theta)$, given the event "decision maker chooses policy θ ," is at least as great for the decision maker as the desirability of participating in the gamble $T(\theta')$, given the event "decision maker chooses policy θ' ," if and only if the choice of policy θ is at least as desirable to the decision maker as the choice of policy θ' . (Intuitively, the "tighter" the probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]\}$, the closer the gambles $\{T(\theta)\}$ approximate the gambles the decision maker believes he would participate in for each choice of θ ; hence the more plausible condition 2) becomes.)

Finally, conditions 3) - 5) can be compared to standard axioms in the von Neumann-Morgenstern tradition. Condition 3) resembles Savage's "sure thing principle" (see [4, page 21 and page 114]) and can be given a similar defense. Condition 4) is a typical Archimedean constraint. Condition 5) states that the decision maker is indifferent between a one stage and a two stage gamble as long as both offer him the same expected return.

Although conditions 1) - 5) in Axiom III become intuitively plausible under this gamble-preference order interpretation for $\langle MQ, \succ^* \rangle$, Axiom III does not impose this interpretation for two reasons: it is not necessary; and more importantly, the underlying assumption that the decision maker can order in preference all the hypothetical, nonrealizable "gambles" in MQ is clearly strong.

4.3 LEMMA [1, 8.4, page 112]. Let M be the mixture set for a set K. Let $\geq'$ be a weak order on M, and let $>'$ be defined on M by $[B >' D] \equiv [B \geq' D$ and not $(D >' B)]$.[3] Then for all $B, D, R \in M$, the following two conditions

(a) $D >' B,\ 0 < t < 1 \Rightarrow tD + [1 - t]R >' tB + [1 - t]R$;

(b) $R >' D >' B \Rightarrow tB + [1 - t]R >' D >' rB + [1 - r]R$ for some $t, r \in (0, 1)$;

are necessary and sufficient for the existence of a function $W : M \to R$, unique up to positive linear transformation, satisfying

$$W(B) \geq' W(D) \Leftrightarrow B \geq' D ;$$

$$W(tB + [1 - t]D) = tW(B) + [1 - t]W(D) ,$$

for all $B, D \in M$ and $t \in [0, 1]$.

4.4 THEOREM. Let Axioms I and II hold. Then for each policy $\theta \in \Theta$ there exists a utility function $u(\cdot|\theta): \Omega_\theta \to R$ satisfying

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \succsim_\theta \omega', \tag{4}$$

for all $\omega, \omega' \in \Omega_\theta$, such that

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succsim \theta', \tag{5}$$

for all $\theta, \theta' \in \Theta$, if and only if Axiom III holds.

<u>Proof</u>. Assume Axioms I, II, and III hold. Then by Axiom III - 3), 4) and 4.3 there exists a function $U^*: MQ \to R$ satisfying

$$U^*(td + [1-t]b) = tU^*(d) + [1-t]U^*(b)\ ; \tag{6}$$

$$U^*(d) \geq U^*(b) \Leftrightarrow d \succsim^* b\ , \tag{7}$$

for all $d, b \in MQ$ and for all $t \in [0,1]$. By Axiom III - 2) and (7), for all $\theta', \theta'' \in \Theta$,

$$U^*(T(\theta')|\theta') \geq U^*(T(\theta'')|\theta'') \Leftrightarrow \theta' \succsim \theta'', \tag{8}$$

where $T(\theta)$, $\theta \in \Theta$, is as defined in 4.1 By Axiom III - 5), (7), and repeated use of (6),

$$U^*(T(\theta)|\theta) = \sum_{i=1}^{n_\theta} U^*(\omega_\theta^i|\theta)\ \sigma(\{\omega_\theta^i\}|\theta),\ \theta \in \Theta\ . \tag{9}$$

For each $\theta \in \Theta$, define a function $u(\cdot|\theta) : \Omega_\theta \to R$ by

$$u(\omega|\theta) = U^*(\omega|\theta)\ , \quad \omega \in \Omega_\theta\ . \tag{10}$$

By Axiom III - 1), (7), and (8),

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \succsim_\theta \omega'\ ,$$

for all $\omega, \omega' \in \Omega_\theta$, $\theta \in \Theta$. By (8), (9), (10) and Axiom II,

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succsim \theta'\ ,$$

for all $\theta, \theta' \in \Theta$.

Conversely, assume Axioms I and II hold, and functions $\{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\}$ exist satisfying (4) and (5). Define $U^o : MQ \to R$ by

$$U^o(\Sigma_i\ (\Sigma_{j=1}^{n_i}\ \omega^j_{\theta_i}\ r^j_i | \theta_i) \cdot t_i) = \Sigma_i\ (\Sigma_{j=1}^{n_i}\ u(\omega^j_{\theta_i} | \theta_i) \cdot r^j_i) \cdot t_i\ .$$

Clearly U^o is a well-defined function. Define a weak order $\succsim^*$ on MQ by

$$a \succsim^* b \Leftrightarrow U^o(a) \geq U^o(b)\ , \quad a, b \in MQ\ .$$

By (4), $\langle MQ, \succsim^* \rangle$ satisfies Axiom III - 1); and by (5) and Axiom II, $\langle MQ, \succsim^* \rangle$ satisfies Axiom III - 2). Finally, conditions 3), 4), and 5) in Axiom III can be verified for $\langle MQ, \succsim^* \rangle$ by straightforward calculation.

Q.E.D.

5. AXIOMATIZATION: PROBABILITY

Two alternative sets of conditions for the weak orders $\{\langle \Omega_\theta, \mathcal{E}_\theta, \geq_\theta \rangle \mid \theta \in \Theta\}$ will be presented which guarantee the existence of finitely additive probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] \mid \theta \in \Theta\}$ as in Axiom I, in a manner consistent with Axioms II and III (see section 4). The first set of conditions, although necessary and sufficient for the desired probability representations, will not guarantee their uniqueness. As discussed in 4.2, if the weak order $\langle MQ , \succ^* \rangle$ appearing in Axiom III is interpreted as a preference order over extraneous gambles, then the plausibility of the consistency requirement 2) in Axiom III varies directly with the "tightness" of the obtained representations. For this reason a second, sufficient set of conditions is presented which ensures the uniqueness of the probability representations. Since uniqueness for a probability representation over a finite set is unusual, it is not surprising that the representations obtained under the second set of conditions are somewhat rigid.

The first set of conditions will be obtained as a corollary of the following representation theorem, a reformulation by D. Scott of a result established by C. Kraft, J. Pratt, and A. Seidenberg [2]. Scott's proof (not given) involves passing by means of "indicator functions" from an algebra of subsets to a finite dimensional vector space representation for which a separating hyperplane theorem (a variant of the Hahn-Banach

Theorem) becomes applicable; hence the somewhat strange appearance of condition (iv) in the statement of the theorem.

Given an algebra $\mathcal{E}$ of subsets of a set Ω, $1_E : \Omega \to \{0, 1\}$ will denote the indicator function for E, defined by

$$1_E(\omega) = \begin{cases} 1, & \omega \in E ; \\ 0, & \omega \notin E . \end{cases}$$

A function $P : \mathcal{E} \to [0, 1]$ will be said to represent a binary relation $\succcurlyeq$ on $\mathcal{E}$ if $[E \succcurlyeq E'] \Leftrightarrow [P(E) \geq P(E')]$, for all $E, E' \in \mathcal{E}$.

5.1 THEOREM [5, Theorem 4.1, page 246]. Let $\mathcal{E}$ be an algebra of subsets of a finite set Ω, and let $\succcurlyeq$ be a binary relation on $\mathcal{E}$. Then for $\succcurlyeq$ to be representable by a finitely additive probability function P on $\mathcal{E}$ it is necessary and sufficient that the conditions

(i) $\Omega > \phi$;

(ii) $E \succcurlyeq \phi$;

(iii) $E \succcurlyeq E'$ or $E' \succcurlyeq E$;

(iv) $1_{E^0} + \ldots 1_{E^{n-1}} = 1_{D^0} + \ldots 1_{D^{n-1}}$ implies $D^0 \succcurlyeq E^0$,

hold for all $E, E', E^i, D^i \in \mathcal{E}$, $i=0, \ldots, n-1$, where $E^i \succcurlyeq D^i$ for $0 < i < n$.

<u>Remark</u>. As Scott notes, condition (iv) is an "unpleasant feature" since the sum $1_A + 1_B$ of two indicator functions cannot be identified with an element of $\mathcal{E}$ except when

$A \cap B = \phi$. Hence the theorem establishes the representation by placing restrictions on objects outside of the proper domain of events $\mathcal{E}$. Nevertheless, the interpretation of the equation in (iv) is straightforward: every element of Ω belongs to exactly the same number of the E^i as the D^i.

A second objection which might be raised to condition (iv) is its testability. (Although Ω is finite, condition (iv) entails an infinite set of restrictions; for repetition of the indicator functions is allowed.) However, the proof of 5.1 presented in Reference [2] includes an algorithm for checking in a finite number of steps whether condition (iv) holds.

According to Scott, 5.1 can be extended to infinite Ω by appropriate use of the Hahn-Banach Theorem.

5.2 COROLLARY. Assume each state flow set Ω_θ, $\theta \in \Theta$, is finite. Then the following three conditions are necessary and sufficient for the existence of finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ satisfying

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E',$$

for all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$:

1) $\Omega_\theta >_\theta \phi$, $\theta \in \Theta$;

2) $E \geq_\theta \phi$ for all $E \in \mathcal{E}_\theta$, $\theta \in \Theta$;

3) $1_{E^o} + \dots 1_{E^{n-1}} = 1_{D^o} + \dots 1_{D^{n-1}} \Rightarrow D^o \geq_\theta E^o$, for all $E^i, D^i \in \mathcal{E}_\theta$, $i = 0, \dots, n - 1$, with $E^i \geq_\theta D^i$, $0 < i < n$, for all $\theta \in \Theta$.

A second, alternative set of conditions sufficient for the existence of probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ as in Axiom I will be obtained as a corollary of the following theorem, due to Krantz et. al. We distinguish between <u>necessary</u> conditions which are implied by the existence of the desired representation, and <u>structural</u> conditions which are sufficient but not necessary for the existence of the desired representation.

5.3 THEOREM [3, Theorem 2, page 208]. Let $\mathcal{E}$ be an algebra of sets on a set Ω, and let $\geq^*$ be a relation on $\mathcal{E}$ such that for every $A, B, C, D \in \mathcal{E}$:

1. (Necessary) $\langle \mathcal{E}, \geq^* \rangle$ is a weak order;
2. (Necessary) $\Omega >^* \phi$ and $A \geq^* \phi$;
3. (Necessary) If $A \cap B = A \cap C = \phi$, then $B \geq^* C$ if and only if $A \cup B \geq^* A \cup C$;
4. (Structural) Ω is finite;
5. (Structural) If $A \cap B = \phi$, $A \geq^* C$ and $B \geq^* D$, then there exist $C', D', E \in \mathcal{E}$ such that:
   (i) $E \sim^* A \cup B$;
   (ii) $C' \cap D' = \phi$;
   (iii) $E \supseteq C' \cup D'$;
   (iv) $C' \sim^* C$ and $D' \sim^* D$,

where $[A \sim^* B] \equiv [A \succsim^* B$ and $B \succsim^* A]$ and $[A >^* B] \equiv [A \succsim^* B$ and not $(B \succsim^* A)]$, $A, B \in \mathcal{E}$. Then there exists a unique order-preserving measure P on $\mathcal{E}$ such that $(\Omega, \mathcal{E}, P)$ is a finitely additive probability space.

<u>Discussion</u>. In place of condition 4, the original Krantz <u>et. al</u>. representation theorem imposes a weaker, necessary Archimedean condition which is compatible with infinite algebras $(\Omega, \mathcal{E})$.

In 1949 Bruno de Finetti questioned whether conditions 5.3 - 1, 2, and 3 were sufficient as well as necessary for the existence of a finitely additive probability representation over a finite algebra $(\Omega, \mathcal{E})$. A counterexample to this conjecture, involving a Boolean algebra generated by five elements, is established in Reference [2]. The nonsufficiency of conditions 5.3 - 1, 2, and 3 for infinite algebras $(\Omega, \mathcal{E})$ is discussed by L. Savage [4, Chapter III, especially page 40]).

As Krantz <u>et. al</u>. note, it is difficult to give a simple interpretation for structural condition 5. Yet, in the presence of conditions 1, 2, and 3, condition 5 is strictly weaker than Savage's postulate P6′, which states: If $B, C \in \mathcal{E}$, and $C >^* B$, then there exists a partition $\{D_1, \ldots D_n\}$ of Ω such that $C >^* B \cup D_i$ for each i (see [4, pages 38 - 39] and [3, pages 206 - 207]). For example, Savage's P6′ forces Ω to be infinite, whereas conditions 5.3 - 1, 2, 3, and 5 are compatible with certain finite Ω (e.g., $\Omega = \{a, b, c, d\}$, with $\text{Prob}(a) = \text{Prob}(b) = \text{Prob}(c) = .2$, and $\text{Prob}(d) = .4$).

Since uniqueness is an extremely strong condition for probability representations over finite algebras, some rigidity in the Krantz et. al. representing function P is to be expected. Specifically, the probabilities assigned by P are integer multiples of a certain minimal fraction $1/n$. (To verify that this restriction holds, see the constructive proof for Theorem 4 [3, pages 44 - 52] on which the proof of 5.3 is based.) The rigidity of this restriction could be somewhat alleviated if the algebra $\mathcal{E}$ were assumed to contain an event such as "N tosses of a fair coin results in N heads," for some arbitrarily large N.

5.4 COROLLARY TO 5.3. Let conditions 2 - 5 in 5.3 hold for each weak order $\langle \Omega_\theta, \mathcal{E}_\theta, \geq_\theta \rangle$, $\theta \in \Theta$. Then there exist unique finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ satisfying

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E' ,$$

for all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$.

6. THE MAIN REPRESENTATION THEOREM

By combining 5.2 with 4.4, the following representation theorem is obtained.

6.1 THEOREM. Let a gc-model $(\langle \Theta, \succ \rangle, \{\langle \Omega_\theta, \succ_\theta \rangle | \theta \in \Theta\}, \{\langle \mathcal{E}_\theta, \geq_\theta \rangle | \theta \in \Theta\})$ be given, and assume each state flow set Ω_θ is finite, with $\mathcal{E}_\theta = 2^{\Omega_\theta}$ (Axiom II). Then conditions 5.2 - 1), 2), 3) and Axiom III are necessary and sufficient for the existence of finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ and utility functions $\{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\}$ satisfying for all policies $\theta, \theta' \in \Theta$:

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E', \quad \text{for all } E, E' \in \mathcal{E}_\theta ;$$

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \geq \omega', \quad \text{for all } \omega, \omega' \in \Omega_\theta ;$$

$$\int_{\Omega_\theta} u(\omega|\theta)\, \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\, \sigma(d\omega|\theta') \Leftrightarrow \theta \succ \theta' .$$

<u>Remark</u>. In the presence of Axiom II, conditions 5.2 - 1), 2), 3) are equivalent to Axiom I (this is the content of Theorem 5.2). Hence Axiom III - 2) is well defined.

FOOTNOTES

[1]A binary relation $\succeq$ on a set D is a <u>weak order</u> if for all a, b, c $\in$ D

(i) $a \succeq b$ or $b \succeq c$
(i.e., $\succeq$ is connected);

(ii) $a \succeq b$ and $b \succeq c$ implies $a \succeq c$
(i.e., $\succeq$ is transitive).

Weak orders have also been referred to as "complete preorderings."

[2]A collection F of subsets of a nonempty set X is said to be an <u>algebra</u> in X if F has the following three properties:

(1) $X \in F$;

(2) If $A \in F$, then $A^c \in F$, where A^c is the complement of A relative to X ;

(3) If $A, B \in F$, then $A \cup B \in F$.

[3]Fishburn's original proposition is stated in terms of a binary relation R which he requires to be a "weak order" in the sense that R is asymmetric and negatively transitive [1, Definition 2.1, page 11]. As is easily verified, the assumption that $\geq'$ is a weak order over M in the sense used in this paper (see Footnote 1) implies that $>'$ is a "weak order" over M in the sense of Fishburn.

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