WEALTH AND THE VALUE OF GENERALIZED LOTTERIES

by

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In two now famous articles, K. J. Arrow [2] and J. W. Pratt [7] demonstrate the existence of correspondences between certain simple risk aversion functions and the demand for and valuation of random economic prospects. Each of these authors simplifies his analysis by assuming that an individual's utility may be written as a function of his money or wealth. While this assumption appears fairly restrictive, Arrow notes that it is consistent with individual preferences which have as their domain a multidimensional commodity space. Specifically he observes that in perfect competition if prices are constant one may interpret money or wealth as a well-defined composite good.

Recently J. E. Stiglitz and R. Deschamps have commented explicitly on the implications of the composite good interpretation suggested by Arrow. Stiglitz [9] analyzes the relation between commodity demand functions and the Pratt-Arrow measures of risk aversion for a von Neumann-Morgenstern utility of money function. Deschamps [4] pursues this same general topic with particular emphasis placed on problems arising from the choice of numeraire.

Spence and Zeckhauser[1] also discuss interpreting von Neumann-Morgenstern utility of money functions as indirect representations of preferences for goods purchasable with money holdings. They argue that the protasis of the composite good theorem is generally contradicted in an uncertain environment. That is, prices are not generally constant over states of the world and even if they were the pattern of intertemporal evolution of random economic processes might make free trading of commodities in each state of the world impossible.

This last point is most easily illustrated via a constructed example. Consider two lotteries, the first denoted "Buy Now, Know Now" (BNKN) and the second "Buy Now, Know Later" (BNKL). An individual purchasing one chance in either lottery will win $100 with probability one-half and $0 with probability one-half. The prizes for both lotteries will be awarded one year hence, however, BNKN prize winners will be notified immediately while the BNKL results will be revealed on the same day prizes are to be distributed.

Since the prizes and probabilities are identical for these two gambles, a strict expected utility of wealth maximizer would be indifferent between them. If one considers an individual who is concerned instead with his lifetime consumption sequence, then the possibility that one lottery might be preferred to the other arises. This follows from the fact that the first year's consumption expenditures are constrained to be independent of the lottery outcome when BNKL tickets are held. No such constraint exists when BNKN tickets are owned.[2] The specific time structure of a lottery is thus an important determinant of its value if what is consumed and when enter the utility function as opposed to the amount spent to finance this consumption.

The aim of the present paper is to elucidate some interconnections between an individual's utility function and his valuation of random economic processes when the realizations of those processes are not necessarily representable as composite goods. I shall normally refer to such processes as lotteries where the lottery "prizes" are commodity bundles having no restrictions on their composition other than a nonnegativity constraint. The lottery concept is thus sufficiently general for the results obtained below to be applicable to economic phenomena ranging from oil exploration to health insurance.

The bulk of the paper is devoted to the derivation of a result which is a partial extension of an important theorem due to Pratt [7]. His theorem[3] establishes an if and only if relationship between the form of an individual's utility of money function and his valuation of fixed lotteries for varying levels of initial wealth. In Theorem 1 below I present a similar "if and only if" relationship for many commodity lotteries when preferences are restricted to be of the additive von Neumann-Morgenstern variety.[4]

The multicommodity lottery is then viewed as a representation of the random consumption sequence obtained by employing some intertemporal strategy in a stochastic environment. The current money value of such strategies is related to an individual's utility function and initial wealth endowment in Theorem 2. This theorem again posits an additive utility function; however, an essentially identical result is eatablished in Theorem 3 for a class of non-additive utility functions. All quasi-concave utility functions homogeneous of degree $\lambda \in (0, 1)$ form a proper subset of this class.

The last few pages contain extensions of two results provided by J. Stiglitz [9] on the relation between attitudes toward risk and demand functions. Stiglitz demonstrates that the form an individual's commodity demand functions may take is not independent of his attitudes toward risks of the "BNKN" variety. I show in Corollaries 1, 1′ and 2 that certain revealed attitudes toward arbitrary BNKN-type lotteries tell us a great deal about the individual's valuation of BNKL lotteries. One intriguing implication of these results is that an individual who is risk neutral toward arbitrary wealth (BNKN) lotteries regardless of commodity prices will display positive and decreasing absolute risk aversion toward all nondegenerate BNKL lotteries.

<u>Notation</u>

The following notation will be used in the remainder of this paper.

R^n	represent the n-dimensional space of reals;
R^{n^+}, $\overline{R^{n^+}}$	represent the positive and nonnegative orthants of R^n respectively;
$X = (x_1, x_2, \ldots, x_T)$	is a T-dimensional vector of commodities;
$P = (p_1, \ldots, p_T)$	is a T-dimensional vector of prices;
A	may be read as stock of wealth, assets or money;
Ω	is a set of events;
ω	is an element of Ω;
$\mathcal{F}$	is a σ-field of subsets of Ω;
Q	is a well-defined probability measure on $\mathcal{F}$;
$\mathcal{L}$	is the set of measurable functions from Ω into $\overline{R^{T^+}}$;
L	is an arbitrary element of $\mathcal{L}$;
$E(f(\cdot), Q)$	represents the mathematical expectation of the function f for the probability measure Q.

Throughout the remainder of the paper individual preferences are required to satisfy the following postulate.

Postulate I

$\mathcal{L}$ is completely preordered by $\succcurlyeq$. There exists a twice continuously differentiable, increasing strictly quasi-concave function, V, such that

$$L^{\bullet}, L^{1} \varepsilon \mathcal{L}, L^{\bullet} \succcurlyeq L^{1} \Leftrightarrow E(V \circ L^{\bullet}(\cdot), Q) \geq E(V \circ L^{1}(\cdot), Q) \quad .$$

Unlike Postulate I, the statements denoted as "Conditions" below are employed only as explicitly noted in the statement of a result. The first condition deals with the concept of additive utility.

Definition 1: V is additive if there exist functions u_i, $i = 1, 2, \ldots, T$ such that

$$V(x) = a + \sum_{i=1}^{T} u_i (x_i) \; \forall \; x = (x_1, x_2, \ldots, x_T) \; \varepsilon \; \overline{R^{T+}} \quad .$$

Condition 1: V is additive.

Note that for $T > 1$ Postulate I implies that $\frac{du_i (x_i)}{dx_i} > 0$ for all $x_i \geq 0$. This postulate also implies that at least $(T - 1)$ of the u_i are strictly concave throughout (strict quasi-concavity of $V(\cdot)$). Condition 2 extends this property to all T of the u_i's.

Condition 2: Each u_i is a strictly concave function of x_i, $i = 1, 2, \ldots, T$.

Definition 2: $x^M (A, p) \; \varepsilon \; \overline{R^{T+}}$ is the $x \; \varepsilon \; \{x \; \varepsilon \; \overline{R^{T+}} : p \circ x \leq A\}$ which maximizes $V(\cdot)$ on this set.

Note that Postulate I further implies $x^M(A, p)$ is unique and is a continuous function of A and p on $\overline{R^{1+}} \times R^{T+}$. It is also easily established that for any $x' \in R^{T+}$ there exists an (A', p') $\in \overline{R^{1+}} \times R^{T+}$ such that $x' = x^M(A', p')$.

<u>Definition 3</u>: For any $L^{\bullet} \in \mathcal{L}$ and $p^{\bullet} \in R^{T+}$

$$A^*(L^{\bullet}, p^{\bullet}) = \min A \in \{A \in \overline{R^{1+}} : V(x^M(A, p^{\bullet})) \geq E[V \circ L^{\circ}(\cdot), Q]\} .$$

<u>Definition 4</u>: For any $L^{\bullet} \in \mathcal{L}$ and $p^{\bullet} \in R^{T+}$

$$\Pi(L^{\bullet}, p^{\bullet}) = p^{\bullet} \circ E[L^{\bullet}(\cdot), Q] - A^*(L^{\bullet}, p^{\bullet}) .$$

One may interpret $L^{\bullet}$ as a lottery with prizes denominated in commodities and $p^{\bullet}$ as a vector of certain commodity prices. $A^*(\circ)$ is called the present certain dollar value of the specified lottery evaluated at existing prices. $\Pi(\cdot)$ is denoted the risk premium associated with the specified lottery and prices. From Condition 2 it is immediate that $\Pi(\cdot)$ is nonnegative for any (L, p) and strictly positive for nondegenerate lotteries.

The notation provided in the next definitions will be employed in the statement and proof of Theorem 1.

<u>Definition 5</u>: For any $L^{\bullet} \in \mathcal{L}$, $p^{\bullet} \in R^{T^+}$ and $\delta \in R^{1+}$, $\Delta x^M(L^{\bullet}, p^{\bullet}, \delta)$ will denote the vector

$$x^M(A^*(L^{\bullet}, p^{\bullet}) + \delta, p^{\bullet}) - x^M(A^*(L^{\bullet}, p^{\bullet}), p^{\bullet}) .$$

Conditions 1 and 2 imply that $\Delta x^M(L^{\bullet}, p^{\bullet}, \delta) \geq 0$, i.e., all commodities are normal goods.

<u>Definition 6</u>: Let $L \in \mathcal{L}$ and $\bar{x} \in \overline{R^{T+}}$. The lottery $L + \bar{x}$ is the function on Ω for which

$$(L + \bar{x})(\omega) = L(\omega) + \bar{x} \quad \forall\, \omega \in \Omega \ .$$

As one might well expect, the individual's attitudes toward risk are a prime determinant of the dollar value he places on a given lottery. A measure of "single commodity risk aversion" is useful in describing these attitudes when preferences are additive.

<u>Definition 7</u>:

$$r_{u_j}(x_j^{\bullet}) = \left. -\frac{d^2 u_j(x_j)}{(dx_j)^2} \bigg/ \frac{du_j(x_j)}{dx_j} \right|_{x_j = x_j^{\bullet}}$$

is referred to as the measure of single commodity risk aversion of the function u_j evaluated at $x_j^{\bullet}$.

For $T = 1$ this measure is simply the well known Pratt-Arrow measure of absolute risk aversion. Pratt's Theorem 2 provides a complete description of the relation between initial wealth and an individual's dollar evaluation of a given lottery for $T = 1$ and monotonic $r_u(\cdot)$. Specifically, his theorem asserts (in our notation) that:

$$\Pi(L + \Delta x^M(L, p, \delta), p) < (\leq)\ \Pi(L, p)$$

for all nondegenerate one dimensional lotteries $L \in \mathcal{L}$ and any $(p, \delta) \in R^{1+} \times R^{1+}$ if and only if $r_u(x)$ is a decreasing (nonincreasing) function of x on $\overline{R^{1+}}$. The same implications hold if "increasing" is substituted for decreasing throughout.

The following theorem provides some insight on the correspondence between lotteries and risk premiums for $T > 1$.

<u>Theorem 1</u>: If Conditions 1 and 2 are satisfied,

$$\Pi \; (L + \Delta \, x^M \, (L, \, p, \, \delta), \, p) < (\leq) \; \Pi \; (L, \, p)$$

for all nondegenerate $L \in \mathcal{L}$, any $p \in R^{T+}$ and any $\delta > 0$ if and only if $r_{u_i} (x_i)$ is a strictly decreasing (nonincreasing) function of x_i on $[0, \infty)$ for all $i = 1, 2, \ldots, T$. Also,

$$\Pi \; (L + \Delta \, x^M \, (L, \, p, \, \delta), \, p) > \Pi \; (L, \, p)$$

for all nondegenerate $L \in \mathcal{L}$, any $p \in R^{T+}$ and any $\delta > 0$ only if $r_{u_i} (x_i)$ is a strictly increasing function of x_i on $[0, \infty)$ for all $i = 1, 2, \ldots, T$.

The lotteries dealt with in this result are more structured than the one mentioned in my introductory remarks. The prizes obtained here are strictly exogenously determined, whereas the holder of a winning BNKN or BNKL ticket can choose what commodities constitute his prize. It is therefore appropriate to elaborate on Theorem 1 in a slightly different framework. BNKN and BNKL type lotteries are dealt with in subsequent results, but for the present, let us consider a special television game show.

The show has a very simple format. An individual is picked at random from the audience and asked to choose and open one of many identical doors. Behind each door is a prize which the contestant wins if that door is chosen. All of these prizes are nontransferable, e.g., a round trip ticket to Hawaii which is valid only for the contestant winning it and which may not be exchanged for cash or any other gift.

An individual owning a pass to this show, which he finds just as entertaining on television as live, has a minimum price for which he would sell it. The difference between this price and the expected market value of the winnings of anyone in the audience is the risk premium, Π, referred to in Theorem 1.[5]

The theorem suggests that if the individual's preferences are additive this risk premium will be inversely related to his certain wealth holdings regardless of the number of doors or types of prizes offered if and only if his single commodity measures of absolute risk aversion are all decreasing. Also, each single commodity measure of absolute risk aversion must be increasing to have a positive relation between nonrisky asset holdings and risk premium for all prize and door combinations which the show's producer might devise.

<u>Proof of Theorem 1</u>: First I shall establish the "only if" portion of the theorem. That is, it will be proven that if

$$\Pi\ (L + \Delta\, x^M\ (L,\ p,\ \delta),\ p) < (\leq,\ >)\ \Pi\ (L,\ p)$$

for all $L \varepsilon \mathcal{L}$, any $p \varepsilon R^{T^+}$ and any $\delta > 0$, then $r_{u_i}(x_i)$, $i = 1, \ldots, T$, is decreasing (nonincreasing, increasing) on R^{1^+}.

As an initial step an arbitrary nonnegative random variable on $(\Omega, \mathcal{F}, Q)$, $L_i(\cdot)$, and an arbitrary positive real number, a, are selected. One may then define the number $\bar{x}$ implicitly by the equation

$$E\ [u_i \circ L_i\ (\cdot),\ Q] = u_i\ (\bar{x}) \quad .$$

Next find an $(A^2, \tilde{p})$ pair such that

$$x_i^M\ (A^2, \tilde{p}) = \bar{x} + a \quad .$$

Since $x_i^M\ (\cdot\ , \tilde{p})$ is continuous and $x_i^M\ (0, \tilde{p}) = 0$ there exists an $A^1 < A^2$ such that

$$x_i^M\ (A^1, \tilde{p}) = \bar{x} \quad .$$

Then define the lottery $L^i \in \mathcal{L}$ by the rule:

$$L^i\ (\omega) = (x_1^M\ (A^1, \tilde{p}),\ \ldots,\ x_{i-1}^M (A^1, \tilde{p}),\ L_i\ (\omega),\ x_{i+1}^M (A^1, \tilde{p}),\ \ldots,\ x_T^M\ (A^1, \tilde{p}))$$

for all $\omega \in \Omega$.

It should be immediately clear that

$$V\ (x^M\ (A^1, \tilde{p})) = E\ [V \circ L^i\ (\cdot)\ ,\ Q]$$

and

$$\Pi\ (L^i, \tilde{p}) = \tilde{p}_i\ [E\ (L_i^i\ (\cdot)\ ,\ Q) - \bar{x}] \quad .$$

If $\Pi\ (L^i + \Delta\ x^M\ (L^i, \tilde{p}, A^2 - A^1), \tilde{p}) < (\leq, >)\ \Pi\ (L^i, \tilde{p})$ then by definition we must have

$$V\ (x^M\ (A^2, \tilde{p})) < (\leq, >)\ \ E\ [V \circ (L^i + \Delta\ x^M\ (L^i, \tilde{p}, A^2 - A^1))(\cdot), Q]$$

or

$$\sum_{j=1}^{T} u_j \, (x_j^M \, (A^2, \tilde{p})) < (\leq, >) \sum_{\substack{j \,\varepsilon\, \{1,\, 2,\, \ldots,\, T\} \\ j \neq i}} u_j \, (x_j^M \, (A^2, \tilde{p}))$$

$$+ \, E \, [u_i \circ (L_i^i + a)(\cdot), \, Q]$$

or

$$u_i \, (x_i^M \, (A^2, \tilde{p})) = u_i \, (\bar{x} + a) < (\leq, >) \, E \, [u_i \circ \, (L_i^i + a)(\cdot), \, Q] \quad .$$

Since $a > 0$ and L_i were arbitrarily chosen Pratt's Theorem 2, cited above, guarantees that $r_{u_i}(x_i)$ is decreasing (nonincreasing, increasing) .

I shall now verify the "if" portion of the theorem. That is, it will be proven that if the $r_{u_i}(x_i)$, $1 = 1, 2, \ldots, T$, are decreasing (nonincreasing) on R^{1+} then

$$\Pi \, (L + \Delta \, x^M \, (L, \, p, \, \delta), \, p) < (\leq) \, \Pi \, (L, \, p)$$

for all $L \,\varepsilon\, \mathcal{L}$, any $p \,\varepsilon\, R^{T+}$ and any $\delta > 0$. The $T = 2$ case is established first and then the result is extended by induction to $T > 2$.

Recall that when $T = 2$

$$E \, [V \circ L \, (\omega), \, Q] = E \, [u_1 \circ L_1(\cdot), \, Q] + E \, [u_2 \circ L_2(\cdot), \, Q] \quad .$$

We may, for any $L \,\varepsilon\, \mathcal{L}$, find $\tilde{x}_1$ and $\tilde{x}_2$ such that

$$E \, [u_i \, \circ \, L_i(\cdot), \, Q] = u_i \, (\tilde{x}_i) \, , \qquad i = 1, \, 2 \quad .$$

By Definition 3 we have

$$u_1 \, (x_1^M \, (A^* \, (L, \, p^\bullet), \, p^\bullet)) + u_2 \, (x_2^M \, (A^* \, (L, \, p^\bullet), \, p^\bullet)) = u_1 \, (\tilde{x}_1) + u_2 \, (\tilde{x}_2)$$

for arbitrarily chosen $p^{\bullet} \varepsilon R^{T+}$. We now have two possibilities which I shall treat as separate cases.

<u>Case I</u>: $x_i^M (A^* (L, p^{\bullet}), p^{\bullet}) = \tilde{x}_i$, $i = 1, 2$.

For any $\delta > 0$ we have $\Delta x_i^M (L, p^{\bullet}, \delta) \geq 0$, $i = 1, 2$, with strict inequality holding for either one or both of them. Pratt's Theorem 2 ensures that

$$E [V \circ (L + \Delta x^M (L, p^{\bullet}, \delta))(\cdot), Q] = E [u_1 \circ (L_1 + \Delta x_1^M (L, p^{\bullet}, \delta))(\cdot), Q] + E [u_2 \circ (L_2 + \Delta x_2^M (L, p^{\bullet}, \delta))(\cdot), Q]$$

$$> (\geq, <) \; u_1 (\tilde{x}_1 + \Delta x_1^M (L, p^{\bullet}, \delta)) + u_2 (\tilde{x}_2 + \Delta x_2^M (L, p^{\bullet}, \delta))$$

$$= V (x^M (A^* (L, p^{\bullet}) + \delta, p^{\bullet}))$$

if r_{u_1} and r_{u_2} are decreasing (nonincreasing, increasing). Thus,

$$A^* (L + \Delta x^M (L, p^{\bullet}, \delta), p^{\bullet}) > (\geq, <) \; A^* (L, p^{\bullet}) + \delta$$

since V and $x^M (\cdot, p^{\bullet})$ are increasing. Therefore,

$$\Pi (L + \Delta x^M (L, p^{\bullet}, \delta), p^{\bullet}) < (\leq, >) \; \Pi (L, p^{\bullet})$$

if r_{u_1} and r_{u_2} are decreasing (nonincreasing, increasing).

<u>Case II</u>: $x_i^M (A^* (L, p^{\bullet}), p^{\bullet}) \neq \tilde{x}_i$, $i = 1, 2$.

The following Lemma is instrumental in establishing the desired result when these inequalities obtain.

<u>Lemma 1</u>: If $\bar{x}_i \neq x_i^M (A^* (L, p^\bullet), p^\bullet)$, $i = 1, 2$, then

$$u_1 (\bar{x}_1 + \Delta x_1^M (L, p^\bullet, \delta)) + u_2 (\bar{x}_2 + \Delta x_2^M (L, p^\bullet, \delta))$$

$$> (\geq, <) \; u_1 (x_1^M (A^* (L, p^\bullet), p^\bullet) + \Delta x_1^M (L, p^\bullet, \delta))$$

$$+ u_2 (x_2^M (A^* (L, p^\bullet), p^\bullet) + \Delta x_2^M (L, p^\bullet, \delta))$$

if r_{u_1} and r_{u_2} are decreasing (nonincreasing, increasing and $x_i^M (A^* (L, p^\bullet), p^\bullet) \neq 0$, $i = 1, 2$) .

Lemma 1 is proven in the appendix to this paper. A diagrammatic representation of the result is provided in Figure 1. Notice that decreasing, constant and increasing absolute risk aversion of the u_i 's is associated respectively with decreasing, constant and increasing curvature of the indifference curves as one moves along any income-consumption expansion path.

Pratt's Theorem 2 guarantees that if r_{u_1} and r_{u_2} are decreasing (nonincreasing), then

$$E [u_1 \circ (L_1 + \Delta x_1^M (L, p^\bullet, \delta))(\cdot), Q] + E [u_2 \circ (L_2 + \Delta x_2^M (L, p^\bullet, \delta))(\cdot), Q]$$

$$> (\geq) \; u_1 (\bar{x}_1 + \Delta x_1^M (L, p^\bullet, \delta)) + u_2 (\bar{x}_2 + \Delta x_2^M (L, p^\bullet \; \delta)) \quad .$$

Application of Lemma 1 thus yields

$$E [u_1 \circ (L_1 + \Delta x_1^M (L, p^\bullet, \delta))(\cdot), Q] + E [u_2 \circ (L_2 + \Delta x_2^M (L, p^\bullet, \delta))(\cdot), Q]$$

$$> (\geq) \; u_1 (x_1^M (A^* (L, p^\bullet) + \delta, p^\bullet)) + u_2 (x_2^M (A^* (L, p^\circ) + \delta, p^\bullet))$$

whenever r_{u_1} and r_{u_2} are decreasing (nonincreasing). Thus,

$$A^* (L + \Delta x^M (L, p^\bullet, \delta), p^\bullet) > (\geq) A^* (L, p^\bullet) + \delta \quad ,$$

and

$$\Pi (L + \Delta x^M (L, p^\bullet, \delta), p^\bullet) < (\leq) \Pi (L, p^\bullet)$$

for r_{u_1} and r_{u_2} decreasing (nonincreasing) as asserted.

Extension of the result to $T > 2$ is accomplished by induction.

We first introduce some new notation which will hopefully simplify the argument.

$${}_n x^M (a, p^\bullet) = ({}_n x_1^M (a, p^\bullet), \ldots, {}_n x_n^M (a, p^\bullet))$$

is the maximizer for

$$\sum_{i=1}^{n} u_i (x_i) \quad \text{on} \quad \{(x_1, \ldots, x_n) \in \overline{R^{n+}} : \sum_{i=1}^{n} x_i \, p_i^\bullet \leq a\} \quad .$$

$$\Delta_n \, x^M (a^\bullet, \delta^\bullet, p^\bullet) \equiv {}_n x^M (a^\bullet + \delta^\bullet, p^\bullet) - {}_n x^M (a^\bullet, p^\bullet) \quad .$$

$${}_n \hat{u} (a^\bullet, p^\bullet) \equiv \sum_{i=1}^{n} u_i \, ({}_n x_i^M (a^\bullet, p^\bullet)) \quad .$$

It should be readily apparent that

$$\max_{a,\, x_{n+1}} \; {}_n \hat{u} (a, p^\bullet) + u_{n+1} (x_{n+1})$$

subject to $\qquad = {}_{n+1} \hat{u} (\bar{A}, p^\bullet) \quad ,$

$$a + p_{n+1}^\bullet \, x_{n+1} \leq \bar{A}$$

and that

$${}_T \hat{u}(a^\bullet, p^\bullet) = V (x^M (a^\bullet, p^\bullet)) \quad .$$

Next let

$$r_{{}_n\hat{u}}(a, p^\bullet) = -\frac{\partial^2 ({}_n\hat{u}(a, p^\bullet))}{(\partial a)^2} \Bigg/ \frac{\partial({}_n\hat{u}(a, p^\bullet))}{\partial a} .$$

Neave [5, Theorem 1, page 46] has proven that if the r_{u_i}, $i = 1, \ldots, n$, are decreasing (nonincreasing), then $r_{{}_n\hat{u}}(a, p^\bullet)$ is decreasing (nonincreasing) in a.

Our induction hypothesis for $n \geq 2$, $IH \cdot (n)$, is: "If r_{u_i}, $i = 1, \ldots, n$, are decreasing (nonincreasing) then

$${}_n\hat{u}(\bar{a}, p^\bullet) = \sum_{i=1}^{n} E\,[u_i \circ L_i^\bullet(\circ), Q]$$

$$\Rightarrow {}_n\hat{u}(\bar{a} + \delta^\bullet, p^\circ) < (\leq) \sum_{i=1}^{n} E\,[u_i \circ (L_i^\bullet + \Delta_n x_i^M(\bar{a}, \delta^\circ, p^\circ))(\cdot), Q]$$

for $\delta^\bullet > 0$ and at least one nondegenerate $L_i^\bullet(\circ)$, $i = 1, \ldots, n$. If no $L_i^\bullet(\cdot)$ is nondegenerate only weak inequality is assured."

Now we simply find the x_{n+1}^* such that

$$E\,(u_{n+1} \circ L_{n+1}^\bullet(\cdot), Q) = u_{n+1}(x_{n+1}^*)$$

and the a^* such that

$$\sum_{i=1}^{n} E\,(u_i \circ L_i^\bullet(\circ), Q) = {}_n\hat{u}(a^*, p^\circ)$$

Neave's result in combination with Lemma 1 insures that if r_{u_i}, $i = 1, \ldots, n + 1$ are decreasing (nonincreasing) then

$$_{n+1}\hat{u}\,(A^{\bullet}, p^{\bullet}) = {}_{n}\hat{u}\left(\sum_{i=1}^{n} {}_{n+1}x_i^M\,(A^{\bullet}, p^{\bullet})\,p_i^{\bullet}, p^{\bullet}\right) + u_{n+1}\left({}_{n+1}x_{n+1}^M\,(A^{\bullet}, p^{\bullet})\right)$$

$$= {}_{n}\hat{u}\,(a^{*}, p^{\bullet}) + u_{n+1}(x_{n+1}^{*}) ,$$

implies

$$_{n+1}\hat{u}\,(A^{\bullet} + \delta^{\bullet}, p^{\bullet}) = {}_{n}\hat{u}\left(\sum_{i=1}^{n} {}_{n+1}x_i^M\,(A^{\bullet}, p^{\bullet})\,p_i^{\bullet} + \sum_{i=1}^{n} \Delta_{n+1}x_i^M\,(A^{\bullet}, \delta^{\bullet}, p^{\bullet})p_i^{\bullet}, p^{\bullet}\right)$$

$$+ u_{n+1}\left({}_{n+1}x_{n+1}^M\,(A^{\bullet}, p^{\bullet}) + \Delta_{n+1}\,x^M\,(A^{\bullet}, \delta^{\bullet}, p^{\bullet})\right)$$

$$\leq {}_{n}\hat{u}\left(a^{*} + \sum_{i=1}^{n} \Delta_{n+1}x_i^M\,(A^{\bullet}, \delta^{\bullet}, p^{\bullet})p_i^{\bullet}, p^{\bullet}\right)$$

$$+ u_{n+1}\left(x_{n+1}^{*} + \Delta_{n+1}x_{n+1}^{*}\,(A^{\bullet}, \delta^{\bullet}, p^{\bullet})\right) \quad .$$

This implication together with IH•(n) implies IH•(n + 1) , a restatement of IH•(n) with (n + 1) inserted wherever n appeared. IH•(2) has been proven and therefore IH•(T) is valid. IH•(T) is simply the "if" assertion of Theorem 1.

It is quite clear that few if any economic phenomena would conform to the basic specifications of Theorem 1. That is, Theorem 1 provides no options for decision making or choice. Given the opportunity to allocate the increment to wealth, $\$\delta$, it is doubtful that augmenting the initial lottery, $L^{\bullet}$, by $\Delta\, x^M\,(L^{\bullet}, p^{\bullet}, \delta^{\bullet})$ would be chosen. Also note that the value of a specific lottery is compared before and after changes in wealth thus neglecting the lottery choice problem.

In order to discuss some problems of individual decision making the concept of a consumption strategy is introduced. A consumption strategy, s , associates with each state of the world, $\omega \in \Omega$, a vector $(x_1, \ldots, x_T)$. Restricting our attention to those strategies which are measurable w.r.t. Q we have $s \in \mathcal{L}$. The fact that consumption strategy connotes some notion of maximization does not obviate its status as a lottery.

The following definitions will be utilized in the remainder of the discussion.

Definition 8: s is a nonnegative function from Ω into $\overline{R^{T+}}$ which is measurable with respect to $\mathcal{F}$. A function s is referred to as a consumption strategy.

Definition 9: $\mathcal{L}(A^{\bullet}, p^{\bullet}) = \{s \in \mathcal{L}: s$ is feasible for wealth $A^{\bullet}$ and certain prices $p^{\bullet}\}$.

Reference to "feasible" in Definition 9 is left vague since the restrictions associated with feasibility vary greatly from problem to problem. Such factors as the timing of consumption decisions, availability of investment opportunities, and debt constraints would all enter into a definition of feasibility.

An additional condition is required for the subsequent discussion.

Condition 3: $\mathcal{L}(A^{\bullet}, p^{\bullet}) + Z(\delta, p^{\bullet}) \subseteq \mathcal{L}(A^{\bullet} + \delta, p^{\bullet})$ for any $A^{\bullet} \in \overline{R^{1+}}$, $p^{\bullet} \in R^{T+}$ and $\delta \in \overline{R^{1+}}$ where $Z(\delta, p^{\bullet}) = \{z \in \overline{R^{T+}} : p^{\bullet} \cdot z \leq \delta\}$.

The condition requires that if an individual's initial wealth is increased, he may use all or part of it to finance additional purchases of the T commodities at the prevailing market prices.

Theorem 2: Given Conditions 1, 2, and 3

$$\max_{\substack{s \in \mathcal{L}(A^{\bullet}, p^{\circ}) \\ s \neq x^M(\bar{A}, p^{\bullet})}} E[V(s(\cdot)), Q] = V(x^M(\bar{A}, p^{\bullet}))$$

implies

$$\underset{s \,\varepsilon\, \mathcal{L}(A^{\bullet} + \delta,\ p)}{\text{Max}} \quad E\,[V(s(\cdot)),\ Q] > (\geq)\ V(x^M(\bar{A} + \delta,\ p^{\bullet}))$$

if $r_{u_i}(\cdot)$ is decreasing (nonincreasing) for all $i = 1, \ldots, T$.

The interpretation of the theorem is straightforward. An individual is asked to determine the minimum expenditure, at prevailing market prices, required to finance a riskless consumption plan which he would accept in exchange for his current uncertain prospects. The individual is then presented with δ dollars. He is now asked to calculate the minimum cost of a riskless consumption plan that he would be willing to trade for the consumption strategy he could pursue with his augmented initial wealth. The theorem asserts that the cost of procuring an acceptable safe consumption plan will have risen by more than (at least) δ dollars if the individual's single commodity measures of absolute risk aversion are decreasing (nonincreasing).

Though each strategy, s , may also be thought of as a lottery, there is a subtle difference between this result and Theorem 1. The individual's strategy has been chosen by him. It is his plan of action for market activities now and in the future. If his initial endowments were to change he might revise the portfolio of stocks he wants to hold. This may result from a change in opportunities (his original wealth might not have been sufficient to meet the margin requirements of his revised portfolio) or simply because his attitudes toward uncertain prospects depend on his initial wealth.

Thus the lottery may change with variations in initial endowments. Referring back to the discussion of Theorem 1, the prizes in the game show would not change if the contestant received a completely unexpected bequest from Uncle Fred. Such a bequest might, however, induce the same individual to trade his BNKN lottery ticket for an available BNKL lottery ticket. Such portfolio adjustments may take place when he chooses a strategy from $\mathcal{L}(A^\bullet + \delta, p^\bullet)$ instead of $\mathcal{L}(A^\bullet, p^\bullet)$.

Proof of Theorem 2: Since any $s \in \mathcal{L}(A^\bullet, p^\bullet)$ is also an element of $\mathcal{L}$, we know from Theorem 1 that if

$$E\,[V(s'(\cdot)),\ Q] = V(x^M(\bar{A}, p^\bullet))$$

then for any $s' \in \mathcal{L}(A^\bullet, p^\bullet)$

$$E[V(s'(\cdot) + x^M(\bar{A} + \delta, p^\bullet) - x^M(\bar{A}, p^\bullet)),\ Q] > (\geq)\ V(x^M(\bar{A} + \delta, p^\bullet))$$

if $r_{u_i}(\cdot)$ is decreasing (nonincreasing) for all $i = 1, 2, \ldots, T$. By Condition 3

$$s' + x^M(\bar{A} + \delta, p^\bullet) - x^M(\bar{A}, p^\bullet) \in \mathcal{L}(A^\bullet + \delta, p^\bullet) \quad .$$

Up to this point the analysis has rested on the assumption that preferences may be represented by an additive von Neumann-Morgenstern utility function (Condition 1). Despite the highly restrictive nature of this condition, I believe in many instances it is a perfectly acceptable simplifying assumption. Nevertheless it would be nice to be able to say something about generalized commodity lotteries when preferences are not additive.

Happily there are conditions which, if satisfied by a nonadditive utility function, imply a result analogous in content to Theorem 2 above. These restrictions, unlike Conditions 1 and 2, are price specific. That is, for every price vector, $p^{\bullet}$, there is a set of utility functions, $\mathcal{V}_{p^{\bullet}}$, which satisfy these new conditions. Specifically, $V(\cdot) \varepsilon \mathcal{V}_{p}\bullet$ if and only if,

1) $x^M (A, p^{\bullet})$ is a linear function of A, and

2) $V(x^M(A^{\bullet}, p^{\bullet}) + z) = V(x^M(A^{\bullet}, p^{\bullet}))$

$$\Rightarrow V(x^M(A^{\bullet} + \delta, p^{\bullet}) + z) \geq V(x^M(A^{\bullet} + \delta, p^{\bullet})) \quad .$$

Recall that Theorem 2 relates the current certain dollar value of strategies to initial wealth and the characteristics of the r_{u_i}'s. Since $V(\cdot)$ is no longer assumed to be additive these r_{u_i}'s are no longer meaningful measures. I, therefore, introduce the following measure.

$$r_v(A, p) = -\left(\frac{\partial^2 V(x^M(A, p))}{(\partial A)^2}\right) \Bigg/ \frac{\partial V(x^M(A, p))}{\partial A} \quad .$$

<u>Theorem 3</u>: Given Condition 3 and $V(\cdot) \varepsilon \mathcal{V}_p$,

$$\underset{\substack{s \,\varepsilon\, \mathcal{L}(A^{\bullet}, p) \\ s \neq x^M(\bar{A}, p)}}{\text{Max}} E[V(s(\cdot)), Q] = V(x^M(\bar{A}, p))$$

$$\Rightarrow \underset{s \,\varepsilon\, \mathcal{L}(A^{\bullet} + \delta, p)}{\text{Max}} E[V(s(\cdot)), Q] > (\geq) V(x^M(\bar{A} + \delta, p))$$

if $r_v(A, p)$ is positive and decreasing (nonincreasing).[6]

Proof of Theorem 3: First let us define the indirect utility function, $u: \overline{R^{1+}} \times R^{T+} \to R^1$, by the rule

$$u(A, p) = V(x^M(A, p)) \quad .$$

Since $x^M(A, p)$ is continuous and increasing in A and $V(\cdot)$ is continuous and increasing in x, $u(\cdot)$ is continuous and increasing in A. Also since

$$r_v(A, p) = -\left(\frac{\partial^2 u(A, p)}{(\partial A)^2}\right) \bigg/ \frac{\partial u(A, p)}{\partial A}$$

is positive, $u(A, p)$ is a strictly concave function of A.

Now let the maximizing $s \varepsilon \mathcal{L}(A^{\bullet}, p)$ be denoted $\bar{s}$ and $A(s(\omega)) = \min A$ such that $u(A, p) \geq V(s(\omega))$. We therefore have

$$E[u(A(\bar{s}(\cdot)), p), Q] = u(\bar{A}, p)$$

$$\Rightarrow E[u(A(\bar{s}(\cdot)) + \delta, p), Q] > (\geq)\ u(\bar{A} + \delta, p)$$

as a direct consequence of Pratt's Theorem 2 since $r_v(A, p)$ is decreasing (nonincreasing). Also note that

$$u(A(s(\omega)) + \delta, p) \leq V(s(\omega) + x^M(\bar{A} + \delta, p) - x^M(\bar{A}, p))$$

by the definition of γ_p.

Finally, $\bar{s}' = (\bar{s} + x^M(\bar{A} + \delta, p) - x^M(\bar{A}, p))$ is not necessarily the maximizing $s \varepsilon \mathcal{L}(A^{\bullet} + \delta, p)$ but is feasible by Condition 3. This completes the proof of Theorem 3.

Notice that lotteries which merely add to or subtract from some initial wealth holdings, $A^{\bullet}$, can only represent a limited class of commodity lotteries. Outcomes are of the "Know Now" variety and will lie along the expansion path of $x^M(\cdot, p^{\bullet})$. Therefore, measures of attitudes toward wealth or money lotteries associated with $u(A, p^{\bullet})$, like $r_v(A, p^{\bullet})$, need not in general convey all the information we would like regarding attitudes toward a larger set of commodity lotteries or strategies. Theorem 3 indicates that this situation is to a large degree ameliorated when $u(A, p^{\bullet})$ is an indirect representation of a $V(\cdot) \varepsilon \mathcal{V}_{p^{\bullet}}$.

The relation between an individual's attitudes toward risks and his preference ordering of the commodity space has been central to the analysis. It was shown in Lemma 1 that a certain measure of risk aversion for additive utility functions is linked closely to the shape of an individual's indifference curve map. In Theorem 3 restrictions have been placed directly on the form of the indifference sets. A result is then obtained on how the dollar value of very general gambles are affected by changes in initial wealth holdings.

The link between the curvature of indifference sets and attitudes toward risk has also been explored by Stiglitz [9]. He has considered properties of $V(\cdot)$ which are implied by attitudes toward lotteries, s^*, of the form $s^*(\omega) = x^M(A, p)$ for appropriate A, for all $\omega \varepsilon \Omega$. He presents the intriguing finding for risk averters[7] that (in our notation):

If $r_v(A, p)$ is a constant function of $A \varepsilon (0, \infty)$ for any $p > 0$ or if $(r_v(A, p) \cdot (A))$ is a constant function of $A \varepsilon (0, \infty)$ for any $p > 0$ then $V(\cdot)$ is a homothetic function.

One may easily verify that strict quasi-concavity and homotheticity of $V(\cdot)$ imply $V \varepsilon \mathcal{V}_p$ for all p.[8] Thus we obtain for strictly quasi-concave V :

<u>Corollary 1</u>: If $r_v(A, p)$ is a constant function of $A \varepsilon (0, \infty)$ for any $p > 0$ then

$$\operatorname*{Max}_{s \,\varepsilon\, \mathcal{L}(A^{\bullet}, p)} E[V(s(\cdot)), Q] = V(x^M(\bar{A}, p))$$

$$\Rightarrow \operatorname*{Max}_{s \,\varepsilon\, \mathcal{L}(A^{\bullet} + \delta, p)} E[V(s(\cdot)), Q] \geq V(x^M(A^{\bullet} + \delta, p))$$

and

<u>Corollary 2</u>: If $(r_v(A, p))\cdot(A)$ is a constant function of $A \varepsilon (0, \infty)$ for any $p > 0$ then

$$\operatorname*{Max}_{\substack{s \,\varepsilon\, \mathcal{L}(A^{\bullet}, p) \\ s \neq x^M(\bar{A}, p)}} E[V(s(\cdot)), Q] = V(x^M(\bar{A}, p))$$

$$\Rightarrow \operatorname*{Max}_{s \,\varepsilon\, \mathcal{L}(A^{\bullet} + \delta, p)} E[V(s(\cdot)), Q] > V(x^M(\bar{A} + \delta, p)) \quad .$$

Finally if one restricts his attention to more complex lotteries Corollary 1 may be somewhat strengthened to:

<u>Corollary 1′</u>: If $r_v(A, p)$ is a constant function of $A \varepsilon (0, \infty)$ for any $p > 0$ and if there exists a maximizing $\bar{s} \varepsilon \mathcal{L}(A^{\bullet}, p^{\bullet})$ for $E[V(\cdot), Q]$ such that $Q\{\omega \varepsilon \Omega: \bar{s}(\omega) \neq x^M(A, p^{\bullet})$ for any $A\} > 0$

then

$$\underset{s \, \varepsilon \, \mathcal{L} \, (A^{\bullet}, \, p^{\bullet})}{\text{Max}} \quad E \, [V(s(\cdot)), \, Q] = V(x^{M} \, (\bar{A}, \, p^{\bullet}))$$

$$\Rightarrow \underset{s \, \varepsilon \, \mathcal{L} \, (A^{\bullet} + \delta, \, p)}{\text{Max}} \quad E \, [V(s(\cdot)), \, Q] > V(x^{M} \, (\bar{A} + \delta, \, p^{\bullet})) \quad .$$

The additional restriction in this corollary insures that one of the expected utility maximizing strategies has positive probability of outcomes lying off the asset expansion path for p . This restriction yields the desired result since in Condition 2 for $v \, \varepsilon \, \mathcal{V}_p$ "$>$" replaces "$\geq$" for strictly quasi-concave homothetic functions. Thus "$\leq$" may be replaced by "$<$" in the proof of Theorem 3 establishing the result (refer to page 21, last inequality).

Corollary 1′ has a particularly interesting interpretation. Let an individual with strictly convex preference sets be confronted with every possible lottery of the BNKN variety. If the price he is willing to pay for any such lottery is independent of his initial wealth for any given commodity prices, then the price he is willing to pay for any BNKL-type lottery will be an increasing function of his initial wealth.

Thus, if an individual was risk neutral toward every BNKN lottery, he would not only display risk aversion but also decreasing absolute risk aversion toward lotteries of the BNKL type.

Many economists (see for example Alchian [1] and Sandmo [8]) have asserted that completely certain prospects are almost nonexistent in the economic sphere. Since the results derived here pertain to comparisons of such certain prospects with uncertain prospects, does not this assertion imply that there are no opportunities for applying them? I believe the answer to this question is no in light of the fact that economists commonly employ simplified models to investigate economic behavior.

In such a model the fact that the consequences of a particular act are not known with complete certainty may or may not be critical. An individual may have two courses of action available to him which differ primarily in terms of the degree of uncertainty associated with their respective consequences. Should this be the case a characterization of his options as a certain and an uncertain outcome may capture the essence of the individual's choice problem.

Economists often employ such characterizations in models of individual behavior. By doing so they are able to focus on the salient features of various economic choices. The monetary economist regularly abstracts from uncertain labor income when he studies an individual's allocation of wealth between a risky and a nonrisky asset. Likewise, when the labor economist investigates an individual's search for a job the random return on his stock portfolio is ignored and accepting a job is considered a riskless proposition.[9] Thus while the real world economic sphere contains few if any examples of completely certain outcomes, they are common in the simplified models with which economists work.

Footnotes

[1]See Zeckhauser and Spence [9] for a more detailed discussion of these restrictions.

[2]One might view a trip to a "reliable" fortune-teller as a means of exchanging a BNKL- for a BNKN-type ticket. Likewise the United States Weather Service attempts to perform such a function.

[3]See Theorem 2, page 130 of [6].

[4]For a discussion of the implications of this assumption see [5].

[5]Notice that the term "risk premium" as used here is not necessarily the same as the concept discussed by Pratt and Arrow. Π will most likely be positive even if the outcome of the game has been previously determined, and the individual is certain that he will win a ticket to Hawaii. This would be the case if he were willing to trade the ticket for commodities costing less than the price of the ticket.

[6]Recall that in proving Theorem 1 Neave [4] was cited for a proof that if V is additive and r_{u_i} decreasing, $i = 1, \ldots, T$, then $r_V(\cdot, p)$ is decreasing for arbitrary positive p.

[7]This is actually a combination of two results in [8].

[8]If V is homothetic, $x^M(A, p^\circ) = \frac{A}{A^\circ} x^M(A^\circ, p^\circ)$ for $A^\circ > 0$, $p^\circ > 0$. Also by homotheticity $V(x^\circ) = V(x^\circ + Z^\circ) \Rightarrow V(\alpha x^\circ) = V(\alpha x^\circ + \alpha z^\circ)$ for $\alpha > 0$. By strict quasi-concavity $V(\beta(\alpha x^\circ) + (1 - \beta)(\alpha x^\circ + \alpha z^\circ)) > V(\alpha x^\circ)$ for $0 < \beta < 1$. Take $\alpha > 1$ and $\beta = \frac{1}{\alpha}$ and we have $V(\alpha x^\circ + z^\circ) > V(\alpha x^\circ)$. Finally, notice that if V is homogeneous of degree $\lambda \in (0, 1]$ then it is homothetic and

$$r_V(A, p) = \frac{(\lambda - 1)}{A} \cdot g(p) \quad .$$

[9]I have applied Theorem 2 of this paper to just such a problem in my "Individual Labor Market Behavior Under Uncertainty".

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Appendix

Proof of Lemma 1: Since along any indifference curve in the x_1, x_2 plane $(u_1(x_1) + u_2(x_2) = \text{constant})$ x_2 is an implicit differentiable function of x_1, we may write:

$$(1) \qquad \left.\frac{d\left(\frac{-u_1'(x_1)}{u_2'(x_2)}\right)}{dx_1}\right|_{u_1(x_1)+u_2(x_2)=\text{constant}}$$

$$= \frac{-u_1'' u_2' + u_2'' \frac{dx_2}{dx_1} u_1'}{[u_2']^2} = \frac{-u_1'' u_2' - u_2'' \frac{u_1'}{u_2'} u_1'}{[u_2']^2}$$

$$= \frac{-u_1''}{u_2'} - \frac{u_2''}{u_2'}\left(\frac{u_1'}{u_2'}\right)^2 = \left(\frac{-u_1''}{u_1'}\right)\cdot\left(\frac{u_1'}{u_2'}\right) + \left(\frac{-u_2''}{u_2'}\right)\left(\frac{u_1'}{u_2'}\right)^2$$

$$= r_{u_1}(x_1)\left(\frac{u_1'(x_1)}{u_2'(x_2)}\right) + r_{u_2}(x_2)\left(\frac{u_1'(x_1)}{u_2'(x_2)}\right)^2$$

$$= \left.\frac{d\left(\frac{dx_2}{dx_1}\right)}{dx_1}\right|_{u_1(x_1)+u_2(x_2)=\text{constant}.}$$

Without loss of generality let $\tilde{x}_1 > x_1^{M^\bullet} \equiv x_1^M(A^*(L, p^\bullet), p^\bullet)$ and $\tilde{x}_2 < x_2^{M^\bullet} \equiv x_2^M(A^*(L, p^\bullet), p^\bullet)$. Also denote the convex combination $\lambda\tilde{x}_1 + (1-\lambda)x_1^{M^\bullet}$ as $x_1(\lambda)$. For any $\lambda \in [0, 1]$ and level of utility, $c = u_1(x_1) + u_2(x_2)$, x_2 becomes a function of $x_1(\lambda)$ and c $(x_2 = x_2(x_1(\lambda), c))$. With $c^\bullet = u_1(x_1^{M^\bullet}) + u_2(x_2^{M^\bullet})$ we have:

$$(2) \qquad \tilde{x}_2 - x_2^{M^\bullet} = x_2(x_1(1), c^\bullet) - x_2(x_1(0), c^\bullet)$$

$$= \int_0^1 \frac{dx_2(x_1(\lambda), c^\bullet)}{d\lambda}\, d\lambda \quad .$$

Simple application of the chain rule for differentiation yields:

$$(3)\qquad \left.\frac{\partial x_2 (x_1 (\lambda), c^{\cdot})}{\partial \lambda}\right|_{\lambda=\bar{\lambda}} = \left.\frac{\partial x_2 (x_1 (\lambda), c^{\cdot})}{\partial x_1}\right|_{x_1=x_1(\bar{\lambda})} (\tilde{x}_1 - x_1^{M^{\cdot}}) \quad .$$

Of course $\left.\frac{\partial x_2(x_1 (\lambda), c^{\cdot})}{\partial x_1}\right|_{x_1=x_1(\bar{\lambda})}$ is merely the marginal rate of substitution evaluated at $x_1 = \bar{\lambda}\ \tilde{x}_1 + (1 - \bar{\lambda})\ x_1^{M^{\cdot}}$.

For $\lambda = 0$ we have two possibilities for the magnitude of (3) since $(x_1^{M^{\cdot}}, x_2^{M^{\cdot}})$ is a utility maximizing consumption pair for $(p_1^{\cdot}, p_2^{\cdot})$. That is, if both $x_1^{M^{\cdot}}$ and $x_2^{M^{\cdot}}$ are positive

$$(3a)\qquad \frac{p_1^{\cdot}}{p_2^{\cdot}} = \frac{\left.\frac{du_1(x_1)}{dx_1}\right|_{x_1^{M^{\cdot}} = x_1}}{\left.\frac{du_2(x_2)}{dx_2}\right|_{x_2^{M^{\cdot}} = x_2}} = \left.\frac{-\ \partial x_2 (x_1 (\lambda), c^{\cdot})}{\partial x_1}\right|_{x_1 = x_1 (0)}$$

and if $x_1^{M^{\cdot}} = 0$, then

$$(3b)\qquad \frac{p_1^{\cdot}}{p_2^{\cdot}} \geq \frac{\left.\frac{du_1(x_1)}{dx_1}\right|_{x_1 = 0}}{\left.\frac{du_2(x_2)}{dx_2}\right|_{x_2^{M^{\cdot}} = x_2}} = \left.\frac{-\ \partial x_2 (x_1 (\lambda), c^{\cdot})}{\partial x_1}\right|_{x_1 = x_1 (0)} \quad .$$

Now if $(x_1^{M^{\cdot}}, x_2^{M^{\cdot}})$ was an interior solution then $\Delta x_1^{M^{\cdot}} \equiv \Delta x_1^{M} (L, p^{\cdot}, \delta) > 0$ and $\Delta x_2^{M^{\cdot}} \equiv \Delta x_2^{M} (L, p^{\cdot}, \delta) > 0$, and

$$(3c)\qquad \frac{p_1^{\cdot}}{p_2^{\cdot}} = \frac{\left.\frac{du_1(x_1)}{dx_1}\right|_{x_1^{M^{\cdot}} + \Delta x_1^{M^{\cdot}} = x_1}}{\left.\frac{du_2(x_2)}{dx_2}\right|_{x_2^{M^{\cdot}} + \Delta x_2^{M^{\cdot}} = x_2}}$$

$$= \left.\frac{-\ \partial x_2 (x_1, c^1)}{\partial x_1}\right|_{x_1 (0) + \Delta x_1^{M^{\cdot}} = x_1}$$

where $c^1 = u_1 (x_1^{M^\cdot} + \Delta x_1^{M^\cdot}) + u_2 (x_2^{M^\cdot} + \Delta x_2^{M^\cdot})$.

If $x_1^{M^\cdot} = 0$ but $\Delta x_1^{M^\cdot} > 0$ then of course

(3d) $$\frac{p_1^\cdot}{p_2^\cdot} = \left.\frac{-\partial x_2 (x_1 , c^1)}{\partial x_1}\right|_{x_1 (0) + \Delta x_1^{M^\cdot} = x_1}$$

$$\geq \left.\frac{-\partial x_2 (x_1 , c^\cdot)}{dx_1}\right|_{x_1 (0) = x_1} .$$

Finally if $x_1^{M^\cdot} = 0$ and $\Delta x_1^{M^\cdot} = 0$ then

(3e) $$\frac{p_1^\cdot}{p_2^\cdot} \geq \left.\frac{-\partial x_2 (x_1, c^1)}{\partial x_1}\right|_{x_1 = 0}$$

$$= \frac{\left.\frac{du_1(x_1)}{dx_1}\right|_{x_1 = 0}}{\left.\frac{du_2(x_2)}{dx_2}\right|_{x_2^{M^\cdot} + \Delta x_2^M = x_2}}$$

which by concavity of $u_2(\cdot)$ is greater than

$$\frac{\left.\frac{du_1(x_1)}{dx_1}\right|_{x_1 = 0}}{\left.\frac{du_2(x_2)}{dx_2}\right|_{x_2^{M^\cdot} = x_2}} = \left.\frac{-\partial x_2 (x_1 , c^\cdot)}{\partial x_1}\right|_{x_1 = 0} .$$

Therefore (3) and (3a) through (3e) yield,

(4) $$\left.\frac{\partial x_2 (x_1 (\lambda) , c^\circ)}{\partial \lambda}\right|_{\lambda = 0} \geq \left.\frac{\partial x_2 (x_1 (\lambda) + \Delta x_1^{M^\cdot} , c^1)}{\partial \lambda}\right|_{\lambda = 0}$$

with inequality holding only if $x_1^{M^\cdot} = 0$. Recall that we have restricted the present discussion to the case where $x_1^{M^\bullet} < \tilde{x}_1$, $x_2^{M^\bullet} > \tilde{x}_2$ and hence $x_2^{M^\bullet} \neq 0$.

Observe also that if there exists a $\lambda^* \varepsilon [0, 1]$ such that

$$(5) \qquad \left.\frac{\partial x_2 (x_1, c^\bullet)}{\partial x_1}\right|_{x_1 (\lambda^*) = x_1} = \left.\frac{\partial x_2(x_1, c^1)}{\partial x_1}\right|_{x_1 (\lambda^*) + \Delta x_1^{M^\bullet} = x_1}$$

then

$$(6) \qquad \left.\frac{\partial\left(\dfrac{\partial x_2 (x_1 (\lambda), c^\bullet)}{\partial \lambda}\right)}{\partial \lambda}\right|_{\lambda^* = \lambda} = \left.\frac{\partial\left(\dfrac{\partial x_2 (x_1, c^\bullet)}{\partial x_1}\right)}{\partial x_1}\right|_{x_1 (\lambda^*) = x_1} (\tilde{x}_1 - x_1^{M^\bullet})^2$$

$$> (=, <) \quad \left.\frac{\partial\left(\dfrac{\partial x_2 (x_1 (\lambda) + \Delta x_1^{M^\bullet}, c^1)}{\partial \lambda}\right)}{\partial \lambda}\right|_{\lambda^* = \lambda}$$

$$= \left.\frac{\partial\left(\dfrac{\partial x_2 (x_1. c^1)}{\partial x_1}\right)}{\partial x_1}\right|_{x_1 = x_1 (\lambda^*) + \Delta x_1^{M^\bullet}} (\tilde{x}_1 - x_1^{M^\bullet})^2$$

for r_{u_1} and r_{u_2} decreasing (constant, increasing).

This inequality follows directly from (1) when the equality

$$\frac{\left.\dfrac{du_1(x_1)}{dx_1}\right|_{x_1 (\lambda^*) = x_1}}{\left.\dfrac{du_2(x_2)}{dx_2}\right|_{x_2 (x_1 (\lambda^*), c^\bullet) = x_2}} = \frac{\left.\dfrac{du_1(x_1)}{dx_1}\right|_{x_1 (\lambda^*) + \Delta x_1^{M^\bullet} = x_1}}{\left.\dfrac{du_2(x_2)}{dx_2}\right|_{x_2 (x_1 (\lambda^*) + \Delta x_1^{M^\bullet}, c^1) = x_2}}$$

is recognized (notice specifically line four of (1)). (4) and (6) taken together give us:

$$(7) \qquad \tilde{x}_2 - x_2^{M^\bullet} > (\geq) \int_0^1 \frac{dx_2 (x_1 (\lambda) + \Delta x_1^{M^\bullet}, c^1)}{d\lambda} \, d\lambda$$

for r_{u_1}, r_{u_2} decreasing (nonincreasing). (Notice that the left- and right-hand side of (7) are negative.) (7) may in turn be re-written as:

$$(8) \qquad x_2\,(x_1(1),\, c^{\cdot}) - x_2\,(x_1(0),\, x^{\cdot}) > (\geq)$$
$$x_2\,(x_1(1) + \Delta\, x_1^{M^{\cdot}},\, c^1) - x_2\,(x_1(0) + \Delta\, x_1^{M^{\cdot}},\, c^1) \quad .$$

The lemma is established by observing:

$$\begin{aligned}(9) \qquad c^1 &= u_1\,(x_1^{M^{\cdot}} + \Delta\, x_1^{M^{\cdot}}) + u_2\,(x_2^{M^{\cdot}} + \Delta\, x_2^{M^{\cdot}}) \\ &= u_1\,(\tilde{x}_1 + \Delta\, x_1^{M^{\cdot}}) + u_2\,(x_2^{M^{\cdot}} + \Delta\, x_2^{M^{\cdot}} \\ &\qquad + x_2\,(x_1(1) + \Delta\, x_1^{M^{\cdot}},\, c^1) \\ &\qquad - x_2\,(x_1(0) + \Delta\, x_1^{M^{\cdot}},\, c^1)) \\ &< (\leq)\; u_1\,(\tilde{x}_1 + \Delta\, x_1^{M^{\cdot}}) + u_2\,(x_2^{M^{\cdot}} + \Delta\, x_2^{M^{\cdot}} + (\tilde{x}_2 - x_2^{M^{\cdot}})) \\ &= u_1\,(\tilde{x}_1 + \Delta\, x_1^{M^{\cdot}}) + u_2\,(\tilde{x}_2 + \Delta\, x_2^{M^{\cdot}})\end{aligned}$$

for r_{u_1}, r_{u_2} decreasing (nonincreasing).

If r_{u_1} and r_{u_2} are increasing <u>and equality holds in (4)</u>, i.e., $(x_1^{M^{\cdot}}, x_2^{M^{\cdot}})$ is an interior solution, then we have

$$(7') \qquad \tilde{x}_2 - x_2^{M^{\cdot}} < \int_0^1 \frac{dx_2\,(x_1\,(\lambda) + \Delta\, x_1^{M^{\cdot}},\, c^1)}{d\,\lambda}\, d\,\lambda$$

and

$$(8') \qquad x_2\,(x_1\,(1),\, c^{\cdot}) - x_2\,(x_1\,(0),\, c^{\cdot})$$
$$< x_2\,(x_1\,(1) + \Delta\, x_1^{M^{\cdot}},\, c^1) - x_2\,(x_1\,(0) + \Delta\, x_1^{M^{\cdot}},\, c^1) \quad .$$

These inequalities in turn imply

$$(9') \quad u_1\ (\tilde{x}_1 + \Delta\ x_1^{M^{\bullet}}) + u_2\ (\tilde{x}_2 + \Delta\ x_2^{M^{\bullet}}) < u_1\ (x_1^{M^{\bullet}} + \Delta\ x_1^{M^{\bullet}}) + u_2\ (x_2^{M^{\bullet}} + \Delta_2^{M^{\bullet}})\ .$$

This completes the proof of the lemma.