PRICE FIXING SCHEMES AND OPTIMAL BUFFER STOCK POLICIES*

by

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ABSTRACT

This paper examines the feasibility of price fixing and the welfare implications of government buffer stock programs in a stochastic general equilibrium framework. It is found that regardless of the price set by the government and regardless of the initial level of buffer stocks, a price fixing scheme will eventually fail with probability one. In a competitive equilibrium with storage by individuals, prices will be random even if the nature of the stochastic processes are known to everyone. In the absence of contingent contracts such an equilibrium can be second best.

I. Introduction

In actual economies uncertainty as to the direction and extent of future price movements is pervasive. This has lead some government policy makers to argue that government buffer stock programs which fix prices could reduce uncertainty and increase economic welfare. Though economists may respond with the long held belief that prices play a crucial role in allocating resources efficiently, the feasibility of price fixing and the welfare implications of government buffer stock programs cannot be examined in the static, nonstochastic model of traditional price theory. There is need of a model which is dynamic and explicitly stochastic.

The model adopted here is a pure exchange economy in which the endowments of two goods, (x) and (y), are stochastic. In each period the aggregate endowments of (x) and (y) are independently distributed, and each series is assumed to be independent and identically distributed over time. Each of m individuals maximize expected utility over an infinite horizon and treat prices as parameters, and each is risk averse. Both goods can be stored with no storage costs and no depreciation. In each time period each individual must determine his consumption of each good and the optimal level of stocks of each good to carry over to the following period. The rate at which (x) exchanges for (y) in each period is the price upon which the analysis focuses. It is assumed that there are no contingent commodity markets.

The simplicity of the model is one of its desirable features. The government is more apt to set the "right" price if the economy and the stochastic processes of the economy are not complicated. It turns out that under specified assumptions, regardless of the price set by the government and regardless of the initial level of buffer stocks, a price fixing scheme will eventually fail with probability one. In a competitive equilibrium with storage, prices will be random even if the nature of the stochastic processes are known to everyone.

Jacques Drèze [3] has suggested that the only reason government buffer stock programs are considered is the absence of contingent commodity markets in which agricultural output could be traded contingent on yields and other determinants of supply and demand. This paper suggests that even if one found a class of feasible government buffer stock policies, such policies may not be needed; the outcome with storage by individuals may be a second best allocation.

The paper proceeds as follows. Section II gives the assumptions and technology of the model and describes the properties of individual storage decisions given that the government is fixing prices. Section III presents a proof that price fixing schemes will eventually fail with probability one. Section IV discusses the properties and welfare implications of the model without government, and in a special case makes some comparisons to the model with all storage prohibited. Section V presents some conclusions and suggestions for further research.

II. Assumptions and Technology of the Model

The stochastic nature of the endowments is now made precise. In each time period there are S possible states of the world. Let Ω be the set of all such states. Let F be the set of all subsets of Ω. Let P be a probability measure on F with $0 < P(\omega)$ for each $\omega \varepsilon \Omega$. Then (Ω, F, P) is a discrete probability space. The economy's total endowment of commodity (x) at time t is X_t, a random variable from (Ω, F) to (R, B), where B is the collection of all Borel sets of the real line R. Similarly, the economy's total endowment of (y) at time t is a random variable Y_t from (Ω, F) to (R, B). In each time period t, X_t and Y_t are independently distributed. Further, $\{X_t, t = 1, 2, \ldots\}$ is a sequence of independent, identically distributed random variables, as is the sequence $\{Y_t, t = 1, 2, \ldots\}$. To simplify the analysis, it is assumed that there exists some $\bar{\omega} \varepsilon \Omega$ such that $X_t(\bar{\omega}) = Y_t(\bar{\omega}) = 0$. As will be explained, this ensures that individual buffer stocks are never zero. Also, without loss of generality, let

$$X_t = E(X_t) + \epsilon_{xt} \text{ and } Y_t = E(Y_t) + \epsilon_{yt}$$

where ϵ_{xt}, ϵ_{yt} are restricted to integer values with span one, $X_t \geq 0$, $Y_t \geq 0$.[1/]

Each of the m individuals of the economy is assumed to have a fixed proportion of the aggregate endowment of each good. That is

$\delta_x^j X_t$ = the j^{th} individual's endowment of (x) at time t

$\delta_y^j Y_t$ = the j^{th} individual's endowment of (y) at time t

where $0 \leq \delta_x^j < 1$, $0 \leq \delta_y^j < 1$, $\sum_{j=1}^{m} \delta_x^j = 1$, $\sum_{j=1}^{m} \delta_y^j = 1$.

Hereafter, superscripts refer to individual j, unless otherwise indicated.

The randomness is introduced in this way since, for purposes of the paper, price is viewed as determined by economic aggregates and not by the sum of many individual components. Bad weather will diminish wheat crops for all farmers.

Each individual is assumed to act as if maximizing expected utility over an infinite horizon. Let X_t^j, Y_t^j be the quantities of (x) and (y) consumed by individual j at time t. Utility functions are assumed separable over time with a discount rate β, $0 < \beta < 1$. Hence, each individual acts as if maximizing

$$E_o \sum_{t=1}^{\infty} \beta^{t-1} \hat{U}^j[X_t^j, Y_t^j]$$

with respect to X_t^j, Y_t^j, t = 1,2, ... Hereafter, $E_t(\cdot)$ denotes the expectation conditioned on all realized variables up to and including time t. $\hat{U}^j(\cdot,\cdot)$ is assumed to have the following properties.

I. $\hat{U}^j(\cdot,\cdot)$ is of class C^2, with strictly positive first derivatives everywhere in the positive orthant. $U_1^j(0,0) = \infty$.

II. $\hat{U}^j(\cdot,\cdot)$ is strictly concave.

III. $\hat{U}^j(\cdot,\cdot) = g^j[W^j(\cdot,\cdot)]$ where $W^j(\cdot,\cdot)$ is homogeneous of degree one, $g_1^j(\cdot) \geq 0$, and $\lim_{w\to\infty} g_1^j(w) = 0$.

Property (II) means that individuals are risk averse. Property (III) means that preferences are homothetic, and that the marginal utility of income has limit zero as income tends to infinity. These properties are satisfied by a large class of utility functions.

The analysis of this section is facilitated by assuming that the government engages in price fixing. The government maintains such a policy by its willingness to exchange (x) for (y) with the public at a specified rate. In general, such a policy requires that the government maintain buffer stocks of both goods. The assumption that prices will remain fixed for all time does not eliminate uncertainty from the decision problem of the individual; endowments are still stochastic. Yet each individual need be concerned only with the total value of inventories of (x) and (y) in terms of one of the goods, say (x). The exchange of (y) for a predetermined amount of (x) in any future period is guaranteed by the government. Hence, the individual's decision problem may be viewed at two stages. In a given period and state, the individual has available a known amount of savings in terms of (x) of the previous period and the realized value of his endowment in the specified state. He decides on his optimal amount of savings in terms of (x) and consequently on his current consumption in terms of (x). With this timing problem solved, the individual then chooses his current consumption of (x) and (y) at the specified price.

To state this more formally, the following notation is needed. Let

$\overline{R}$ = the relative price of (y) in terms of (x) fixed by the government.

$r' = [1,\overline{R}]$

K_t^j = the j^{th} individual's stocks of (x) and (y) in terms of (x) at the end of period t, to be choosen during period t.

$z_t^{j\prime} = [\delta_x^j X_t, \ \delta_y^j Y_t]$

I_t^j = the total value of consumption in terms of (x) at time t of individual j.

Then $\quad I_t^j = K_{t-1}^j + r'z_t^j - K_t^j$. Let $C_t^j = X_t^j + \bar{R}\, Y_t^j$.

h_{xt}^j = the j^{th} individual's demand for (x) at time t

h_{yt}^j = the j^{th} individual's demand for (y) at time t

$h_{xt}^j = \hat{h}_x^j(I_t^j,\bar{R})$

$h_{yt}^j = \hat{h}_y^j(I_t^j,\bar{R})$

The analysis is facilitated by the indirect utility function $\hat{V}^j(\cdot,\cdot)$ where

$$\hat{V}^j[I_t^j,\bar{R}] = \hat{U}^j[\hat{h}_x^j(I_t^j,\bar{R}),\ \hat{h}_y^j(I_t^j,\bar{R})]$$

Then the j^{th} individual will act to maximize

$$E_0\sum_{t=1}^{\infty} \beta^{t-1}\hat{V}^j(K_{t-1}^j + r'z_t^j - K_t^j,\bar{R})$$

with respect to K_t^j, t=1,2, ..., given K_0^j.

Establishing the existence and properties of the solution to the individual's optimization problem is non-trivial.[2/] The analysis consists of establishing a series of lemmas. The statements and proofs of these, lemmas 1 through 10, are in the appendix, but it is hoped that the text makes clear the principle lines of the argument.

Let $W_t^j = K_{t-1}^j + r'z_t^j$. Then W_t^j represents the wealth of individual j at time t in terms of (x). The idea is to find a function $f^j(\cdot,\cdot)$ such that

$$f^j(W_t^j,\bar{R}) = \max_{0\le K_t^j \le W_t^j} \left\{\hat{V}^j(W_t^j-K_t^j,\bar{R}) + \beta\, E_t f^j(r'z_{t+1}^j + K_t^j,\ \bar{R})\right\} . \tag{1}$$

$f^j(\cdot,\cdot)$ is found by assuming the horizon is finite and taking the appropriate limit. Let

$$f^{j,N}(W_N^j,\bar{R}) = \max_{0\le K_N^j\le W_N^j} \left\{\hat{V}(W_N^j - K_N^j,\bar{R}) + \beta E_N f^{j,N-1}(K_N^j + r'z_{N-1}^j,\bar{R})\right\} \tag{2}$$

where $f^{j,N}$ denotes the maximum of expected utility of individual j given wealth W_N^j with N periods left in the individual's planning horizon.

From the properties of $\hat{V}^j(\cdot,\bar{R})$ and Eq. (2), it can be established by induction that $f^{j,N}(W,\bar{R})$ is strictly concave for each N and monotone increasing with respect to N for each W. From the discount rate β and the property that Y_t and X_t are bounded from above, it follows that $f^{j,N}(W,\bar{R})$ is uniformly bounded with respect to N. Hence for each $W \geq 0$, $\lim_{N\to\infty} f^{j,N}(W,\bar{R})$ exists, and this limit is denoted $f^j(W,\bar{R})$. It is shown that $f^j(W,\bar{R})$ satisfies the functional equation (1) and is therefore strictly concave and continuous with finite partial derivatives from the left, $f^j_{1-}(\cdot,\bar{R})$, and from the right $f^j_{1+}(\cdot,\bar{R})$, $f^j_{1-}(\cdot,\bar{R}) < f^j_{+}(\cdot,\bar{R})$.

As $f^j(\cdot,\bar{R})$ and $\hat{V}^j(\cdot,\bar{R})$ are strictly concave and continuous, there does exist on the compact set $[0, W^j_t]$ a unique K^j_t which maximizes the right side of the functional equation (1). This optimum is denoted by $\tilde{K}^j_t = \varphi^j[W^j_t,\bar{R}]$. There is an optimum level of consumption in terms of (x), and this optimum is denoted by $\tilde{C}^j_t = \psi^j[W^j_t,\bar{R}]$.

Given that $\{\tilde{K}^j_t;t=1,2,\ldots\}$ is a sequence of current actions and contingent plans which maximize

$$\hat{V}^j(W^j_t-K^j_t,\bar{R}) + \beta E_t f^j(K^j_t + r'Z^j_{t+1},\bar{R}),$$

then it is shown that $\{\tilde{K}^j_t\}$ also maximize

$$\hat{V}^j(W^j_t-K^j_t,\bar{R}) + E_t\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\hat{V}^j(r'Z^j_\tau + K^j_{\tau-1} - K^j_\tau,\bar{R})$$

This justifies the functional equation approach.

The remainder of this section is devoted to establishing the properties of the Markov processes

$$\tilde{K}^j_t = \varphi^j(\tilde{K}^j_{t-1} + r'Z^j_t,\bar{R}) \quad \cdot \; j = 1,2,\ldots m$$

These properties are stated in Proposition I below. First, however, some additional remarks are needed.

It can be shown that $0 < \tilde{K}_t^j < W_t^j$. Suppose that $\tilde{K}_t^j = 0$. Then there exists a state $\bar{\omega}$ at time t+1 such that $W_{t+1}^j(\bar{\omega}) = 0 \Rightarrow C_{t+1}^j(\bar{\omega}) = 0$. But by property (I) of $\hat{U}^j(\cdot,\cdot)$, $\hat{V}_1^j(0,\bar{R}) = \infty$. Hence, this could not be optimal. Similarly, if $\tilde{K}_t^j = W_t^j$, then $C_t^j = 0$, and this could not be optimal. Hence, individual buffer stocks are never zero.

The differentiability of $f^j(\cdot,\bar{R})$ everywhere has not been established. But necessary and sufficient conditions for an optimum with respect to K_t^j are

$$-\hat{V}_1^j(W_t^j - \tilde{K}_t^j,\bar{R}) + \beta E_t f_{1-}^j(r'z_{t+1}^j + \tilde{K}_t^j, \bar{R}) \geq 0$$

$$-\hat{V}_1^j(W_t^j - \tilde{K}_t^j, \bar{R}) + \beta E_t f_{1+}^j(r'z_{t+1}^j + \tilde{K}_t^j, \bar{R}) \leq 0 \qquad (3)$$

The following lemmas are needed.

<u>Lemma 11</u>

$\varphi^j(\cdot,\bar{R})$ and $\psi^j(\cdot,\bar{R})$ are non-decreasing functions

Proof:

$V^j(\cdot,\bar{R})$ and $f^j(\cdot,\bar{R})$ are strictly concave and strictly increasing.

<u>Lemma 12</u>

$$\lim_{W\to\infty} \varphi^j(W,\bar{R}) = \infty$$

Proof: see appendix.

<u>Lemma 13</u>

$\varphi^j(\cdot,\bar{R})$ and $\psi^j(\cdot,\bar{R})$ are continuous functions

proof: the proof follows Brock and Mirman [2 , p. 490]. See the appendix.

<u>Lemma 14</u>

$$\lim_{w\to\infty} f_{1+}^j(w,\bar{R}) = 0$$

Proof: see appendix

Also, there will be need of an infinite product space. Let $\Omega_t = \Omega$, $F_t = F$, $t = 1,2,\cdots$. Then consider the probability space

$$\left(\prod_{t=1}^{\infty} \Omega_t, \prod_{t=1}^{\infty} F_t, \mu\right)$$

where

$\prod_{t=1}^{\infty} \Omega_t$ is the set of all sequences $(\omega_1, \omega_2, \ldots)$ such that $\omega_t \epsilon \Omega_t$ $t = 1,2, \ldots$

$\prod_{t=1}^{\infty} F_t$ = the smallest sigma-field containing the cylinder sets. Then $\tilde{K}_t^j$ is a measurable function from $(\prod_{\tau=1}^{t} \Omega_\tau, \prod_{\tau=1}^{t} F_\tau)$ to (R,B), and for each $a \geq 0$, $\mu\left\{(\omega_1,\omega_2,\ldots) \epsilon \prod_{t=1}^{\infty} \Omega_t : \tilde{K}_t^j(\omega_1,\omega_2,\ldots) \leq a\right\} = \sum_{\tilde{K}_t(\omega_1,\omega_2,\ldots)\leq a} P(\omega_1)P(\omega_2)\cdot \quad \cdot P(\omega_t)$

The important results are contained in Proposition 1. It is useful to refer to Figure 1.

<u>Proposition 1</u>

Given $\bar{R} > 0$, let $M^j = \underset{w\epsilon\Omega}{\text{Max}}\; r'Z_t^j$. Recall that $\underset{w\epsilon\Omega}{\text{Min}}\; r'Z_t^j = 0$. Then

(i). For every $\bar{R} > 0$, there exist K_*^j and K_{**}^j depending on $\bar{R}$ such that

$$\varphi^j(K_{**}^j + M^j, \bar{R}) = K_{**}^j$$

$$\varphi^j(K_*^j, \bar{R}) = K_*^j$$

where $0 = K_*^j < K_{**}^j$.

(ii). For every $\bar{R} > 0$, let $A^j = [K_*^j, K_{**}^j]$. Then if $\tilde{K}_t^j \epsilon A^j, \tilde{K}_\tau \epsilon A^j$ for all $\tau > t$.

(iii). For all $K_0^j \geq 0$, $\mu\{\tilde{K}_t^j \notin A^j \text{i.o.}\} = 0$

proof of (i):[3/]

as $\hat{V}_1^j(0, \bar{R}) = \infty$, $\varphi^j(0 + M^j, \bar{R}) > 0$. This establishes a positive intercept in

Figure 1. Now, suppose that

$\varphi^j(K^j_{t-1} + M^j, \bar{R}) > K^j_{t-1}$ for all $K^j_{t-1} > 0$ so that $\varphi^j(\cdot, R)$ stays above the 45° line of Figure 1. Then it would be the case that for all $K^j_{t-1} \geq 0$

$$-\hat{V}^j_1(M^j, \bar{R}) + E_t\beta\, f^j_{1+}[r'z^j_{t+1} + K^j_{t-1}, \bar{R}] > 0$$

Then

$$\lim_{K^j_{t-1}\to\infty} E_t\beta f^j_{1+}[r'z^j_{t+1} + K^j_{t-1}, \bar{R}] \geq \hat{V}^j_1(M^j, \bar{R}) > 0$$

But by the monotone convergence theorem and lemma 14, this establishes a contradiction. Therefore, there exist some K^j such that

$$\varphi^j(K^j + M^j, \bar{R}) \leq K^j$$

By lemma 13 and the intermediate value theorem there exists a K^j_{**} such that

$$\varphi^j(K^j_{**} + M^j, \bar{R}) = K^j_{**}$$

By a similar proof, there exists a K^j_* such that

$$\varphi^j(K^j_*, \bar{R}) = K^j_*$$

That $K^j_* \leq K^j_{**}$ is established by lemma 11. Suppose $K^j_* > 0$. Then, despite a positive level of stocks, consumption is zero. This violates property I of $U^j(\cdot,\cdot)$. By a similar argument, $K^j_{**} > K^j_*$.

Proof of (ii):

Suppose $K^j_t \in A^j$. Then by lemma 11

$$\varphi^j(K^j_t + r'z^j_{t+1}, \bar{R}) \leq \varphi^j(K^j_{**} + M^j, \bar{R}) = K^j_{**}$$

$$\varphi^j(K^j_t + r'z^j_{t+1}, \bar{R}) \geq \varphi^j(K^j_* + 0, \bar{R}) = K^j_*$$

Proof of (iii): see appendix.

From Proposition I, it is clear that $\tilde{K}_t^j$ eventually enters the set A^j with probability one. Hence, it is assumed that $K_0^j \in A^j$, and hence $\tilde{K}_t^j$ is bounded for all t.

It is possible to say something more about the distributions of $\tilde{K}_t^j$. Following Brock and Mirman [2], let $P^j(k,B)$ be the probability that inventories of individual j will be in the set B at time t+1 given that $K_t^j = k$. $P^j(k,B)$ is the probability transition function on the process $\{K_t^j,\ t = 1,2,\ldots\}$. Define a distribution on $\tilde{K}_t^j$ as follows

$$m_t^j(B) = \int_{R+} P(\zeta,B)\ m_{t-1}^j(d\zeta)$$

It can be shown that there exists a stationary measure m^j such that

$$m^j(B) = \int_{R+} P(\zeta,B) m^j(d\zeta) \tag{4}$$

Hence, given that K_0^j has the distribution m^j, then by (4) m^j is a distribution for all $\tilde{K}_t^j$, $t \geq 1$.

Proposition II

There exist a stationary measure m^j as defined by Eq. (4).

The proof of Proposition II uses theorems given in Kushner [6]. See lemmas 15, 16, 17, and the proof in the appendix.

III. The Eventual Failure of Price Fixing

Having shown that there exists a solution to each individual's optimization problem given that the government is maintaining a fixed price, there will be determined some I_t^j, $j=1,2,\ldots m$, $t = 1,2,\cdots$. Given I_t^j, individual j will purchase utility maximizing quantities of (x) and (y) at the fixed price $\bar{R}$. For the policy to be feasible it is specified that $\bar{R}$ be an equilibrium price in the sense that any excess demand for (x) by individuals be matched by an excess supply on the part of the government. This section shows that under specified assumptions no price can be maintained indefinitely far into the future regardless of the initial level of government buffer stocks.

This result is best analyzed by supposing first that the government sets a price which would correspond to a competitive equilibrium with aggregate private consumption of (x) at $E(X_t)$ and aggregate private consumption of (y) at $E(Y_t)$. In some sense the price corresponding to this outcome is the most likely;[4/] given the stationary measure m^j as

$$K_t^j = K_{t-1}^j - C_t^j + \bar{R}\ \delta_y^j Y_t + \delta_x^j X_t$$

then

$$\sum_{j=1}^{m} E(C_t^j) = \bar{R}\ E(Y_t) + E(X_t)$$

<u>Proposition III</u>

Let $\bar{R}$ correspond to a competitive equilibrium with aggregate consumption of (x) and (y) at $E(X_t)$ and $E(Y_t)$ respectively.

Let G_0 = the stock of (Y) held by the government at t=0.

Let A_t = the cumulative net amount of (y) sold to individuals in exchange for (x) up through and including time t.

Then with probability one there exists some $T^* < \infty$ such that $A_{T^*} > G_0$

Proof:

Let $S_{yT} = \sum_{t=1}^{T} \epsilon_{yt}$, $S_{xT} = \sum_{t=1}^{T} \epsilon_{xt}$, $S_T = [S_{xT}, S_{yT}]$

Then S_T is a random walk in R^2 and has the property that for integers B_x and B_y there exists some $T^* < \infty$ such that $S_{xT^*} = B_x$ and $S_{yT^*} = B_y$ with probability one.[5/] For the proof, let B_y = the largest integer less than or equal to

$$-G_0 - 1 - \sum_{j=1}^{m} (K_{**}^j/\bar{R})$$

Then let B_x be such that $\sum_{j=1}^{m} \hat{h}_y^j(\delta_x^j B_x + \delta_y^j \bar{R} B_y - K_{**}^j) \geq 0$. This last choice is possible as $\hat{h}_y^j(\cdot, R)$ is linear, $j = 1,2, \ldots m$, by homotheticity. In period t the government must sell

$$\sum_{j=1}^{m} \hat{h}_y^j(I_t^j, R) - Y_t^*$$

units of (Y) to the public, where Y_t^* is the aggregate of (y) over all individuals made available for consumption from private wealth in period t. Then the cumulative net sales of (y) at time T^* will be

$$A_{T*} = \sum_{t=1}^{T*} \left\{ \sum_{j=1}^{m} \hat{h}_y^j(I_t^j, \bar{R}) - Y_t^* \right\} = \sum_{j=1}^{m} \hat{h}_y^j \left[\sum_{t=1}^{T*} I_t^j, \bar{R} \right] - \sum_{t=1}^{T*} Y_t^*$$

using the linearity of $\hat{h}_y^j(\cdot, \bar{R})$. Now

$$\sum_{t=1}^{T*} Y_t^* \leq \sum_{t=1}^{T*} Y_t + \sum_{j=1}^{m} (K_{**}^j / \bar{R}) \qquad \sum_{t=1}^{T*} I_t^j \geq \sum_{t=1}^{T*} (\delta_y^j Y_t \bar{R} + \delta_x^j X_t) - K_{**}^j$$

therefore

$$A_{T*} \geq \sum_{j=1}^{m} \{\hat{h}_y^j[\delta_x^j E(X_t) T^* + \delta_x^j S_{xT*} + T^* \delta_y^j \bar{R} E(Y_t) + \delta_y^j \bar{R} S_{yT*} - K_{**}^j]\} - T^* E(Y_t)$$
$$- S_{yT*} - \sum_{j=1}^{m} (K_{**}^j / \bar{R})$$

$$\therefore A_{T*} \geq T^* \{ \sum_{j=1}^{m} \hat{h}_y^j [\delta_x^j E(X_t) + \delta_y^j \bar{R} E(Y_t)] - E(Y_t) \} + \sum_{j=1}^{m} \hat{h}_y^j [\delta_x^j B_x + \delta_y^j \bar{R} B_y - K_{**}^j]$$
$$- B_y - \sum_{j=1}^{m} (K_{**}^j / \bar{R})$$

But by the choice of $\bar{R}$,

$$\sum_{j=1}^{m} \hat{h}_y^j [\delta_x^j E(X_t) + \bar{R}\, \delta_y^j E(Y_t)] - E(Y_t) = 0$$

Hence by the choice of B_x, B_y,

$$A_{T*} \geq G_0 + 1 > G_0.$$

Proposition IV

If tastes are identical and homothetic for all individuals, then regardless of the fixed price and regardless of the initial level of government buffer stocks, those stocks will be insufficient to maintain the fixed price with probability one.

Proof:

If tastes are identical and homothetic, then in a competitive equilibrium without government intervention

$$R_t = \theta(\sum_{j=1}^{m} X_t^j / \sum_{j=1}^{m} Y_t^j)$$

where

R_t is the price of (y) in terms of (x) at time t

$\theta(\cdot)$ is strictly monotone increasing and continuous.

let $R^* = \inf[\theta(\cdot)]$
$R^{**} = \sup[\theta(\cdot)]$

with

$R^* \geq 0$, $R^{**} \leq \infty$

Clearly, if $\bar{R} \notin [R^*, R^{**}]$, then the policy will fail with probability one as government purchases will be in one direction only.

Suppose that $R^* < \bar{R} < \theta\left[\frac{E(X_t)}{E(Y_t)}\right]$. Then there exist

a $\gamma > 0$ with $0 < \gamma \leq E(X)$ such that $\bar{R} = \theta\ [E(X_t) - \gamma/E(Y_t)]$

let $X_t^\gamma = E(X_t) - \gamma$. Then

$$X_t = X_t^\gamma + (\gamma + \epsilon_{xt}) \text{ with } E(\epsilon_{xt} + \gamma) > 0.$$

let $S_{yT} = \sum_{t=1}^{T} \epsilon_{yt}$, $S_{xT} = \sum_{t=1}^{T} (\epsilon_{xt} + \gamma)$

As before let B_y = the largest integer $\leq - G_0 - 1 - \sum_{j=1}^{m} (K_{**}^j/\overline{R})$

Then by Feller [4 , p. 203] S_{xt} visits $(-\infty, a)$ a finite number of times for all $a>0$. Hence, there exists some $T^* > 0$ such that

$$S_{yT*} = B_y$$

and

S_{xT*} is large enough that

$$\sum_{j=1}^{m} \hat{h}_y^j (\delta_x^j S_{xT*} + \delta_y^j S_{yT*}\overline{R} - K_{**}^j) \geq 0$$

Then as before

$$A_{T*} \geq T^*\{\sum_{j=1}^{m} \hat{h}_y^j [\delta_x^j X^Y + \overline{R}\delta_y^j E(Y_t)] - E(Y_t)\} + \sum_{j=1}^{m} \hat{h}_y^j (\delta_x^j S_{xT*} + \overline{R}\, \delta_y^j S_{yT*} - K_{**}^j)$$

$$- S_{yT*} - \sum_{j=1}^{m} (K_{**}^j/\overline{R}) > G_0.$$

Clearly, if $\overline{R} > \theta[E(X_t)/E(Y_t)]$, it can be shown that with probability one the government will find its initial stock of (x) insufficient to maintain the price. The proof is entirely similar. 6/

IV. Competitive Equilibrium without Government

The purpose of this section is to examine the properties of a competitive equilibrium when there is no government. Unfortunately, the model is difficult to analyze in the general case, but it is argued by way of an example that certain properties of the equilibrium can be anticipated.

It is clear from Section III that, in the absence of government, individuals will not act in such a way as to fix a price. (Let $G_0 = 0$.) Hence, in general, individuals may not be indifferent between maintaining stocks of (x) and stocks of (y). In the absence of all types of forward contracts, one might anticipate relations of the form

$$\tilde{K}^j_{x,t} = \varphi^j_x(K^j_{x,t-1}, K^j_{y,t-1}, \cancel{Z}^j_t, R_t, R_{t+1}, \ldots)$$

$$\tilde{K}^j_{y,t} = \varphi^j_y(K^j_{x,t-1}, K^j_{y,t-1}, \cancel{Z}^j_t, R_t, R_{t+1}, \ldots)$$

where $\tilde{K}^j_{x,t}, \tilde{K}^j_{y,t}$ are the optimal choices of storage of (x) and (y) respectively at time t by individual j as a function of previous stocks of (x) and (y), current endowment, current price, and all future prices in all states. Yet in an equilibrium with rational expectations, the anticipated distributions of future prices should be the distributions consistent with current choices of stocks by all individuals for each good. Establishing the existence of such a fixed point in an infinite dimensional price space is the analytical difficulty refered to above. Assuming existence, one might anticipate that the model could be described by reduced form equations

$$[\tilde{K}^j_{y,t}, \tilde{K}^j_{x,t}] = \varphi^j(K^1_{x,t-1}, K^1_{y,t-1}, K^2_{x,t-1}, \ldots K^m_{y,t-1}; X_t, Y_t)$$

One might establish that $\tilde{K}^j_{y,t}, \tilde{K}^j_{x,t}$ are bounded and possess stationary measures, and then derive the properties of consumption and price over time.

There is, however, a special case which makes it appropriate to use many of the results derived in Section II. Suppose all individuals are specialized with respect to endowments; group A has all of (x) in every state at all times, and group B has all of (y). Preferences of all individuals are identical and homothetic and can be represented by the logarithm of a Cobb-Douglas utility function. As such a function displays constant relative risk aversion, the model can be treated as if consisting of two individuals with specialized endowments both of whom take prices as parameters. In general, necessary conditions for optimal storage of (x) and (y) by individual j at time t=1 are that

$$\hat{V}^j_1[r_1 \cancel{Z}^j_1 + K^j_{x,0} - \tilde{K}^j_{x,1} + R_1(K^j_{y,0} - \tilde{K}^j_{y,1}), R_1] = E_1\beta^{t-1}\hat{V}^j_1[r_t Z^j_t + \tilde{K}^j_{x,t-1} - \tilde{K}^j_{x,t} + \qquad (5)$$

$$+ R_t(\tilde{K}^j_{y,t-1} - \tilde{K}^j_{y,t}), R_t] \qquad t = 2,3,4, \ldots$$

and

$$R_1\hat{V}_1^j[\cdot,R_1] = E_1\{\beta^{t-1}\hat{V}_1^j[\cdot,R_t]R_t\} \qquad t = 2,3,4 \ldots \tag{6}$$

Equations (5) and (6) state that the discounted expected marginal utility of (x) for individual j be the same for all periods, and similarly for (y).

Suppose group A stores only (x). If

$$\hat{U}^A(X_t^A, Y_t^A) = \ln (X_t^A)^{\alpha}(Y_t^A)^{1-\alpha}, \ 0 < \alpha < 1, \text{ then}$$

$$\hat{V}^A(I_t^A,R_t) = \alpha \ln \alpha + (1-\alpha) \ln (1-\alpha) - (1-\alpha) \ln R_t + \ln (X_t - K_{x,t}^A + K_{x,t-1}^A)$$

Clearly, decisions with respect to $\{K_{x,t}^A;\ t = 1,2, \ldots\}$ will be independent of $\{R_t;\ t = 1,2, \ldots\}$.

Suppose group B stores only (y). Then

$$\hat{V}^B(I_t^B,R_t) = \alpha \ln \alpha + (1-\alpha) \ln (1-\alpha) - (1-\alpha) \ln R_t + \ln R_t + \ln (Y_t + K_{y,t-1}^B - K_{y,t}^B)$$

and current actions and plans are independent of current and future prices. Hence, given the restrictions on storage, one may take prices as fixed in deriving optimal behavior, and many results of Section II apply. In particular, optimal choices do exist. Price distributions can then be derived given optimal behavior.

It remains to show that the constraint that each individual stores only the good with which he is endowed is not binding in equilibrium.

$$\text{Suppose } \tilde{K}_{y,t-1}^A = \tilde{K}_{y,t}^A = \tilde{K}_{x,t-1}^B = \tilde{K}_{x,t}^B = 0 \qquad t = 1,2,3, \ldots$$

then

$$R_t = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{X_t + \tilde{K}_{x,t-1}^A - \tilde{K}_{x,t}^A}{Y_t + \tilde{K}_{y,t-1}^B - \tilde{K}_{y,t}^B}\right) \tag{7}$$

Suppose A is in equilibrium with respect to storage of (x). Then

$$1/(X_1 + K_{x,0}^A - \tilde{K}_{x,1}^A) = E_1[\beta^{t-1}/(X_t + \tilde{K}_{x,t-1}^A - \tilde{K}_{x,t}^A)] \qquad t = 2,3, \ldots$$

From (7)

$$1/R_1(Y_1+K^B_{y,0} - \tilde{K}^B_{y,1}) = E_1[\beta^{t-1}/R_t(Y_t + \tilde{K}^B_{y,t-1} - \tilde{K}^B_{y,t})] \qquad t = 2,3, \ldots$$

On the assumption that B is in equilibrium with respect to storage of (y),

$$R_1/R_1(Y_1 + K^B_{y,0} - \tilde{K}^B_{y,1}) = E_1[\beta^{t-1}R_t/R_t(Y_t - \tilde{K}^B_{y,t} + \tilde{K}^B_{y,t-1})] \quad t = 2,3,4, \ldots$$

Then from (7)

$$R_1/(X_t + \tilde{K}^A_{x,0} - \tilde{K}^A_{x,1}) = E_1[\beta^{t-1}R_t/X_t - \tilde{K}^A_{x,t} + \tilde{K}^A_{x,t-1}] \qquad t = 2,3, \ldots$$

Hence, A and B are both in equilibrium with respect to storage of (x) and (y).[7/]

Hence, in the special case, decisions are independent of prices, and notationally the optimal choices may be written

$$\tilde{K}^A_{x,t} = \varphi^A(\tilde{K}^A_{x,t-1} + X_t)$$

$$\tilde{K}^B_{y,t} = \varphi^B(\tilde{K}^B_{y,t-1} + Y_t).$$

By application of the results of Section II, there exist self-contained sets for each process, entry into such sets occurs with probability one, and there exist stationary measures for each process.

It is interesting to compare (7) to the distribution of prices which would prevail if the government prohibited the holding of buffer stocks by individuals. Such a prohibition might be based on the belief that individual "speculation" was "destabilizing", a proposition which has been much discussed in the literature. In the model all agents speculate in the sense that their actions are based on probability distributions of the random variables of the system.

If no buffer stocks were held by the government or by individuals, the price would be

$$R^*_t = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{X_t}{Y_t}\right) \tag{8}$$

for X_t and Y_t not both zero.

Equation (8) describes a sequence of independent random variables. While (7) does have a stationary distribution, the effect of optimal inventory accumulation by individuals is to introduce correlation into the price series. Also, R_t^* has no finite mean as Y_t may be zero with positive probability. In contrast, individual buffer stocks preclude zero consumption of either good at any time. In this example, then, optimal inventory accumulation has in some sense stabilized prices. Yet a major point of this paper is that price instability may be a misleading indicator of economic welfare. There is a more natural criterion for evaluating the welfare effect of inventory accumulation--the maximization of expected utility.

In general, in the absence of contingent commodity markets, the solutions to the model with and without storage are not Pareto optimal. Yet necessary conditions for a second best solution in the absence of all forward markets can still be derived.[8/]

$$\text{Max } \{E_0\sum_{t=1}^{\infty} \beta^{t-1}\hat{U}^1(X_t^1,Y_t^1) + \sum_{j=2}^{m} a^j[E_0\sum_{t=1}^{\infty} \beta^{t-1}\hat{U}^j(X_t^j,Y_t^j) - \overline{U}^j] + \sum_{t=1}^{\infty} b_t[X_t + \sum_{j=1}^{m}$$

$$(K_{x,t-1}^j - K_{x,t}^j - X_t^j)] + \sum_{t=1}^{\infty} c_t[Y_t + \sum_{j=1}^{m} (K_{y,t-1}^j - K_{y,t}^j - Y_t^j)]\}$$

with respect to $\{X_t^j,\ Y_t^j,\ K_{x,t}^j,\ K_{y,t}^j;\ j = 1,2,\ \ldots\ m;\ t = 1,2,\ \ldots\}$

where $\overline{U}^j$, $K_{x,0}^j$, $K_{y,0}^j$ are parameters. Necessary conditions for a maximum are

$$E_0\beta^{t-1} \frac{\partial \hat{U}^1(\cdot,\cdot)}{\partial X_t^1} - b_t = 0 \qquad t = 1,2,\ \ldots$$

$$E_0\beta^{t-1} \frac{\partial \hat{U}^1(\cdot,\cdot)}{\partial Y_t^1} - c_t = 0 \qquad t = 1,2,\ \ldots \tag{9}$$

$$a^j E_0\beta^{t-1} \frac{\partial \hat{U}^j(\cdot,\cdot)}{\partial X_t^j} - b_t = 0 \qquad t = 1,2,\ \ldots$$

$$a^j E_0 \beta^{t-1} \frac{\partial \hat{U}^j(\cdot,\cdot)}{\partial Y_t^j} - c_t = 0 \qquad t = 1, 2, \ldots$$

$$- c_t + c_{t+1} = 0 \qquad t = 1, 2, \ldots$$

$$- b_t + b_{t+1} = 0 \qquad t = 1, 2, \ldots$$

These necessary conditions for a maximum state among other things that the discounted expected marginal utility of (x) for individual j be equal in all periods, and similarly for (y). With storage prohibited such conditions will not, in general, be satisfied.

In the specific example with the logarithm of Cobb-Douglas utility functions and with specialization of endowments, the solution will satisfy conditions (9), conditions which are now necessary for Pareto optimality.

A troublesome aspect of the solution is the absence of contingent commodity markets. Yet it can be shown that in the specific example forward markets would not be active even if allowed.

In some models with infinite horizons it is possible to dominate a competitive equilibrium path by increasing consumption in at least some periods. In the specific example, suppose $\tilde{C}^A_{x,t}$ increases by some $\lambda > 1$ at each state in each time period. Let $\tilde{\tilde{C}}^A_{x,t}$, $\tilde{\tilde{K}}^A_{x,t}$ denote the new paths of consumption of A in terms of (x) and stocks of (x) respectively. Implicit in this analysis is a redistribution of (x) and (y) each period as if the economy were in a new competitive equilibrium. This allocation will satisfy the necessary conditions (9). Now

$$\tilde{\tilde{K}}^A_{x,\tau} = \tilde{K}^A_{x,\tau} - \sum_{t=0}^{\tau} (\tilde{\tilde{C}}^A_{x,t} - \tilde{C}^A_{x,t}) \leq K^A_{**} - (\lambda-1)\sum_{t=0}^{\tau} \tilde{C}^A_{x,t}$$

A necessary condition for the convergence of the series $(\lambda-1)\sum_{t=0}^{\infty} \tilde{C}^A_{x,t}$ is that $\lim_{t\to\infty} \tilde{C}^A_{x,t} = 0$. Let $\omega^* \varepsilon \Omega$ be such that $X_t(\omega^*)$ is the least positive realization of X_t. Denote $\psi^A[0 + X_t(\omega^*)] = b > 0$. For convergence of $\tilde{C}^A_{x,t}$ to zero it is required that $X_t \geq X_t(\omega^*)$

only a finite number of times. Hence, for convergence it is required that there exists some T* such that for each $t > T^*$, $X_t = 0$. This event occurs with probability zero. Hence, convergence occurs on a set of measure zero, and $(\lambda-1)\sum_{t=0}^{\infty} \tilde{C}^A_{x,t} = \infty$ with probability one. There exist with probability one some T such that $\tilde{K}^A_{x,T} < 0$. Hence, the new rule is incompatible with conditions (9). By a similar argument, it can be established that the path $\tilde{C}^B_{y,t}$ cannot be dominated. Clearly, paths with less consumption of (x) and (y) are inferior to the competitive equilibrium path. Hence, the solution to the specific example is Pareto optimal.

V. Conclusions and Suggestions for Further Research

The model has shown that under simple assumptions about an economic system and the stochastic processes of that system, price fixing schemes will fail with probability one. In a stochastic general equilibrium framework in which individual behavior relations are derived from the maximization of expected utility, a random price will be a property of the equilibrium. It is reasonable then to question the feasibility of fixing a relative price in actual economies. If it is not the intent of government buffer stock programs to fix a relative price, then it is up to policy makers to make precise what they have in mind.

The suggestion by Drèze that government buffer stock programs compensate in part for the absence of contingent contracts can be misleading. Complete contingent markets make possible an efficient allocation of risk bearing for a given time period, a function which should be distinguished from an optimal timing of consumption. In the absence of forward markets, the outcome with storage by individuals may be a second best allocation. Feasible government buffer stock programs may limit price variability, but price variability is not an appropriate measure of economic welfare.

The assumption that expectations are rational removes from the model one argument for government buffer stock programs -- limitations on individual knowledge or foresight. Yet those who argue along those lines must explain why the government has access to more information or processes information more efficiently that the private sector. This is a topic for further research.

The restriction that there be no forward markets was imposed exogenously for much of the paper. This was done in part to make the analysis tractable. Yet, in principle, coherent economic models should make the choice of market structure endogenous. This is being pursued in further research.

APPENDIX

Lemma 1

If $f^{j,N-1}(r'Z^j_{N-1} + K^j_N,\overline{R})$ is a strictly concave function of K^j_N and $\hat{V}^j(\cdot,\overline{R})$ is strictly concave, then

$\hat{V}^j(W^j_N - K^j_N, \overline{R}) + \beta E_N f^{j,N-1}(r'Z^j_{N-1} + K^j_N,\overline{R})$ is a strictly concave function of K^j_N, W^j_N.

Proof: let $0 < \lambda < 1$ $\quad W' \neq W''$ $\quad K' \neq K''$. Then

$$\hat{V}^j[\lambda W' + (1-\lambda)W'' - \lambda K' - (1-\lambda)K'', R] + \beta E_N f^{j,N-1}[\lambda K' + (1-\lambda)K'' + r'Z^j_{N-1},\overline{R}]$$

$$> \lambda\hat{V}^j(W' - K',\overline{R}) + (1-\lambda)\hat{V}^j(W''-K'',\overline{R}) + \beta\lambda\, E_N f^{j,N-1}(K' + r'Z^j_{N-1},\overline{R})$$

$$+ (1-\lambda)\,\beta E_N f^{j,N-1}(K'' + r'Z^j_{N-1},\overline{R})$$

Lemma 2 : Given the hypothesis of lemma 1, if a maximum to the right side of (2) exists, then $f^{j,N}(W^j_N,\overline{R})$ as defined by (2) is a strictly concave function of $W^j_N, W^j_N \geq 0$.

Proof: see Bellman [1, lemma 1, p. 21].

Lemma 3

Let $g^j[w,\overline{R}]$ be strictly concave w.r.t. w, $w \geq 0$. Then for each $W > 0$, $g^j(\cdot,\overline{R})$ is continuous, has finite partial derivatives from the left, $g^j_{1-}(\cdot,\overline{R})$, and from the right, $g^j_{1+}(\cdot,\overline{R})$, where

$$g^j_{1-}(\cdot,\overline{R}) \leq g^j_{1+}(\cdot,\overline{R})$$

Proof: see Katzner [5, lemma B. 2-4, p. 187].

Lemma 4

$f^{j,N}(\cdot,\bar{R})$ is strictly concave for all N.

Proof: $f^{j,1}(W_1^j,\bar{R}) = \underset{0 \le K_1^j \le W_1^j}{\text{MAX}} \{\hat{V}^j(W_1^j - K_1^j,\bar{R})\}$. The strict concavity and continuity of $\hat{V}^j(W_1^j - K_1^j,\bar{R})$ w.r.t. K_1^j on the compact set $[0,W_1^j]$ ensures the existence and uniqueness of a maximum. Assume $f^{j,N}(W_N^j,\bar{R})$ is strictly concave w.r.t. W_N^j. Then $f^{j,N}(\cdot,\bar{R})$ is continuous by lemma 3. Define

$$f^{j,N+1}(W_{N+1}^j,\bar{R}) = \underset{0 \le K_{N+1}^j \le W_{N+1}^j}{\text{MAX}} \{\hat{V}^j(W_{N+1}^j - K_{N+1}^j,\bar{R}) + \beta E_{N+1} f^{j,N}(K_{N+1}^j + r' Z_N^j,\bar{R})\}.$$

The strict concavity and continuity of $\hat{V}^j(\cdot,\bar{R})$ and $f^{j,N}(\cdot,\bar{R})$ w.r.t. K_{N+1}^j on the compact set $[0,W_{N+1}^j]$ ensures the existence and uniqueness of a maximum. By lemma 1 and lemma 2 $f^{j,N+1}(\cdot,\bar{R})$ is strictly concave.

Lemma 5

For each W, $W \ge 0$, $f^{j,N}(W,\bar{R})$ is uniformly bounded in N.

Proof: The highest level of discounted expected utility for individual j is dominated by that associated with individual j actually receiving all of (x) and all of (y) of the economy in each state in every period, the maximum of aggregate income over all states to be realized in each state in every period, and individual j acting as if he knew in the current decision period that he would receive such endowments in the future. Then the endowment of individual j in each state in every period would be

$$M = \underset{\omega\epsilon\Omega}{\text{MAX}}\ [X_t(\omega) + \bar{R}Y_t(\omega)].$$

Given current wealth, W^j, there is a maximum amount of previous savings compatible with this wealth, namely $W^j - \underset{\omega\epsilon\Omega}{\text{MIN}}\ [X_t(\omega) + \bar{R}Y_t(\omega)] = \dot{W}^j$. Assume, in addition to the endowment, individual j receives $\dot{W}^j$ each period. Then there would be no savings in any period, and letting $\hat{V}^j(0,\bar{R}) = 0$, for all N, $f^{j,N}(W^j,\bar{R}) < \sum_{t=1}^{N} \beta^{t-1}\hat{V}^j(\dot{W}^j + M,\bar{R})$

$$< \sum_{t=1}^{\infty} \beta^{t-1}\hat{V}^j(\dot{W}^j + M,\bar{R}) = \frac{\hat{V}^j(\dot{W}^j + M,\bar{R})}{1-\beta}$$

Lemma 6

$f^{j,N}(W,\overline{R})$ is monotone increasing w.r.t. N.

Proof: $f^{j,1}(W,\overline{R}) = \underset{0 \leq K_1^j \leq W}{\mathrm{MAX}} \{\hat{V}^j(W-K_1^j,\overline{R})\} = \hat{V}^j(W,\overline{R})$

$$f^{j,2}(W,\overline{R}) + \underset{0 \leq K_2^j \leq W}{\mathrm{MAX}} \{\hat{V}^j(W-K_2^j,\overline{R}) + \beta E_2 f^{j,1}(r'Z_1^j + K_2^j,\overline{R})\} \geq \hat{V}^j(W,\overline{R}) + \beta E_2 f^{j,1}(r'Z_1^j,\overline{R})$$

$$> f^{j,1}(W,\overline{R})$$

Assume $f^{j,N}(W,\overline{R}) > f^{j,N-1}(W,\overline{R})$, and let the maximizing value of K_N^j be denoted $\tilde{K}_N^j$. Then

$$f^{j,N+1}(W,\overline{R}) = \underset{0 \leq K_{N+1}^j \leq W}{\mathrm{MAX}} \{\hat{V}^j(W - K_{N+1}^j,\overline{R}) + \beta E_{N+1} f^{j,N}(K_{N+1}^j + r'Z_N^j,\overline{R})\}$$

$$\geq \hat{V}^j(W - \tilde{K}_N^j,\overline{R}) + \beta E_{N+1}\, f^{j,N}(\tilde{K}_N^j + r'Z_N^j,\overline{R})$$

$$\geq \hat{V}^j(W - \tilde{K}_N^j,\overline{R}) + \beta E_N\, f^{j,N-1}(\tilde{K}_N^j + r'Z_{N-1}^j,\overline{R}) = f^{j,N}(W,\overline{R})$$

Lemma 7

For each $W \geq 0$, the $\lim\limits_{N\to\infty} f^{j,N}(W,\overline{R})$ exists and is concave w.r.t. W. Denote this limit as $f^j(W,\overline{R})$.

Proof: existence follows from lemma 5 and lemma 6. Concavity follows from lemma 4.

Lemma 8

$f^j(W,\overline{R})$ satisfies the functional equation (1).

Proof: see Bellman [1, pp. 12-13].

Lemma 9

$f^j(\cdot,\overline{R})$ is strictly concave.

Proof: this follows from lemma 4 and the functional equation (1).

<u>lemma 10</u>

Given that $\{\tilde{K}^j_t,\ t=1,2,\ \ldots\}$ is a sequence of current actions and contingent plans which maximize

$$\hat{V}^j(W^j_t - K^j_t,\bar{R}) + \beta E_t f^j(K^j_t + r'z^j_{t+1},\bar{R})$$

then $\{\tilde{K}^j_t,\ t=1,2,\ \ldots\}$ also maximizes

$$\hat{V}^j(W^j_t - K^j_t,\bar{R}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\hat{V}^j(r'z^j_\tau + K^j_{\tau-1} - K^j_\tau,\bar{R})$$

proof: (the proof follows a rough outline of T. Muench). It is sufficient to argue the case at t=1.

$$E_1\sum_{t=1}^{T} \beta^{t-1}\hat{V}^j(r'z^j_t + K^j_{t-1} - K^j_t,\bar{R})$$

$$= E_1\{f^j(W^j_1,\bar{R}) + \sum_{t=1}^{T} \beta^{t-1}[\hat{V}^j(W^j_t - K^j_t,\bar{R}) + \beta f^j(W^j_{t+1},\bar{R}) - f^j(W^j_t,\bar{R})] - \beta^T f^j(W^j_{T+1},\bar{R})\}$$

$$= f^j(W^j_1,\bar{R}) + E_1\sum_{t=1}^{T} \beta^{t-1}E_t[\hat{V}^j(W^j_t - K^j_t,\bar{R}) + \beta f^j(W^j_{t+1},\bar{R}) - f^j(W^j_t,\bar{R})] - E_1\beta^T f^j(W^j_{T+1},\bar{R})$$

Note that as

$$f^j(W^j_t,\bar{R}) = \max_{0\le K^j_t\le W^j_t} \{\hat{V}^j(W^j_t - K^j_t,\bar{R}) + \beta E_t f^j(W^j_{t+1},\bar{R})\}$$

then

$$\hat{V}^j(W^j_t - K^j_t,\ \bar{R}) + \beta E_t f^j(W^j_{t+1},\bar{R}) - f^j(W^j_t,\bar{R}) \le 0.$$

Therefore, for all feasible plans where $I^j_t \geq 0$, and $\hat{V}^j(0,\bar{R}) = 0$, by the monotone convergence theorem $E_1\sum_{t=1}^{\infty} \beta^{t-1}\hat{V}^j(r'z^j_t + K^j_{t-1} - K^j_t,\bar{R})$

$$= f^j(W^j_1,\bar{R}) + E_1\sum_{t=1}^{\infty} \beta^{t-1}E_t[\hat{V}^j(W^j_t - K^j_t,\bar{R}) + \beta f^j(W^j_{t+1},\bar{R}) - f^j(W^j_t,\bar{R})] - \lim_{T\to\infty} E_1\beta^T f^j(W^j_{T+1},\bar{R})$$

Hence, for all K^j_t, $f^j(W^j_1,\bar{R}) \geq E_1\sum_{t=1}^{\infty} \beta^{t-1}\hat{V}^j(W^j_t - K^j_t,\bar{R})$, so that $f^j(W^j,\bar{R})$ is an upper bound on discounted expected utility for all feasible plans.

Furthermore, choose $\tilde{K}_t^j = \varphi^j(W_t^j,\bar{R})$. Then, as shown in Proposition 1, W_t^j is bounded from above without loss of generality.

Hence, $\lim_{T\to\infty} E_1\beta^T f^j(W_{T+1}^j,\bar{R}) = 0$. Therefore,

$$f^j(W_1^j,\bar{R}) = E_1\sum_{t=1}^{\infty} \beta^{t-1}\hat{V}^j(W_t^j - K_t^j,\bar{R})$$

<u>lemma 12</u>

$$\lim_{W\to\infty} \varphi^j(W,\bar{R}) = \infty$$

proof: Suppose $\lim_{W\to\infty} \varphi^j(W,\bar{R}) = B$. As $f^j(\cdot,\bar{R})$ is strictly concave and strictly increasing, the right derivative is strictly monotone decreasing. Then, from Eqs. (3) of the text

$0 < \beta E_t f_{1+}^j (r'\not{z}_{t+1}^j + B,\bar{R}) \le \beta f_{1+}^j [r'\not{z}_{t+1}^j + \varphi^j(W,\bar{R}),\bar{R}] \le \hat{V}_1^j[W - \varphi^j(W,\bar{R}),\bar{R}]$. But as $\hat{U}^j(\cdot,\cdot)$ is of class C^2, $\hat{V}_1^j(\cdot,\bar{R})$ is continuous and $\lim_{W\to\infty} \hat{V}_1^j[W - \varphi^j(W,\bar{R}),\bar{R}] = 0$

<u>lemma 13</u>

$\varphi^j(\cdot,\bar{R})$, $\psi^j(\cdot,\bar{R})$ are continuous.

proof: following Brock and Mirman [2 , p. 490], as $\psi^j(\cdot,\bar{R})$, $\varphi^j(\cdot,\bar{R})$ are monotone increasing, they possess, at most, jump discontinuities.

Hence,

$$\lim_{W\uparrow W_0} \psi^j(W,\bar{R}) \le \lim_{W\downarrow W_0} \psi^j(W,\bar{R})$$

$$\lim_{W\uparrow W_0} \varphi^j(W,\bar{R}) \le \lim_{W\downarrow W_0} \varphi^j(W,\bar{R})$$

But $\lim_{W\uparrow W_0} [\psi^j(W,\bar{R}) + \varphi^j(W,\bar{R}) = \lim_{W\uparrow W_0} (W) = W_0$

$$\lim_{W\downarrow W_0} [\psi^j(W,\bar{R}) + \varphi^j(W,\bar{R}) = \lim_{W\downarrow W_0} (W) = W_0$$

Hence,

$$[\lim_{W\uparrow W_0} \varphi^j(W,\bar{R}) - \lim_{W\downarrow W_0} \varphi^j(W,\bar{R})] + [\lim_{W\uparrow W_0} \psi^j(W,\bar{R}) - \lim_{W\downarrow W_0} \psi^j(W,\bar{R})] = 0$$

As both terms are non-positive, it follows that each is zero.

<u>lemma 14</u>

$$\lim_{W\to\infty} f^j_{1+} (W,\bar{R}) = 0$$

proof: As $f^j_{1+} (\cdot,\bar{R})$ is strictly monotone decreasing, suppose

$f^j_{1+} (\cdot,\bar{R}) \geq b > 0$. From lemma 5 and lemma 7, $f^j(W,\bar{R}) \leq \dfrac{\hat{V}^j(W + M,\bar{R})}{1-\beta}$

As $\lim\limits_{W\to\infty} \hat{V}^j_1 (W + M,\bar{R}) = 0$, there exists some W* such that for all $W \geq W^*$, $\dfrac{\hat{V}_1(W + M,\bar{R})}{1-\beta} \leq \dfrac{b}{2}$

Let $g(W,\bar{R}) = \dfrac{\hat{V}^j(W^* + M,\bar{R})}{1-\beta} + \dfrac{b}{2} (W-W^*)$.

Then, by construction for every $W > W^*$, $\dfrac{\hat{V}^j(W + M,\bar{R})}{1-\beta} < g(W,\bar{R})$

Let $h(W,\bar{R}) = f^j(W^*,\bar{R}) + b(W - W^*)$

Then, for every $W > W^*$, $f^j(W,\bar{R}) > h(W,\bar{R})$.

But, by the linearity of $g(\cdot,\bar{R})$, $h(\cdot,\bar{R})$, there exists some W** such that

$h(W^{**},\bar{R}) = g(W^{**},\bar{R})$. Then, by construction $f^j(W^{**},R) > \dfrac{V^j(W^{**},\bar{R})}{1-\beta}$

which is the desired contradiction.

<u>Proposition I - (iii)</u>

For all $K^j_0 \geq 0$, $\mu\{\tilde{K}^j_t \notin A^j \text{i.o.}\} = 0$

proof: Suppose $K^j_{t-1} > K^j_{**}$. Then if for $\tau > t-1$, $\tilde{K}^j_\tau < K^j_{**}$, part (ii) applies and

and (iii) is proved. Suppose that

$\tilde{K}_t^j > K_{**}^j$ for all t. It can be shown that $\lim_{t\to\infty} \tilde{K}_t^j = K_{**}^j$. For, suppose that

$\tilde{K}_t^j \geq K_{t-1}^j > K_{**}^j$.

Then

$$- \hat{V}_1^j(M,\bar{R}) + \beta E_t f_{1+}^j(K_{**}^j + r'z_{t+1}^j,\bar{R}) > 0$$

But,

$$\varphi^j(M + K_{**}^j,\bar{R}) = K_{**}^j \Rightarrow - \hat{V}_1^j(M,\bar{R}) + E_t\beta f_{1+}^j(K_{**}^j + r'z_{t+1}^j,R) \leq 0$$

Therefore, by contradiction

$$\tilde{K}_t^j < K_{t-1} \text{ and } \lim_{t\to\infty} \tilde{K}_t^j = K_{**}^j.$$

Now, $\varphi^j(K_{**}^j + r'z_t^j,\bar{R}) - K_{**}^j < 0$ for all $r'z_t^j < M$. By the continuity of $\varphi^j(\cdot,\bar{R})$ there exist some $\delta > 0$ s.t. for all K^j : $K_{**}^j < K^j < K_{**}^j + \delta$,

$$\varphi^j(K^j + r'z_{t+1}^j,\bar{R}) - K_{**}^j < 0 \text{ if } r'z_{t+1}^j < M$$

As $\lim_{t\to\infty} \tilde{K}_t^j = K_{**}^j$, there exists some T such that for all $t > T, \tilde{K}_t^j < K_{**}^j + \delta$. Let ω^* be such that $r'z_{t+1}^j(\omega^*) = M$, and, without loss of generality, let $T = 0$. Then,

$$\sum_{t=1}^{\infty} \mu[\tilde{K}_t^j > K_{**}^j] = \sum_{t=1}^{\infty} [P(\omega^*)]^t < \infty.$$ Hence, by the Borel Cantelli lemma

$$\mu\{\tilde{K}_t^j > K_{**}^j \text{ i.o.}\} = 0.$$

<u>lemma 15</u>

Let $J(K_t^j) \geq 0$ satisfy

$$E_0 J[\varphi^j(r'z_1^j + K_0^j,\bar{R})] - J(K_0^j) = -g^j(K_0^j) + c^j$$

where

$g^j(K_0^j) \geq 0$ and $g^j(K_0^j) > c^j + 1$ on $H - D$, H is the range space of $\varphi^j(\cdot,\bar{R})$, a subset of Euclidian space, and D is a compact set. Also, let $h(\cdot)$ be a bounded, continuous function such that $\lim_{u\to\infty} |h(u)| = 0$.

If $E_0 h[\varphi^j(K_0^j + r'z_1^j,\bar{R})$ is bounded, continuous, and

$\lim_{|K_j^0|\to\infty} E_0 h[\varphi^j(K_0^j + r'z_1^j,\bar{R}) = 0$, then there is at least one finite invariant measure for $\tilde{K}_t^j$.

proof: see Kushner [6 , theorem 4, p. 208, and theorem 7, p. 211].

<u>lemma 16</u>

A set A is self-contained if $\tilde{K}_t^j \in A$, then $K_\tau^j \in A$ for all $\tau \geq t$. Suppose there are not two or more disjoint self-contained subsets of H. Then there is, at most, one invariant probability measure for $\tilde{K}_t^j$.

proof: see Kushner [6 , theorem 5, p. 207].

<u>lemma 17</u>

$$\lim_{W\to\infty} \psi^j(W,\bar{R}) = \infty$$

proof: Suppose by lemma 11 that $\lim_{W\to\alpha} \psi^j(W,\bar{R}) = \bar{\psi}^j < \infty$. By construction for each $W \geq 0$,

$$E_t f_{1-}^j [r'z_{t+1}^j + \varphi^j(W,\bar{R}),\bar{R}] \geq \hat{V}_1^j[\psi^j(W,\bar{R}),\bar{R}]$$

Then

$$\lim_{W\to\infty} E_t f_{1-}^j [r'z_{t+1}^j + \varphi^j(W,\bar{R}),\bar{R}] \geq \hat{V}_1^j[\bar{\psi}^j,\bar{R}] > 0$$

From lemma 14, $\lim_{W\to\infty} f_{1-}^j(W,\bar{R}) = 0$. With lemma 12, this establishes a contradiction.

Proposition II

There exists a unique stationary measure for $\tilde{K}_t^j$ as defined by Eq. (4) in the text.

proof: let $J(K_t^j) = K_t^j \geq 0$. Let $L^j(K_0^j,\bar{R}) = E_0\varphi^j(K_0^j + r'z_1^j,\bar{R}) - K_0^j$. As

$\lim_{W\to\infty} \psi^j(W,\bar{R}) = \infty$ by lemma 17, then for all $z_1^j \geq 0$,

$\lim_{K_0^j\to\infty} E_0[\varphi^j(r'z_1^j + K_0^j,\bar{R}) - K_0^j] = -\infty$. Also, $L^j(K_0^j,\bar{R})$ is monotone decreasing; given

$\bar{\bar{K}} > \bar{K} > 0$, $L^j(\bar{\bar{K}},\bar{R}) - L^j(\bar{K},\bar{R})$

$$= \sum_{\omega\in\Omega} P(\omega)\{\varphi^j[\bar{\bar{K}} + r'z_1^j(\omega),R] - \varphi^j[\bar{K} + r'z_1^j(\omega),\bar{R}] - (\bar{\bar{K}} - \bar{K})\} \leq 0$$

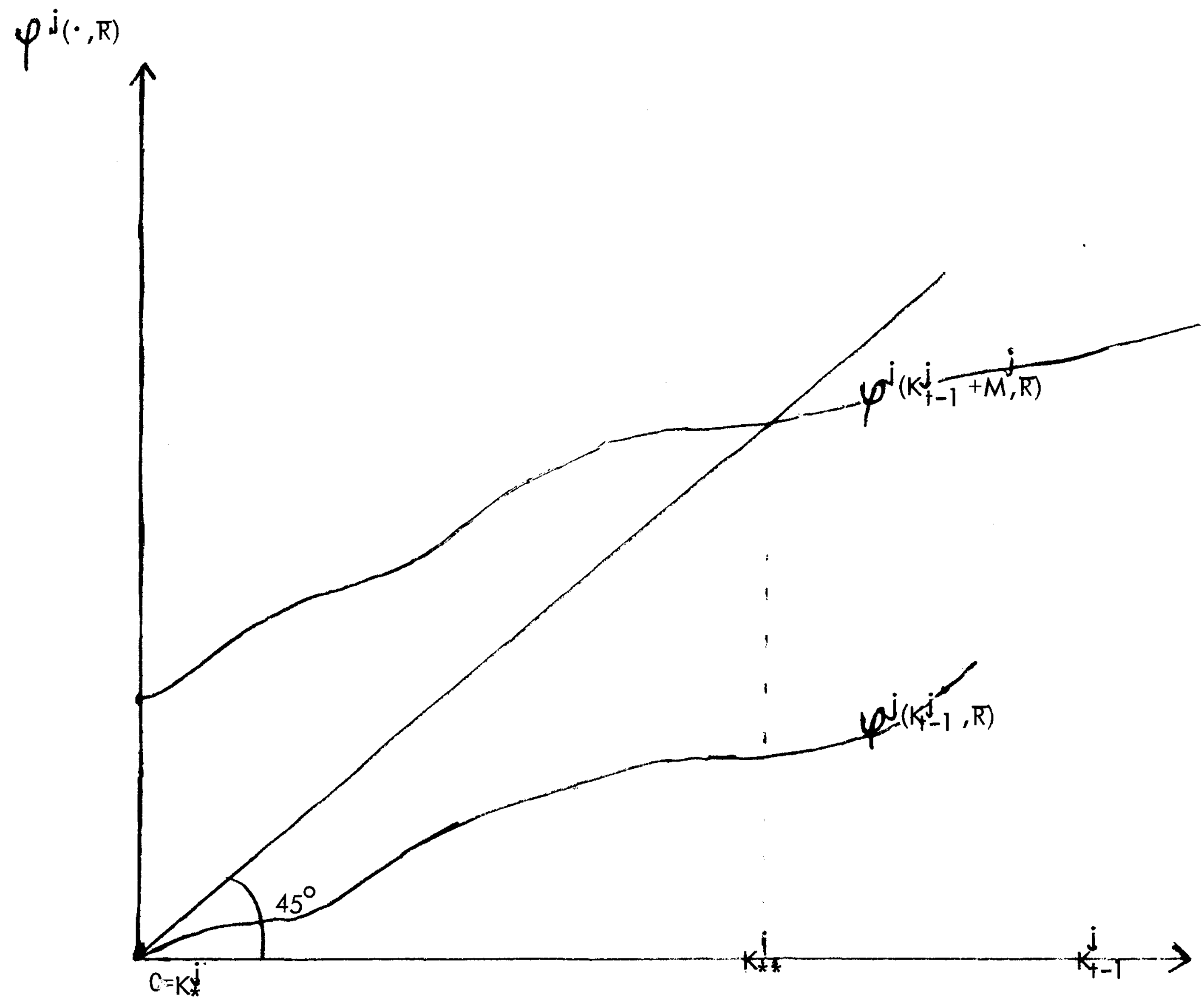
Hence, there exist some k such that $L^j(K,\bar{R}) < -1$ for $K \in [0,k]^c$. Let

$$c^j = E_0\ \varphi^j(r'z_1^j,\bar{R})$$

then

$L^j(K_0^j,\bar{R}) = -g^j(K_0^j) + c^j$ where $g^j(K_0^j) \geq 0$ and $g^j(K_0^j) > c^j + 1$ except on $[0,k]$, a compact set.

Also, if $h(\cdot)$ is bounded, continuous, and $\lim_{u\to\infty} |h(u)| = 0$, then $E_0h[\varphi^j(K_0^j + r'z_1^j),\bar{R}]$ is bounded and continuous w.r.t. K_0^j, and tends to zero as K_0^j tends to ∞, by lemmas 12 and 13. By lemma 15, there exist at least one stationary measure. Uniqueness follows from lemma 16 and Proposition I.



$\varphi^j(\cdot, \overline{R})$
$\varphi^j(K^j_{t-1} + M^j, \overline{R})$
$\varphi^j(K^j_{t-1}, \overline{R})$
45°
$0 = K^j_*$
K^j_{**}
K^j_{t-1}

FIGURE 1

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FOOTNOTES

1. For a definition of span see Feller [4, p. 200].

2. This stochastic dynamic programming problem is really a choice between consumption and savings in a one good model in which future income is subject to random shocks. This distinguishes it from the literature in which the return from saving is random and enters in a multiplicative rather than additive way. See Brock and Mirman [2]. The results established below cast some doubt on Friedman's contention that the marginal propensity to consume out of transitory income is one.

3. Elements of this proof were first suggested by John Danforth, but the author alone is responsible for any errors.

4. The price corresponding to $E(X_t)$, $E(Y_t)$ is not the expected price unless the correspondence is linear.

5. See Feller [4, theorems 3 and 4, pp. 202-3].

6. The text does not deal with the case $\bar{R} = R^*$. If R^* is never attained on $\theta(\cdot)$, then a policy of $\bar{R} = R^*$ will fail immediately. If $R^* = \min\ \theta(\cdot)$, then the proof of the text applies.

7. Equations (6) and (7) can fail to be sufficient conditions for a competitive equilibrium with storage if there is an excess accumulation of either (x) or (y) over time. It is shown below that this does not occur in the special case at issue.

8. In the model

$$E_0\sum_{t=1}^{\infty} \beta^{t-1}\hat{U}^j(X_t^j,Y_t^j) < (1/1-\beta)\hat{U}^j[\max_{\omega\in\Omega} X_t(\omega),\ \max_{\omega\in\Omega} Y_t(\omega)] < \infty$$