INCOMPLETE FORWARD MARKETS IN A PURE EXCHANGE ECONOMY WITH STOCHASTIC ENDOWMENTS*

by

Robert M. Townsend

Discussion Paper No. 74-47, November 1974

* Helpful comments from John Chipman, John Danforth, Clifford Hildreth, and Steve Salant are gratefully acknowledged. I am especially indebted to Neil Wallace. The author alone is responsible for any errors. I would like to thank the Federal Reserve Bank of Minneapolis and the Board of Governors of the Federal Reserve System for financial support.

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

ABSTRACT

Jacques Drèze has stressed the need for research into the functions and shortcomings of existing institutions and for the application of standard welfare economics based on Pareto optimality to limited exchange opportunities for the allocation of risk bearing. The objective of the paper is to examine the workings and welfare implications of actual forward markets and to place those markets in the context of complete contingent commodity markets. Second best government policies in the absence of complete markets are explored.

I. Introduction

Complete contingent commodity markets provide a convenient framework in which problems involving choices under uncertainty can be analyzed. Yet the absence of such complete markets has been noted by many authors. Among them, Jacques Drèze [3] has stressed the need for research into the functions and shortcomings of existing institutions. The objective of this paper is to examine the workings and welfare implications of actual forward markets and to place those markets in the context of complete contingent commodity markets.

In actual forward markets a typical forward contract is one which is binding independent of the state of the world at the time of maturity. Forward currency and commodity futures contracts are examples of such unconditional contracts. Section III examines the existence and properties of equilibria in models restricted exogenously to unconditional contracts. It is shown by way of some examples that risk averse participants use unconditional contracts to hedge against the randomness of both spot prices and their exogenous endowments; with active spot markets, optimal behavior does not involve the "elimination of risk" by purchasing the consumption bundle forward.

Section IV provides a welfare analysis of unconditional contracts. It is shown that in a two state world, unconditional contracts with active spot markets may achieve Pareto optimal allocations. Contrary to the comments by Radner [6], subsequent spot markets need not introduce externalities which affect the optimality of competitive equilibria. With more states than two, the outcome with spot and unconditional contracts is in general Pareto non-optimal though, with identical homothetic preferences, Pareto superior to the outcome with all forward contracts prohibited. Hence futures contracts do not usually compensate for the absence of complete contingent contracts, but they do make possible beneficial hedging by all parties. Second best government policies in the absence of complete markets are also explored.

The welfare analysis of Section IV would lead one to expect more contingent contracts than are observed in actual markets. Though it is not the purpose of this paper to explain the absence of such contracts, Section V does offer some thoughts on the preponderance of unconditional contracts. If contracts are contingent on information not revealed to everyone, moral hazard would seem to require third party verification of states. Unconditional contracts would seem to be costless relative to the certification schemes associated with contingent contracts.

II. Structure and Technology of the Model

The model is a two good, pure exchange economy with stochastic endowments. Before the random endowments are realized, each individual can decide on the quantity and type of forward contracts to purchase or issue. After endowments are realized, forward contracts are executed, and trade and consumption takes place in spot markets. In the model no actual transfer or consumption of resources takes place in forward markets; only claims are traded. There is no uncertainty in spot markets; every individual knows his endowment and the market price. An individual's consumption is not limited to the claims acquired in forward markets--retrading is possible in the spot markets.

The exogenous endowment of each of m individuals is assumed to have a discrete probability distribution with n possible realizations. A state of the world specifies for each individual the realized endowment of each of the two goods (X) and (Y). Let Ω be the set of all possible states of the world with typical element w. Then w is an m-dimensional row vector specifying a state for each individual. By construction Ω has n^m elements. Let (Ω, F, g) be a discrete probability space where F is the set of all subsets of Ω and g(w) is the probability that state w will occur. Let

$Z^i(w)' = [Z_x^i(w), Z_y^i(w)]$ where $Z_j^i(w)$ is the i^{th} individual's exogenous endowment of commodity (j) in state w.

In the planning period, each individual is assumed to maximize expected utility. Let $c_j^i(w)$ be the i^{th} individual's consumption of commodity (j) in state w. Then each individual acts as if to maximize

$$\sum_{w\epsilon\Omega} g(w)U^i[c_x^i(w), c_y^i(w)].$$

Each individual's subjective probability distribution of states of the world is the actual probability measure $g(\cdot)$ of the model. In this manner, $U^i(\cdot,\cdot)$ is the utility of individual i in state w. State dependent utility functions $U^{i,w}[c_x^i(w),c_y^i(w)]$ could be assumed without affecting the analysis which follows. It is further assumed that

(I) $U^i(\cdot,\cdot)$: $R^2 \rightarrow R^1$ with continuous partial derivatives $U_1^i(\cdot,\cdot)$, $U_2^i(\cdot,\cdot)$, strictly positive for $c_x^i(w) > 0$, $c_y^i(w) > 0$; and $U_1^i(0,0) = \infty$.

(II) $U^i(\cdot,\cdot)$ is strictly concave.

The properties of $U^i(\cdot,\cdot)$ are sufficient to ensure the existence of a competitive equilibrium in each state w. At this point, uniqueness of the competitive equilibrium may be taken as an assumption, though in the special cases discussed below, uniqueness follows from additional assumptions. Property (II) ensures that each individual is risk averse.

The analysis is facilitated by the use of the indirect utility function. Let

$I^i(w)$ = the income of individual i in terms of (X) in state w

h^i_{xw} = the demand for good (X) in state w by individual i

h^i_{yw} = the demand for good (Y) in state w by individual i

$P(w)$ = the price of (Y) in terms of (X) in state w

$r(w)' = [1, P(w)]$

Then $[h^i_{xw}, h^i_{yw}]$ maximize $U^i[c^i_x(w), c^i_y(w)]$ subject to $I^i(w) = P(w)c^i_y(w) + c^i_x(w)$. Define the indirect utility function V^i as

$$V^i[I^i(w), P(w)] = U^i\{h^i_{xw}[I^i(w), P(w)], h^i_{yw}[I^i(w), P(w)]\}$$

Complete forward markets are described as follows. Each individual can issue or purchase contingent commodity claims, each of which entitles the holder to one unit of the specified commodity if a particular state of the world occurs. Let

$X^i(w)$ = claims on (X) contingent on state w acquired by individual i in the market for claims

$Y^i(w)$ = claims on (Y) contingent on state w acquired by individual i in the market for claims

Equilibrium in the market for claims exists when

$$\sum_{i=1}^{m} X^i(w) = 0 \text{ and } \sum_{i=1}^{m} Y^i(w) = 0 \text{ for each } w\epsilon\Omega.$$

Let $P_x(w)$ = the price of one claim on one unit of (X) in state w,

$P_y(w)$ = the price of one claim on one unit of (Y) in state w.

As yet, the numeraire is unspecified. The budget constraint for individual i in the market for claims is that the net value of excess demand for claims be zero, or

$$\sum_{w\epsilon\Omega} [X^i(w)P_x(w) + Y^i(w)P_y(w)] = 0 \qquad i = 1,2, \ldots m \qquad (1)$$

With insurance schemes an individual's income in state w is the value of his claims on (X) and on (Y) in state w, and the value of his exogenous endowment in state w.

$$I^i(w) = r(w)'Z^i(w) + P(w)Y^i(w) + X^i(w)$$

To prevent bankruptcy it is specified that $I^i(w) \geq 0$, for each $w \epsilon \Omega$. However, by property (I) of $U^i(\cdot,\cdot)$, $V_1^i[0, P(w)] = +\infty$ for each $w \epsilon \Omega$ and, therefore, the bankruptcy constraint will never be binding in equilibrium.

In summary, the objective of individual i is to maximize

$$\begin{aligned} &\sum_{w \epsilon \Omega} g(w)V^i[r(w)'Z^i(w) + P(w)Y^i(w) + X^i(w), P(w)] \\ &\quad - \lambda\{\sum_{w \epsilon \Omega} [P_x(w)X^i(w) + P_y(w)Y^i(w)\} \end{aligned} \tag{2}$$

with respect to $\{X^i(w), Y^i(w); w \epsilon \Omega\}$ where λ is a lagrange multiplier. This yields first order conditions (3).

$$\begin{aligned} &g(w)V_1^i[I^i(w)^*, P(w)] - \lambda P_x(w) = 0 && w \epsilon \Omega \\ &g(w)P(w)V_1^i[I^i(w)^*, P(w)] - \lambda P_y(w) = 0 && w \epsilon \Omega \\ &\sum_{w \epsilon \Omega} [P_x(w)X^i(w)^* + P_y(w)Y^i(w)^*] = 0 \end{aligned} \tag{3}$$

where here and below the superscript * denotes maximizing quantities.

In equilibrium $P_y(w)/P_x(w) = P(w)$ for each state w as otherwise riskless arbitrage would be possible. In general the choice of $\{X^i(w)^*, Y^i(w)^*; w \epsilon \Omega\}$ is <u>not</u> unique. To see this, note that the indirect utility function is strictly concave with respect to $\{I^i(w); w \epsilon \Omega\}$ as

$$\begin{bmatrix} g(1)\, V^i_{11}[I^i(1),\, P(1)] & & 0 \\ & \ddots & \\ & g(w) V^i_{11}[I^i(w),\, P(w)] & \\ 0 & & \ddots \end{bmatrix}$$

is a diagonal matrix with strictly negative diagonal elements. The budget constraint may be written as

$$\sum_{w \in \Omega} [P_x(w) I^i(w) - P_x(w) r(w)' Z^i(w)] = 0$$

As $I^i(w) \geq 0$ for each $w\Omega$, the income possibilities set is strictly convex. Hence, the choice of $\{I^i(w)^*;\ w \in \Omega\}$ is unique, but without additional constraints, the choice of $\{X^i(w)^*,\ Y^i(w)^*;\ w \in \Omega\}$ is not.

An additional restriction consistent with the choice of incomes is that there be no trading in spot markets, that is, that each individual consume the goods acquired with claims. Though this restriction is typical of many formulations of complete Arrow-Debreu markets, Arrow [2] proved in his seminal paper that complete contingent commodity contracts are not needed; n securities yielding a dollar in each of n states is a consistent restriction. This result suggests that restrictions of a different nature should be examined. Sections III and IV examine the restrictions associated with unconditional contracts.

III. Equilibrium with Unconditional Contracts

Forward contracts which are binding independent of the state of the world at the time of maturity are common in actual markets. Such unconditional contracts can be described in terms of contingent commodity markets. Additional constraints are that each individual acquire or issue in the market the same number of claims independent of the state,

$$X^i(w) = X^i_f \qquad w\epsilon\Omega \qquad i = 1,2, \dots m$$

$$Y^i(w) = Y^i_f \qquad w\epsilon\Omega \qquad i = 1,2, \dots m$$

Then X^i_f, Y^i_f are forward purchases of (X) and (Y) respectively, independent of the state. Denote $\sum_{w\epsilon\Omega} P_y(w)/\sum_{w\epsilon\Omega} P_x(w) = F$. Then F is the forward price of (Y) in terms of (X). The budget constraint (1) of individual i is then of the form

$$X^i_f + FY^i_f = 0 \tag{4}$$

From (2), individual i seeks to maximize with respect to Y^i_f

$$\sum_{w\epsilon\Omega} g(w)V^i\{r(w)'\mathcal{Z}^i(w) + Y^i_f[P(w) - F],\ P(w)\} \tag{5}$$

The purpose of this section is to examine the properties of a competitive equilibrium in the model restricted exogenously to unconditional forward contracts. Existence of a competitive equilibrium is not proved in the general case. Rather, the approach adopted here is to reveal the essential workings of unconditional contracts by way of some particular examples.

Though each individual treats all spot and forward prices as parameters, spot market prices are not necessarily independent of the existence and direction of forward contracts. This complicates the analysis. For the remainder of this section, the following property is assumed.

(III) Preferences of all individuals are identical and homothetic.

Where unambiguous the superscript ^{i} on utility functions is now deleted. Under property (III), the spot price will depend on the ratio of the aggregate endowment of (X) to the aggregate endowment of (Y).

$$P(w) = f[\sum_{i=1}^{m} \mathcal{Z}^i_x(w)/\sum_{i=1}^{m} \mathcal{Z}^i_y(w)]$$

Hence, spot market prices will be independent of the existence and direction of forward contracts.

Under property (III) some characteristics of the demand for unconditional forward contracts can be derived. Let $E = \{F\colon \min_{w \in \Omega} P(w) < F < \max_{w \in \Omega} P(w)\}$

<u>lemma 1</u>. For each $F \in E$

$$\sum_{w \in \Omega} V\{r(w)'\mathbf{z}^i(w) + Y_f^i[P(w)-F],\ P(w)\} \text{ is strictly concave with respect to } Y_f^i.$$

Lemma 1 follows immediately from property (II), and the proof is not given here.

(5) is continuous with respect to Y_f^i as h_{xw}^i, h_{yw}^i are continuous with respect to $I^i(w)$. Given $F \in E$, let

$$S_1 = \{w\colon F - P(w) > 0\} \qquad\qquad S_2 = \{w\colon F - P(w) < 0\}.$$

Then from the bankruptcy constraints

$$\max_{w \in S_2}[r(w)'\mathbf{z}^i(w)/F-P(w)] \le Y_f^i \le \min_{w \in S_1}[r(w)'\mathbf{z}^i(w)/F-P(w)]$$

Hence, for each $F \in E$, a maximizing choice of Y_f^i does exist as a continuous function or a compact set must achieve a maximum on that set. By lemma 1, the choice must be unique. This optimum value of Y_f^i will be denoted $Y_f^{i*} = \varphi^i(F)$. From (5) a necessary and sufficient condition for a maximum is that

$$\sum_{w \in \Omega} g(w)[P(w)-F]\ V_1\{r(w)'\mathbf{z}^i(w) + Y_f^{i*}[P(w)-F],\ P(w)\} = 0 \tag{6}$$

Riskless arbitrage will ensure that Y_f^{i*} is infinite in absolute value if $F \notin E$. Hence, the search for a competitive equilibrium can be limited to $F \in E$.

<u>lemma 2.</u> Given properties (I)-(III) of $U(\cdot,\cdot)$, $\varphi^i(F)$ has continuous derivatives.

<u>Proof</u>:

As noted, $I^i(w)^* > 0 \quad w \epsilon \Omega$. Hence, Y_f^{i*} will always be an interior solution. Let

$$G^i(F, Y_f^i) = \sum_{w \epsilon \Omega} g(w)[P(w)-F]\ V_1\{r(w)'z^i(w) + Y_f^i[P(w)-F],\ P(w)\}$$

Then, for each $F \epsilon E$, $G^i[F, \varphi^i(F)] = 0$; $G_1^i(\cdot,\cdot)$, $G_2^i(\cdot,\cdot)$ are continuous with respect to F and Y_f^i; $G_2^i(\cdot,\cdot) < 0$ by lemma 1. Hence, the implicit function theorem applies. Q.E.D.

Denote $\underset{w \epsilon \Omega}{\text{Min}}\ P(w) = P'$, $\underset{w \epsilon \Omega}{\text{Max}}\ P(w) = P''$

<u>lemma 3</u>. Under properties (I)-(III) of $U(\cdot,\cdot)$ $\lim_{F \to P'} \varphi^i(F) = +\infty$ and $\lim_{F \to P''} \varphi^i(F) = -\infty$

<u>Proof</u>:

Suppose it were not the case that $\lim_{F \to P'} \varphi^i(F) = +\infty$. Then one can construct a sequence $\{F_n\} \epsilon E$ such that $\lim_{n \to \infty} F_n = P'$ and the corresponding sequence $\{\varphi^i(F_n)\}$ is always less than some positive number K. Then

$$\lim_{F_n \to P'} \sum_{w \epsilon \Omega} g(w) V_1\{r(w)'z^i(w) + \varphi^i(F_n)[P(w)-F_n], P(w)\}\ [P(w)-F_n]$$

$$\geq \lim_{F_n \to P'} \sum_{w \epsilon \Omega} g(w) V_1\{r(w)'z^i(w) + K[P(w)-F_n], P(w)\}[P(w)-F_n]$$

$$= \sum_{w \epsilon \Omega} g(w) V_1\{r(w)'z^i(w) + K[P(w)-P'],\ P(w)\}[P(w)-P'] > 0$$

This is the desired contradiction as for each $F \epsilon E$, $G^i[F, \varphi^i(F)] = 0$. The proof that $\lim_{F \to P''} \varphi^i(F) = -\infty$ follows similarly. Q.E.D.

A fundamental existence property can now be proved.

<u>Proposition I</u>. Under properties (I)-(III) of $U(\cdot,\cdot)$, the model restricted exogenously to unconditional forward contracts possesses a competitive equilibrium.

<u>Proof</u>:

Define $\varphi(F) = \sum_{i=1}^{m} \varphi^i(F)$. By lemma 2, $\varphi^i(F)$ is continuous on E. By lemma 3, there exist F' and F" in E such that $\varphi(F') > 0$, $\varphi(F") < 0$. Therefore, by the intermediate value theorem there exists some F^* such that $\varphi(F^*) = 0$. This establishes existence. Q.E.D.

An Example - Correlation of Economic Aggregates

It is assumed that all individuals can be classified as either receivers of (X), group A, or receivers of (Y), group B.

$$Z^i_x(w) = S^i_x X(w), \qquad i = 1,2, \ldots m_1$$

$$Z^j_y = S^j_y Y \qquad j = 1,2, \ldots m_2 \qquad 0 < S^i_x < 1,\ 0 < S^j_y < 1$$

$$\sum_{i=1}^{m_1} S^i_x = 1,\ \sum_{j=1}^{m_2} S^j_y = 1 \qquad m_1 + m_2 = m.$$

Hence, X(w) represents the aggregate over all individuals of the economy's endowment of (X) in state w. Y is non-stochastic and represents the aggregate endowment of (Y). One can interpret the model as consisting of two groups. One group, farmers, produce only wheat, the output of which is exogenous and stochastic. Bad weather is assumed to diminish the crop to the same extent for all farmers. The other group produces only manufactured goods, the output of which is certain. In terms of the technology of section II, individuals of group B have degenerate distributions of holdings of (Y), and members of group A have identically distributed holdings of (X) up to a constant of proportionality. Without loss of generality it is assumed that X(w) is strictly increasing.

In this example an additional property is assumed.

(IV) All individuals have constant relative risk aversion.

A coefficient of relative risk aversion, C^i, is defined in terms of the indirect utility function.

$$C^i = C^i[I^i(w),\ P(w)] = -V_{11}[I^i(w),\ P(w)]I^i(w)/V_1[I^i(w),\ P(w)]$$

Constancy means that C^i is independent of both prices and incomes for all states.

A coefficient of absolute risk aversion, D^i, is defined as follows:

$$D^i[I^i(w),\ P(w)] = -\ V_{11}[I^i(w),\ P(w)]/V_1[I^i(w),\ P(w)]$$

Then $D^i[I^i(w),\ P(w)] = C^i/I^i(w)$ and D^i is strictly decreasing in $I^i(w)$.

<u>lemma 4</u>. Given properties (I)-(IV) of $U(\cdot,\cdot)$, $\varphi^i(F)$ is strictly monotone decreasing.

<u>Proof</u>:

by lemma 2 $\varphi^i(F)$ is differentiable, and

$$\frac{d\varphi^i(F)}{dF} = \frac{-G_1^i(F,Y_f^{i*})}{G_2^i(F,Y_f^{i*})}$$

$$G_1^i(F,Y_f^{i*}) = -Y_f^{i*}\sum_{w\epsilon\Omega} g(w)[P(w)-F]V_{11}[I^i(w)^*,\ P(w)] - \sum_{w\epsilon\Omega} g(w)V_1[I^i(w)^*,\ P(w)]$$

$$G_2^i(F,Y_f^{i*}) = \sum_{w\epsilon\Omega} g(w)[P(w)-F]^2V_{11}[I^i(w)^*,\ P(w)] < 0$$

It remains to establish that $G_1^i(F,Y_f^{i*}) < 0$. For suppose $[P(w)-F] \geq 0$ and that $Y_f^{i*} \geq 0$. Then

$$D^i\{r(w)'z^i(w) + Y_f^{i*}[P(w)-F] \leq D^i[r(w)'z^i(w)]$$

Hence,

$$[P(w)-F]V_{11}\{r(w)'z^i(w) + Y_f^{i*}[P(w)-F],\ P(w)\} \geq -\ [P(w)-F]\ \cdot\ D^i[r(w)'z^i(w)]$$

$$\cdot\ V_1\{r(w)'z^i(w) + Y_f^{i*}[P(w)-F],\ P(w)\}$$

If $[P(w)-F] \leq 0$, this relationship still holds. Hence,

$$\sum_{w\in\Omega} g(w)[P(w)-F]V_{11}[I^i(w)^*,P(w)] \gtrless -\sum_{w\in\Omega} g(w)[P(w)-F]D^i[r(w)\cdot\mathcal{Z}^i(w)]V_1[I^i(w)^*,P(w)]$$

For members of group A, $D^i[r(w)\cdot\mathcal{Z}^i(w)] = D^i[S_x^iX(w)]$ is strictly monotone decreasing with respect to w. For individuals of group B, $D^i[S_y^iY,P(w)]$ is strictly monotone decreasing with respect to w. Hence, from (6)

$$\sum_{w\in\Omega} g(w)[P(w)-F]V_{11}[I^i(w)^*, P(w)] > 0$$

and $\frac{d\varphi^i(F)}{dF} < 0$. The case of $Y_f^{i*} < 0$ is similar with appropriate changes in sign. Q.E.D.

Following Arrow, if relative risk is constant, the willingness to accept a bet should remain unchanged as the bet and income are increased proportionately. Formally, this leads to lemma 5.

<u>lemma 5</u>. In this example, with properties (I)-(IV) of $U(\cdot,\cdot)$, let Y_f^{i*} denote an optimal choice given exogenous endowment $r(w)\cdot\mathcal{Z}^i(w)$, and some fixed $F\in E$. Then, if the endowment changes to $k[r(w)\cdot\mathcal{Z}^i(w)]$, the optimizing choice will increase to kY_f^{i*} for the same fixed F.

<u>Proof</u>:

Following Stiglitz [7], constant relative risk averse functions are of one of the following forms where $a(\cdot)$ and $d(\cdot)$ are functions of P(w):

$$V[I^i(w), P(w)] = a[P(w)] \ln [I^i(w)] + d[P(w)]$$

$$V[I^i(w), P(w)] = a[P(w)][I^i(w)]^{c+1} + d[P(w)]$$

The proof follows immediately from (6). Q.E.D.

<u>Corollary to lemma 5</u>. If $Y_f^{i*} = 0$ initially, then as $r(w)\cdot\mathcal{Z}^i(w)$ changes to $k[r(w)\cdot\mathcal{Z}^i(w)]$, Y_f^{i*} will remain zero.

From lemma 5 one may treat the economy as if consisting of two individuals in which A has all of (X) in all states and B has all of (Y). Each individual treats prices as parameters.

As tastes are identical and homothetic, let $\sigma(w)$ denote the elasticity of substitution in consumption of each individual in state $w \varepsilon \Omega$. Also let $R(w)$ denote the ratio of the share of A of national income in terms of (X) to the share of B. Then, $R(w) = X(w)/P(w)Y$. The findings are summarized in the following proposition.

Proposition II. Under the assumptions of the example with properties (I)-(IV) of $U(\cdot,\cdot)$, suppose $\sigma(w) = \sigma^*$ for each $w \varepsilon \Omega$.

(i) If $0 < \sigma^* < 1$ then $Y_f^{A*} > 0$.

(ii) If $\sigma^* = 1$ then $Y_f^{A*} = Y_f^{B*} = 0$.

(iii) If $\sigma^* > 1$ then $Y_f^{A*} < 0$.

Proof:

Case (i) will be considered in detail. In the argument which follows, it is useful to refer to Figure 1.

Let

$$G^A(F,\ Y_f^A) = \sum_{w \varepsilon \Omega} g(w)[P(w)-F]V_1\{X(w) + Y_f^A]P(w)-F],\ P(w)\}.$$

By lemma 2 and lemma 3 there exists some F" such that $G^A(F'', 0) = 0$. Define $s \varepsilon \Omega$ such that $P(s) \leq F'' < P(s+1)$. Let

$$G^B(F,\ Y_f^B) = \sum_{w \varepsilon \Omega} g(w)[P(w)-F]V_1\ \{P(w)Y + Y_f^B[P(w)-F],\ P(w)\}.$$

By homotheticity, with $0 < \sigma^* < 1$, $R(w)$ is a strictly decreasing function of w. Let $k = 1/R(s)$. Then $X(w)/R(w) > kX(w)$, $w = s + 1, \dots n$. By lemma 1, $V_{11}(\cdot,\cdot) < 0$, and by construction $P(w)-F'' > 0$ for $w = s + 1, \dots n$.

$$\therefore \sum_{w=s+1}^{n} g(w)[P(w)-F'']V_1[X(w)/R(w),P(w)] < \sum_{w=s+1}^{n} g(w)[P(w)-F'']V_1[kX(w),P(w)]$$

Similarly, for $w = 1,2, \dots s$, $X(w)/R(w) \leq kX(w)$ and $P(w) - F'' \leq 0$.

$$\therefore G^B(F'',0) < \sum_{w=1}^{n} g(w)[P(w)-F'']V_1[kX(w),P(w)]$$

But by the corollary to lemma 5

$$G^A(F'',0) = 0 \Rightarrow \sum_{w=1}^{n} g(w)[P(w)-F'']V_1[kX(w),P(w)] = 0$$

Therefore, $G^B(F'',0) < 0$. By a similar argument there exists an F' such that $G^B(F',0) = 0$ and $G^A(F',0) > 0$. By lemma 1, $G^B(F'',0) < 0 \Rightarrow \varphi^B(F'') < 0$ and $G^A(F',0) > 0 \Rightarrow \varphi^A(F') > 0$. Define $\varphi(F) = \varphi^A(F) + \varphi^B(F)$. By lemma 2 $\varphi(F)$ is continuous. As $\varphi(F'') < 0$, $\varphi(F') > 0$, there exists an F* such that $F' < F^* < F''$ and $\varphi(F^*) = 0$. F* is an equilibrium foward rate with $\varphi^A(F^*) > 0$. Uniqueness follows immediately from monotonicity of lemma 4.

Case (ii). If $\sigma^* = 1$, then $P(w)Y/X(w) = k$ and by lemma 5, A and B will always be on the same side of the market. The equilibrium solution must be $Y_f^{A*} = Y_f^{B*} = 0$.

Case (iii) follows from case (i) with appropriate changes in sign. Q.E.D. Roughly speaking, if $0 < \sigma^* < 1$, the value of terms of (X) of the exogenous endowment of B increases more as w increases than does the exogenous endowment of A. In the terminology of Hildreth [5], for a given forward rate, B is more anxious to engage in a venture, the outcome of which is negatively correlated with w. This difference in insurance values creates a forward market despite agreement on the probability distribution of future spot rates. With $\sigma^* = 1$, A's and B's exogenous endowments in terms of (X) are perfectly correlated, and a forward contract has the same insurance value for each. It should be noted that with $\sigma^* > 1$, A will buy (X) forward even though A is endowed with (X) only and can anticipate consuming (Y) in all states. With active spot markets, optimal behavior under risk aversion does not necessarily involve the "elimination of risk" by purchasing the consumption bundle forward. Classification schemes which distinguish "speculators" from "hedgers" may be misleading in some circumstances.1/ Though such a distinction is possible in a model in which participants differ with respect to attitudes toward risk or access to information. The example illustrates that active forward markets do not require this distinction.

IV. Welfare Implications of Unconditional Contracts

The objective of this section is to analyze the welfare implications of unconditional contracts. It was shown by Arrow [2] that complete contingent commodity markets are sufficient but not necessary for Pareto optimal allocations; with securities yielding one dollar in each state, participants would face the same market opportunitites as with complete markets. It is shown in this section that, in some circumstances, contracts conditioned on subsets of states may also suffice. The strongest result is the following.

Proposition III. Suppose in a two state world there exists a competitive equilibrium with complete contingent contracts and with no trade in spot markets in which the spot price of (Y) in terms of (X) differs in each of the two states. Then one equilibrium to the model with complete contingent markets and subsequent spot markets has the property that all contracts are unconditional.

It will be shown that in each of the two situations described in Proposition III the same market opportunities are available. As a matter of notation, prices and quantities of the equilibrium with contingent contracts are denoted with a superscript *.

With complete markets

$$I^i(w)^* = r(w)^* Z^i(w) + P(w)^* Y^i(w)^* + X^i(w)^* \qquad w = 1,2$$

It will be shown that individual i can achieve the same income in terms of (X) with appropriately chosen unconditional contracts. That is, there exists a solution $\{X_f^{i*}, Y_f^{i*}\}$ to the following equations:

$$\begin{bmatrix} 1 & P(1)^* \\ 1 & P(2)^* \end{bmatrix} \begin{bmatrix} X_f^{i*} \\ Y_f^{i*} \end{bmatrix} = \begin{bmatrix} X^i(1)^* + P(1)^* Y^i(1)^* \\ X^i(2)^* + P(2)^* Y^i(2)^* \end{bmatrix} \tag{7}$$

Recall that $P(w)^* = P_y(w)^*/P_x(w)^*$ $w \epsilon \Omega$. Initially, $\sum_{w=1}^{2} [P_x(w)^* X^i(w)^* + P_y(w)^* Y^i(w)^*] = 0$. Therefore the elements of the matrix on the right side of (7) are equal only if both are zero. Hence, if $P(1)^* \neq P(2)^*$, (7) is consistent and has a unique solution. Then each individual has the same consumption opportunity set in each state as with contingent contracts. It remains to show that all budget constraints are satisfied and all markets clear with the same spot market prices. Let $F^* = \sum_{w=1}^{2} P_y(w)^* / \sum_{w=1}^{2} P_x(w)^*$. Then from (7)

$$P_y(1)^* Y^i(1)^* + P_x(1)^* X^i(1)^* = P_y(1)^* Y_f^{i*} + P_x(1)^* X_f^{i*}$$
$$P_y(2)^* Y^i(2)^* + P_x(2)^* X^i(2)^* = P_y(2)^* Y_f^{i*} + P_x(2)^* X_f^{i*} \qquad (8)$$

By summing equation (8)

$$F^* Y_f^{i*} + X_f^{i*} = 0$$

and the budget constaint of individual i for unconditional contracts is satisfied.

Given equilibrium in contingent markets

$$\sum_{i=1}^{m} [X^i(w)^* + P(w)^* Y^i(w)^*] = 0 \qquad w \epsilon \Omega.$$

By construction

$$P(w)^* Y^i(w)^* + X^i(w)^* = Y_f^{i*}[P(w)^* - F^*] \qquad w \epsilon \Omega.$$

Then $\sum_{i=1}^{m} Y_f^{i*} [P(w)^* - F^*] = 0$ for each $w \epsilon \Omega$.

By hypothesis $[P(w)^* - F^*] \neq 0$ for every $w \epsilon \Omega$. Then $\sum_{i=1}^{m} Y_f^{i*} = 0$ and unconditional markets are in equilibrium. This leads immediately to Proposition IV.

Proposition IV. Given the hypotheses of Proposition III, the outcome in the model restricted exogenously to unconditional contracts and subsequent spot markets is Pareto optimal.

Proof:

The outcome with complete contingent markets and no active spot markets is Pareto optimal, and under the stated conditions each individual can acquire the same consumption bundles. Q.E.D.

It is necessary that spot market prices differ across states in order that the returns from unconditional contracts be random, a randomness which is used to offset the randomness of exogenous endowments. This has strong implications for a second best policy by government in the absence of contingent markets. An effort in the name of 'stability' to fix spot market prices may make allocations with unconditional contracts Pareto non-optimal.

If $P(1)^* = P(2)^*$, then equations (7) are consistent if and only if in the initial equilibrium there is no trading of claims across states. In this special case, an exogenous restriction to trading in spot markets only would not affect the optimality of the equilibrium.

Proposition IV suggests that if there are significant transactions costs associated with contingent contracts, then, in a two state world, unconditional contracts may emerge endogenously. In this respect, it is unfortunate that an extension of Proposition IV does not hold if there are more states than two. It is instructive to examine an n-state world. The analog of (7) is

$$\begin{bmatrix} 1 & P(1)^* \\ 1 & P(2)^* \\ \vdots & \vdots \\ 1 & P(n)^* \end{bmatrix} \begin{bmatrix} X_f^{i*} \\ Y_f^{i*} \end{bmatrix} \quad \begin{bmatrix} X^i(1)^* + P(1)^* Y^i(1)^* \\ X^i(2)^* + P(2)^* Y^i(2)^* \\ X^i(n)^* + P(n)^* Y^i(n)^* \end{bmatrix}$$

In general, such systems will be inconsistent.

It may suffice, however, to have n distinct types of claims, each of which is conditioned on several states. In this way, the need to verify the occurrence of states may be minimized while achieving optimal allocations. However, such contracts may require that spot market prices vary across states upon which contracts are conditioned.

The above results suggest that with more states than two, the outcome of the model restricted to spot markets and unconditional contracts is Pareto non-optimal. To pursue this, the following lemma is needed.

Lemma 6. Suppose in an n-state world there exists an equilibrium in the model restricted to unconditional contracts and subsequent spot markets. In general, for more states than two, all individuals will not have the same rates of commodity substitution across states.

Proof:

The case n = 3 will be considered. Let the superscript * denote prices of the initial equilibrium. Suppose the conclusion of the lemma is false, and let R' denote the rate of commodity substitution of (X) in state one for (Y) in state two, and let R" denote the rate of substitution of (X) in state one for (Y) in state three. Then it is sufficient to show that in general the following equations are inconsistent.

$$R' = \frac{g(1)V_1^i\{r(1)'^{*}Z^i(1) + Y_f^i[P(1)^{*}-F^{*}],\ P(1)^{*}\}}{g(2)P(2)^{*}V_1^i\{r(2)'^{*}Z^i(2) + Y_f^i[P(2)^{*}-F^{*},\ P(2)^{*}\}}$$

$$R'' = \frac{g(1)V_1^i\{r(1)'^{*}Z^i(2) + Y_f^i[P(1)^{*}-F^{*}],\ P(1)^{*}\}}{g(3)P(3)^{*}V_1^i\{r(3)'^{*}Z^i(3) + Y_f^i[P(3)^{*}-F^{*}],\ P(3)^{*}\}}$$

Recall that $V_1^i[\cdot,\ P(w)]$ is continuous with respect to Y_f^i, and $V_1^i[0,\ P(w)] = \infty$. Inconsistency follows from the bankruptcy constraints for unconditional contracts and the intermediate value theorem; in general there is no Y_f^i which solves both equations. Extensions for $n > 3$ follow similarly. Q.E.D.

Proposition V.[2/] With more states than two, the outcome of the model restricted exogenously to unconditional contracts and subsequent spot markets is in general Pareto non-optimal. There exists in general an allocation which is Pareto superior and which could be supported, with a state-dependent income redistribution program, by a competitive equilibrium in contingent commodity markets.

Proof:

The necessary and sufficient conditions for Pareto-optimal allocations can be derived in the usual way by the maximization of the expected utility of one individual subject to fixed expected utility levels of the other m-1 individuals. These conditions state that the rates of substitution of contingent commodities be equal for all individuals. In general, the rates of commodity substitution across states are not equal in the initial equilibrium. This establishes that the given equilibrium is not Pareto optimal.

Now fix the expected utilities of all but one of the m-1 individuals at the levels of the initial equilibrium. Then it is possible to find an allocation which increases the expected utility of the remaining individual. By construction that allocation is optimal and could be supported by a competitive equilibrium in contingent markets. Let the government be committed to the state dependent confiscations and transfers necessary to achieve that allocation. Q.E.D.

Proposition V suggests that in the absence of complete markets it may be optimal to have a compulsory insurance program. Yet the ability of a government to carry out the state dependent transfers associated with this program may depend on its ability to distinguish among realized states. This will be pursued in Section V.

On the assumption that tastes are identical and homothetic, it is possible to make sharper welfare comparisons. If tastes are identical and homothetic, then spot market prices are independent of the direction and type of forward contracts. If there are active forward markets, and if an individual chooses not to participate in such markets, then his consumption possibility set is precisely what it would have been had there been no forward markets at all. In this sense, the possibility of forward transactions can only make him better off.

<u>Proposition VI</u>. If tastes are identical and homothetic, the outcome in the model restricted to unconditional forward contracts with subsequent spot markets is Pareto non-inferior and possibly Pareto superior to the outcome with all forward markets prohibited.

V. Conclusions and Suggestions for Further Research

The model suggests that in some circumstances a distinction between speculators and hedgers may be misleading. Though such a distinction may be appropriate in a model in which participants differ with respect to attitudes toward risk or access to information, active forward markets do not require such a distinction.

An important welfare proposition of the paper is that the competitive equilibrium allocation of the model restricted exogenously to unconditional contracts with subsequent spot markets is in general Pareto non-optimal. This leads one to ponder again the dearth of contingent contracts. One explanation may be the inability of agents to distinguish among states. In the model, offsetting movements of endowments among individuals may leave spot price unchanged among states. Suppose individual i has made a contract to deliver one unit of (X) to individual j contingent on state w. Individual j could always claim that state w had occurred, and if individual i were unable to verify such through spot prices or otherwise, such a contract would not be made. As has been noted by Arrow [1], among others, this is the problem of moral hazard.

This suggests that with contingent contracts there may be associated particular state verification schemes. In some cases the costs of state verification may be large relative to the additional benefits of contingent contracts, and contingent contracts will not emerge. In contrast, unconditional futures contracts, though providing some of the benefits of contingent contracts, do not require state verification. This may explain their preponderance.

It should be stressed that in the paper the choice of market structure is not endogenous. On the one hand, the welfare analysis often takes as given an equilibrium with complete contingent markets. Yet, as was noted above, state verification may be a problem. On the other hand, the model is at times restricted to unconditional contracts even though spot prices may reveal to all individuals the state of the world. The exogenous imposition of market structure has been a useful and at times essential analytic technique in explaining the workings and welfare implications of unconditional contracts. Yet, in principle, coherent economic models should make the choice of market structure endogenous. This is being pursued in further research.

In the absence of complete contingent markets there may be an optimal policy for government. One such policy might be to make revelation of information mandatory. Dissemination of information may encourage the development of more contingent markets. Yet, if state revelation by government requires resources, private markets may already indicate the efficient solution. A similar qualification applies to compulsory government insurance programs. The model also suggests that a concern with price variability may not be justified. An effort to fix spot market prices may diminish the extent to which incomplete forward markets can compensate for the absence of complete markets.

FOOTNOTES

1. Hieronymus [4] cites a U.S.D.A., C.E.A. report on the holding of corn futures on January 27, 1967 which reveals that farmers held more than one-third of the net long speculative position and were also hedging in the wrong direction.
2. Steve Salant has been particularly helpful with the proposition.

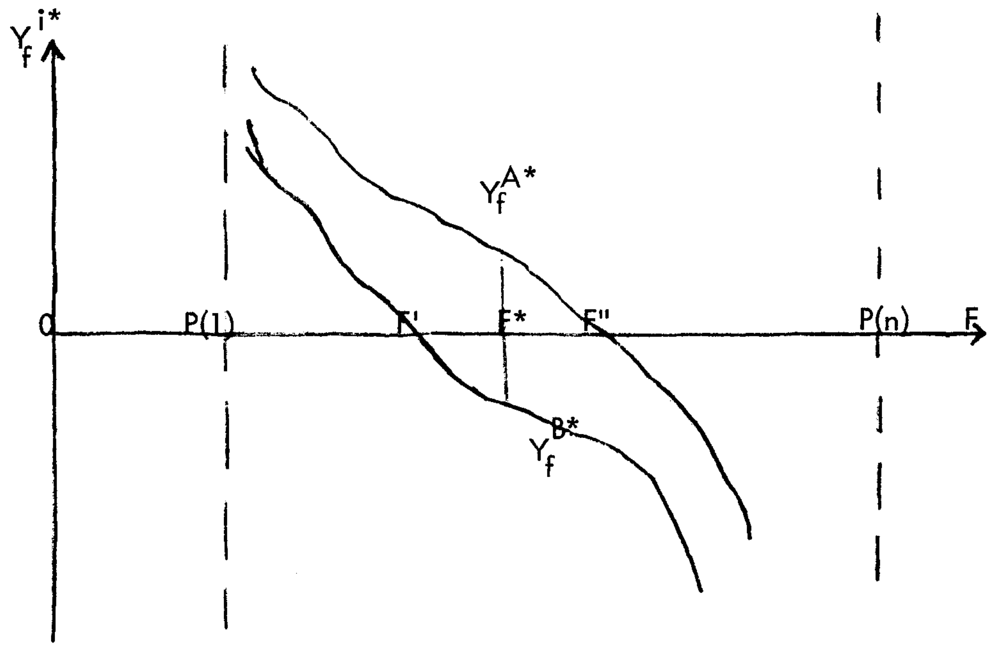

Figure 1

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