EXPECTED UTILITY, INFINITE HORIZON AND JOB SEARCH

by

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Discussion Paper No. 74-42, June 1974

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In my paper "Expected Utility, Mandatory Retirement and Job Search" [3] an optimal search strategy is characterized for an unemployed expected utility maximizer. As is evident from the title, the individual considered is allowed to participate in the labor force for only a finite number of periods. The relation between the individual's attitudes toward risks and particular aspects of the job search strategy is investigated in some detail.

The objective of the paper was to demonstrate that important differences exist between the optimal search strategies of expected income maximizers and those of expected utility maximizers. In order to realize that objective economically I chose to alter the assumptions relating to objective function of some existing models of job search and to re-solve the intertemporal optimization problem. Essentially the procedure may be described as placing a new individual in an old environment and watching him do his thing.

In the next section of this paper I use the same basic technique to compare search strategies for expected income and expected utility maximizers when the work horizon is infinite. Consumption allocation as well as job acceptance or rejection decisions must be considered simultaneously by the expected utility maximizer. I find that the way these decisions are made in the infinite horizon formulation is strikingly similar to the way they are made in the finite horizon model.

While the economics of the infinite horizon do not in this case differ significantly from those encountered in the finite horizon one would be incorrect in portraying the shift from finite to infinite horizon as a purely technical exercise. Certain dynamic aspects of the search strategy may be highlighted by approximating a finite but long work life with an infinite one. When the unemployed individual knows he must retire in the foreseeable future the passage of time is accompanied not only by gains in information and wealth depletion, but also by a reduction in feasible career length. By allowing one to ignore reductions in career length, the infinite horizon model facilitates analysis of the influence of asset depletion and learning on an individual's search strategy.

Of course, one would not expect asset depletion to have any effect on an expected present value of income maximizing strategy. If at the same time learning does not take place (independently distributed wage offers) and the horizon is eliminated, then there is no reason to suspect any change in the individual's strategy due to the passage of time. The validity of this suspicion has been demonstrated by Gordon and Hynes [5] and McCall [8].

In [3] I show that asset holdings may well effect the strategy chosen by an unemployed expected utility maximizer. I prove that certain reasonable assumptions on the individual's attitudes toward risk imply that more and more job offers become acceptable as asset holdings are reduced.[1] If this result carries over to the infinite horizon model one might expect intertemporal asset depletion to be associated with declining wage aspirations.

In the last section of the present paper I show that this is indeed the case when the labor force participant's environment is completely stationary. The term stationarity as used here implies that prices and interest rates are constant over time, wage offers are generated as a purely random process and the preference ordering of infinite consumption sequences is unchanged with the passage of time. The only thing which changes from one period to the next in such an environment is the individual's endogenously determined wealth. The individual's readiness to accept lower income jobs increases in this model solely in response to a falling bank balance.

This result helps to resolve certain important inconsistencies between observed reservation wage dynamics and the predictions of existing models of the job search process. For instance, there exists extensive empirical evidence that individual reservation wages do fall over time.[2] Heretofore this strategy characteristic has been obtained solely as the result of a mandatory deadline being set for the termination of the search activity.[3] The data reveals, however, a relationship between the rate of decline of the reservation wage and age (alternatively, years until retirement) which contradicts the hypothesis that an approaching horizon is the explanation for declining wage aspirations.[4]

Job Search, An Infinite Horizon Strategy

The following definitions and assumptions are employed in the characterization of an infinitely long lived individual's search strategy.

Definitions

$c(t)$ is a composite consumption good for period t ;

$p(t)$ is the certain price in period t of a unit of $c(t)$ measured in dollars;

$r(t)$ is the certain one period rate of interest for borrowing and lending applicable to period t ;

$A(t)$ is the dollar value of physical assets held at the beginning of period t ;

t denotes a point in time or a time interval with end points t , $t+1$. The specific context will determine which case is applicable;

$\Gamma(\cdot) = \{y(0,\cdot), y(1,\cdot), \ldots\}$ is a random process which is well defined and measurable on the probability space (Ω, F, Q) ;

$y(t,\omega)$ is the value taken on by the random variable $y(t,\cdot) = y_{t+1}(t,\cdot), y_{t+2}(t,\cdot), \ldots)$ when the the state of the world is $\omega \in \Omega$;

$y^{*}(t)$ is the wage stream offer actually observed in period t ;

$\gamma(t)$ is the observed current event as of date t , $(y^{*}(0), y^{*}(1), \ldots, y^{*}(t-1))$;

$$Y_t^s(\gamma(t)) = \max_{i \,\in \{t-s, t-s+1, \ldots, t-1\}} \left(y^{*}_{i+1}(i) + \sum_{j=2}^{\infty} \frac{y^{*}_{i+j}(i)}{\prod_{K=t}^{t+j-2} (1+r(K))} \right)$$

is the highest present value of income available to the individual given that he can accept any single wage stream offer received in the last s periods;

$G^t(\bar{\gamma}(t),B) \equiv Q\{\omega:y(t,\omega) \in B|\bar{\gamma}(t)\}$

$dG^t(\gamma(t),b) \equiv Q\{\omega:y(t,\omega) \in [b, b + db) \mid \bar{\gamma}(t)\}$

$b(t)$ is the dollar magnitude of unemployment benefits in period t ;

$s(t)$ is the dollar magnitude of search costs in period t ;

$V(c(0),c(1),\ldots)$ is a function representing the consumers' preferences over consumption-leisure vectors. This function is denoted as the individual's utility function.

Assumptions

A1) If the individual has not accepted a job prior to date $t = 0, 1, \ldots,$ then $c(t)$ is chosen prior to observing the value of the random variable $y(t,\cdot)$.

A2) If the individual accepts an offer during period $t = 0, 1, \ldots,$ labor income commences in period $t + 1$ and continues forever. No observations on the random variables $y(t+1,\cdot), y(t+2,\cdot), \ldots,$ are obtained subsequent to job acceptance.

A3) If the individual is unemployed at date $t = 0, 1, \ldots,$ $A(t+1) = [A(t)-s(t)+b(t)-p(t)c(t)](1+r(t))$. If the individual has accepted the wage stream offer $y^*(t-j)$ during period $(t-i), i \leqslant j$, then

$$A(t+1) = [A(t) + y_{t-j+i+1}(t-j)-p(t)c(t)](1+r(t)) .$$

A4) The range of $\gamma(t)$ is denumerable and $Y_t^s(\gamma(t))$ is bounded, $t=0,1,\ldots$.

A5) $V(c(0),c(1),\ldots) = \sum_{i=0}^{\infty} u_i(c(i))$. $u_i(0) = 0$, $u_i'(c(i)) > 0$, $u_i(c(i))$ is strictly concave, and

$$|u_i(c(i))| < K, \; i=0, 1, \ldots, c(i)\in[0,\infty) .$$

Also, for any $\epsilon>0$ there exists an $n(\epsilon)$ such that

$$\left| \sum_{i=n(\epsilon)}^{\infty} u_i(c'(i))-u_i(c^*(i))\right| < \epsilon$$

for any $c'(i) \geqslant 0$ and $c^*(i) \geqslant 0$, $i = n(\epsilon),n(\epsilon)+1,\ldots$

A6) A plan which specifies a particular value for $c(j)$ and employment status as of date j for each $\gamma(j)\in\Psi(0)\times\ldots\times\Psi(j-1)$, $j=t,t+1,\ldots,$ is referred to as a strategy. Such a strategy is feasible if and only if corresponding to each $\gamma(j)$, $j=t,t+1,\ldots$:

a) $c(j) \geq 0$

b) $$A(j) \geq -[b(j)-s(j) + \sum_{i=j+1}^{\infty} \frac{b(i)-s(i)}{\prod_{k=j}^{i-1}(1+r(k))}] \equiv B_j$$

if the individual is unemployed at date j, and

c) $$A(j) \geq -[y_{j-m+k}(j-m) + \sum_{i=j-m+k+1}^{\infty} \frac{y_i(j-m)}{\prod_{n=j}^{1+m-k-1}(1+r(n))}]$$

if wage offer $y(j-m)$ has been accepted in period $j-k$, $k \geq m$.

Using these definitions and assumptions a description of the individual's choice problem as of an arbitrary date $t \geq 0$ may be given. I shall be primarily interested in the behavior of an unemployed individual and hence I shall assume that employment has not been accepted prior to date t . This labor force participant may be characterized by his current asset endowments, $A(t) \geq B_t$, and his previously observed wage offers, $\gamma(t)\epsilon\Psi(0)\times\ldots\times\Psi(t-1)$. Associated with these initial conditions is a non-empty set of feasible strategies available to him.

For each feasible strategy future consumption levels are well defined random variables on the probability space $(\Omega,\mathfrak{F},Q)$. Since V is defined on the sequence of single period consumption levels, utility becomes a random variable on this same probability space when a particular strategy is specified.

<u>Postulate</u>: In each period the individual ranks alternative random consumption sequences according to the expected value

of utility each of them would provide. The individual acts as if he were attempting to choose the feasible strategy which yields the highest expected utility.

The procedure I employ to derive an optimal strategy within this framework is a variant of Bellman's "method of successive approximations" discussed in [2]. First, a sequence of objective functions is specified which converges to V. For each element of this sequence a strategy which maximizes its expected value may then be determined. Finally, the resulting sequence of maximizing strategies and expected values is used to characterize a strategy which maximizes the expected value of V.

The sequence of functions alluded to in the preceding paragraph is the sequence of finite truncations of V, $\{V^{(q)}\}$.

<u>Definition</u> The function $V^{(N)}:\overline{R^{1+}}x\overline{R^{1+}}x\ldots\rightarrow R^1$ with

$$V^{(N)}(c(0),c(1),\ldots) = \sum_{i=0}^{N} u_i(c(i)),\ 0 \leq c(i),\ i=0,\ 1,\ \ldots,$$

is referred to as "the Nth truncation of V."

This particular sequence of functions is utilized because each of its elements belongs to a class of utility functions for which the analysis of expected utility maximizing job search strategies is an accomplished fact. Within an environment formally identical to the one advanced here I have derived an optimal search strategy for an individual whose preferences can be represented by a finite truncation of V.[5]

The following two definitions and four lemmas have been extracted from [3] and are presented here for easy reference.

<u>Definition 1)</u> The function $E_{j,N}:R^1\rightarrow R^1$ is defined for $x\epsilon[0,\infty)$ by

$$E_{j,N}(x) = \max_{c(j),\ldots,c(N)} \sum_{i=j}^{N} u_i(c(i))$$

subject to: $p(j)c(j) + \sum_{i=j+1}^{N} \frac{p(i)c(i)}{\prod_{k=j}^{i-1}(1+r(k))} \leq x$,

and $c(i) \geq 0$, $i=j, \ldots, N$.

For $x\varepsilon(-\infty,0)$

$$E_{j,N}(x) = -1 .$$

Lemma 1) $E_{j,N}(\cdot)$ is continuously differentiable and strictly concave on $[0,\infty)$.

Proof: This result is a special case of a result stated and proved in [3] as Lemma 2//

Definition 2) The function $S_{j,N}:[B_j,\infty)x\Psi(0)x\ldots x\Psi(j-1)\to R^1$ is defined for $j < N$ by

$$S_{j,N}(A(j),\gamma(j)) = \max_{c(j),A(j+1)} u_j(c(j))$$

$$+\int_{\Psi(j)} \max[E_{j+1,N}(A(j+1)+Y^s_{j+1}(\gamma(j),v)),S_{j+1,N}(A(j+1),(\gamma(j),v))]dG^j(\gamma(j),v)$$

subject to: $c(j) \geq 0$, $B_{j+1} \leq A(j+1) = [A(j)+b(j)-s(j)-p(j)c(j)](1+r(j))$.

For $j=N$,

$$S_{N,N}(A(N),\gamma(N)) = \max_{c(N)} u_N(C(N))$$

subject to: $p(N)c(N) \leq A(N)$.

Lemma 2) $S_{j,N}(\cdot,\gamma(j))$ is continuous and increasing on $[B_j,\infty)$. $S_{j,N}(A(j),\cdot)$ is bounded on $\Psi(0)x\ldots x\Psi(j-1)$ and is measurable with respect to Q.

Proof: This lemma is a special case of results stated and proved in [3] as Lemma 6//

Due to the additive form of the V and $V^{(N)}$ functions, previous levels of consumption, $\overline{c(0)}, \ldots, \overline{c(j-1)}$, will play no role in the individual's decisions as of date j .

I have, therefore, chosen to eliminate references to previous consumption levels in the following analysis. I use the phrase "the expected value of V ($V^{(N)}$) from date j forward" to indicate the expected value of V ($V^{(N)}$) minus $\sum_{i=0}^{j-1} u_i(\overline{c(i)})$. It should be immediately apparent that if a particular strategy maximizes the expected value of V or $V^{(N)}$ from date j forward it also maximizes the expected value of V or $V^{(N)}$ for any previously determined $\overline{c(0)},\ldots,\overline{c(j}-1)$.

The next two results suggest an interpretation for the functions defined and characterized above in terms of the expected value of $V^{(N)}$ from date j forward. Current asset endowment and past wage offer observations will be denoted simply as $A(j)$ and $\gamma(j)$ respectively.

<u>Lemma 3</u>) If $Y_j^s(\gamma(j)) + A(j) \geq 0$ the individual may accept employment commencing at date j. Should this option be chosen, the maximum attainable value of $V^{(N)}$ from date j forward equals

$$E_{j,N}(A(j) + Y_j^s(\gamma(j))) .$$

Proof: This result is contained on page 22 of [ ]//

<u>Lemma 4</u>) If the individual does not accept employment on or before date j, the maximum attainable expected value of $V^{(N)}$ from date j forward equals

$$S_{j,N}(A(j),\gamma(j))$$

as of date j.

Proof: This is a special case of a result stated and proved in [3] on pages 24-29//

Two additional results follow immediately from the particular form of the successive truncations of V .

Lemma 5) For any $A(j) \geq B_j$ and $\gamma(j) \in \Psi(0) \times \ldots \times \Psi(j-1)$,

$$S_{j,N+1}(A(j),\gamma(j)) \geq S_{j,N}(A(j),\gamma(j)) \text{ , and}$$

$$E_{j,N+1}(A(j)+Y_j^s(\gamma(j)) \geq E_{j,N}(A(j)+Y_j^s(\gamma(j))) \text{ .}$$

Proof: Since $u_i(0) = 0$, $i=0, 1, \ldots$ and $c(N+1) = 0$ is feasible whether or not employment has been accepted as of date $N+1$, the inequalities are clearly satisfied//

Lemma 6) $\lim_{N\to\infty} S_{j,N}(A(j),\gamma(j))$ and

$$\lim_{N\to\infty} E_{j,N}(A(j)+Y_j^s(\gamma(j)))$$

exist and are finite for any $A(j) \geq B_j$ and $\gamma(j) \in \Psi(0) \times \ldots \times \Psi(j-1)$.

Proof: The sequences $\{S_{j,N}(A(j),\gamma(j))\}$ and $\{E_{j,N}(A(j)+Y_j^s(\gamma(j)))\}$ are monotonic (Lemma 5) and are uniformly bounded (A5)//

I shall denote the limiting functions indicated in Lemma 6 as

$$S_{j,*}(A(j),\gamma(j)) \quad \text{and} \quad E_{j,*}(A(j)+Y_j^s(\gamma(j)))$$

respectively.

Lemma 7) If the individual has not accepted employment on or before date j , has current asset holdings $A(j) \geq B_j$ and past wage offer observations $\gamma(j)$, then the maximum attainable expected value of V from date j forward does not exceed

$$S_{j,*}(A(j),\gamma(j))$$

as of date j .

Proof: (By contradiction) Assume there exists a feasible strategy for which the expected value of V from date j forward equals $Z > S_{j,*}(A(j),\gamma(j))$ as of date j . Then,

$$Z-S_{j,*}(A(j),\gamma(j)) = \epsilon > 0$$

and there exists, by A5, an $n(\epsilon/2)$ with

$$\sum_{i=n(\epsilon/2)}^{\infty} u_i(c(i)) < \epsilon/2$$

for any $c(i) \geqslant 0$, $i = n(\epsilon/2), \ldots$. Since the strategy associated with Z is feasible, the expected value of $V^{(n(\epsilon/2))}$ from date j forward when this strategy is employed must be less than or equal to $S_{j,n(\epsilon/2)}(A(j),\gamma(j))$ by Lemma 4. Thus,

$$Z < S_{j,n(\epsilon/2)}(A(j),\gamma(j)) + \epsilon/2$$

By Lemma 5

$$S_{j,*}(A(j),\gamma(j)) \geqslant S_{j,n(\epsilon/2)}(A(j),\gamma(j))$$

hence,

$$\epsilon/2 > Z-S_{j,*}(A(j),\gamma(j)) = \epsilon > 0 .$$

This last inequality is clearly impossible and the proof is complete//

Lemma 8) If $Y_j^s(\gamma(j)) + A(j) \geqslant 0$ the individual may accept employment commencing at date j . Should this option be chosen, the maximum attainable value of V from date j forward does not exceed

$$E_{j,*}(A(j) + Y_j^s(\gamma(j))) .$$

Proof: The proof precisely parallels the proof given for Lemma 7//

Lemmas 7 and 8 establish $S_{j,*}(A(j),\gamma(j))$ and $E_{j,*}(A(j) + Y_j^s(\gamma(j)))$ as upper bounds for the expected value of V from date j forward if the individual is unemployed or employed respectively with current assets $A(j)$ and wage offer observations $\gamma(j)$. These values are in fact least upper bounds since we can always determine feasible strategies for which the expected value of V from date j forward is equal to $S_{j,N}(A(j),\gamma(j))$ and/or $E_{j,N}(A(j) + Y_j^s(\gamma(j)))$ for any $N \geq j$, and these values converge to $S_{j,*}(A(j),\gamma(j))$ and $E_{j,*}(A(j) + Y_j^s(\gamma(j)))$ as $N\to\infty$. The next two results indicate that the characteristics of the $S_{j,N}$ and $E_{j,N}$ functions delineated in Lemmas 1 and 2 are shared by the least upper bound functions, $S_{j,*}$ and $E_{j,*}$.

<u>Lemma 9</u>) $S_{j,*}(\cdot,\gamma(j))$ is a continuous and increasing function on $[B_j,\infty)$, and $S_{j,*}(A(j),\cdot)$ is measurable with respect to Q on $\Psi(0)\times\ldots\times\Psi(j-1)$.

Proof: $S_{j,N}(\cdot,\gamma(j))$ is continuous and increasing for all $N \geq j$. The sequence $\{S_{j,N}(\cdot,\gamma(j))\}$ converges uniformly to $S_{j,*}(\cdot,\gamma(j))$ since for any $\varepsilon > 0$ there exists an $n(\varepsilon)$ such that

$$\sum_{i=n(\varepsilon)}^{\infty} u_i(c(i)) < \varepsilon \quad \text{for any} \quad c(i) \geq 0, \ i=n(\varepsilon), \ldots$$

Thus,

$$|S_{j,*}(A(j),\gamma(j))-S_{j,q}(A(j),\gamma(j))| < \epsilon$$

for all $q \geq n(\epsilon)$ and any $A(j) \geq B_j$ by A5. The limit of a uniformly convergent sequence of continuous functions is continuous. Also, the limit of a sequence of increasing functions is nondecreasing. This may be strengthened to increasing since $u_i'(c(i)) > 0$ on $[0,\infty)$ for all $i=j, j+1, \ldots$.

Finally, Lemma 2 insures that $S_{j,N}(A(j),\cdot)$ is measurable on $\mathcal{Y}(0)x\ldots x\mathcal{Y}(j-1)$ with respect to Q for all $N \geq j$. The limit of a convergent sequence of measurable functions is measurable//

<u>Lemma 10</u>) $E_{j,*}(\cdot)$ is a continuous and increasing function on $[0,\infty)$.

Proof: $E_{j,q} \to E_{j,*}$ uniformly since $V^{(q)} \to V$ uniformly. Therefore, since $E_{j,q}(\cdot)$ is continuous and increasing, $q = j, j+1, \ldots,$ $E_{j,*}(\cdot)$ is continuous and nondecreasing. $E_{j,*}(\cdot)$ is in fact strictly increasing since $u_j'(c(j)) > 0$ for $c(j) \epsilon [0,\infty)$//

Two new functions are introduced here in order to simplify some subsequent expressions.

$$f_{j,N}(A(j),\gamma(j)) = \max[E_{j,N}(A(j)+Y_j^s(\gamma(j))) , \quad S_{j,N}(A(j),\gamma(j))] ,$$

and

$$f_{j,*}(A(j),\gamma(j)) = \max[E_{j,*}(A(j)+Y_j^s(\gamma(j))) , \quad S_{j,*}(A(j),\gamma(j))] .$$

Both $f_{j,N}$ and $f_{j,*}$ map

$$[B_j,\infty)x\mathcal{Y}(0)x\ldots x\mathcal{Y}(j) \text{ into } R^1 .$$

<u>Lemma 11</u>) $f_{j,N}(\cdot,\gamma(j))$ and $f_{j,*}(\cdot,\gamma(j))$ are continuous and increasing functions on $[B_j,\infty)$, $f_{j,N}(A(j),\cdot)$ and $f_{j,*}(A(j),\cdot)$ are measurable with respect to Q on $\Psi(0)\times\ldots\times\Psi(j-1)$, and $f_{j,q}\to f_{j,*}$ uniformly as $q\to\infty$.

Proof: If $B_j \geq 0$, both $f_{j,N}(\cdot,\gamma(j))$ and $f_{j,*}(\cdot,\gamma(j))$ are maxima of continuous and increasing functions and are themselves continuous and increasing. If $B_j < 0$ and $Y_j^s(\gamma(j)) + B_j < 0$,

$$f_{j,N}(\cdot,\gamma(j)) = S_{j,N}(\cdot,\gamma(j)) \quad \text{and}$$

$$f_{j,*}(\cdot,\gamma(j)) = S_{j,*}(\cdot,\gamma(j)) \quad \text{on} \quad [B_j, Y_j^s(\gamma(j)) + B_j] .$$

On $[Y_j^s(\gamma(j)) + B_j,\infty)$ $f_{j,N}(\cdot,\gamma(j))$ and $f_{j,*}(\cdot,\gamma(j))$ are the maxima of continuous and increasing functions. Thus, $f_{j,N}(\cdot,\gamma(j))$ and $f_{j,*}(\cdot,\gamma(j))$ are continuous and increasing on the entire interval $[B_j,\infty)$. Also, $f_{j,N}(A(j),\cdot)$ and $f_{j,*}(A(j),\cdot)$ are the maxima of measurable functions and are therefore measurable.

Finally, since $E_{j,q}$ and $S_{j,q}$ converge uniformly to $E_{j,*}$ and $S_{j,*}$ respectively, $f_{j,q}\to f_{j,*}$ uniformly as $q\to\infty$//

The expression for $S_{j,N}(A(j),\gamma(j))$ found in Definition 2 may now be given a more convenient form.

$$S_{j,N}(A(j),\gamma(j)) = \max_{c(j),A(j+1)} u_j(c(j)) + \int_{\Psi(j)} f_{j+1,N}(A(j+1),(\gamma(j),v))dG^j\ (\gamma(j),v)$$

subject to: $c(j) \geq 0$ and

$$B_{j+1} \leq A(j+1) = [A(j)+b(j)-s(j)-p(j)c(j)](1+r(j)) .$$

Since $f_{j+1,N}(\cdot,\gamma(t+1))$ is not necessarily concave there may be several feasible $(c(j),A(j+1))$ pairs which yield a maximum value for the right-hand side of the above expression. I denote the set of all such maximizing pairs as $M_{j,N}(A(j),\gamma(j))$.

As N takes on successively larger values a sequence of sets of maximizers may be obtained. These sets must be contained in a common compact set (a closed rectangle) due to the form of the budget constraint. Thus if one chooses a unique $(c(j),A(j+1))$ pair from each $M_{j,q}(A(j),\gamma(j))$, $q = j+1,\ j+2,\ \ldots$, a sequence is generated which must possess a convergent subsequence. Denoting one such convergent subsequence as $\{(c^{q'}(j),A^{q'}(j+1))\}$ and the corresponding subsequential limit as $(c^*(j),A^*(j+1))$, we have the following important lemma.

Lemma 12)

$$S_{j,*}(A(j),\gamma(j)) = u_j(c^*(j)) + \int_{\Psi(j)} f_{j+1,*}(A^*(j+1),(\gamma(j),v))dG^j(\gamma(j),v)$$

$$= \max_{c(j),A(j+1)} u_j(c(j)) + \int_{\Psi(j)} f_{j+1,*}(A(j+1),(\gamma(j),v))dG^j(\gamma(j),v)$$

subject to: $c(j) \geqslant 0$ and

$$B_{j+1} \leqslant A(j+1) = [A(j)+b(j)-s(j)-p(j)c(j)](1+r(j)) .$$

Proof: First, observe that

$$f_{j+1,q'}(A^{q'}(j+1),(\gamma(j),y(j))) \to f_{j+1,*}(A^*(j+1),(\gamma(j),y(j)))$$

for all $y(j)\varepsilon\Psi(j)$ since $f_{j+1,q'} \to f_{j+1,*}$ uniformly and $A^{q'}(j+1) \to A^*(j+1)$ as $q' \to \infty$. Also, due to the boundedness of V the sequence

$$\{f_{j+1,q'}(\cdot,(\gamma(j),y(j)))\} \text{ is uniformly}$$

bounded on $\Psi(j)$. The Lebesque Dominated Convergence Theorem may thus be applied to obtain,

$$\lim_{q'\to\infty} \int_{\Psi(j)} f_{j+1,q'}(A^{q'}(j+1),(\gamma(j),v))dG^j(\gamma(j),v)$$

$$= \int_{\Psi(j)} f_{j+1,*}(A^*(j+1),(\gamma(j),v))dG^j(\gamma(j),v) .$$

Since $S_{j,q'}(A(j),\gamma(j)) \to S_{j,*}(A(j),\gamma(j))$ the first equality is established.

The second equality would fail to hold only if there existed a $c'(j) \geq 0$ and $A'(j+1) = [A(j)+b(j)-s(j)-p(j)c'(j)](1+r(j)) \geq B_{j+1}$ with

$$S_{j,*}(A(j),\gamma(j)) < u_j(c'(j)) + \int_{\Psi(j)} f_{j+1,*}(A'(j+1),(\gamma(j),v))dG^j(\gamma(j),v) .$$

This inequality is clearly impossible, however, since the right-hand side of this inequality is the limit of the sequence

$$\{u_j(c'(j)) + \int_{\Psi(j)} f_{j+1,n}(A'(j+1),(\gamma(j),v))dG^j(\gamma(j),v)\} \; n=j+1,j+2,\ldots$$

which is uniformly bounded from above by $S_{j,*}(A(j),\gamma(j))$//

The reader will quite naturally wonder if Lemma 12 tells us anything about the unemployed labor force participant's behavior. For instance, may we conclude that $c^*(j)$ is the optimal choice for current consumption? The following lemma indicates that, while one cannot be certain that $c^*(j)$ is a unique maximizer, the individual will confine his choice to a particular set of which $c^*(j)$ is an element.

Definition: $M_j^*(A(j),\gamma(j))$ is the set of feasible $(c(j),A(j+1))$ for which

$$S_{j,*}(A(j),\gamma(j)) = u_j(c(j)) + \int_{\Psi(j)} f_{j+1,*}(A(j+1),(\gamma(j),v))dG^j(\gamma(j),v) .$$

Lemma 13) If the individual is unemployed as of date j with asset holdings $A(j) \geqslant B_j$ and previous wage offer observations $\gamma(j) \epsilon \Psi(0) x \ldots x \Psi(j-1)$, he will consume $\overline{c(j)}$ during period j only if $(\overline{c(j)},\overline{A}(j+1)) \epsilon M_j^*(A(j),\gamma(j))$.

Proof: If the individual were to choose some feasible $(c'(j),A'(j+1)) \notin M_j^*(A(j),\gamma(j))$, then the maximum expected value of V from date j forward would be bounded from above by

$$u_j(c'(j)) + \int_{\Psi(j)} f_{j+1,*}(A'(j+1),(\gamma(j),v))dG^j(\gamma(j),v) < S_{j,*}(A(j),\gamma(j)) .$$

For any $(c(j),A(j+1)) \epsilon M_j^*(A(j),\gamma(j))$ the least upper bound for the expected value of V from date j forward is $S_{j,*}(A(j),\gamma(j))$//

After choosing his current level of consumption the unemployed individual is free to sample from the wage offer distribution. Once this sample has been taken he must decide whether or not to become employed in the following period. He should, it seems, accept his best available job offer if the utility from doing so exceeds the expected utility associated with remaining unemployed. Likewise, if the expected utility from continuing job search exceeds that associated with accepting any of his available offers, one would expect the individual to remain unemployed. This decision rule is stated formally in the next lemma with the aid of the following two definitions.

Definition:

$$G_j^*(A(j+1),\gamma(j)) = \{y(j)\varepsilon\mathcal{Y}(j):E_{j+1,*}(A(j+1)+Y^s_{j+1}(\gamma(j),y(j))) > S_{j+1,*}(A(j+1),(\gamma(j),y(j)))\}$$

Definition: $\overline{G_j^*}(A(j+1),\gamma(j))$ is the closure in $\mathcal{Y}(j)$ of $G_j^*(A(j+1),\gamma(j))$.

Lemma 14) If the unemployed individual has determined that his asset holdings at date $j+1$ will be $\overline{A(j}+1)$, then he will accept employment to commence at $j+1$ if

$$y(j)\varepsilon G_j^*(\overline{A(j}+1),\gamma(j))$$

and he will choose to be unemployed at $j+1$ if

$$y(j)\varepsilon\mathcal{Y}(j) - \overline{G}_j^*(\overline{A(j}+1),\gamma(j)) .$$ [6]

Proof: Simply observe that if employment is not accepted when $y(j)\varepsilon G_j^*(\overline{A(j}+1),\gamma(j))$ then the expected value of V from date $j+1$ forward is bounded from above by $S_{j+1,*}(\overline{A(j}+1),(\gamma(j),y(j)))$. The maximum value of V from $j+1$ forward if employment is accepted, $E_{j+1,*}(\overline{A(j}+1)+Y^s_{j+1}(\gamma(j),y(j)))$, exceeds that upper bound by definition of $G_j^*(\overline{A(j}+1),\gamma(j))$.

When $y(j)\varepsilon\mathcal{Y}(j) - \overline{G}_j^*(\overline{A(j}+1),\gamma(j))$ the maximum expected value of V from date $j+1$ forward is bounded from above by $E_{j+1,*}(\overline{A(j}+1)+Y^s_{j+1}(\gamma(j),y(j)))$ if employment is accepted. This is, of course, less than the least upper bound for the expected value of V from date $j+1$ forward if job search is continued, $S_{j+1,*}(\overline{A(j+1)},(Y(j),y(j)))$, by definition of $\psi(j) - \overline{G}^*_j(\overline{A(j+1)},Y(j))$//

Notice that, since j is an arbitrary date, Lemma 13 and Lemma 14 are applicable as long as the individual remains unemployed. Thus, for any evolution of the random process generating job offers these two results place restrictions on both the successive levels of consumption and the successive criteria for the termination of job search.

The results of this section indicate that an intimate relationship will exist between the job choice and consumption choice decisions of an expected utility maximizer. I have referred to this strategy characteristic elsewhere ([3]) as the nondecomposability property. That is, the individual's strategy cannot in general be decomposed into separate income maximization and consumption allocation problems. Such a decomposition is of course presumed if a simple model of expected present value of income maximization is used to characterize a utility maximizer's job search behavior.

Search Strategy in a Stationary Setting

We now turn to a very interesting special case of the problem considered in the preceding section. Certain restrictions are placed on the environmental parameters of the model as well as on the individual's objective function. Within this restricted formulation only the individual's endogenously determined asset holdings can change as time passes.

The following restrictions are assumed satisfied in the remainder of the paper.

R1) $r=r(i)$, $p=p(i)$, $b=b(i)$ and $s=s(i)$ for all $i=0, 1, \ldots$

R2) $u_i(c(i)) = \beta^i u(c(i))$ for all $i=0, 1, \ldots,$ where $0 < \beta\ 1$ and $u(\cdot)$ is twice continuously differentiable and strictly concave on $[0,\infty)$.

R3) $y(i)$'s are identically and independently distributed random variables, $i=0, 1, \ldots$

$(Q\{\omega\epsilon\Omega:y(i,\omega)\epsilon E\} \equiv G(E)$ and

$Q\{\omega\epsilon\Omega:y(i,\omega)\epsilon [b,b+db)\} \equiv dG(b))$.

R4) $Y_i^s(\gamma(i)) = Y_i^o(\gamma(i))$ (only the most recent job offer is available to the individual).

R5) $\beta(1+r) < 1$.

R6) Let Y^M be the infimum of

$$\{b\epsilon R^1:Q\{\omega\epsilon\Omega:Y_i^o(\gamma(i,\omega)) > b\} = 0\} .$$

$$Y^M > \sum_{i=0}^{\infty} \frac{b-s}{(1+r)^i} = \left(\frac{1+r}{r}\right)(b-s) .$$

This last restriction has a very simple interpretation. R6 asserts that there is a positive probability of observing a wage offer which is financially more rewarding than staying unemployed forever. If such a restriction was not satisfied the term "job search" would not be applicable since the individual would never expect to find a job paying more than search itself.

These restrictions afford us the opportunity to simplify some of our cumbersome notation without introducing any ambiguity into the analysis. First, due to R4 and the definition of

$Y_t^s(\gamma(t))$, the present value of income associated with the observed wage stream offers $\gamma(t)$ may be written as:

$$Y(y(t-1)) = Y_t^o(\gamma(t)) = \sum_{i=t}^{\infty} \cancel{\frac{y_i(t-1)}{(1+r)^{i-t}}} .$$

Second, due to R1, the minimum value for asset holdings when the individual is unemployed, B_t, may be written simply as

$$B \equiv B_t = \sum_{i=0}^{\infty} \frac{s-b}{(1+r)^i} = \frac{1+r}{r} (s-b) .$$

An additional reduction in our notation may be realized as a result of the following lemma. The result merely states that when past wage offers have no informational content (R3) and are not bankable (R4) the previously observed wage offers have no effect on the attainable expected utility of an unemployed individual.

<u>Lemma 15</u>) $S_{t,*}(A(t),\cdot)$ is constant on $\psi(0)x\ldots x\psi(t-1)$.

Proof: I have shown in [ ] (Lemma 7) that R3 together with R4 imply that $S_{t,N}(A(t),\cdot)$ is constant on $\psi(0)x\ldots x\psi(t-1)$. Since $S_{t,*}(A(t),\cdot)$ is $\lim_{N\to\infty} S_{t,N}(A(t),\cdot)$, Lemma 15 is an immediate consequence of that earlier result//

Due to this constancy, nothing is lost by eliminating reference to $\gamma(t)$ and writing $S_{t,*}(A(t),\gamma(t))$ as $S^*_{t,*}(A(t))$ from this point onward.

The stationary nature of our restricted problem is reflected in the following two lemmas.

<u>Lemma 16</u>) $E_{t+i,*}(\bar{x}) = \beta^i E_{t,*}(\bar{x})$ for all $\bar{x} \geq 0$, $t=0, 1, \ldots,$ and $i=0, 1, \ldots$.

Proof: The result follows immediately from the definition of $E_{t,*}$, R1 and R2//

<u>Lemma 17</u>) $S^*_{t+i,*}(\bar{x}) = \beta^i S^*_{t,*}(\bar{x})$ for all $\bar{x} \geq B$, $t=0, 1, \ldots$ and $i=0, 1, \ldots$.

Proof: The result follows immediately from the definition of $S^*_{t,*}$, R1, R2, and Lemma 15//

Within this simplified environment, just as within the more general one considered earlier, there is a set of wage offers during each period of unemployment which if received will induce the individual to accept employment. This set, previously denoted $G^*_t(A(t+1),\gamma(t))$, has some special attributes in the stationary setting. For instance, it is independent of $\gamma(t)$ and t . Also, if a particular wage offer, $\overline{y(t)}$, is an element of the set then any wage offer yielding at least as large a present value of income as $\bar{y}(t)$ is also an element of the set. These properties are stated precisely in Lemma 18.

<u>Lemma 18</u>) There exists a continuous function,

$$Y^*:[B,\infty) \rightarrow \left[\frac{(1+r)(b-s)}{r}, \frac{Y^M}{(1+r)} + (b-s)\right]$$

such that for any $t=0, 1, \ldots,$ and $A(t+1) \geq B$

$$y(t)\epsilon\overline{G^*_t}(A(t+1),\gamma(t))$$

if and only if

$$Y(y(t)) \geq Y^*(A(t+1)) .$$

Proof: Employing our simplified notation we may rewrite the definition of the "acceptance set".

$$\bar{G}^*_t\ (A(t+1),\gamma(t))$$

$$= \{y(t)\varepsilon\,\mathcal{Y}(t) : S^*_{t+1,*}(A(t+1)) \leqslant E_{t+1,*}(A(t+1)+Y(y(t)))\}\ .$$

From the definition of Y^M given in R6 one may easily establish by induction that for any $A(t+1) \geqslant B$ and finite $N \geqslant t+1$,

$$E_{t+1,N}(A(t+1)+Y^M) > S_{t+1,N}(A(t+1),\gamma(t))$$

and hence in the limit

$$E_{t+1,*}(A(t+1)+Y^M) \geqslant S_{t+1,*}(A(t+1),\gamma(t)) = S^*_{t+1,*}(A(t+1))\ .$$

Likewise, one may verify that for any $A(t+1) \geqslant B$ and finite $N \geqslant t+1$,

$$E_{t+1,N}(A(t+1)+(\frac{1+r}{r})(b-s)) < S_{t+1,N}(A(t+1),\gamma(t))$$

and in the limit we have

$$E_{t+1,*}(A(t+1)+\frac{(1+r)}{r}(b-s)) \leqslant S^*_{t+1,*}(A(t+1))\ .$$

These inequalities together with the definition of $S^*_{t+1,*}$ imply that

a) $$S^*_{t,*}(A(t)) \geqslant \max_{c(t),A(t+1)} \beta^t u(c(t))+E_{t+1,*}(A(t+1)+\frac{(b-s)(1+r)}{r})$$

subject to $c(t) \geqslant 0$ and

$$B \leqslant A(t+1) = [A(t)+b-s-pc(t)](1+r),$$

and

b) $$S^*_{t,*}(A(t)) \leqslant \max_{c(t),A(t+1)} \beta^t u(c(t))+E_{t+1,*}(A(t+1)+Y^M)$$

subject to $c(t) \geqslant 0$ and

$$-Y^M \leqslant A(t+1) = [A(t)+b-s-pc(t)](1+r)\ .$$

The right-hand side of inequality a is, by definition, simply

$$E_{t,*}(A(t) + \frac{(1+r)(b-s)}{r}) \quad \text{and}$$

the right-hand side of b is, again by definition,

$$E_{t,*}(A(t) + {Y^M}/{(1+r)} + b-s) \ .$$

Since $E_{t,*}(\cdot)$ and $S_{t,*}(\cdot)$ are strictly increasing and continuous, these inequalities ensure the existence of a function $Y^*(\cdot)$ with the properties asserted in Lemma 18//

Much attention has been focused on the Y^* function in the job search literature. It is interpretable as the reservation or acceptance income for the unemployed labor force participant. The property that its value may depend on current asset holdings is generally absent from formal job search models. This is not to say that such a dependence is a surprising consequence of the present formulation. Several economists* have noted the intuitive appeal of an asset sensitive reservation wage, however, formal models have typically failed to generate such a prediction.

Actually, more than mere dependence is predicted on intuitive grounds. A positive relation between acceptance wage and assets is suggested as "reasonable". Within the present model such reasonable predictions can be obtained only after making some reasonable assumptions regarding attitudes toward risk. Specifically, we must consider how an individual's willingness to take gambles changes as his asset holdings increase or decrease.

This particular question has been widely discussed in the literature on individual attitudes toward uncertainty. Perhaps the most significant aspect of the discussion has been the lack of controversy regarding what wealth-risk aversion relationship is reasonable. Both formal and casual empiricism have been advanced to substantiate the view that an individual's propensity to engage in risky ventures is positively related to his wealth. Stated differently, it is generally believed that the dollar value of insurance against any specific risk increases as asset holdings are reduced.

The following assumption on the individual's utility function is both necessary and sufficient to ensure that the price he is willing to pay to completely insure himself against all subsequent uncertain prospects will decline as his current assets increase.[7]

Assumption 7) $\frac{-u''(c)}{u'(c)}$ is a strictly decreasing function of c on $[0,\infty)$. (The reader should note that this assumption is simply the Pratt-Arrow concept of decreasing absolute risk aversion applied to the single period utility indicators.)

The next lemma is a natural and important consequence of the assumed attitudes toward risks. The result merely observes that as the individual's asset holdings decline (increase) he will find at least as many (no more) job offers acceptable as when he possessed more (less) wealth.

<u>Lemma 19</u>) $Y^*(\cdot)$ is a monotonic non-decreasing function.
I shall strengthen this result to read "monotonic increasing" below (see the proof of Theorem 1). Lemma 19 is, however, an essential element in the proof of that stronger result.

Proof: Define $Y_{t,N}(A(t))$ implictly by the equality,

$$S_{t,N}(A(t),\gamma(t)) = E_{t,N}(A(t)+Y_{t,N}(A(t))) .$$

I have shown as Theorem 1 of [ ] that Assumption 7 implies $Y_{t,N}(\cdot)$ is a strictly increasing function of $A(t)$ for any finite N . Since,

$$S_{t,N} \to S^*_{t,*} \text{ and } E_{t,N} \to E_{t,*}$$

uniformly as $N \to \infty$, $Y_{t,N} \to Y^*$ and Y^* is, therefore, non-decreasing//

Several properties of the u_i functions have been shown to carry over to the $E_{t,*}$ function. For instance, $E_{t,*}$ is continuous, increasing and concave as a result of the assumption that each u_i, i=0, 1, ..., is continuous, increasing and concave. The next lemma asserts that decreasing absolute risk aversion is likewise carried over from the u_i's to the $E_{t,*}$'s .

<u>Lemma 20</u>) $\dfrac{-E''_{t,*}(x)}{E'_{t,*}(x)}$ is a strictly decreasing function of x on $(0,\infty)$.

I have chosen to place the proof of this lemma in an appendix primarily because of its length. Also, the result's importance extends beyond the present work, and restating and proving it in an appendix serves to make it more accessible to those not interested in job search.

The preceding analysis provides a foundation for the investigation of two dynamic aspects of an individual's search strategy in a stationary setting. The first of these pertains to the pattern of accumulation or decumulation of assets associated with an optimal search strategy. The second dynamic aspect to be considered is the shape of the time path of reservation incomes implied by an optimal sequence of consumption and savings decisions.

The next lemma indicates that the unemployed labor force participant's bank balance will be drawn down over the course of his job search.

Lemma 21) If $\overline{A}(t) > B$ and $(c^*(t),A^*(t+1)) \in M_t^*(\overline{A}(t),\gamma(t))$ then $\overline{A}(t) > A^*(t+1)$.

The proof of this result relies heavily on the assumption that the product of the psychological discount rate, β , and the market interest rate, r , is less than one. Despite this rather simple underlying motivation for the lemma, the proof is quite lengthy and I have chosen to place it in an appendix (see Appendix 2).

The following theorem culminates our discussion of an optimal search strategy under stationarity restrictions. As time passes the theorem asserts that the acceptance or reservation income falls. This declining reservation income results purely from the dynamic process of asset depletion.

Theorem 1) If an individual is currently unemployed with asset holdings $A(t) > B$, then his reservation income in the preceding period was greater than it is in the current period. If $A(t) = B$ then the reservation income in the preceding period was at least as large as it is in the current period, and it will remain constant in all subsequent periods.

Proof: From the definition of $Y^*(\circ)$ we may write for any $\overline{A(t)} \geqslant B$;

$$S^*_{t,*}(\overline{A(t)}) = \beta^t u(\overline{c(t)}) + \int_{\overline{G}^*_t(\overline{A(t+1)},\gamma(t))} E_{t+1,*}(\overline{A(t+1)}+Y(v))dG(v)$$

$$+ G(\psi(t)-\overline{G}^*_t(\overline{A}(t+1),\gamma(t)))E_{t+1,*}(\overline{A}(t+1)+Y^*(\overline{A}(t+1))) ,$$

where $(\overline{c(t)},\overline{A(t+1)}) \varepsilon M^*_t(\overline{A(t)},\gamma(t))$.

Also,

$$S^*_{t,*}(\overline{A(t)}) = E_{t,*}(\overline{A(t)} + Y^*(\overline{A(t)})) .$$

Next, observe that the following expression may be obtained from the definition of $E_{t,*}$:

$$E_{t,*}(x) = \max_{c(t),A(t+1)} \beta^t u(c(t)) + E_{t+1,*}(A(t+1))$$

subject to $c(t) \geqslant 0$ and $0 \leqslant A(t+1) = [x-pc(t)](1+r)$.

Since $E_{t+1,*}(\cdot)$ and $u(\cdot)$ are strictly concave, the maximizing $c(t)$ and $A(t+1)$ are non-decreasing functions of X. Let $c^1(t),A^1(t+1)$ be the maximizers for $x^1 = \overline{A(t)}+Y^*(\overline{A(t)})$, and let $c^2(t)$, $A^2(t+1)$ be the maximizers for $x^2 = \overline{A(t)}+Y^*(\overline{A(t)}) + d$, $d > 0$. As noted, $c^2(t)-c^1(t)=\Delta c \geqslant 0$ and $A^2(t+1)-A^1(t+1) = \Delta A \geqslant 0$.

Finally, Lemma 19 states that $Y^*(\cdot)$ is non-decreasing, hence for $d > 0$

$$S^*_{t,*}(\overline{A(t)}+d) \geq \beta^t u(\overline{c(t)}+\Delta c) + \int_{\bar{G}^*_t(\overline{A(t+1)},\gamma(t))} E_{t+1,*}(\overline{A(t+1)}+\Delta A+Y(v))dG(v)$$

$$+ G(\psi(t)-\bar{G}^*_t(\overline{A(t+1)},\gamma(t)))E_{t+1,*}(\bar{A}(t+1)+\Delta A+Y^*(\overline{A(t+1)}))$$

I have demonstrated in Theorem 1 of [4] that $\frac{-u''(c)}{u'(c)}$ and $\frac{-E''_{t+1,*}(x)}{E'_{t+1,*}(x)}$ strictly decreasing in c and x respectively guarantees that the right-hand side of this inequality is strictly greater than

$$\beta^t u(c^2(t)) + E_{t+1,*}(A^2(t+1)) = E_{t,*}(\overline{A(t)}+Y^*(\overline{A(t)})+d) .$$

Since $E_{t,*}(\cdot)$ is strictly increasing,

$$E_{t,*}(\overline{A(t)} + Y^*(\overline{A(t)})+d) < S^*_{t,*}(\overline{A(t)}+d)$$

implies

$$Y^*(\overline{A(t)}+d) > Y^*(\overline{A(t)})$$

for any $\overline{A(t)} \geq B$ and $d > 0$.

By Lemma 21 and the definition of B, $A(t) > B$ if and only if $A(t-1) > A(t)$. Therefore, $Y^*(A(t)) < Y^*(A(t-1))$ when $A(t) > B$, $Y^*(A(t)) \leq Y^*(A(t-1))$ when $A(t) = B$ and, of course, $Y^*(A(t)) = Y^*(A(t+1))$ if $A(t) = B$//

Appendix I: Proof of Lemma 20.

Lemma 20)

$$\frac{-E''_{t,*}(x)}{E'_{t,*}(x)} \text{ is a strictly decreasing function of } x \text{ on } (0,\infty) .$$

Proof: I consider the cases when $u'(0) < \infty$ and when $u'(0) = +\infty$ separately.

Case I: $u'(0) < \infty$.

Recall that

$$E_{t,*}(x) = \max_{c(t),\ldots} \sum_{i=t}^{\infty} \beta^i u(c(i))$$

subject to $c(i) \geq 0$, $i=t, \ldots,$ and $\sum_{i=t}^{\infty} \frac{pc(i)}{(1+r)^{i-t}} \leq x$.

Necessary conditions for this maximum are simply:

1. $\beta^t u'(c(t)) = (1+r)^{i-t}\beta^i u'(c(i))$

   for any $i > t$ such that $c(i) > 0$, and

2. $\beta^t u'(c(t)) \geq (1+r)^{i-t}\beta^i u'(c(i))$

   for any $i > t$ such that $c(i) = 0$.

If $u'(0)$ is finite, R5 ensures that for any $\bar{x} < \infty$ there exists an $n(\bar{x}) < \infty$ such that $c^*(i) = 0$, $i \geq n(\bar{x})$ where $(c^*(t), c^*(t+1), \ldots)$ is the unique maximizer for $E_{t,*}(\bar{x})$. Also, since u is strictly concave, $n(x)$ is non-decreasing in x . Therefore, on any interval $[0,M]$, $x \leq M < \infty$

$$E_{t,*}(x) = \max_{c(t),\ldots,c(n(M))} \sum_{i=t}^{n(M)} \beta^i u(c(i))$$

subject to $c(i) \geq 0$, $i=t, \ldots, n(M)$ and

$$\sum_{i=t}^{n(M)} \frac{pc(i)}{(1+r)^{i-t}} \leq x .$$

Neave ([9] Lemma 1, page 46) has shown that if $E_{t,*}$ has this interpretation then Assumption 7 implies $\frac{-E''_{t,*}(x)}{E'_{t,*}(x)}$ is strictly decreasing on $[0,M]$. Since M is an arbitrary finite number the proof is complete for Case I.

Case II: $u'(0) = +\infty$.

Necessary conditions for a maximum in this case are:

$$\beta^t u'(c(t)) = (1+r)^{i-t}\beta^i u'(c(i)) \qquad i=t+1, \dots .$$

For any $x > 0$ there is a unique vector, $(c(t), c(t+1), \dots)$, which satisfies these conditions as well as the budget constraint,

$$\sum_{i=t}^{\infty} \frac{pc(i)}{(1+r)^{i-t}} = x .$$

We may define the functions $f^i : \overline{R^{1+}} \to \overline{R^{1+}}$, $i=1, 2, \dots$ by

$$u'(c(t)) = (1+r)^i \beta^i \, u'(f^i(c(t))) .$$

From this definition it is immediate that

$$u'(f^i(c(t))) = (1+r)\beta u'(f^{i+1}(c(t)))$$

and hence

$$f^{i+1}(c(t)) = f \cdot f^i(c(t))$$

where f^1 and f are synonymous.

The function f has two important properties. First, $0 < f(x) < x$ for all $x > 0$ since $\beta(1+r) < 1$. Second, $1 > f'(x) > 0$. We obtain this second property by the following three steps.

Step 1: $u'(c) = (1+r)\beta u'(f(c))$ for all $c > 0$

implies $u''(c) = (1+r)\beta u''(f(c))f'(c)$.

Step 2: $\dfrac{-\beta(1+r)u'(f(c))}{-u'(c)} = 1$

hence $$\frac{\;\frac{-u''(c)}{u'(c)}\;}{\frac{-u''(f(c))}{u'(f(c))}} = f'(c) .$$

Step 3: $f(c) < c$ and $\dfrac{-u''(c)}{u'(c)}$ is decreasing and positive implying $1 > f'(c) > 0$ Q.E.D.

Next define the partial sum

$$g_n(c(t)) = pc(t) + \sum_{i=1}^{n} \frac{pf^i(c(t))}{(1+r)^i} .$$

$$g_n'(c(t)) = p + \frac{pf'(c(t))}{(1+r)} + \dots + \frac{pf'(f^{n-1}(c(t)))\dots f'(c(t))}{(1+r)^n}$$

$$< p + \frac{p}{1+r} + \dots + \frac{p}{(1+r)^n}$$

since $f'(c) < 1$ for $c > 0$. This implies $\{g_n'(c(t))\}$ converges uniformly for $c(t) > 0$ (see Rudin [11], p. 134), and that $\{g_n\}$ converges to a function g on $(0,\infty)$ with

$$g'(c(t)) = \lim_{n\to\infty} g_n'(c(t)) \quad \text{for} \quad c(t) > 0$$

(see Rudin p. 140).

By our necessary conditions and budget equation we observe that

$$g(c(t)) - x = 0$$

for the optimal feasible value of $c(t)$. Since g is increasing and differentiable this equation yields the optimal $c(t)$ as an implicit differentiable function of x , $c_t(x)$, with

$$c_t'(x) = 1 \big/ g'(c_t(x)) \quad .$$

Now observe that

$$\lim_{n\to\infty} \sum_{i=t}^{n} \beta^i (1+r)^{i-t} u'(f^{i-t}(c_t(x))) \frac{df^{i-t}(c_t(x))}{dc_t(x)} \frac{c_t'(x)}{(1+r)^{i-t}}$$

$$= u'(c_t(x))c_t'(x) \left[\lim_{n\to\infty} \sum_{i=t}^{n} \frac{df^{i-t}(c_t(x))}{dc_t(x)} \frac{1}{(1+r)^{i-t}}\right]$$

$$= u'(c_t(x))c_t'(x)\; g'(c_t(x)) = u'(c_t(x)) \;.$$

Since the convergence is uniform and since

$$E_{t,*}(x) = \lim_{n\to\infty} \sum_{i=t}^{n} \beta^i u(f^{i-t}(c_t(x)))$$

we have $E_{t,*}'(x) = u'(c_t(x))$. By parallel arguments we obtain

$$E_{t,*}''(x) = u''(c_t(x))\; c_t'(x)$$

$$= \beta(1+r)u''(c_t(x))f'(c_t(x))c_t'(x) \big/ (1+r)$$

$$= \ldots \;.$$

Thus we obtain

$$\frac{-E_{t,*}''(x)}{E_{t,*}'(x)} = \frac{-u''(c_t(x))\; c_t'(x)}{u'(c_t(x))}$$

$$= \frac{-u''(f(c_t(x)))\; f'(c_t(x))\; c_t'(x)}{u'(f(c_t(x)))(1+r)}$$

$$= \ldots \;.$$

Finally, notice that $\dfrac{-u''(f^{i-t}(c_t(x)))}{u'(f^{i-t}(c_t(x)))}$

is decreasing for all $i = t, t+1, \ldots$ and for at least one

$i=t, t+1, \ldots$

$$\frac{df^{i-t}(c_t(x))}{dc_t(x)} \quad \frac{c_t'(x)}{(1+r)^{i-t}}$$

is decreasing since

$$1 = \sum_{i=t}^{\infty} \frac{df^{i-t}(c_t(x))}{dc_t(x)} \frac{c_t'(x)}{(1+r)^{i-t}} .$$

Therefore, $-E''_{t,*}(x) / E'_{t,*}(x)$ is decreasing for Case II and

the proof is complete//

Appendix 2: Proof of Lemma 21.

Lemma 21) If $\overline{A(t)} > B$ and $(c^*(t), A^*(t+1)) \in M_t^*(\overline{A(t)}, \gamma(t))$ then $\overline{A(t)} > A^*(t+1)$.

Proof: Recall that R1 through R6 imply that $S_{t,N}(A(t),\gamma(t))$ is constant on $\psi(0) x \ldots x \psi(t-1)$. We may thus define the function $S^*_{t,N}: R^1 \to R^1$ by,

$$S^*_{t,N}(A(t)) = S_{t,N}(A(t),\gamma(t)) .$$

Also, for any function $f: R^1 \to R^1$ the difference

$$f(x+d) - f(x)$$

shall be expressed simply as

$$\Delta f(x;d) .$$

Employing this new notation the first step of the proof is verification of the assertion that

(1) $$\Delta S^*_{t,*}(A(t);d) \leq \Delta E_{t,*}(A(t)-B;d)$$

for any $A(t) \geq B$ and $d > 0$. We begin by advancing the following induction hypothesis.

IH(N): $\Delta S^*_{t,N}(A(t);d) \leq \Delta E_{t,N}(A(t)-B;d)$

for any $A(t) \geq B$ and $d > 0$.

Now let

$$Z_{t,N}(X) = \int_{\psi(t)} \max[E_{t,N}(X+Y(v)), S^*_{t,N}(X)]dG(v)$$

Since $E_{t,N}(A(t)+Y(y')) = S^*_{t,N}(A(t))$ implies $\Delta S^*_{t,N}(A(t);d) > \Delta E_{t,N}(A(t)+Y(y');d)$ (see the proof of Lemma 19) and since $E_{t,N}$ is strictly concave,

(2) $$\Delta Z_{t,N}(A(t);d) < \Delta S^*_{t,N}(A(t);d)$$

for all $A(t) \geq B$ and $d > 0$.

Notice that

$$\frac{1}{\beta}S^*_{t,N+1}(A(t-1)) = S^*_{t-1,N}(A(t-1)) = \max_{c(t-1),A(t)} \beta^{t-1}u(c(t-1)) + Z_{t,N}(A(t))$$

subject to: $c(t-1) \geq 0$ and

$$B \leq A(t) = [A(t-1)+b-s-pc(t-1)](1+r) ,$$

and

$$\frac{1}{\beta}E_{t,N+1}(A(t-1)-B) = E_{t-1,N}(A(t-1)-B) = \max_{c(t-1),A(t)} \beta^{t-1}u(c(t-1)) + E_{t,N}(A(t)-B)$$

subject to: $c(t-1) \geq 0$ and

$$B \leq A(t) = [A(t-1)+b-s-pc(t-1)](1+r) .$$

Thus, IH(N+1) i.e. $\Delta E_{t,N+1}(A(t)-B;d) \geq \Delta S^*_{t,N+1}(A(t);d)$ for any $A(t) \geq B$ and $d > 0$, is equivalent to the statement that $\Delta E_{t-1,N}(A(t-1)-B;d) \geq \Delta S^*_{t-1,N}(A(t-1);d)$ for any $A(t-1) \geq B$ and $d > 0$.

For any $\bar{A} \geq B$ and $d > 0$ there exist four consumption-asset pairs, (c^i,A^i) $i = 1, 2, 3, 4$, which satisfy the following six conditions:

a) $\beta^{t-1}u(c^1) + Z_{t,N}(A^1) = S^*_{t-1,N}(\bar{A})$,

b) $\beta^{t-1}u(c^2) + E_{t,N}(A^2-B) = E_{t-1,N}(\bar{A}-B)$,

c) $c^i \geq 0$ and $B \leq A^i = [\bar{A}+b-s-pc^i](1+r)$, $i = 1, 2,$

d) $\beta^{t-1}u(c^3) + Z_{t,N}(A^3) = S^*_{t-1,N}(\bar{A}+d)$.

e) $\beta^{t-1}u(c^4) + E_{t,N}(A^4-B) = E_{t-1,N}(\bar{A}+d-B)$, and

f) $c^i \geqslant 0$ and $B \leqslant A^i = [\bar{A}+d+b-s-pc^i](1+r)$, $i = 3, 4$.

I shall partion the proof of IH(N+1) by the two mutually exclusive possibilities, $A^3 \geqslant A^2$ and $A^3 < A^2$.

Case I: $A^3 \geqslant A^2$.

Since (c^1,A^1) is clearly a maximizing pair

$$\beta^{t-1}[u(c^1)-u(c^2)] \geqslant Z_{t,N}(A^2)-Z_{t,N}(A^1) .$$

Also, by (2) and the assumption that $A^3 \geqslant A^2$.

$$Z_{t,N}(A^3)-Z_{t,N}(A^2) \leqslant E_{t,N}(A^3-B) - E_{t,N}(A^2-B) .$$

Therefore since

$$\beta^{t-1}u(c^3) + E_{t,N}(A^3-B) \leqslant E_{t-1,N}(\bar{A}+d-B) ;$$

$$\begin{aligned}\Delta E_{t-1,N}(\bar{A}-B;d) &> \beta^{t-1}u(c^3)+E_{t,N}(A^3-B)-\beta^{t-1}u(c^2)-E_{t,N}(A^2-B)\\ &= \beta^{t-1}[u(c^3)-u(c^1)+u(c^1)-u(c^2)]+E_{t,N}(A^3-B)-E_{t,N}(A^2-B)\\ &\geqslant \beta^{t-1}(u(c^3)-u(c^1))+Z_{t,N}(A^2)-Z_{t,N}(A^1)+Z_{t,N}(A^3)-Z_{t,N}(A^2)\\ &= \Delta S^*_{t-1,N}(\bar{A};d) .\end{aligned}$$

Case II: $A^3 < A^2$.

First let $\varepsilon = A^3-A^1$. One may easily prove that since $u(\cdot)$ is strictly concave, $d > 0$ and (c^1,A^1) , (c^3,A^3) are maximizers, $\varepsilon \geqslant 0$. Of course we have

$$S^*_{t-1,N}(\bar{A}) \geqslant \beta^{t-1}u(c^1-\frac{\varepsilon}{p(1+r)}) + S^*_{t,N}(A^1+\varepsilon)$$

or equivalently,

(3) $\beta^{t-1}(u(c^1)-u(c^1-\varepsilon/p(1+r))) \geq \Delta S^*_{t,N}(A^1;\varepsilon)$.

Notice that $A^3 < A^2$ implies $A^2-A^1 > A^3-A^1$ or ,

$$p(1+r)(c^1-c^2) > \varepsilon \Rightarrow c^1 - \varepsilon/p(1+r) > 0 .$$

We now require a second partitioning of our proof into the subcases of $d-\varepsilon/(1+r) > 0$ and $d-\varepsilon/1+r \leq 0$. First, if $d-\varepsilon/1+r > 0$ the strict concavity of $u(\cdot)$ implies that

$$u(c^2+\varepsilon/(1+r)p)-u(c^2) \geq u(c^1)-u(c^1-\varepsilon/(1+r)p) ,$$

and

$$u(c^2+ \frac{d}{p})-u(c^2+\varepsilon/(1+r)p) \geq u(c^1+ \frac{d}{p} -\varepsilon/(1+r)p)-u(c^1) .$$

Summing these two inequalities and applying (3) we obtain,

$$\beta^{t-1}[u(c^2+ \frac{d}{p}) - u(c^2+\varepsilon/(1+r)p)] + \beta^{t-1}[u(c^2+\varepsilon/p(1+r)) - u(c^2)]$$

$$\geq \beta^{t-1}[u(c^1+ \frac{d}{p} - \varepsilon/p(1+r)) - u(c^1)] + \Delta S^*_{t,N}(A^1;\varepsilon) .$$

Adding

$$E_{t-1,N}(\bar{A}+d-B) - \beta^{t-1}u(c^2+ \frac{d}{p}) - E_{t,N}(A^2-B) \geq 0$$

to the previous inequality and rearranging terms yields,

$$\Delta E_{t-1,N}(\bar{A}-B;d) > \Delta S^*_{t-1,N}(\bar{A};d) .$$

If $d - \varepsilon/1+r \leq 0$, i.e. $c^3 < c^1$,

$$\beta^{t-1}[u(c^1-u(c^3- \frac{d}{p})] \geq \Delta S^*_{t,N}(A^1;\varepsilon)$$

since (C^1,A^1) is a maximizer and $c^3 - \frac{d}{p} = c^1 - \varepsilon/p(1+r)$.

Subtracting $\beta^{t-1}[u(c^1)-u(c^3)]$ from both sides of the preceding inequality, we obtain:

$$\beta^{t-1}[u(c^3)-u(c^3-\frac{d}{p})] \geq \Delta S^*_{t-1,*}(\bar{A};d) .$$

Due to the strict concavity of u and the presumed inequality, $A^3 < A^2$ (implying $c^3-\frac{d}{p} \geq c^2$) ,

$$\beta^{t-1}[u(c^2+\frac{d}{p}) - u(c^2)] \geq \Delta S^*_{t-1,N}(\bar{A};d) .$$

Also,

$$\beta^{t-1}u(c^4) + E_{t,N}(A^4-B) > \beta^{t-1}u(c^2+\frac{d}{p}) + E_{t,N}(A^2-B)$$

hence

$$\Delta E_{t-1,N}(\bar{A}-B;d) > \beta^{t-1}[u(c^2+\frac{d}{p}) - u(c^2)] .$$

Therefore,

$$\Delta E_{t-1,N}(\bar{A}-B;d) > \Delta S^*_{t-1,N}(\bar{A};d)$$

in this final case and $IH(N+1)$ is thus implied by $IH(N)$. Since

$$S^*_{t,t}(A(t)) = E_{t,t}(A(t)-B)$$

by definition for all $A(t) \geq B$, $IH(n)$ is valid for arbitrary n . In the limit as $n \to \infty$,

(4) $$\Delta E_{t,*}(A(t)-B;d) \geq \Delta S^*_{t,*}(A(t);d)$$

for all $A(t) \geq B$ and $d > 0$.

(4) is a principle ingredient in the remainder of our proof. Let $(c^{\bullet},A^{\bullet})$ be the maximizer for

$$E_{t,*}(\overline{A(t)}-B) = \max_{c,A} \beta^t u(c) + E_{t+1,*}(A-B)$$

subject to: $c > 0$ and

$$B \leq A = [\overline{A(t)}+b-s-pc](1+r) .$$

Also let (c^*,A^*) be an arbitrary element of $M_t^*(\overline{A(t)},\gamma(t))$. If $c^* \neq c^\circ$ then

(5) $\beta^t[u(c^\circ)-u(c^*)] > E_{t+1,*}(A^*-B) - E_{t+1,*}(A^\circ-B)$.

The concavity of $E_{t+1,*}$, monotonicity of $Y^*(\cdot)$ and (4) imply that if $c^* < c^\circ$,

(6) $E_{t+1,*}(A^*-B) - E_{t+1,*}(A^\circ-B)$

$$\geq \int_{\psi(t)} \max[E_{t+1,*}(A^*+Y(v)),S^*_{t+1,*}(A^*)]dG(v)$$

$$- \int_{\psi(t)} \max[E_{t+1,*}(A^\circ+Y(v)),S^*_{t+1,*}(A^\circ)]dG(v) .$$

However, $(c^*,A^*) \in M_t^*(\overline{A(t)},\gamma(t))$ implies

(7) $\beta^t[u(c^\circ)-u(c^*)] \leq \int_{\psi(t)} \max[E_{t+1,*}(A^*+Y(v)),S^*_{t+1,*}(A^*)]dG(v)$

$$- \int_{\psi(t)} \max[E_{t+1,*}(A^\circ+Y(v)),S^*_{t+1,*}(A^\circ)]dG(v) .$$

Thus $c^* < c^\circ$ implies a contradiction hence $c^* \geq c^\circ$.

Finally, observe that c° is the first element of the vector $(c^\circ(t),c^\circ(t+1),\ldots)$ which maximizes

$$\sum_{i=t}^{\infty} \beta^t u(c(i))$$

$$\text{subject to: } \overline{A(t)}-B \geq \sum_{i=t}^{\infty} \frac{pc(i)}{(1+r)^{i-t}} .$$

$\beta(1+r) < 1$ implies $c^\circ > c^\circ(t+1)$ if $\overline{A(t)} > B$ and in general $c^\circ(i) \geq c^\circ(i+1)$, $i = t+1, \ldots$. Therefore,

$$\overline{A(t)} - B > \sum_{i=t+1}^{\infty} \frac{pc^\circ(i)}{(1+r)^{i-t-1}}$$

and hence

$$pc^{\bullet} > \bar{A}(t)\ (1 - \frac{1}{(1+r)})) + (b-s)\ .$$

Because $c^* \geqslant c^{\bullet}$,

$$A^* < [\bar{A}(t)+b-s-\bar{A}(t)\ (1 - \frac{1}{(1+r)})-b+s](1+r) = \bar{A}(t)//$$

FOOTNOTES

1/ The "reasonable" assumption I refer to here is that risky ventures are normal goods. Arrow defends this assumption in [1] on the basis of casual empiricism. Also, Projector and Weiss [11] provide data on individual allocation of savings which is supportive of this assumption.

2/ See, for example, Sheppard and Belitsky's Job Hunt [14] or Kasper's The Relation Between the Duration of Unemployment and the Change in Asking Wage [7].

3/ Both Gronau [6] and Salop [13] employ an arbitrary deadline to obtain a strategy with a declining aspiration level. The only other models I have encountered which "predict" a declining acceptance wage depend on the assumption that people consistently overestimate the market demand for their services when they commence the job search activity and then revise these estimates downward as time passes. I do not find models which are based on such consistent naiveté appealing.

4/ Kasper [7], for example, includes age in his asking wage equation as an independent variable. He finds that the hypothesis that the rate of decline of asking wage is positively related to age can be rejected at the 93% confidence level. A positive relationship between rate of acceptance wage decline and age is, however, predicted by a model of expected income maximizing job search with a finite horizon if that horizon is associated with age. If the horizon is not related to age what is it related to? Running out of funds may be the response but then the simple expected income maximization model is clearly not the appropriate model.

5/ I refer here to the assumptions underlying the analysis presented in [3]. This is most easily seen by comparing the criteria for feasibility of strategies in the present paper and in [3].

6/ If $y(j) \varepsilon \bar{G}^*_j(\overline{A(j}+1),\gamma(j))-G^*_j(\overline{A(j}+1),\gamma(j))$, i.e.

$S_{j+1,*}(\overline{A(j}+1),(\gamma(j),y(j))) = E_{j+1,*}(\overline{A(j}+1) + Y^s_{j+1}(\gamma(j),y(j)))$,

no strictly preferred action need exist. I have adopted the convention of assuming the individual accepts employment whenever this situation arises. No subsequent results rely on this assumption.

7/ This claim is substantiated in Theorem 1 of [4].

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