EXPECTED UTILITY, MANDATORY RETIREMENT AND JOB SEARCH

by

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The theory of the labor market behavior of firms and individuals has been in a constant state of flux in recent years. The classical assumptions of a perfectly competitive market for labor services, which were for so long accepted by labor economists, have been critically reviewed and found lacking. Theoretical abstractions which may be classified as examples of "The New Microeconomic Foundations of Employment and Inflation Theory", for instance, discard the perfect knowledge assumption of earlier models. In its place we find varying degrees of imperfect information assumed available to the hypothesized market participants.

The explicit incorporation of uncertainty in this approach represents a fundamental departure from the classical view of the functioning of the labor market. There can be no doubt that this new view has significantly enhanced our understanding of price and employment dynamics. Nevertheless there remain many opportunities to refine existing analyses of the labor market participant's behavior in his newly inhabited stochastic environment.

Such opportunities to a large extent result from the failure of labor economists to incorporate in their models certain basic theoretical developments relating to individual behavior under uncertainty. An important example of this is the assumption, typically found in this literature, that individuals who are unemployed and searching for work attempt to maximize the expected present value of their income. While this assumption greatly simplifies analysis, it has been found to yield implications inconsistent with behavior in other market situations involving uncertainty. Portfolio diversification and a thriving insurance industry are, for example, quite difficult to explain in a world populated by expected income maximizers. For that reason economists studying these phenomena long ago opted for an assumption of expected utility maximization.

Two compelling arguments can be advanced for likewise adopting the assumption of expected utility maximization in a model of job search.[1] First, the extent to which existing characterizations of optimal search strategies strictly depend on the risk neutrality assumption can be established only after the logical implications of an alternative assumption are fully analyzed. Second, specific knowledge regarding individual attitudes toward risks acquired in other fields, e.g. estimates for demand functions for risky and nonrisky assets or for various types of insurance, can yield no insights into job search behavior in a model which presupposes risk neutrality.

These arguments comprise the raisons d'être for the following analysis wherein I obtain the optimal job search strategy for an expected utility maximizing unemployed labor force participant. This strategy not only involves decisions regarding the acceptability of observed wage offers, but also the simultaneous determination of consumption expenditures and savings. I find that because of the interrelated nature of the consumption and job choice decisions, expected income is not in general maximized by the optimal strategy. In fact, expected income as well as expected duration of unemployment depend crucially on attitudes toward risk and initial wealth.

The paper is organized in three sections. First, an abbreviated review of the related literature is provided. In the second section I present a verbal description and analysis of a model of job search. The third section includes a mathematical representation of the model with proofs of results stated in Section II. The second and third sections are close but not perfect substitutes for one another and the reader should allocate his time between the two depending on the availability of this resource as well as on his relative propensities for reading words versus equations.

The Literature

Finding a job is trivially accomplished in classical labor supply models. There is a unique known wage rate associated with each potential labor supplier's abilities which he may obtain simply by placing himself on the market. The notion of job search is entirely out of place in this pristine setting where the first wage offer received is sure to be the best (and worst) offer available.

The assumption of perfect knowledge, upon which classical market models of this sort are built, appeared to some economists to be a singularly poor representation of the real market for labor services. The terms of individual wage contracts are not generally well known to all labor force participants.

Thus, market models wherein the perfect information assumption has been eliminated have been of particular interest to labor economists.

Kenneth Arrow [3] was one of the earliest theorists to consider the implications of imperfect information regarding market prices. He found that disequilibrium situations could develop with many prices existing for the same good. More recently Mortensen [15] and Lucas and Prescott [10] have demonstrated the possibility of having nondegenerate price distributions even in equilibrium[2] when market participants possess imperfect information. These papers are important for labor economists because they provide logically consistent grounds for expecting non-unique wage rates for identical tasks and laborers to persist over time. Earlier models predicted that wage rate differentials could be explained entirely by factors like the quality of the worker and the unpleasantness of the task.

Some economists did not wait for such logically consistent market models to begin studying markets with many prices existing for identical products. In particular, George Stigler's "Economics of Information" [19] assumed price dispersion, without proving that it could arise or persist, and then proceeded to analyze individual behavior in the face of it. He turned his attention to individual behavior when wage dispersion exists in "Information in the Labor Market" [20].

In these papers Stigler considers an individual who is desirous of paying a low price for a given commodity and an individual who wants a high price for his labor services respectively.

The latter individual seeks to maximize his expected income with respect to his choice of the number of samples he will take from the wage distribution. It is shown that expected income maximization may require substantial sampling, i.e. job search.

Subsequent to Stigler's contributions several economists have considered the job search activity in the context of optimal stopping rule theory. J.J. McCall's "Economics of Information and Job Search" [11] is an excellent example of this approach. The distinguishing characteristic of McCall's analysis and other optimal stopping rule analyses is the emphasis on determining the best form for the job search strategy. That is, while Stigler's laborer wants to maximize expected labor income he is restricted to a strategy (choice of number of searches without revision) which is not proven to be optimal and, in fact, is not.

Optimal search strategies may, of course, take many specific forms depending on the environmental assumptions made. One characteristic is, however, common to all of these strategies. At each date a set of wage offers is determined which, if received, will be accepted. Frequently these sets consist of all the wage offers above some critical value. When this is the case the individual sets may be completely described by this critical value which is referred to as the reservation or acceptance wage.

The primary emphasis of economists working on job search has been related to the way this reservation wage or more generally, acceptance set, is altered due to changes in search costs, wage distribution parameters, time horizon and interest rates. This preoccupation with reservation wages stems from the fact that important labor market aggregates depend on the individually determined acceptance sets. The nature of this dependence has been illuminated by Mortensen [14], Holt [9], and Gronau [8], all of whom incorporate acceptance wage equations in models of the labor market. These models have value as tools for policy evaluation as well as bases for predicting the impact of endogenous structural changes in our economy.

Job Search, A New Look

The essential difference between the model I present here and the models of job search mentioned above relates to the objectives of the individual considered and not to the assumptions regarding environment. I shall attempt to "cover all the bases" with respect to environmental assumptions in an effort to highlight the effect of considering a different individual objective function. Note that this comparison is, therefore, between expected income maximization and expected utility maximization, where utility is not necessarily linear in income, within comparable environments.

As an initial step let us characterize the economic actor whose behavior will be studied below. All of the information about the individual which is relevant for our analysis is embodied in a statement of his preferences, past experience, expectations and endowments. Stigum [22] presents a similar but not identical characterization of an economic agent.

The individual's preferences are related to the quantity of market goods and leisure consumed in every period of his life. That is, he derives enjoyment from consuming the commodities available in the market place as well as from spending time in their consumption, "leisure" time. These preferences give rise to a continuous utility function which assigns a number to each lifetime sequence of consumption and leisure.

The utility function is assumed to have a special form which allows us to consider the value of consumption and leisure in any period independently of the magnitude of these variables in any other period. This property of the utility function is termed intertemporal separability or intertemporal additivity. Thus, lifetime utility may be viewed as the sum of separate single period utility functions.

The individual may be confronted with a situation where his future consumption sequence may take on many values with associated probabilities. Such a situation is referred to as a gamble. I assume that his preference ranking over gambles is the same as the numerical ranking provided by taking the mathematical expectation of his utility for each gamble. I also assume that each single period utility function is strictly concave. This implies that the expected utility derived from a fifty-fifty chance at two consumption sequences is less than the utility of the sum of one half of each consumption sequence.

Given any set of options available to this individual he is assumed to choose one which offers the maximum utility. When some of these options are gambles he chooses one with the highest expected value of utility. Such an individual is commonly referred to as an expected utility maximizer.

In addition to preferences this economic unit has a memory. As time passes he collects and saves data on the way the world is. In this paper his observations are confined to a single wage offer in every period he looks for work. Thus, his memory consists of a list of all past wage offers.

The values observed for past wage offers may influence his expectations regarding the probability of encountering any given wage offer in the future. It is assumed that for any given sequence of previously observed wage offers, he has a well defined subjective probability distribution over the values wage offers might take in future periods.

The final characteristic of the individual relates to his endowments. These may be thought of as physical wealth holdings, educational attainments, skills or aptitudes and time until retirement or death. As time passes these endowments may change, but at each date they provide a fixed point of departure. The concept of endowments used here corresponds to the notion of state variables in standard optimal control theory.

Possessing a description of the individual's more salient characteristics, we are now prepared to place some restrictions on his behavior. First, within existing expected income maximization models, the time allotted to work and search is treated as given. This assumption is embodied in the present model by eliminating leisure as a choice variable and fixing it at some predetermined level for each period.

For our individual to have any decision at all now income or wages must be related to feasible consumption. That is, wages as such are not arguments of the individual's utility function. A standard assumption is that market goods are purchased using income and asset holdings. I assume specifically that there exist certain market prices for consumption commodities in each period and that expenditures on these commodities must be financed by earned income and initial wealth. A capital market is also posited such that money can be borrowed or lent at a known positive rate of interest in each period. At the beginning of any period the individual's net asset position is determined by first taking the preceding period's net assets less consumption expenditure plus unemployment benefits less physical search costs if he is unemployed or plus labor income if he is employed and then multiplying this sum by one plus last period's rate of interest on borrowing or lending. In addition lending institutions are assumed to enforce a finite lower bound on the individual's bequests by judiciously limiting his allowable debt during each period. The precise nature of borrowing limits is somewhat arbitrary but in their absence the individual's behavior is quite predictable and uninteresting.

The assertion that there exists a nondegenerate distribution of wages and imperfect information as to their location is embodied in the assumption on the individual's expectations regarding the wage offers he will observe in the future if he samples. Recall that the individual has a subjective probability distribution defined over the single wage offer he will observe in any period of search. Past and current wage offers sampled may or may not effect his expectations about subsequent observations of wage offers.

Finally, in the spirit of other job search models, the individual is assumed to keep any job which he accepts until some mandatory date of retirement.[3]

Given the assumptions provided thus far we may obtain a dynamic programming iterative solution to the individual's expected utility maximization problem. This solution corresponds to an optimal search strategy for the individual. The individual pursuing this search strategy is seen to make two decisions whenever he is unemployed. He chooses the level of his consumption of market goods for the current period and he decides upon a set of wage offers which, if observed, will be accepted. This set will be denoted as the acceptance set. If during some period a wage offer is observed which is an element of the acceptance set, then he will become employed and remain so until he retires.

The acceptance set in any period is a function of his expectations regarding future wage offers given past and current wage offers, the number of years remaining before retirement, unemployment benefits less search costs and the current level of asset holdings. This set may have gaps. That is, while an individual may find a modest wage acceptable he could choose to reject a more lucrative offer because it leads him to drastically elevate his expectations regarding future prospects. Of course, given a high enough current offer the probability that subsequent job offers will exceed it may be quite low and it would be accepted. M. Rothschild [18] (Proposition 3, page 11) has established conditions for an expected income maximizer which insure that no such gaps exist. He refers to the absence of gaps as the "reservation price property" while I use the terminology monotone acceptance set to refer to this property.

Notice that the individual's choice of job offers is not restricted to the current offer. That is, the individual may have the opportunity to take any wage offer which he has previously observed (a la Alchian [1] and Stigler [20]) or he may be constrained to accept or reject forever the current offer (a la Gronau [8], and Gordon and Hynes [7]). Under either of these assumptions the general form of the strategy discussed will remain intact. I do find, however, that when no learning takes place then the acceptance set will be monotone if only current offers may be accepted.

I believe it would be beneficial to pause here and consider the distinguishing characteristics of this formulation. There are two fundamental properties of the strategy obtained above which are not shared by strategies generated from other models of job search. I denote the first as the general nondecomposability property and the second as the endowment sensitive property.

The property of nondecomposability as used here refers to the joint nature of the consumption allocation and job choice decisions. A decomposition into expected income maximization and then consumption allocation only leads to a maximum expected utility when either the wage rate is certain, or when the utility function is linear in each period's consumption.

By the endowment sensitive property I refer to the characteristic that endowments of assets may well effect the individual's determination of acceptable wage offers. This sensitivity of the current acceptance set to current asset endowments gives rise to a surprising difference in the willingness of the individual to make actuarily fair bets when he is employed versus when he is unemployed. Once a job has been accepted the individual will shun all opportunities to make bets with expected returns less than or equal to zero. While unemployed, though, the individual may be quite willing to engage in an actuarily fair (or even unfair) bet despite the strict concavity of his single period utility functions.

This is by no means all we can say about the relation between asset endowments and search strategy. However, before any additional results may be presented, a common notion regarding the definition and measurement of attitudes toward risk within this model must be established. I refer the reader to the work of Pratt [16] and Arrow [2] on this subject. These authors consider measures of absolute and relative or proportional risk aversion. These measures apply to a utility of money or wealth function which has not been employed in describing the preferences of the hypothetical laborer of this paper.

Stiglitz [21] has utilized the Pratt-Arrow analysis in a multiperiod or multigood setting where the utility of assets is viewed as an indirect indicator of the satisfaction provided by the goods purchased with these assets. Unfortunately such a procedure is not feasible when sequential uncertainty is involved. An explanation of this infeasibility is given in [5] where I have developed a generalized notion of these measures which reduces to the Pratt-Arrow-Stiglitz concept when the gambles are restricted to the type considered by them.

In this generalization I consider random sequences of income as opposed to considering initial wealth as a random variable. When the individual maximizes expected utility for some a random sequence of income, initial wealth, and fixed values for leisure in each period a random consumption plan results. There is some level of assets that when allocated optimally over the lifetime with the fixed leisure values will yield the same level of utility as that associated with the above random consumption sequence and the same fixed leisure. If we subtract the initial wealth from this level of assets we obtain the certain dollar value of the random income sequence given the fixed leisure values and initial wealth.

If initial wealth is changed the certain dollar value of the random income sequence may change. If there is a positive relationship between the value of random income sequences and initial wealth we say that sequences of risks are normal goods. If the relationship is negative we say that sequences of risks are inferior goods. I have shown the intuitive but nontrivial result that all sequences of risk are normal goods if and only if each single period measure of absolute risk aversion is decreasing. Also, if all sequences of risks are inferior goods then each single period measure of absolute risk aversion must be increasing.[4]

Arrow [2] appeals to "everyday observation" in defending the hypothesis that risks are normal goods. Additional evidence relating to this hypothesis is provided by Projector and Weiss [17]. They have collected data on the financial characteristics of consumers which suggest a definite positive relationship between wealth holdings and what would generally be characterized as risky investments. Thus, their study provides clear support for the normal good hypothesis. This evidence as well as casual empiricism on my part has led me to adopt the assumption that sequences of risks are normal goods as defined above.

This assumption, used in the above model of job search, yields the implication that the acceptance set is negatively related to asset endowments. That is, if assets are increased the set of acceptable job offers will be reduced and the probability of remaining unemployed for another period will increase. Charles Holt [9] suggests that such a relationship between assets and acceptable jobs has definite intuitive appeal.[5]

One should notice that the successive acceptance sets determine not only the expected duration of unemployment, but also the expected present value of labor income. I have been able to obtain two fundamental results regarding the relation between the expected length of search and asset endowments and the expected present value of labor income and these endowments.

First, the expected duration of search is a nondecreasing function of physical asset endowments. Therefore, the individual would tend to remain unemployed for a longer period of time if his asset holdings were augmented by some positive amount. Put differently, if two individuals differ only in terms of wealth, the richer of the two will, on average, be unemployed longer than the poorer.

Second, I find that assets are positively related to the expected present value of labor income. Thus, not only would the interest income increase if an individual's wealth was increased but also his expected income from labor services. Again considering two individuals who are identical in everything but initial wealth, we are able to conclude that the rich are expected to get richer. This conclusion fits very well with the assertion made by Friedman [6] that initial endowments should effect the results of an individual's confrontation with choices involving risk over his lifetime. Note again that asset endowments are irrelevant to a model of expected income maximization.

There is a type of inequality preserving mechanism at work when sequences of risks are normal goods. As two individuals differing only in initial wealth endowment move through life sequentially maximizing expected utility we find the expected disparity of wealth growing. One can heuristically liken this case to an unstable system. If wealth differs for two individuals at a point in time the wealth levels tend to diverge.

As yet no mention has been made of the influence of the size of search costs or unemployment benefits on the search strategy. Under a system where unemployment benefits persist until employment is obtained the expected utility of any search strategy is a function of the difference between benefits and costs. I find that a ceteris paribus decline in the magnitude of this difference will be non-negatively related to the probability that a job will be accepted in any given period. McCall [11] has shown that this relationship also holds under the assumption of risk neutrality. It is of some interest to note the impact of varying search costs and unemployment benefits on the individual's expected present value of labor earnings. Under the assumption of expected income maximization any movement away from equality of search cost and benefits leads to a reduction in expected labor income. In the present formulation this is no longer true. An increase in benefits so they exceed search costs may lead to higher expected labor earnings than are associated with equal search costs and benefits.

Insofar as wages reflect the social value of the individual's product, this result has some implications for government policy's effect on social welfare. Expected income maximization models of job search unambiguously predict that the expected value of social product is maximized if unemployment benefits just offset physical search costs. The model I present suggests that the expected value of the individual's social product is maximized when unemployment benefits exceed search costs. In both frameworks, though, the value of the individual's expected social product can be increased by raising unemployment benefits when they are less than search costs.

The Mathematical Model

In this section a precise formulation of the model discussed in the body of the paper is provided. I shall begin by presenting a set of definitions. These definitions are then utilized in setting forth assumptions regarding the individual economic agent and his environment. Certain aspects of the individual's behavior are then considered, via some lemmas, propositions and examples.

Definitions

$C(t)$ is a composite consumption good for period t ;

$p(t)$ is the certain price in period t of a unit of $C(t)$ measured in dollars;

$r(t)$ is the certain one period rate of interest for borrowing and lending applicable to period t ;

$A(t)$ is the dollar value of physical assets held at the beginning of period t ;

t denotes a point in time or a time interval with end points t , $t + 1$. The specific context will determine which case is applicable;

$\ell(t)$ is the number of hours not used in market activities during period t ;

T is the last period for which work is feasible. $N \geq T$;

$N+1$ is the date of expiration for the individual;

$\Gamma(\cdot) = \{y(0,\cdot),y(1,\cdot),\ldots\}$ is a random process which is well defined and measurable on the probability space (Ω,F,Q) ;

$y(t,\omega)$ is the value taken on by the random variable $y(t,\cdot) = (y_{t+1}(t,\cdot), y_{t+2}(t,\cdot), \ldots, y_T(t,\cdot))$ when the state of the world is $\omega \in \Omega$;

$y^{\bullet}(t)$ is the wage stream offer actually observed in period t ;

$\Upsilon(t)$ is the observed current event as of date t , $(y^{\bullet}(0), y^{\bullet}(1), \ldots, y^{\bullet}(t-1))$;

$$Y_t^s(\Upsilon(t)) = \max_{i \in \{t-s, t-s+1, \ldots, t-1\}} \left(y^{\bullet}_{i+1}(i) + \sum_{j=2}^{T-t+1} \frac{y^{\bullet}_{i+j}(i)}{\prod_{K=t}^{t+j-2} (1+r(K))} \right)$$

is the highest present value of income available to the individual given that he can accept any single wage stream offer received in the last s periods;

$$G^t(\bar{\Upsilon}(t), B) \equiv Q\{\omega : y(t,\omega) \in B \,|\, \bar{\Upsilon}(t)\}$$

$$dG^t(\Upsilon(t), b) \equiv Q\{\omega : y(t,\omega) \in [b, b + db) \mid \bar{\Upsilon}(t)\}$$

$b(t)$ is the dollar magnitude of unemployment benefits in period t ;

$s(t)$ is the dollar magnitude of search costs in period t ;

B_{N+1} is a finite lower bound for the dollar value of bequests;

$V(C(0), \ell(0), C(1), \ell(1), \ldots, C(N), \ell(N))$ is a function representing the consumers' preferences over consumption-leisure vectors. This function is denoted as the individual's utility function.

The following assumptions, noted in the body of this paper, are utilized in what follows.

A1) If the individual has not accepted a job prior to date $t = 0, 1, \ldots, T$, then $C(t)$ is chosen prior to observing the value of the random variable $y(t,\cdot)$.

A2) If the individual accepts an offer during period $t = 0 , 1, \ldots, T - 1$, labor income commences in period $t + 1$ and continues through period T . No observations on the random variables

$y(t+1,\cdot)\ y(t+2,\cdot)\ \ldots,\ y(T,\cdot)$ are obtained subsequent to job acceptance.

A3) If the individual is unemployed at date $t = 0, 1, \ldots, T$
$A(t+1) = [A(t) - s(t) + b(t) - p(t)C(t)](1 + r(t))$.
If the individual has accepted the wage stream offer $y^{\circ}(t-j)$ during period $(t - i)$, $i \leqslant j$, then

$$A(t+1) = [A(t) + y_{t-j+i+1}(t-j) - p(t)C(t)](1+r(t)) \quad t \leqslant T .$$

For $t = T+1, \ldots, N$

$$A(t+1) = [A(t) - p(t)C(t)](1+r(t)) .$$

A4) $\ell(t) = \ell'$, $t = 0, 1, \ldots, T$, and $\ell(t) = \ell^{\circ}$, $t = T+1, \ldots, N$.

A5) Utility is intertemporally separable, i.e.

$$V(C(0),\ell(0), \ldots, C(N),\ell(N)) = \sum_{i=0}^{N} V_i(C(i),\ell(i)) .$$

Because of A4) we may eliminate the $\ell(i)$'s from our notation by defining the N+1 functions

$$u_i(C(i)) \equiv \begin{cases} V_i(C(i),\ell') & \text{for } i = 0, 1, \ldots T \\ \text{or} \\ V_i(C(i),\ell^{\circ}) & \text{for } i = T+1, \ldots, N . \end{cases}$$

Each $u_i(\cdot)$ is a continuously differentiable, strictly concave and increasing function of $(i) \in [0, \infty)$.

A6) $\mathcal{Y}(t)$, the range of the function $y(t,\cdot)$, is a bounded denumerable set, $t = 0, 1, \ldots, T$.[6]

Using the definitions provided here, an individual at date t may be characterized by his endowments at date t, $A(t)$, his observed current event, $\gamma(t)$, his subjective expectations for future wage offers given $\gamma(t)$, $G^t(\gamma(t), B)$, and his preferences, $V(C(0), \ell(0), \ldots, (N), \ell(N))$.

Postulate I:[7] In each period the individual ranks alternative random consumption-leisure sequences according to the expected value of the utility each of them would provide. The individual behaves as if he were attempting to choose the feasible strategy which yields the highest expected utility.

This postulate contains two terms which are as yet undefined, strategy and feasible.

1) A strategy at date t^o is a plan which specifies a sequence $C(t^o)$, $C(t^o + 1), \ldots, C(N)$, and whether the current or some previous wage offer will be accepted for each sequence of $\gamma(t^o), \gamma(t^o + 1), \ldots, \gamma(N)$. If in period $t^o + s$ an offer is observed which the strategy specifies as being acceptable, no further offers will be observed and $\gamma(t^o + s + i) = (y^o(0), \ldots, y^o(t^o + s), y^o(t^o + s), \ldots, y^o(t^o + s))$.

2) A strategy is feasible if and only if for every $(\gamma(0), \gamma(1), \ldots, \gamma(N))$:

a) $C(t) \geqslant 0 \quad \forall \quad t=0, 1, \ldots, N$

b) $A(N+1) \geqslant B_{N+1}$

c) $$A(t) \geqslant -\left[b(t)-s(t) + \sum_{i=t+1}^{T} \frac{b(i)-s(i)}{\prod_{j=t}^{i-1}(1+r(j))} + \frac{B_{N+1}}{\prod_{j=t}^{N}(1+r(j))}\right] \equiv B_t^s$$

when the individual is unemployed at date $t-1$.

d) $$A(t) \geqslant -\left[y_{t-j+k}(t-j) + \sum_{i=t-j+k+1}^{T-j+k} \frac{y_i(t-j)}{\prod_{j=t}^{i-1}(1+r(j))}\right] + \frac{B_{N+1}}{\prod_{j=t}^{N}(1+r(j))}$$

when wage stream offer $y(t-j)$ has been accepted in period $(t-k)$, $k \leqslant j$.

Given these assumptions and definitions, a dynamic programming backward solution to the individual's sequential expected utility maximization may be obtained. The essential feature of this method is that it embodies Bellman's "Principle of Optimality" at each decision node or date.

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. [4].

In the present framework it is best to consider first the individual's utility maximizing strategy upon entering one of the absorbing states of the model, retirement and employment (A2, A3). The job search strategy then consists of a sequential decision rule specifying conditions under which such a state will be entered as well as consumption levels at each date.

The individual must retire effective at date $(T+1)$. After retiring his only decisions relate to the allocation of his asset holdings among his remaining years. The utility he obtains from date $T+1$ forward thus depends solely on $A(T+1)$, and is representable by a function, R_{T+1}, with

$$R_{T+1}(A(T+1)) = \max_{C(T+1),\ \ldots,\ C(N)} \sum_{i=T+1}^{N} u_i(C(i))$$

subject to

$$A(i+1) = [A(i)-P(i)C(i)](1+r(i)) \quad i = T+1, \ldots, N, \quad A(N+1) \geq B_{N+1}$$

and

$$C(i) \geq 0 \ \forall \ i = T+1, \ldots, N.$$

<u>Lemma 1</u>. $R_{T+1}(\cdot)$ is a continuously differentiable, increasing and strictly concave function of $A(T+1)$ on $[B^s_{T+1}, \infty)$.

Proof: Bellman's proofs of Lemma 1 and Theorem 6 on pages 21-23 of [4] are sufficient to establish this result//

Notice that the form of the utility function renders that which has transpired prior to $(T+1)$ only important through its effect on $A(T+1)$ at this point in time. Similarly the individual's strategy subsequent to his acceptance of employment is simply an optimal allocation of his certain assets and guaranteed income. The maximum utility attainable from date t forward given current asset holdings, $A(t)$, and having accepted the most lucrative wage offer available in the preceding period is represented by the function E_t. On $[B_{N+1}/\prod_{i=t}^{N}(1+r(i)), \infty)$,

$$E_t(A(t)+Y_t^s(\gamma(t))) = \max_{C(t), \ldots, C(T), A(T+1)} \sum_{i=t}^{T} u_i(C(i)) + R_{T+1}(A(T+1))$$

subject to

$$A(t)+Y_t^s(\gamma(t)) \geq C(t) + \sum_{i=t+1}^{T} \frac{P(i)C(i)}{\prod_{j=t}^{i-1}(1+r(j))} + \frac{A(T+1)}{\prod_{j=t}^{T}(1+r(j))}$$

$$C(i) \geq 0 \quad i = t, \ldots, T, \quad A(T+1) \geq B_{T+1}^s ,$$

elsewhere

$$E_t(A(t)+Y_t^s(\gamma(t))) = \sum_{i=t}^{T} u_i(0) + R_{T+1}(B_{T+1}^s) - 1 .$$

Lemma 2. $E_t(\cdot)$ is a continuously differentiable, increasing and concave function of $[A(t) + Y_t^s(\gamma(t))]$ on $[B_{N+1}/\prod_{i=t}^{N}(1+r(i)), \infty)$.

Proof: Same as for Lemma 1//

Utilizing Lemma 1 and Lemma 2 we may now consider the individual's strategy prior to date $T+1$ when he has not entered the absorbing employment state. At date T the individual once again has no options regarding becoming employed since he must retire before any job taken in this period could be commenced. The function S_T represents the maximum utility attainable if period T is entered unemployed with assets $A(T)$ and past wage observations $\gamma(T)$.

$$S_T(A(T),\ \gamma(T)) = \max_{C(T),\ A(T+1)} u_T(C(T)) + R_{T+1}(A(T+1))$$

subject to

$$A(T+1) = [A(T) + b(T) - s(T) - p(T)\ C(T)](1+r(T)),\ C(T) \geq 0$$

and

$$A(T+1) \geq \frac{B_{N+1}}{\prod_{j=T+1}^{N} (1+r(j))}\ .$$ [8]

Lemma 3. S_T is continuously differentiable, increasing and concave in $A(t)$ on $[B_t^S,\infty)$ and measurable in $\gamma(T)$ with respect to Q.

Proof: Bellman's proofs cited in Lemma 1 establish "continuously differentiable, increasing and concave in $A(T)$." Measurability holds trivially since $S_T(\cdot)$ is a constant function of $\gamma(T)$ //

During any period prior to T the individual may accept employment if he has not already done so. Thus, if at any date prior to T the individual is unemployed he must make the joint decisions on consumption and whether or not to accept employment. In Lemmas 4 through 6 the general form of as well as the expected utility associated with an optimal consumption-employment choice strategy are detailed.

The function $S_t(A(t), \gamma(t))$, $t=0, 1, \ldots, T$, represents the maximum expected utility associated with entering period t unemployed, with assets equal to $A(t) \geqslant B_t^s$ and having past wage stream observations $\gamma(t)$. This function and its properties are obtained in an inductive fashion. We take as an induction hypothesis that S_t, $t < T$, is increasing, and continuous in $A(t) \geqslant B_t^s$, is a measurable and bounded function of $\gamma(t)$ on $\Psi(0)x\ldots x\Psi(t-1)$, and that $S_t(A(t), \gamma(t))$ is the maximum expected utility associated with entering period t unemployed with assets $A(t)$ and past wage observations $\gamma(t)$. The lemmas then establish these properties for S_{t-1}. Since the induction hypothesis is clearly valid for $S_T(\cdot)$ by Lemma 3, S_{t-1}, $t=0, 1, \ldots, T$, has the properties and interpretation claimed.

The interpretation given to $S_t(\cdot)$ makes the job acceptance decision subsequent to the receipt of wage stream offer $y(t-1)$ an immediate consequence of the relative magnitudes of $E_t(A(t) + Y_t^s(\gamma(t)))$ and $S_t(A(t), \gamma(t))$. In keeping with the criterion of expected utility maximization, if $E_t(\cdot)$ exceeds $S_t(\cdot)$ employment is chosen while the converse relationship leads to continued search. Should equality obtain no definite choice is indicated.

The following decision rule provides for a determinate choice for any $y(t-1)$ and previously observed wage stream offers $\gamma(t-1)$ and chosen level of $A(t)$.

Accept employment if

(1) $$y(t-1) \in G_{t-1}(A(t), \gamma(t-1))$$

and,

continue search if

$$y(t-1) \in \Psi(t-1) - G_{t-1}(A(t),\ \gamma(t-1)) \quad \text{where}$$

$$G_{t-1}(A(t),\gamma(t-1)) = \{y(t-1)\in\Psi(t-1): E_t(A(t) + Y_t^s(\gamma(t-1),y(t-1))) \geq S_t(A(t),(\gamma(t-1),y(t-1)))\}$$

(1) belongs to the set of "measurable decision rules".

Definition: A measurable decision rule, MDR, is a measurable function, g , which associates with each $\bar{y}(t-1) \in \Psi(t-1)$ a probability, $g(\bar{y}(t-1))$, that employment will be accepted during period $(t-1)$ if $\overline{y}(t-1)$ is observed.

Note that an MDR may be either random, the probability that a given job will be accepted and the probability that it will be rejected both positive, or pure, probability of accepting employment given any particular wage observation either one or zero.

Lemma 4 indicates that (1) is an expected utility maximizing MDR.

Lemma 4. For any feasible $C(t-1)$, $A(t)$ and $\gamma(t-1)$ there exists no MDR for employment acceptance or rejection yielding greater expected utility at date $t-1$ than (1).

Proof: We associate with (1) the measurable decision rule

$$K_{G_{t-1}(A(t),\gamma(t-1))}(y(t-1)) = \begin{cases} 1 & \text{if } y(t-1) \in G_{t-1}(A(t),\gamma(t-1)) \\ 0 & \text{if } y(t-1) \notin G_{t-1}(A(t),\gamma(t-1)) . \end{cases}$$

That is, the probability that employment will be accepted upon observing $\overline{y}(t-1) \in \Psi(t-1)$ is given by the characteristic function of $G_{t-1}(A(t),\gamma(t-1))$. Since $E_t(\cdot)$ and $S_t(\cdot)$ are measurable, $G_{t-1}(\cdot)$ is measurable and hence (1) satisfies the definition of a measurable decision rule.

The expected utility at date $(t-1)$ for any feasible $\overline{C(t-1)}$, $\overline{A(t)}$, $\overline{\gamma(t-1)}$ and arbitrary admissible decision rule, $g(\cdot)$, is:

$$(2)\quad u_{t-1}(\overline{C(t-1)}) + \int_{\Psi(t-1)} g(v)E_t(\overline{A(t)}+Y_t^S(\overline{\gamma(t-1)},v))dG^{t-1}(\overline{\gamma(t-1)},v)$$

$$+ \int_{\Psi(t-1)} [1-g(v)]S_t(\overline{A(t)},(\overline{\gamma(t-1)},v))dG^{t-1}(\overline{\gamma(t-1)},v) .$$

One may easily verify that the existence of an admissible function, g, yielding a larger value for (2) than $K_{G_{t-1}}(\cdot)$ implies the contradiction that either

$$S_t(\overline{A(t)},(\overline{\gamma(t-1)},\overline{y(t-1)})) > E_t(\overline{A(t)}+Y_t^S(\overline{\gamma(t-1)},\overline{y(t-1)}))$$

and $y(t-1) \in G_{t-1}(\overline{A(t)},\overline{\gamma(t-1)})$ or

$$S_t(\overline{A(t)},(\overline{\gamma(t-1)},\overline{y(t-1)})) \leq E_t(\overline{A(t)}+Y_t^S(\overline{\gamma(t-1)},\overline{y(t-1)}))$$

and $y(t-1) \notin G_{t-1}(\overline{A(t)},\overline{\gamma(t-1)})$. It is also apparent that $K_{G_{t-1}}(\cdot)$ is not necessarily a unique maximizer since rejecting employment when $E_t(\cdot) = S_t(\cdot)$ would have no effect on expected utility. Likewise an M.D.R. which differs from $K_{G_{t-1}}(\cdot)$ on a set of measure zero would yield the same expected utility//

Knowing what wage stream offers he will accept or reject in the upcoming period for each state of nature allows the individual to calculate his expected utility from next period onward as a function of next period's initial wealth holdings and previously observed wage stream offers. This expected utility is represented by the function Z_t .

$$(3)\quad Z_t(A(t),\gamma(t-1)) = \int\limits_{G_{t-1}(A(t),\gamma(t-1))} E_t(A(t)+Y_t^s(\gamma(t-1),v))dG^{t-1}(\gamma(t-1),v)$$

$$+ \int\limits_{\Psi(t-1)-G_{t-1}(A(t),\gamma(t-1))} S_t(A(t),(\gamma(t-1),v))dG^{t-1}(\gamma(t-1),v)$$

$$= \int\limits_{\Psi(t-1)} \max[E_t(A(t)+Y_t^s(\gamma(t-1),v)) \text{ or } S_t(A(t),(\gamma(t-1,v))]dG^{t-1}(\gamma(t-1),v)$$

<u>Lemma 5</u>. $Z_t(\cdot)$ is a continuous and strictly increasing function of $A(t)$ on $[B_t^s,\infty)$, and is a measurable function of $\gamma(t-1)$ on $\Psi(0)\times\ldots\times\Psi(t-2)$.

<u>Proof</u>: First let us define a function

$$f_t:[B_t^s,\infty)\times\Psi(0)\times\ldots\times\Psi(t-1) \to R^1 ,$$

$$f_t(A(t),\gamma(t)) = \max[E_t(A(t)+Y_t^s(\gamma(t))) \text{ or } S_t(A(t),\gamma(t))]$$

Since $E_t(\cdot)$ is a continuous function on $[B_{N+1}\prod_{i=t}^{N}(1+r(i)),\ \infty)$ and $Y_t^s(\cdot)$ is in turn continuous in $\gamma(t)$, E_t is measurable. S_t is, by hypothesis, measurable. Hence, as the maximum of two measurable functions, $f_t(A(t),\cdot)$ is a measurable function of $\gamma(t)$ w.r.t. Q. Since $S_t(A(t),\cdot)$ and $E_t(A(t)+Y_t^s(\cdot))$ are bounded on the range of $\gamma(t)$, $f_t(A(t),(\gamma(t-1),\cdot))$ is clearly bounded on $\Psi(t-1)$ (itself a bounded set), and is, therefore, integrable on $\Psi(t-1)$.

Also since E_t and S_t are continuous functions of $A(t)$, so is f_t (max of two continuous functions is continuous).

Hence for any sequence $\{A^q(t)\}$ with $A^q(t) \to A^*(t)$, $f_t(A^q(t), \gamma(t)) \to f_t(A^*(t), \gamma(t))$. This point-wise convergence of the integrand of (3) implies that $Z_t(A^q(t), \gamma(t-1)) \to Z_t(A^*(t), \gamma(t-1))$ by the Lebesgue Dominated ccnvergence theorem. The strictly increasing nature of f_t in $A(t)$ directly implies Z_t increasing in $A(t)$. Boundedness also follows immediately from the boundedness of f_t.

Since $f_t(A(t), \gamma(t))$ is measurable w.r.t. Q, we may find a sequence of measurable simple functions $\{S_n(\gamma(t))\}$ such that $\lim_{n\to\infty} S_n(\gamma(t)) = f(A(t), \gamma(t))$ on $\Psi(0)x\ldots x\Psi(t-1)$. We have

$$\int_{\Psi(t-1)} S_n(\gamma(t-1), \cdot)dG^{t-1}(\gamma(t-1),\cdot) = C_1G^{t-1}(\gamma(t-1), B_1) + \ldots + C_mG^{t-1}(\gamma(t-1), B_m)$$

with $B_i \subset \Psi(t-1)$ and B_i measurable, $i=1, \ldots, m$. Since A6) implies $G^{t-1}(\gamma(t-1), B_i)$ are measurable functions of $\gamma(t-1)$ w.r.t. Q

$$\sum_{i=1}^{m} C_iG^{t-1}(\gamma(t-1), B_i) = g_n(\gamma(t-1))$$

is a measurable function of $\gamma(t-1)$ w.r.t. Q.

$$\lim_{n\to\infty} g_n(\gamma(t-1)) \to \int_{\Psi(t-1)} f_t(A(t), \gamma(t-1),\cdot)dG^{t-1}(\gamma(t-1),\cdot)$$

$$= Z(A(t), \gamma(t-1))$$

is thus a measurable function of $\gamma(t-1)$ w.r.t. Q //

The expected utility from date $t-1$ forward for the unemployed individual may now be determined as the sum of $u_{t-1}(C(t-1))$ and $Z_t(A(t), \gamma(t-1))$ where $C(t-1)$ and $A(t)$

must, of course, satisfy appropriate budget constraints. The maximum expected utility associated with being unemployed at date $t-1$, having asset holdings $A(t-1)$ and having previous wage offer observations $\gamma(t-1)$ is thus

$$(4)\quad S_{t-1}(A(t-1),\gamma(t-1)) = \max_{C(t-1),A(t)} u_{t-1}(C(t-1)) + Z_t(A(t),\gamma(t-1))$$

subject to $A(t) = [A(t-1) + b(t-1)-s(t-1)-p(t-1)C(t-1)](1+r(t-1))$

$$C(t-1) \geqslant 0 \quad \text{and} \quad A(t) \geqslant B_t^s .$$

Lemma 6. $S_{t-1}(\cdot,\gamma(t-1))$ is a continuous and increasing function of $A(t-1)$ on $[B_{t-1}^s,\infty)$. $S_{t-1}(A(t-1),\cdot)$ is a measurable function of $\gamma(t-1)$ on $\Psi(0)\times\ldots\times\Psi(t-2)$ w.r.t. Q. $S_{t-1}(A(t-1),\cdot)$ is bounded on $\Psi(0)\times\ldots\times\Psi(t-2)$.

Proof: First, Bellman's proofs cited in the proof of Lemma 1 may be used once again to establish that $S_{t-1}(\cdot,\gamma(t-1))$ is continuous and increasing in $A(t-1)$ on $[B_{t-1}^s,\infty)$.

Second, note that

$$0 \leqslant C(t-1) \leqslant \frac{\left[\frac{B_t^s}{(1+r(t-1))} - A(t-1)-b(t-1)-s(t-1)\right]}{p(t-1)}$$

since $A(t) \geqslant B_t^s$. Denote the upper bound for $C(t-1)$ in this inequality as M. We now define certain particular feasible values for $C(t-1)$ as

$$C_{nj}(t-1) = \frac{j}{2^n} M \quad j=0,\ 1,\ \ldots,\ 2^n,\ n=0,\ 1,\ \ldots$$

and $A_{nj}(t) = [A(t-1) + b(t-1) - s(t-1) - p(t-1)C_{nj}(t-1)]$. Since $Z(A(t),\gamma(t-1))$ is a measurable function of $\gamma(t-1)$ for any feasible $A(t)$,

$$\{\gamma(t-1):\ Z(A_{nj}(t),\gamma(t-1)) > b-u_{t-1}(C_{nj}(t-1))\}$$

is measurable $(b-u_{t-1}(C_{nj}(t-1))$ is just a real number).

Likewise

$$B_n = \bigcup_{j=0}^{2^n} \{\gamma(t-1) | Z(A_{nj}(t), \gamma(t-1)) > b-u_{t-1}(C_{nj}(t-1))\}$$

is measurable w.r.t. Q. Thus $\bigcup_{n=0}^{\infty} B_n$ is measurable. Claim

$$\bigcup_{n=0}^{\infty} B_n = \{\gamma(t-1) | S_{t-1}(A(t-1),(\gamma(t-1)) > b\} .$$

If $\gamma(t-1) \in B_n$ for some n then

$$b < u_{t-1}(C_{nj}(t-1)) + Z(A_{nj}(t), \bar{\gamma}(t-1)) \quad \text{for some } j=0, \ldots, 2^n$$

$$\leq S_{t-1}(A(t-1), \bar{\gamma}(t-1)) .$$

If $S_{t-1}(A(t-1),\bar{\gamma}(t-1)) > b$ then there exists some n such that $\bar{\gamma}(t-1) \in B_n$. This follows clearly from the continuity $u_{t-1}(\cdot)$ and $Z_t(\cdot)$ in $C(t-1)$ and $A(t)$ respectively since every feasible $(C(t-1), A(t))$ is the limit of some sequence of $\{C_{nj}(t-1), A_{nj}(t)\}$ //

We now know several things about the individual's strategy and its value to him. When unemployed he makes his consumption and job choice decisions jointly. As one might expect, he accepts a job whenever the expected utility from doing so exceeds that attainable by continuing search. This decision rule gives rise to a set of wage offers at each date which will be accepted if received. This set is referred to as the acceptance set and its composition depends on the wage offers which the individual has received in the past as well as his current asset position.

The dependence of the acceptance set on current wealth holdings leads to some rather unexpected findings. In particular, the expected utility of an unemployed individual is not necessarily a concave function of wealth even though the u_i's and the E_i's are strictly concave. Thus, while the individual would never think of accepting a fair gamble when employed or retired, he might accept a fair or slightly unfair gamble when unemployed.

A further consequence of the possible non-concavity of the $S_i(\cdot,\gamma(i))$ and $Z_{i+1}(\cdot,\gamma(i))$ functions is that there may be a multiplicity of maximizing strategies. Though each maximizing strategy yields the same expected utility, the consumption, savings and job acceptance decisions may differ among them. As we shall see, no fundamental difficulties result from having non-unique maximizers.

Before proceeding farther with this analysis the following additional result is easily established for independently distributed wage stream offers.

<u>Lemma 7</u>. The acceptance set is monotone if the $y(t)$'s are independently distributed and $s = 0$ for $Y_t^s(\gamma(t))$.

An acceptance set is termed monotone if when one wage stream offer is in the set this implies all wage stream offers with higher or the same present values are also in the set.

By definition $Y_t^0(\gamma(t))$ is merely the present discounted value of wage stream offer $y(t-1)$. The condition that $s = 0$ thus is the requirement that wage offers must be accepted or rejected during the period in which they are received.

Proof: Again induction is used to establish the result. Let the induction hypothesis be $S_t(A(t), \gamma(t))$ is independent of $\gamma(t)$. Since $E_t(A(t) + Y_t^o(\gamma(t)))$ is increasing in $Y_t^{\bullet}(\gamma(t))$ (specifically in $y(t-1)$), $G_{t-1}(A(t),\gamma(t-1))$ as defined in (1) is monotone. Also given that S_t, E_t and G^{t-1} do not depend on $\gamma(t-1)$, it is clear that Z_t is independent of $\gamma(t-1)$. This implies immediately that S_{t-1} is independent of $\gamma(t-1)$. The proof is completed by observing that the values taken on by $S_T(\cdot)$ are independent of $\gamma(T-1)$ //

Next let us define a concept which shall crop up frequently in our remaining discussion.

Definition: The certain dollar value of job search at date t given $A(t)$ and $\gamma(t)$, $CV_t(A(t), \gamma(t))$, is the value of X for which the following equality holds:

$$S_t(A(t), \gamma(t)) = E_t(A(t) + X) .$$

Lemma 8. $CV_t(\cdot,\gamma(t))$ is a continuous function of $A(t)$.

Proof: Since $\psi(t)$ is bounded, $t=0, \ldots, T$, there exists a number $M > Y_t^s(\gamma(t))$ for all feasible $\gamma(t)$. Also because $E_t(\cdot)$ is a strictly increasing function, it is easily seen that

$$E_t(A(t) + M) > S_t(A(t), \gamma(t))$$

for all feasible $A(t)$, $\gamma(t)$ pairs. Likewise defining

$$m = \frac{B_{N+1}}{\Pi_{j=t}^{N}(1+r(j))} - A(t)$$

it is clear that

$$E_t(A(t) + m) \leq S_t(A(t), \gamma(t))$$

for all feasible $A(t)$, $\gamma(t)$ pairs. Since $E_t(\cdot)$ is continuous and increasing, $CV_t(A(t), \gamma(t))$ is a well-defined function on $[B_t^s, \infty) X \mathcal{Y}(0) X \ldots X \mathcal{Y}(t-1)$.

Next consider any sequence $\{A^q(t)\}$ converging to $A^*(t)$ with $A^q(t) \geqq B_t^s$, $q = 0, 1, \ldots$. Since S_t is continuous

$$S_t(A^q(t), \gamma(t)) \to S_t(A^*(t), \gamma(t)) \ .$$

By the definition of $CV_t(A(t),\gamma(t))$ we must therefore have

$$E_t(A^q(t) + CV_t(A^q(t),\gamma(t))) \to E_t(A^*(t) + CV_t(A^*(t),\gamma(t))) \ .$$

Since E_t is increasing the convergence of E_t implies

$$A^q(T) + CV_t(A^q(t),\gamma(t)) \to A^*(t) + CV_t(A^*(t),\gamma(t))$$

hence

$$CV_t(A^q(t),\gamma(t)) \to CV_t(A^*(t),\gamma(t)) //$$

One may view the job search activity as a type of lottery. Each outcome is a particular income stream. The lottery evolves over time as various unacceptable job offers are encountered and terminates when a specific job is accepted. The individual would be indifferent between the job search lottery and $CV_t(A(t), \gamma(t))$ dollars.

Theorem 1: $CV_t(\cdot,\gamma(t))$ is a nondecreasing (increasing) function of $A(t)$ for any feasible $\gamma(t)$ if

$$r_{u_i}(C(i)) = \frac{-u_i''(C(i))}{u_i'(C(i))}$$

is a nonincreasing (decreasing) function of $C(i)$ on $[0,\infty)$, $i=0, 1, \ldots, N$.

Proof: Theorem 2 page 28 of [5] implies this result directly//

In words this theorem states that if each of the one period utility functions displays nonincreasing (decreasing) absolute risk aversion, then the certain dollar value of job search is a nondecreasing (increasing) function of asset holdings for any previously observed wage stream offers. The assumption of nonincreasing absolute risk aversion upon which Theorem 1 depends shall be utilized in the remainder of this paper.

Assumption 7. $r_{u_i}(C(i)) = \dfrac{-u_i''(C(i))}{u_i'(C(i))}$ is a nonincreasing function of $C(i)$ on $[0,\infty)$ for all $i=0, 1, \ldots, N$.

Corollary 1 is now easily established.

Corollary 1: For any $\delta > 0$, and given $\gamma(t-1)$ and $A(t)$,

$$G_{t-1}(A(t),\gamma(t-1)) \supseteq G_{t-1}(A(t) + \delta,\gamma(t-1))$$

Proof: Assume the contradiction that there exists a $\bar{y}_{(t-1)} \in G_{t-1}(A(t) + \delta, \gamma(t-1))$ and $\bar{y}_{(t-1)} \notin G_{t-1}(A(t), \gamma(t-1))$. From Lemma 4 we have:

$$E_t(A(t) + \delta + Y_t^s(\gamma(t-1),\bar{y}(t-1))) \geq S_t(A(t) + \delta,(\gamma(t-1),\bar{y}(t-1)))$$

and

$$E_t(A(t) + Y_t^s\ (\gamma(t-1),\bar{y}(t-1))) < S_t(A(t),(\gamma(t-1),\bar{y}(t-1)))\ .$$

Since E_t is increasing in $Y_t^s(\gamma(t-1),y(t-1))$, the last inequality implies that $CV_t(A(t),\ (\gamma(t-1),\bar{y}(t-1))) > Y_t^s(\gamma(t-1),\overline{y(t-1)})$

which together with Theorem 1 make the first weak inequality impossible//

Notice that the probability that an individual will no longer be unemployed in the next period is directly related to the size of his current wage stream acceptance set. Corollary 1 therefore provides a link between asset holdings and state transition probabilities. One might suspect that this link could be easily extended to establish an empirically testable relationship between the expected duration of unemployment and wealth. The possible multiplicity of expected utility maximizing strategies mentioned in the discussion of Lemma 6, however, makes this task fairly complex.

In order to resolve this apparent difficulty some additional discussion of the nature of specific search strategies is required.

First let $\Sigma(A(t),\gamma(t))$ represent the nonempty set of expected utility maximizing completely specified strategies for any feasible $(A(t),\gamma(t))$ pair. Recall that each $\sigma \in \Sigma(A(t),\gamma(t))$ specifies a particular feasible sequence of consumption, asset stocks and acceptance sets for each possible sequence of wage stream offers from periods t through T. Thus, if the individual decides at date t to employ strategy $\sigma \in \Sigma(A(t), \gamma(t))$, there is a uniquely determined level of wealth, denoted $A^{\sigma}_{t+i}(\gamma(t+i-1))$, which will be held at date $(t+i)$, $i \geqslant 1$ if wage stream offers $\gamma(t+i-1) = (\bar{\gamma}(t),\bar{y}(t),\ldots,\bar{y}(t+i-2))$ are observed in the intervening periods. Notice that since

$C(t)$ is determined prior to the observation of a $y(t)$, the level of wealth held at date $(t+1)$ for strategy σ does not depend on wage stream offers other than $\gamma(t)$.

The set of wage stream offers which would lead the individual to accept employment in period $(t+i)$, given he is employing strategy $\sigma \in \Sigma(\overline{A(t)},\overline{\gamma}(t))$ and has observed wage stream offers $(\overline{y(t)}, .., \overline{y(t+i-1)})$, is

$$G_{t+i}(A^{\sigma}_{t+i+1}(\gamma(t+i)),\gamma(t+i))$$

given $\gamma(t+i) = (\overline{\gamma(t)},\overline{y(t)}, \ldots, \overline{y(t+i-1)})$. The probability that this individual will be unemployed and searching at date $(t+n)$ may, therefore, be expressed as,

$$(5) \quad P^{\sigma}_{t+n} = Q\{\omega \in \Omega\colon y(t,\omega) \notin G_t(A^{\sigma}_{t+1}(\overline{\gamma(t)}),\overline{\gamma(t)}),$$

$$\ldots, y(t+n-1) \notin G_{t+n-1}(A^{\sigma}_{t+n}(\gamma(t+n-1)),\gamma(t+n-1))\} .$$

That P^{σ}_{t+n} exists (that the set in brackets is measurable) can be shown to follow from denumerability of the $\psi(i)$, $i=0, 1, \ldots, T$.

Given a precise notion of the probability of being unemployed at any date for any particular expected utility maximizing strategy we are now able to consider the question of expected duration of unemployment.

Definition: The expected duration of search unemployment as of date t for an unemployed individual employing strategy $\sigma \in \Sigma(A(t),\gamma(t))$, $E(DS|\sigma)$, is defined by the equality

$$(6) \quad E(DS|\sigma) \equiv \sum_{i=t}^{T}(1+i-t)[P^{\sigma}_{i} - P^{\sigma}_{i+1}] .$$

Theorem 2: If $\sigma^1 \epsilon \Sigma(A^1(t),\overline{y(t)})$ and $\sigma^2 \epsilon \Sigma(A^2(t),\overline{y(t)})$ and $A^2(t) > A^1(t)$, then

$$E(DS|\sigma^1) \leq E(DS|\sigma^2) .$$

This theorem states that, all other things being equal, the expected duration of search unemployment will either increase or remain the same if wealth holdings are increased.

Proof: First notice that since the individual is unemployed at date t, $P_t^\sigma = 1$. Also since retirement commences at T+1 , $P_{T+1}^\sigma = 0$. This allows us to rewrite our equation (6) as

$$E(DS|\sigma) = 1 + \sum_{i=t+1}^{T} P_i^\sigma .$$

I claim that $P_{t+i}^{\sigma^1} \leq P_{t+i}^{\sigma^2}$ $i=t+1, \ldots, T$. This is established by showing that for $j=0, \ldots, T-1-t$,

$$\begin{aligned} &\overline{y(t}+j) \not\leq G_{t+j}(A_{t+j+1}^{\sigma^1}(y(t+j)), y(t+j)) \\ (7) \quad &\Rightarrow \overline{y(t}+j) \not\leq G_{t+j}(A_{t+j+1}^{\sigma^2}(y(t+j)), y(t+j)) . \end{aligned}$$

The following lemma in conjunction with Corollary 1 verifies this implication.

Lemma 9. If $\sigma^1 \epsilon \Sigma(A^1(t),\bar{y}(t))$ and $\sigma^2 \epsilon \Sigma(A^2(t),\bar{y}(t))$ and $A^2(t) > A^1(t)$, then either

$$A_{t+1}^{\sigma^2}(y(t)) > A_{t+1}^{\sigma^1}(y(t)) \quad \text{or} \quad A_{t+1}^{\sigma^2}(y(t)) = A_{t+1}^{\sigma^1}(y(t)) = B_{t+1}^s ,$$

$$t=0, \ldots, T-1 .$$

This lemma has a straightforward interpretation. Regardless of the maximizing search strategies chosen, if current asset holdings are increased the next period's beginning asset holdings will increase with only one exception. The sole exception is that next period's asset holdings may stay the same if they were initially at the lower bound for that period's allowable asset level. The proof of Lemma 9 is contained in the appendix to this paper.

Repeated application of this lemma insures that

$$A^{\sigma^1}_{t+j+1}(\gamma(t+j)) \leq A^{\sigma^2}_{t+j+1}(\gamma(t+j)),\ j=0,\ \ldots,\ T-t-1\ .$$

Thus Corollary 1 establishes the implication given in (7). Referring to the expression for P^{σ}_{t+1} given above it is apparent that this in turn implies $P^{\sigma^1}_{t+i} \leq P^{\sigma^2}_{t+i}$, $i=1, \ldots, T-t$. This completes the proof of Theorem 2//

We shall now consider another type of expectation held by the individual. In particular his expected noninterest income. When search costs and unemployment benefits just equal one another in every period expected noninterest income is simply expected wage earnings. In general, the expected present value of noninterest income for an unemployed individual at any date $t+i$, $i=0, 1, \ldots, T-t$, denoted PVY_{t+i} , depends on $A(t+i) = A^{\sigma}_{t+i}(\gamma(t+i-1))$, $\gamma(t+i)) = (\bar{\gamma}(t), y(t), \ldots, y(t+i-1))$ and, of course, that maximizing strategy which has been chosen $\sigma \in \Sigma(A(t), \bar{\gamma}(t))$.

PVY_t given $A(t)$, $\gamma(t)$ and $\sigma \in \Sigma(A(t), \gamma(t))$ is obtained using the inductive definition

$$(8)\quad (PVY_{t+i}|A^{\sigma}_{t+i}(\gamma(t+i-1)),\gamma(t+i),\sigma) = b(t+1) - s(t+i)$$
$$+ \frac{1}{(1+r(t+i))} \int_{G_{t+i}(A^{\sigma}_{t+i+1}(\gamma(t+i)),\gamma(t+i))} Y^S_{t+i+1}(\gamma(t+i),v)\ dG^{t+i}(\gamma(t+i),v)$$

$$+ \frac{1}{(1+r(t+i))} \int_{\Psi(t+i)-G_{t+i}(A^{\sigma}_{t+i+1}(\gamma(t+i)),\gamma(t+i))} (PVY_{t+i+1}|A^{\sigma}_{t+i+1}(\gamma(t+i)),(\gamma(t+i),v),\sigma)dG^{t}(\gamma(t+i),v),$$

and $(PVY_T|A^{\sigma}_T(\gamma(T-1)),\gamma(T),\sigma) = b(T)-s(T)$.

Lemma 10. For any feasible $A(t)$, $\gamma(t)$ and $\sigma \in \Sigma(A(t),\gamma(t))$

$$(PVY_t|A(t),\gamma(t),\sigma) \geq CV_t(A(t),\gamma(t)),\ t=0,\ \ldots,\ T\ .$$

Proof: Once again an inductive proof is employed. The induction hypothesis for $j+1 > t$ is

$$(9)\quad (PVY_{j+1}|A^{\sigma}_{j+1}(\gamma(j)),\gamma(j+1),\sigma) \geq CV_{j+1}(A^{\sigma}_{j+1}(\gamma(j)),\gamma(j+1))$$

for all feasible $\gamma(j+1)$ given $\overline{\gamma(t)}$.

Now letting

$$C^{\sigma}_j(\gamma(j)) = (A^{\sigma}_j(\gamma(j-1)) - \frac{A^{\sigma}_{j+1}(\gamma(j))}{(1+r(j))} + b(j)-s(j))\ \frac{1}{\beta(i)}$$

and recalling the definition of $CV_t(\cdot)$ and of $S_t(\cdot)$, we obtain the following.

$$E_j(A^{\sigma}_j(\gamma(j-1)) + CV_j(A^{\sigma}_j(\gamma(j-1)),\gamma(j))) = S_j(A^{\sigma}_j(\gamma(j-1)),\gamma(j))$$

$$= u_j(C^{\sigma}_j(\gamma(j))) + \int_{G_j(A^{\sigma}_{j+1}(\gamma(j)),\gamma(j))} E_{j+1}(A^{\sigma}_{j+1}(\gamma(j))+Y^{s}_{j+1}(\gamma(j),v))dG^{j}(\gamma(j),v)$$

$$+ \int_{\Psi(j)-G_j(A^{\sigma}_{j+1}(\gamma(j)),\gamma(j))} S_{j+1}(A^{\sigma}_{j+1}(\gamma(j)),(\gamma(j),v))dG^{j}(\gamma(j),v)$$

$$= u_j(C^{\sigma}_j(\gamma(j))) + \int_{G_j(A^{\sigma}_{j+1}(\gamma(j)),\gamma(j))} E_{j+1}(A^{\sigma}_{j+1}(\gamma(j))+Y^{S}_{j+1}(\gamma(j),v))dG^{j}(\gamma(j),v)$$

$$+ \int_{\Psi(j)-G_j(A^{\sigma}_{j+1}(\gamma(j)),\gamma(j))} E_{j+1}(A^{\sigma}_{j+1}(\gamma(j))+CV_{j+1}(A^{\sigma}_{j+1}(\gamma(j)),(\gamma(j),v)))dG^{j}(\gamma(j),v)\ .$$

These equalities and the strict concavity of E_{j+1} imply

(10)
$$E_j(A_j^\sigma(\gamma(j-1))+CV_j(A_j^\sigma(\gamma(j-1)),\gamma(j)))$$
$$\leq u_j(C_j^\sigma(\gamma(j)))+E_{j+1}(A_{j+1}^\sigma(\gamma(j))+\frac{\int Y_{j+1}^S(\gamma(j),v)dG^j(\gamma(j),v)}{G_j(A_{j+1}^\sigma(\gamma(j)),\gamma(j))}$$
$$+\frac{\int CV_{j+1}(A_{j+1}^\sigma(\gamma(j)),(\gamma(j),v))dG^j(\gamma(j),v))}{\psi(j)-G_j(A_{j+1}^\sigma(\gamma(j)),\gamma(j))} .$$

(8), (9) and (10) taken together yield

(11)
$$E_j(A_j^\sigma(\gamma(j-1)) + CV_j(A_j^\sigma(\gamma(j-1)),\gamma(j)))$$
$$\leq u_j(C_j^\sigma(\gamma(j)))+E_{j+1}(A_{j+1}^\sigma(\gamma(j))+(1+r(j))[(PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma)$$
$$- b(j)+s(j)])$$

Now by the definition of E_t, $t=0, \ldots, T$,

(12)
$$E_j(A_j^\sigma(\gamma(j-1)) + (PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma))$$
$$= \max_{C(j)} u_j(C(j))+E_{j+1}((1+r(j))(A_j^\sigma(\gamma(j-1))+(PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma)$$
$$- p(j)C(j)))$$

subject to $C(j) \geq 0$

and $p(j)C(j) \leq A_j^\sigma(\gamma(j-1))+(PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma) - B_{N+1}/\prod_{i=j}^{N}(1+r(i))$.

Since $C_j^\sigma(\gamma(j))$ satisfies this last pair of constraints and

$$A_{j+1}^\sigma(\gamma(j)) = [A_j^\sigma(\gamma(j-1))-p(j)C_j^\sigma(\gamma(j))+b(j)-s(j)](1+r(j)) ,$$

(11) and (12) insure that

$$E_j(A_j^\sigma(\gamma(j-1))+CV_j(A_j^\sigma(\gamma(j-1)),\gamma(j)))$$
$$\leq E_j(A_j^\sigma(\gamma(j-1))+(PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma)) .$$

E_j is strictly increasing, hence

$$CV_j(A_j^\sigma(\gamma(j-1)),\gamma(j)) \leq (PVY_j|A_j^\sigma(\gamma(j-1)),\gamma(j),\sigma) \ .$$

To complete the proof we need only find some $j+1 > t$ for which our induction hypothesis is valid. For $j+1 = T$

$$CV_T(A_T^\sigma(\gamma(T-1)),\gamma(T)) = b(T)-S(T)$$

and from (8)

$$(PVY_T|A_T^\sigma(\gamma(T-1)),\gamma(T),\sigma) = b(T)-S(T) \ .$$

The proof of Lemma 10 is thus complete//

Notice that Lemma 10 may be strengthened to

$$(PVY_t|A_t^\sigma(\gamma(t-1)),\gamma(t),\sigma) > CV_t(A_t^\sigma(\gamma(t-1)),\gamma(t))$$

when the wage distribution is non-degenerate in periods subsequent to t due to the strict concavity of the E_i, $i=t+1, \ldots, T$.

This lemma is quite helpful in the proof of the next theorem. Theorem 3 asserts that the expected present value of labor force income for an unemployed labor force participant is non-negatively related to initial or current wealth.

Theorem 3: If $\sigma^1 \in \Sigma(A^1(t),\bar{\gamma}(t))$ and $\sigma^2 \in \Sigma(A^2(t),\bar{\gamma}(t))$ and $A^1(t) < A^2(t)$ then

$$(PVY_t|A^1(t),\bar{\gamma}(t),\sigma^1) \leq (PVY_t|A^2(t),\bar{\gamma}(t),\sigma^2) \ .$$

Proof: Take as an induction hypothesis for date $i \leq T$

$$(PVY_i|A_i^{\sigma^1}(\gamma(i-1)),\gamma(i),\sigma^1) \leq (PVY_i|A_i^{\sigma^2}(\gamma(i-1)),\gamma(i),\sigma^2)$$

if

$$A_i^{\sigma^1}(\gamma(i-1)) < A_i^{\sigma^2}(\gamma(i-1)) \ .$$

We are now able to obtain the following relationships.

$$(PVY_{i-1}|A^{\sigma^1}_{i-1}(\gamma(i-2)),\gamma(i-1),\sigma^1)$$

$$= b(i-1)-s(i-1) + \frac{1}{1+r(i-1)}\left(\frac{\int Y_i^s(\gamma(i-1),v)\;dG^{i-1}(\gamma(i-1),v)}{G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))} + \frac{\int (PVY_i|A_i^{\sigma^1}(\gamma(i-1)),(\gamma(i-1),v),\sigma^1)dG^{i-1}(\gamma(i-1),v)}{\psi(i-1)-G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))}\right)$$

by definition,

$$\leqslant b(i-1)-s(i-1) + \frac{1}{1+r(i-1)}\left(\frac{\int Y_i^s(\gamma(i-1),v)\;dG^{i-1}(\gamma(i-1),v)}{G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))} + \frac{\int (PVY_i|A_i^{\sigma^2}(\gamma(i-1)),(\gamma(i-1),v),\sigma^2)dG^{i-1}(\gamma(i-1),v)}{\psi(i-1)-G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))}\right)$$

by the induction hypothesis,

$$\leqslant b(i-1)-s(i-1) + \frac{1}{1+r(i-1)}\left(\frac{\int Y_i^s(\gamma(i-1),v)\;dG^{i-1}(\gamma(i-1),v)}{G_{i-1}(A_i^{\sigma^2}(\gamma(i-1)),\gamma(i-1))} + \frac{\int CV_i(A_i^{\sigma^2}(\gamma(i-1)),(\gamma(i-1),v))dG^{i-1}(\gamma(i-1),v)}{G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))-G_{i-1}(A_i^{\sigma^2}(\gamma(i-1)),\gamma(i-1))} + \frac{\int (PVY_i|A_i^{\sigma^2}(\gamma(i-1)),(\gamma(i-1),v),\sigma^2)dG^{i-1}(\gamma(i-1),v)}{\psi(i-1)-G_{i-1}(A_i^{\sigma^1}(\gamma(i-1)),\gamma(i-1))}\right)$$

by the definitions of $CV_i(\cdot)$ and $G_{i-1}(\cdot)$ and Corollary 1,

$$\leqslant b(i-1)-s(i-1)$$

$$+\frac{1}{1+r(i-1)}\left[\int Y_i^s(\gamma(i-1),v)\,dG^{i-1}(\gamma(i-1),v) \atop G_{i-1}(A_i^{\sigma^2}(\gamma(i-1)),\gamma(i-1))\right.$$

$$\left. + \int (PVY_i|A_i^{\sigma^2}(\gamma(i-1)),(\gamma(i-1),v)\sigma^2)dG^{i-1}(\gamma(i-1),v) \atop \psi(i-1)-G_{i-1}(A_i^{\sigma^2}(\gamma(i-1)),\gamma(i-1))\right]$$

by Lemma 10

$$= (PVY_{i-1}|A_{i-1}^{\sigma^2}(\gamma(i-2)),\gamma(i-1),\sigma^2)\ .$$

Since the induction hypothesis is valid for $i=T$, i.e.

$$(PVY_T|A_T^{\sigma^1}(\gamma(T-1)),\gamma(T),\sigma^1) = b(T)-\ (T)$$

$$= (PVY_T|A_T^{\sigma^2}(\gamma(T-1)),\gamma(T),\sigma^2)\ ,$$

the proof is complete//

Although the expression for the expected utility of a searching unemployed individual, $S_i(A(i),\gamma(i))$ did not include explicitly the terms $b(i)$, $s(i)$, ..., $b(T)$, $s(T)$, its value does, of course, depend on their magnitude. Let $b_{t,T} = (b(t), \ldots, b(T))$ and $s_{t,T} = (s(t), \ldots, s(T))$ and denote the expected utility associated with being unemployed at date t with $b_{t,T} = b^{\bullet}_{t,T}$ and $s_{t,T} = s^{\bullet}_{t,T}$ as $S^{\bullet}_t(A(t),\gamma(t))$. In addition denote the expected utility associated with being unemployed at date t when $b_{t,T} = b'_{t,T}$ and $s_{t,T} = s'_{t,T}$ as $S'_t(A(t),\gamma(t))$.

Lemma 11. If $(b^{\bullet}(i) - s^{\bullet}(i)) \geq (b'(i) - s'(i))$, $i=t, \ldots, T$

then

$$G^{\bullet}_{t-1}(A(t),\gamma(t-1) = \{y(t-1)\epsilon\Psi(t-1)\colon S^{\bullet}_{t}(A(t),(\gamma(t-1),y(t-1)))$$
$$\leq E_t(A(t) + Y^s_t(\gamma(t-1),y(t-1)))\}$$

$$\subseteq G'_{t-1}(A(t),\gamma(t-1)) = \{y(t-1)\epsilon\Psi(t-1)\colon S'_{t}(A(t),(\gamma(t-1),y(t-1)))$$
$$\leq E_t(A(t) + Y^s_t(\gamma(t-1),y(t-1)))\} .$$

This lemma states that the set of wage stream offers which would induce an individual to accept employment will either become larger or remain the same if net unemployment revenue is reduced.

<u>Proof</u>: If $(b^{\bullet}(i) - s^{\bullet}(i)) \geq (b'(i) < s'(i))$ $i=t, \ldots, T$ it follows immediately that $S^{\bullet}_t(A(t),\gamma(t)) \geq S'_t(A(t),\gamma(t))$ for any $A(t)$, $\gamma(t)$. It is also clear that $E_t(A(t) + Y^s_t(\gamma(t))$ is not influenced by $b(i) - s(i)$, $i=t, \ldots, T$ //

It is easily seen from Lemma 11 that the probability of not accepting employment in any particular period, t, is non-negatively related to $(b(i)-s(i))$, $i=t+1, \ldots, T$. The effect of a change in the magnitude of $(b(i)-s(i))$, $i=t+1, \ldots, T$, on the expected present value of labor force income net of search costs, wages, and unemployment benefits, however, may be ambiguous.

<u>Example</u>: Let $y(i)$'s be independently distributed with

$$Q\{\omega\epsilon\Omega\colon y(T-1,\omega) = D\} = \tfrac{1}{2} \text{ and}$$
$$Q\{\omega\epsilon\Omega\colon y(T-1,\omega) = 0\} = \tfrac{1}{2} \text{ and}$$
$$Q\{\omega\epsilon\Omega\colon y(T-2,\omega) = (b,b)\} = 1 .$$

Also assume only the most recent job offer received may be accepted. Finally, let $r(i) = 0$, $i=T-2, T-1, T$.

Now by definition we have

$$S_{T-1}(A(T-1),\gamma(T-1)) = \max_{C(T-1),A(T)} U_{T-1}(C(T-1)) + \tfrac{1}{2} E_T(A(T)+D) + \tfrac{1}{2} S_T(A(T),\gamma(T))$$

subject to $C(T-1) \geqslant 0$ and

$$B_T^s \leq A(T) = [A(T-1) + b(T-1) - s(T-1) - p(t-1)C(T-1)] .$$

Also by definition it is clear that

$$E_T(A(T) + b(T)-s(T)) = S_T(A(T),\gamma(T)) .$$

If $S_{T-1}(A(T-1),\gamma(T-1)) = E_{T-1}(A(T-1)+2b)$, $b(T-1) \leqslant s(T-1)$ and $b(T) \leqslant s(T)$ then expected wage income is $2b$ since $y(T-2)$ is definitely the acceptance set. Nevertheless, $2b < \tfrac{1}{2}D$ = (expected present value of wage income if search is continued) because

$$E_{T-1}(A(T-1) + 2b) = \max_{C(T-1),A(T)} U_{T-1}(C(T-1)) + E_T(A(T) + 2b)$$

and u_{T-1} and E_T are strictly concave. Thus, if $b(T-1) - s(T-1) + b(T) - s(T)$ is increased search will be continued with a resulting higher expected present value of wage income, $\tfrac{1}{2}D$, than the individual had before, $2b$. If it had been the case that $b(T-1) = s(T-1)$, $b(T) = s(T)$ and $\tfrac{1}{2}D < 2b$, we would, of course, observe

$$E_{T-1}(A(T-1) + 2b) > S_{T-1}(A(T-1),\gamma(T-1)) .$$

It is clear, however, that a sufficiently large increase in $b(T-1)$ or $b(T)$ would reverse this inequality. Such a reversal reduces the expected present value of wage income.

Appendix - Proof of Lemma 9

I have chosen to place this proof in an appendix due to its length. It is more of a distraction when included in the body of the paper than is warranted by its content.

Recall that Lemma 9 asserts that: $A^{\bullet}(t) < A^{1}(t), \sigma^{\bullet} \epsilon \Sigma(A^{\bullet}(t), \overline{\gamma(t)})$ and $\sigma^{1} \epsilon \Sigma(A^{1}(t), \overline{\gamma(t)})$ imply $A^{\sigma^{\bullet}}_{t+1}(\overline{\gamma(t)}) < A^{\sigma^{1}}_{t+1}(\overline{\gamma(t)})$ if

$$A^{\sigma^{\bullet}}_{t+1}(\overline{\gamma(T)}) > B^{s}_{t+1} .$$

As a first step in the proof of this assertion new functions, S^{*}_{i}, are defined by induction and their characteristics are noted.

$$S^{*}_{i}(A(i),\gamma(i)) = \max_{C(i),A(i+1)} U_{i}(C(i)) + \int_{G_{i}(A^{\sigma^{\bullet}}_{i+1}(\gamma(i)),\gamma(i))} E_{i+1}(A(i+1)+Y^{s}_{t+1}(\gamma(i),v))dG^{i}(\gamma(i),v) + \int_{\psi(i)-G_{i}(A^{\sigma^{\bullet}}_{i+1}(\gamma(i)),\gamma(i))} S^{*}_{i+1}(A(i+1),(\gamma(i),v))dG^{i}(\gamma(i),v)$$

subject to $B^{s}_{i+1} \leqslant A(i+1) = [A(i)-s(i)+b(i)-p(i)c(i)](1+r(i))$

$C(i) \geqslant 0$

and

$$S^{*}_{T}(A(T),\gamma(T)) = S_{T}(A(T),\gamma(T)) .$$

I take as an induction hypothesis that $S^{*}_{j}(\cdot,\gamma(j))$ is continuously differentiable and concave on $[B^{s}_{j},\infty)$,

a) $S^{*}_{j}(A(j),\gamma(j) \leq S_{j}(A(j),\gamma(j))$ and

b) $S^{*}_{j}(A^{\sigma^{\bullet}}(\overline{\gamma(j-1)}),\overline{\gamma(j)}) = S_{j}(A^{\sigma^{\bullet}}_{j}(\overline{\gamma(j-1)}),\overline{\gamma(j)})$

where $\bar{\gamma}(j) = (\overline{\gamma(j-1)}, y(j-1))$ for $y(j-1) \in \mathcal{Y}(j-1)$.

This hypothesis clearly implies that the function

$$Z_j^*(A(j),\gamma(j-1)) = \int\limits_{G_{j-1}(A_j^{\sigma^\bullet}(\gamma(j-1)),\gamma(j-1))} E_j(A(j)+Y_j^s(\gamma(j-1),v))dG^{j-1}(\gamma(j-1),v)$$

$$+ \int\limits_{\mathcal{Y}(j-1)-G_{j-1}(A_j^{\sigma^\bullet}(\gamma(j-1)),\gamma(j-1))} S_j^*(A(j),(\gamma(j-1),v))dG^{j-1}(\gamma(j-1),v)$$

is a continuously differentiable, strictly concave function of $A(j)$ on $[B_j^s,\infty)$. Also,

a') $Z_j^*(A(j),\overline{\gamma(j-1)}) \leq Z_j(A(j),\overline{\gamma(j)})$ and

b') $Z_j^*(A^{\sigma^\bullet}(\overline{\gamma(j-1)}),\overline{\gamma(j-1)}) = Z_j(A_j^{\sigma^\bullet}(\overline{\gamma(j-1)}),\overline{\gamma(j-1)})$

for arbitrary feasible $\overline{\gamma(j-1)}$.

Bellman's results cited in the proof of Lemma 1 establish the strict concavity and continuous differentiability of the function $S_{j-1}^*(\cdot,\overline{\gamma(j-1)})$ defined by

$$S_{j-1}^*(A(j-1),\gamma(j-1)) = \max_{C(j-1),A(j)} U_{j-1}(C(j-1))+Z_j^*(A(j),\overline{\gamma(j-1)})$$

subject to $C(j-1) \geq 0$,

$$B_j^s \leq A(j) = [A(j-1)+b(j-1)-s(j-1)-p(j-1)c(j-1)](1+r(j-1)) .$$

Since $C_{j-1}^{\sigma^\bullet}(\gamma(j-1)),A_j^{\sigma^\bullet}(\gamma(j-1))$ are maximizers for

$$S_{j-1}(A_{j-1}^{\sigma^\bullet}(\overline{\gamma(j-2)}),\overline{\gamma(j-1)}) = \max_{C(j-1),A(j)} U_{j-1}(C(j-1))+Z_j(A(j),\overline{\gamma(j-1)})$$

subject to $C(j-1) \geq 0$ and

$$B_j^s \leq A(j) = [A_{j-1}^{\sigma^\bullet}(\overline{\gamma(j-2)})+b(j-1)-s(j-1)-p(j-1)C(j-1)](1+r(j-1))$$

We must have

$$S_{j-1}^*(A_{j-1}^{\sigma^\bullet}(\overline{\gamma(j-2)}),\overline{\gamma(j-1)}) = U_{j-1}(C_{j-1}^{\sigma^\bullet}(\overline{\gamma(j-1)})+Z_j^*(A_j^{\sigma^\bullet}(\overline{\gamma(j-1)}),\overline{\gamma(j-1)})$$

by a' and b'. This in turn implies

a") $S^*_{j-1}(A(j-1),\overline{\gamma(j-1)}) \leq S_{j-1}(A(j-1),\overline{\gamma(j-1)})$ and

b") $S^*_{j-1}(A^{\sigma^\bullet}_{j-1}(\overline{\gamma(j-2)}),\overline{\gamma(j-1)}) = S_{j-1}(A^{\sigma^\bullet}_{j-1}(\overline{\gamma(j-2)}),\overline{\gamma(j-1)})$.

This establishes the desired properties for S^*_{j-1}, $j-1 = 0, 1, \ldots,$ $T-1$, since $S^*_T(A(T),\gamma(T)) = S_T(A(T),\gamma(T))$ and $S_T(\cdot,\gamma(T))$ is strictly concave and continuously differentiable. Since $U_t(\cdot)$ and $Z^*_{t+1}(\cdot)$ are strictly concave and continuously differentiable, if $A^{\sigma^\bullet}_{t+1}(\overline{\gamma}(t)) > B^S_{t+1}$, then $A'(t+1) > A^{\sigma^\bullet}_{t+1}(\overline{\gamma}(t))$ where $C'(t),A'(t+1)$ are unique maximizers for $U_t(c(t))+Z^*_{t+1}(A(t+1),\overline{\gamma}(t))$ subject to $C(t) \geqq 0$, $A(t+1) \geqq B^S_{t+1}$, $A^1(t) = A^\bullet(t) + \delta$, and $A(t+1) = [A^\bullet(t)+\delta+b(t)-S(t)-p(t)C(t)](1+r(t))$. Since $A^{\sigma^\bullet}_{t+1}(\overline{\gamma}(t)) = [A^\bullet(t)+\delta-S(t)+b(t)-p(t)(C^{\sigma^\bullet}_t(\overline{\gamma}(t))+\frac{\delta}{p(t)})](1+r(t))$ we must have $U_t(C'(t))+Z^*_{t+1}(A'(t+1),\overline{\gamma}(t))$

$$> U_t(C^{\sigma^\bullet}_t(\overline{\gamma}(t)) + \frac{\delta}{p(t)}) + Z^*_{t+1}(A^{\sigma^\bullet}_{t+1}(\overline{\gamma}(t)),\overline{\gamma}(t))$$

$$= U_t(C^{\sigma^\bullet}_t(\overline{\gamma}(t)) + \frac{\delta}{p(t)}) + Z_{t+1}(A^{\sigma^\bullet}(\overline{\gamma}(t),\overline{\gamma}(t))\ .$$

Since $Z^*_{t+1}(A'(t+1),\overline{\gamma}(t)) \leqq Z_{t+1}(A'(t+1),\overline{\gamma}(t))$ by by (a'), $C^{\sigma^\bullet}_t(\overline{\gamma}(t)) + \frac{\delta}{p(t)}$ and $A^{\sigma^\bullet}_{t+1}(\gamma(t))$ cannot maximize

$$U_t(C(t)) + Z_{t+1}(A(t+1),\overline{\gamma}(t))$$

subject to $C(t) \geqq 0$, $B^S_{t+1} \leqq A(t+1)$ and $A(t+1) = [A^\bullet(t)+\delta+b(t)-S(t)-p(t)C(t)](1+r(t))$.

We may now establish that $A^{\sigma^1}_{t+1}(\overline{\gamma(t)}) \geqslant A^{\sigma^\bullet}_{t+1}(\overline{\gamma(t)})$.

$$U_t(C^{\sigma^\bullet}_t(\overline{\gamma}(t)))+Z_{t+1}(A^{\sigma^\bullet}_{t+1}(\overline{\gamma}(t)),\overline{\gamma}(t)) \geqq U_t(C^{\sigma^1}_t(\overline{\gamma(t)})-\frac{\delta}{p(t)})$$
$$+ Z_{t+1}(A^{\sigma 1}_{t+1}(\overline{\gamma(t)}),\overline{\gamma}(t))$$

Since $(C_t^{\sigma^\bullet}(\bar{\gamma}(t)), A_{t+1}^{\sigma^\bullet}(\bar{\gamma}(t)))$ are maximizers. Rearrangement yields $Z_{t+1}(A_{t+1}^{\sigma^\bullet}(\bar{\gamma}(t)), \bar{\gamma}(t))) - Z_{t+1}(A_{t+1}^{\sigma^1}(\overline{\gamma(t)}), \bar{\gamma}(t))$
$\geqq U_t(C_t^{\sigma^1}(\gamma(t)) - \frac{\delta}{p(t)}) - U_t(C_t^{\sigma^\bullet}(\bar{\gamma}(t))) > U_t(C_t^{\sigma^1}(\overline{\gamma(t)})) - U_t(C_t^{\sigma^\bullet}(\bar{\gamma}(t)) + \frac{\delta}{p(t)})$
if $U_t(C_t^{\sigma^1}(\overline{\gamma(t)}) - \frac{\delta}{p(t)}) - U_t(C_t^{\sigma^\bullet}(\bar{\gamma}(t)))$ is positive, because U_t is strictly concave. However, since $(C_t^{\sigma^1}(\overline{\gamma(t)}), A_{t+1}^{\sigma^1}(\overline{\gamma(t)}))$ are maximizers, the last inequality is impossible. Thus $U_t(C_t^{\sigma^1}(\overline{\gamma(t)}) - \frac{\delta}{p(t)}) - U_t(C_t^{\sigma^\bullet}(\overline{\gamma(t)}))$ is non-positive.

$C_t^{\sigma^1}(\overline{\gamma(t)}) - \frac{\delta}{p(t)} = C_t^{\sigma^\bullet}(\overline{\gamma(t)})$ implies $A_{t+1}^{\sigma^\bullet}(\overline{\gamma(t)}) = A_{t+1}^{\sigma^1}(\overline{\gamma(t)})$, which has been ruled out above. Therefore, since U_t is strictly increasing,

$$C_t^{\sigma^1}(\overline{\gamma(t)}) - \frac{\delta}{p(t)} < C_t^{\sigma^\bullet}(\overline{\gamma(t)}) \quad \text{and}$$

$$A_{t+1}^{\sigma^\bullet}(\overline{\gamma(t)}) < A_{t+1}^{\sigma^1}(\overline{\gamma(t)}) \quad //$$

Footnotes

1/ After this paper was essentially completed I encountered a paper by D. Whipple entitled "A Generalized Model of Job Search" [23]. Though Professor Whipple does consider expected utility rather than expected income maximization he does not derive an optimal search strategy. The individual in his model chooses a unique sequence of acceptance wages and asset holdings at the outset of his planning horizon. The plan of action obtained in this fashion ignores the fact that optimal savings or job choice decisions at any future date depend crucially on that which has transpired between that date and the present. This procedure is counter to that prescribed by Bellman's well known "Principle of Optimality" and will not, in general, yield an expected utility maximizing strategy for the environment and individual he has portrayed.

2/ Equilibrium is, in these papers, consistent with more than one price existing in the market. The situation is an equilibrium in the sense that the distribution of prices tends to persist unchanged over time.

3/ It is easily shown that this assumption is unnecessary, though used, in models of the Gronau [8] type. Even without this assumption the optimal strategy will involve keeping all accepted jobs until retirement.

4/ These results are contained in Theorem 1 of [5].

5/ The intuitive appeal of this result reflects directly on the acceptability of the normal good assumption. If the individual considers risks to be inferior goods we find he is more choosy about what jobs he will take if there is a ceteris paribus decline in his wealth.

6/ Note that all that is used in proving the existence of an optimal search strategy is

A6') Let E be any set of $y(t)$ with $\{\omega\epsilon\Omega : y(t,\omega)\epsilon E\}\epsilon\mathfrak{F}$, then $\{\omega\epsilon\Omega : G^t(\gamma(t,\omega),E) \geq b\}\epsilon\mathfrak{F}$.

Denumerability of the $\mathcal{Y}(t)$ is employed in establishing the measurability of the consumption strategy. In discussing expected duration of unemployment and expected labor income, Theorem 2 and Theorem 3, this measurability is required. An alternative assumption which implies A6') just as does A6) may be used.

A6") For E measurable, $G^t(\gamma(t),E)$ is a continuous function of $\gamma(t)$.

A6") must be accompanied by an additional assumption to insure measurability of the consumption strategy. A sufficient additional assumption would be that the individual always choose a measurable strategy. That this is possible is insured by the fact that choosing the maximum (minimum) element of the set of maximizing consumption levels at each date will yield a measurable strategy. (A6") insures upper semicontinuity of correspondence from $(A(t),\gamma(t))$ to maximizing $(C(t),A(t+1))$, and hence the set of maximizers is closed.)

7/ This postulate is essentially the same as that presented by Stigum in [22].

8/ It is clear that if the individual was allowed to refrain from sampling in this last period thereby avoiding $S(T)$ without sacrificing $b(T)$ he would do so. One may motivate the assertion that search will be conducted in T by supposing $b(T) \geqslant S(T)$ and that benefits cannot be received unless costs are incurred. Alternatively, one may simply interpret my assertion as a simplifying assumption of the model.

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