A MODEL OF CHOICE WITH UNCERTAIN INITIAL PROSPECT

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Discussion Paper No. 74 - 38, January 1974

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In an earlier work, one of the authors [1] developed an abstract model for choice from a one-dimensional family of uncertain ventures and applied some of the results to problems of betting. The principal innovation in the model was the inclusion of an uncertain initial prospect in place of fixed initial wealth. In Sections 1 and 2 of the present paper the abstract model is extended, with particular attention given to relations of comparative statics. In Sections 3 and 4 some of the results from Sections 1 and 2 are applied to simple, hypothetical, economic decision problems.

1. The abstract model

Suppose a decision maker's current prospect is represented by a random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$. Elements ω of Ω are sequences of developments in the decision maker's environment. The probability measure P on σ-field $\mathcal{F}$ represents his personal probabilities. A value $x = X(\omega)$ is the wealth realized by the decision maker (dm) if the sequence ω is realized in his environment and if he carries out his current commitments and plans.

The decision maker has the option of modifying his current prospect by undertaking a new venture Y, also a random variable. If he decides

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* Research supported in part by National Science Foundation grants GS-3317 and GS-3568 to the University of Minnesota.

to undertake an amount α of Y, his new prospect is $X + \alpha Y$. He is assumed to choose α to maximize $E\varphi(X + \alpha Y)$ where φ is his utility of wealth function.

In economic applications α is usually restricted to a subset of the real line. In the one-dimensional case, however, starting with the unrestricted family is an efficient way to approach various restricted families; thus we initially assume that α can be any real number. Since $P(Y = 0) = 1$ would imply that $E\varphi(X + \alpha Y) = E\varphi(X)$ for every α, this trivial case is excluded throughout the paper. Other conditions to be used in various combinations are listed below for convenient reference.

I. $\varphi'(x) = \dfrac{d\varphi(x)}{dx} > 0 \quad \forall\, x \in R$

II. $\lim_{x\to\infty} \varphi(x) = 0$

III. $\varphi''(x) < 0 \quad \forall\, x \in R$

IV. $\varphi''(x)$ is monotonic

V. $E\,|\varphi(X + \alpha Y)|$ finite $\forall\, \alpha \in R$

VI. $E\,|Y\varphi'(X + \alpha Y)|$ finite $\forall\, \alpha \in R$

VII. $E\,|Y^2 \varphi''(X + \alpha Y)|$ finite $\forall\, \alpha \in R$

VIII. $E\,|Y\varphi''(X + \alpha Y)|$ finite $\forall\, \alpha \in R$

For any given X and Y, let $\eta(\alpha) = E\varphi(X + \alpha Y)$. The decision problem is then to maximize $\eta(\alpha)$ for $\alpha \in R$. Since positive linear transformations of φ are also utility functions, we may for convenience take φ so that $E\varphi(X) = 0$. We say that a venture αY is favorable ($X + \alpha Y$ is preferred to X) if $\eta(\alpha) > 0$ and that αY is optimal if $\eta(\alpha) \geq \eta(\alpha^*)$ $\forall$ $\alpha^* \in R$.

One of the most natural and at the same time most important questions to be answered in the model is under what conditions a maximal point $\hat{\alpha}$ exists. Theorem 1 below states that under certain regularity conditions a maximal point will exist if and only if neither Y nor $-Y$ is a sure thing. The following lemma is needed for the proof of Theorem 1, as well as for later theorems.

<u>Lemma 1</u>. Assumptions III, V, and VI imply that $\eta'(\alpha)$ exists and $\eta'(\alpha) = EY\varphi'(X + \alpha Y)$ $\forall$ $\alpha \in R$.

<u>Proof</u>: Choose $\alpha_1 < \alpha_0 < \alpha_2$. By the Mean Value Theorem, for any $\omega \in \Omega$ and $\alpha \in [\alpha_1, \alpha_2]$ $\exists$ $\bar{\alpha}$ $(\alpha, \alpha_0, \omega)$ in (α_1, α_2) $\ni$

$$\left| \frac{\varphi(X + \alpha Y) - \varphi(X + \alpha_0 Y)}{\alpha - \alpha_0} \right| = \left| \frac{(\alpha - \alpha_0) Y \varphi'(X + \bar{\alpha} Y)}{\alpha - \alpha_0} \right|$$

$$= \left| I_{(Y > 0)} Y\varphi'(X + \bar{\alpha}Y) + I_{(Y < 0)} Y\varphi'(X + \bar{\alpha}Y) \right|$$

$$\leq \left| I_{(Y > 0)} Y\varphi'(X + \bar{\alpha}Y) \right| + \left| I_{(Y < 0)} Y\varphi'(X + \bar{\alpha}Y \right|$$

$$\leq \left| I_{(Y > 0)} Y\varphi'(X + \alpha_1 Y) \right| + \left| I_{(Y < 0)} Y\varphi'(X + \alpha_2 Y) \right|$$

where the final inequality follows from III. VI implies that the final sum is integrable so, by Lebesgue's Dominated Convergence Theorem

$$\eta'(\alpha_0) = \lim_{\alpha \to \alpha_0} \frac{\eta(\alpha) - \eta(\alpha_0)}{\alpha - \alpha_0} = \int \lim_{\alpha \to \alpha_0} \frac{\varphi(X+\alpha Y) - \varphi(X+\alpha_0 Y)}{\alpha - \alpha_0} = \int Y\varphi'(X+\alpha_0 Y) \quad .$$

Corollary. Assumptions IV, VI, and VII $\Rightarrow \eta''(\alpha)$ exists and $\eta''(\alpha) = EY^2\varphi''(X + \alpha Y)\ \forall\ \alpha \in R$.

Proof: The proof is the same as in Lemma 1, with φ' and φ'' replacing φ and φ' . In the final inequality α_1 and α_2 will need to be interchanged if, as is usual, φ'' is assumed to be increasing.

Theorem 1. Under I, II, III, V, and VI, $\eta'(\alpha) = 0$ has a unique solution if and only if $P(Y < 0) > 0$ and $P(Y > 0) > 0$

Proof: Theorem 1 gives $\eta'(\alpha) = EY\varphi'(X + \alpha Y)\ \forall\ \alpha \in R$. Suppose $P(Y < 0) = 0$. Then $P(Y > 0) > 0$ by the initial nontriviality assumption, and, for any α , $Y\varphi'(X + \alpha Y) \geq 0$ with strict inequality holding on a set of positive probability. Hence $\eta'(\alpha) = \int Y\varphi'(X + \alpha Y) > 0$. Similarly $P(Y > 0) = 0 \Rightarrow \eta' < 0$.

Now suppose $P(Y > 0)$ and $P(Y < 0)$ are both positive. Write

$$\eta'(\alpha) = \int_{Y>0} Y\varphi'(X + \alpha Y) - \int_{Y\leq 0} [-Y]\,\varphi'(X + \alpha Y) \equiv \mu(\alpha) - \upsilon(\alpha) ,$$

where the decomposition is justified by VI. By I and III μ is positive and decreasing and υ is positive and increasing. Let $\{\alpha_n\}$ be any sequence such that $\alpha_n \to +\infty$. Letting $I_A(\cdot)$ represent the indicator function for A , $I_{[Y>0]}(Y)\, Y\varphi'(X + \alpha_n Y) \leq I_{[Y>0]}(Y)\, Y\varphi'(X + \alpha_1 Y)$ for all n sufficiently large. Thus by Lebesgue's Dominated Convergence Theorem and assumption II,

$\lim_{n\to+\infty} \mu(\alpha_n) = \int_{Y>0} \lim_{n\to+\infty} Y\varphi'(X + \alpha_n Y) = 0$. The proof that

$\lim_{n\to+\infty} \upsilon(-\alpha_n) = 0$ is similar. Since $\{\alpha_n\}$ was arbitrary,

$\lim_{\alpha\to+\infty} \mu(\alpha) = \lim_{\alpha\to-\infty} \upsilon(\alpha) = 0$.

Thus $\eta' \equiv \mu - \upsilon$ is negative for sufficiently large α and positive for sufficiently small α. It follows by Darboux's Theorem[1] that $\exists\ \hat{\alpha} \in R$ such that $\eta'(\hat{\alpha}) = 0$. III implies φ is strictly concave, which yields η strictly concave by a simple check of definition. Hence $\hat{\alpha}$ is unique.

In the course of proving Theorem 1, it was remarked that φ strictly concave implies η strictly concave. The strict concavity of η and its direct consequences are separated out into a theorem for easy reference.

Theorem 2. Assumptions III, V, and VI imply:

i) η is strictly concave;

ii) If $\hat{\alpha}$ exists, $\eta'(0) \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless 0$;

iii) If $\hat{\alpha}$ does not exist, then η is strictly increasing (decreasing) if $\eta'(0) > 0\ (< 0)$.

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[1] Darboux's Theorem states: If $f{:}R \to R$ is differentiable on $[a, b]$, with $f'(a) = A$, $f'(b) = B$, and C lies strictly between A and B, $\exists$ a point $c \in (a, b)$ such that $f'(c) = C$. [For proof, consider the Function $g{:}x \to Cx - f(x)$ which is differentiable on $[a, b]$ and attains a maximum or minimum on (a, b).]

<u>Proof</u>: By III, φ is strictly concave; and by V, η exists and is finite $\forall\ \alpha \in R$. Hence for any $\alpha_1, \alpha_2 \in R$, and $\lambda \in (0, 1)$, $\eta(\lambda\alpha_1 + [1 - \lambda]\alpha_2) \equiv E\varphi(\lambda[X + \alpha_1 Y] + [1 - \lambda][X + \alpha_2 Y]) > E[\lambda\varphi(X + \alpha_1 Y) + [1 - \lambda]\varphi(X + \alpha_2 Y)] = \lambda\eta(\alpha_1) + [1 - \lambda]\eta(\alpha_2)$. Thus η is strictly concave.

Lemma 1 implies $\eta'(\alpha)$ exists and is finite $\forall\ \alpha \in R$ This, together with the strict concavity of η, immediately implies ii) and iii).

<u>Corollary 1</u>. Under III, V, and VI:

i) If $\eta'(0) > 0$, then $\exists\ \alpha^* > 0$ such that αY is favorable if and only if $\alpha \in (0, \alpha^*)$;

ii) If $\eta'(0) < 0$, then $\exists\ \alpha^* > 0$ such that αY is favorable if and only if $\alpha \in (-\alpha^*, 0)$.

Under the conditions of Theorem 2, it is easy to see how knowing $\hat{\alpha}$ and α^* would place the decision maker in a position to quickly evaluate any restricted one-dimensional family of ventures. For convenience, assume $\eta'(0) > 0$. If A, the admissable set for α, includes $\hat{\alpha}$ then $\hat{\alpha}$ is, of course, uniquely optimal for the restricted family. If $A \cap (0, \alpha^*) = \phi$ there are no favorable ventures. If $\hat{\alpha} \notin A$, $A \cap (0, \alpha^*) \neq \phi$, one would look for the maximal element of $A \cap (0, \hat{\alpha})$ and the minimal element of $A \cap (\hat{\alpha}, \alpha^*)$. If both exist one would need to compare expected utilities at these two points. If A is not closed it may happen that there is no optimal venture, but one can then compare ventures corresponding to α in a neighborhood of $\sup\{\alpha \mid \alpha \in A \cap (0, \hat{\alpha})\}$ with those corresponding to α in a neighborhood of $\inf\{\alpha \mid \alpha \in A \cap (\hat{\alpha}, \alpha^*)\}$.

Consider two individuals with respective probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{F}^*, P^*)$, utilities φ and φ^*, and current prospects X and X^* . Assuming performance can be enforced, they can exchange a venture Y by agreeing that the second decision maker will "pay" the first $Y(\omega)$ if ω is realized.[2] Such an exchange is said to be mutually favorable if $E\varphi(X + Y) > 0$ and $E^*\varphi^*(X^* - Y) > 0$. Possible existence of mutually favorable exchanges promises to be a topic of some interest. Clearly an economy has not reached a Pareto optimum if costless mutually favorable exchanges exist. An immediate second corollary to Theorem 2 gives a sufficient condition for the existence of mutually favorable exchanges between two decision makers.

<u>Corollary 2</u>. If, for any venture Y and two individuals as described above, $\eta'(0)$ and $\eta^{*\prime}(0)$ are of opposite signs, then $\exists\ \epsilon > 0$ such that exchanging αY is mutually favorable for any $\alpha \in (0, \epsilon)$.

<u>Proof</u>: If $\eta'(0) > 0$ and $\eta^*(0) < 0$, let $\epsilon = \min\{\delta, \delta^*\}$ where αY is favorable to the first individual for $\alpha \in (0, \delta)$ and αY is favorable to the second individual for $\alpha \in (-\delta^*, 0)$.

In many contexts one would expect to encounter substantial difficulties in verifying the existence of mutually favorable exchanges. Theorem 3 below may sometimes help. According to this theorem it is only necessary to check for the existence of mutually favorable exchanges of a simple kind

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[2]When discussing an exchange of a venture Y , it is implicitly assumed that Y is a measurable function on both $(\Omega, \mathcal{F})$ and $(\Omega, \mathcal{F}^*)$.

called bets.

Let $A \in \mathcal{F} \cap \mathcal{F}^*$ satisfy $0 < P(A) < 1$ and $0 < P^*(A) < 1$. Let $B = \Omega - A$ and $y_1, y_2 > 0$. Then $Y = y_1 I_A - y_2 I_B$ is called a plain venture, and an exchange of Y for $-Y$ is called a bet based on A.

Lemma 3. For A, B as above, if

$$\int_A \varphi'(X)\,dP \Big/ \int_B \varphi'(X)\,dP \neq \int_A \varphi^{*\prime}(X^*)\,dP^* \Big/ \int_B \varphi^{*\prime}(X^*)\,dP^* ,$$

then there exists a mutually favorable bet based on A.

Proof: Suppose $\int_A \varphi'(X)\,dP / \int_B \varphi'(X)\,dP > \int_A \varphi^{*\prime}(X^*)\,dp^* / \int_B \varphi^{*\prime}(X^*)\,dP^*$.

Choose y lying strictly between the two terms, and define

$Y = I_A - yI_B$, $\eta(\alpha) = \int \varphi(X + \alpha Y)\,dP$, and $\eta^*(\alpha) = \int \varphi^*(X^* + \alpha Y)\,dP^*$.

As is easily checked, $\eta'(0) \gtreqless 0 \Leftrightarrow \int_A \varphi'(X)\,dP / \int_B \varphi'(X)\,dP \gtreqless y$, similarly for η^*. Hence $\eta'(0) > 0$, $\eta^{*\prime}(0) < 0$, and by Corollary 2 to Theorem 2 there exists a mutually favorable bet based on A.

The proof for the opposite inequality follows immediately by symmetry.

Theorem 3. There exists a mutually favorable bet between two decision makers if and only if there exists a mutually favorable exchange of a random variable Y, where Y is not constant - a.s. for either P or P^*.

<u>Proof</u>: The proof in the necessity direction is trivial, since a mutually favorable bet is a mutually favorable exchange satisfying the required condition.

For the sufficiency, assume without loss of generality that

$$E\varphi(X) = E^*\varphi^*(X^*) = 0 \quad \text{and} \quad E\varphi'(X) = E^*\varphi'(X^*) = K > 0 \quad .$$

[This is possible, since the invariance of φ and φ^* under positive linear transformations allows two degrees of freedom in the definition of each.] Suppose Y offers a favorable mutual exchange, Y not constant a.s. under either P or P^*. Then

$$\eta(0) = 0, \quad \eta(1) > 0, \quad \eta^*(0) = 0, \quad \eta^*(-1) > 0 \Rightarrow \eta'(0) \equiv EY\varphi'(X) > 0$$

and

$$\eta^{*\prime}(0) \equiv E^*Y\varphi^{*\prime}(X^*) < 0 \Rightarrow \text{either} \quad EY^+\varphi'(X) > E^*Y^+\varphi'(X^*)$$

or

$$- EY^-\varphi'(X) > - E^*Y^-\varphi^{*\prime}(X^*) \quad .$$

Assume the former inequality holds. Define

$$C_{N,k} = \left[\omega : \frac{k}{2^N} \leq Y(\omega) < \frac{k+1}{2^N}\right] \quad \text{and}$$

$$Y_N(\cdot) = \sum_{k=0}^{N2^N} \frac{k}{2^N} I_{C_{N,k}}(\cdot) \quad . \quad \text{Then } Y_N \uparrow Y^+ \Rightarrow \text{by Lebesgue's}$$

Monotone Convergence Theorem,

$$EY_N\varphi'(X) \uparrow EY^+\varphi'(X) \quad \text{and} \quad E^*Y_N\varphi^{*\prime}(X^*) \uparrow E^*Y^+\varphi^{*\prime}(X^*) \quad .$$

Hence $\exists\, \bar{N}$ s.t. for $N \geq \bar{N}$, $EY_N\varphi'(X) > E^*Y_N\varphi^{*\prime}(X^*)$;

and by definition of Y_N, for each such $N \ \exists \ k_N \in$ $\{0, \ldots N\,2^N\}$ such that

$$\int_{C_{N,k_N}} \varphi'(X)\ dP > \int_{C_{N,k_N}} \varphi^*(X^*)\ dP^* \quad .$$

One may always choose $N \geq \bar{N}$ such that

$K > \int_{C_{N,k_N}} \varphi'(X)\ dP$; for Y not constant P - a.s.

$\Rightarrow Y$ takes on at least two distinct values y_2, y_1, $y_2 > y_1$, with positive P-measure. If $y_1 < 0$, then clearly $K \equiv E\varphi'(X) > \int_{C_{N,k_N}} \varphi'(X)\ dP\ \forall N$. If $y_1 > 0$, one may choose N so large that the $\{\frac{k}{2^N} : k = 0, \ldots N\,2^N\}$ grid separates y_1 from y_2. In this case again

$$K > \int_{C_{N,k_N}} \varphi'(X)\ dP \quad .$$

Hence choosing $N \geq \bar{N}$ so that $K > \int_{C_{N,k_N}} \varphi'(X)\ dP$, and defining $A = C_{N,k_N}$ and $B = \Omega - C_{N,k_N}$:

$$\frac{\int_A \varphi'(X)\ dP}{\int_B \varphi'(X)\ dP} \equiv \frac{\int_A \varphi'(X)\ dP}{K - \int_A \varphi'(X)\, dP} > \frac{\int_A \varphi^{*\prime}(X^*)\ dP^*}{K - \int_A \varphi^{*\prime}(X^*)\, dP^*} \equiv \frac{\int_A \varphi^{*\prime}(X^*)\ dP^*}{\int_B \varphi^{*\prime}(X^*)\ dP^*} \quad .$$

It follows by Lemma 3 that there exists a mutually favorable bet based on A.

The proof for $-EY^-\varphi'(X) > -E^*Y^-\varphi^{*\prime}(X^*)$ is similar.

2. Comparative statics

A question of great interest is how the optimal choice $\hat{\alpha}$ responds to various changes in φ, P, X and Y. Write $Y = W - h$ and $X = \bar{X} + Z$, where $EX = \bar{X}$. In what follows we consider the response of $\hat{\alpha}$ to changes in the location parameters $\bar{X}$ and h.

If the decision maker receives an unexpected gift of c dollars and his beliefs are not changed, then $\bar{X}$ is increased by c and Z is unaffected. It thus seems reasonable to call a change in $\hat{\alpha}$ due to a change in $\bar{X}$ the wealth effect. If the venture Y is the purchase of a security, then it is natural to think of W as representing possible returns from the security and h the price. Accordingly we shall refer to responses to changes in h as price effects even though in some cases the decomposition of Y may be arbitrary.

In the theorems which follow, all of the eight conditions listed in Section 1 will be assumed without specific reference.[3] Moreover, Y will always be assumed to satisfy $P[Y > 0] > 0$ and $P[Y < 0] > 0$. Theorem 1 then guarantees the existence of $\hat{\alpha}$ as a well defined function of $(\bar{X}, h)$. When differentiation with respect to $\bar{X}$ or h is being undertaken, all expectations are considered taken with respect to the distribution of (Z, W). Calculations not involving differentiation with respect to $\bar{X}$ or h will be carried out in terms of the distribution of (X, Y), a matter of notational change only since the Jacobian of transformation is 1. In all cases the simpler notation in terms of X and Y will be retained in expressing arguments of functions.

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[3]Theorem 4, basic to all of Section 2, requires all eight conditions to hold if it is to follow directly from the standard statement of the Implicit Function Theorem.

<u>Theorem 4</u>. Let $X = \bar{X} + Z$, $Y = W - h$, $\eta''(\alpha) = EY^2\, \varphi''(X + \alpha Y) = \Delta$. For given Z and W

$$\text{(i)} \quad \frac{\partial\hat{\alpha}}{\partial\bar{X}} = -\Delta^{-1}\, EY\, \varphi''(X + \hat{\alpha}Y)$$

$$\text{(ii)} \quad \frac{\partial\hat{\alpha}}{\partial h} = \Delta^{-1}\, [E\varphi'(X + \hat{\alpha}Y) + EY\, \varphi''(X + \hat{\alpha}Y)] = \Delta^{-1}\, E\varphi'(X + \hat{\alpha}Y) - \hat{\alpha}\frac{\partial\hat{\alpha}}{\partial\bar{X}}$$

<u>Proof</u>: Implicit Function Theorem.

One would like to make conditional assertions about the signs of $\frac{\partial\hat{\alpha}}{\partial h}$ and $\frac{\partial\hat{\alpha}}{\partial\bar{X}}$ which an investigator might be able to distinguish in practice. In many of the theorems below assertions are made concerning the sign of $\frac{\partial\hat{\alpha}}{\partial\bar{X}}$. Although it is not true in general that $\hat{\alpha} \geq 0 \Rightarrow \frac{\partial\hat{\alpha}}{\partial\bar{X}} \geq 0$ or vice versa, the conditions $[\hat{\alpha} \geq 0, \frac{\partial\hat{\alpha}}{\partial\bar{X}} \geq 0]$ here consistently appear together, as do the conditions $[\hat{\alpha} \leq 0, \frac{\partial\hat{\alpha}}{\partial\bar{X}} \leq 0]$. By Theorem 4, $[\hat{\alpha} \geq 0, \frac{\partial\hat{\alpha}}{\partial\bar{X}} \geq 0] \Rightarrow [\frac{\partial\hat{\alpha}}{\partial h} < 0]$ and $[\hat{\alpha} \leq 0, \frac{\partial\hat{\alpha}}{\partial\bar{X}} \leq 0] \Rightarrow [\frac{\partial\hat{\alpha}}{\partial h} < 0]$. Thus the determination of the sign of $\frac{\partial\hat{\alpha}}{\partial h}$ is an implicit consequence of each of the theorems in which these pairs of conditions occur.

<u>Theorem 5</u>. $E[Y|X] \gtreqless 0$ ae $\Rightarrow \hat{\alpha} \gtreqless 0$ and $E[Y|X] = 0$ ae $\Rightarrow \frac{\partial\hat{\alpha}}{\partial\bar{X}} = 0$.

<u>Proof</u>: $\hat{\alpha}$ agrees in sign with $\eta'(0)$ [Theorem 2] and $\eta'(0) = EY\varphi'(X) = E(\varphi'(X)\, E[Y|X]) \gtreqless 0$ as $E[Y|X] \gtreqless 0$ ae. From Theorem 4, (i), $\frac{\partial\hat{\alpha}}{\partial\bar{X}}$ agrees in sign with $EY\varphi''(X + \hat{\alpha}Y)$. If $E[Y|X] = 0$ ae, then $EY\varphi''(X + \hat{\alpha}Y) = EY\varphi''(X) = E(\varphi''(X)\, E[Y|X]) = 0$.

Theorem 6. If $E[Y|X] = EY$, then $EY \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless 0$, and $EY = 0 \Rightarrow \frac{\partial \hat{\alpha}}{\partial \bar{X}} = 0$.

Proof: By Theorem 5, $EY \gtreqless 0 \Rightarrow \hat{\alpha} \gtreqless 0$ and $EY = 0 \Rightarrow \frac{\partial \hat{\alpha}}{\partial \bar{X}} = 0$. By Theorem 2, $0 < \hat{\alpha} \Rightarrow 0 < \eta'(0) = EY\varphi'(X) = E\varphi'(X)\, E\, [Y|X] = E\varphi'(X) \cdot EY \Rightarrow 0 < EY$. Similarly for $\hat{\alpha} = 0$ and $\hat{\alpha} < 0$.

Corollary. If X is independent of Y, $EY \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless 0$.

Proof: X independent of $Y \Rightarrow E[Y|X] = EY$.

Theorems 8, 10 and 12 below relate the wealth effect $\frac{\partial \hat{\alpha}}{\partial \bar{X}}$ to the behavior of absolute risk aversion r, where $r(x) \equiv - \varphi''(x)/\varphi'(x)$ [see Arrow [2], Pratt [3]]. Economists generally regard decreasing absolute risk aversion as typical of most decision makers, and the condition appears as a hypothesis in Lemma 8 and Theorems 8 and 10. It is thus reassuring to first note the existence of a broad, mathematically tractable class of utility functions which satisfy $r' < 0$ over the entire real line.[4]

Theorem 7. Define φ^* to be the class of functions $\varphi : R \to R$ having the form:

$$\varphi(x) = - \sum_{i=1}^{N} a_i e^{-b_i x} + B \quad ,$$

where B is an arbitrary constant, $a_i > 0\ \forall i$, $b_i > 0\ \forall i$, and $b_i \neq b_j$ for $i \neq j$. Then for any $\varphi \in \varphi^*$, $[-1]^{n+1}\, \partial^{(n)}\varphi/\partial x^n > 0$ for $n \geq 1$.

[4] Theorem 7 can also be proved from Pratt's more general Theorem 5 in [3], as has recently been brought to our attention.

Moreover, $r' \leq 0$, and $r' < 0$ if $N > 1$.

Proof: The proof of the first claim is obvious from the definition of φ .

Since $r' \underset{(=)}{<} 0 \Leftrightarrow \varphi'\varphi''' - \varphi''^2 \underset{(=)}{>} 0$, the proof of the second claim will follow if it can be shown that all such functions φ satisfy $\varphi'\varphi''' - \varphi''^2 \geq 0$, strict inequality holding for $N > 1$. The proof will proceed by induction on N .

For $N = 1$ it is well known that $\varphi'\varphi''' - \varphi''^2 \equiv 0$.

Let $\varphi(x) = - a_1 e^{-b_1 x} - a_2 e^{-b_2 x} + B$. Then $\varphi'(x) = a_1 b_1 e^{-b_1 x} + a_2 b_2 e^{-b_2 x}$; $\varphi''(x) = - a_1 b_1^2 e^{-b_1 x} - a_2 b_2^2 e^{-b_2 x}$; and $\varphi'''(x) = a_1 b_1^3 e^{-b_1 x} + a_2 b_2^3 e^{-b_2 x}$. Hence $\varphi'\varphi''' =$

$$a_1^2 b_1^4 e^{-2b_1 x} + a_2^2 b_2^4 e^{-2b_2 x} + a_1 a_2 b_1^3 b_2 e^{-(b_1+b_2)x} + b_1 b_2^3 a_1 a_2 e^{-(b_1+b_2)x}$$

and $\varphi''^2 = a_1^2 b_1^4 e^{-2b_1 x} + a_2^2 b_2^4 e^{-2b_2 x} + 2a_1 a_2 b_1^2 b_2^2 e^{-(b_1+b_2)x}$

$$\Rightarrow \varphi'\varphi''' - \varphi''^2 = a_1 a_2 b_1 b_2 \, [b_1^2 + b_2^2 - 2b_1 b_2] \, e^{-(b_1+b_2)x} > 0$$

since $b_1^2 + b_2^2 - 2b_1 b_2 = [b_1 - b_2]^2$. Thus the second claim holds for $N \leq 2$.

Assume it holds for $N = k$. Let $\varphi(x) = - \sum_{i=1}^{k+1} a_i e^{-b_i x} + B$.

Then $\varphi'(x) = \sum_{i=1}^{k+1} a_i b_i e^{-b_i x}$; $\varphi''(x) = - \sum_{i=1}^{k+1} a_i b_i^2 e^{-b_i x}$;

and $\varphi'''(x) = \sum_{i=1}^{k+1} a_i b_i^3 e^{-b_i x}$. Hence $\varphi'\varphi''' =$

$$\left[\sum_{i=1}^{k} a_i b_i e^{-b_i x}\right] \cdot \left[\sum_{i=1}^{k} a_i b_i^3 e^{-b_i x}\right] +$$

$$\left[a_{k+1} b_{k+1} e^{-b_{k+1} x}\right] \cdot \varphi''' + \varphi' \cdot \left[a_{k+1} b_{k+1}^3 e^{-b_{k+1} x}\right] \text{ ; and } \varphi''^2 =$$

$$\left[-\sum_{i=1}^{k} a_i b_i^2 e^{-b_i x}\right] \cdot \left[-\sum_{i=1}^{k} a_i b_i^2 e^{-b_i x}\right] +$$

$2\varphi'' \cdot \left[- a_{k+1} b_{k+1}^2 e^{-b_{k+1} x}\right]$. Using the induction hypothesis for $N = k$, $\varphi'\varphi''' - \varphi''^2 > 0$ will hold for $N = k + 1$ if $A \equiv \left[a_{k+1} b_{k+1} e^{-b_{k+1} x}\right] \cdot \varphi''' + \varphi' \cdot \left[a_{k+1} b_{k+1}^3 e^{-b_{k+1} x}\right] \geq B \equiv 2\varphi'' \left[- a_{k+1} b_{k+1}^2 e^{-b_{k+1} x}\right]$.

But $A = \sum_{i=1}^{k} \left[a_{k+1} a_i b_{k+1} b_i\right] \left[b_{k+1}^2 + b_i^2\right] e^{-(b_i + b_{k+1})x} + a_{k+1}^2 b_{k+1}^4 e^{-2b_{k+1} x}$ and $B = \sum_{i=1}^{k} \left[2\, a_{k+1} a_i b_{k+1}^2 b_i^2 e^{-(b_i + b_{k+1})x}\right] + a_{k+1}^2 b_{k+1}^4 e^{-2b_{k+1} x} \Rightarrow A - B = \sum_{i=1}^{k} a_{k+1} a_i b_{k+1} b_i \left[b_{k+1}^2 + b_i^2 - 2\, b_{k+1} b_i\right] e^{-(b_1 + b_{k+1})x} > 0$ since $b_{k+1}^2 + b_i^2 - 2\, b_{k+1} b_i = \left[b_{k+1} - b_i\right]^2 > 0 \; \forall i$.

Theorem 8 below provides an important positive check on the soundness of the model. Its corollary states that if X is independent of Y and $r' < 0$, then $EY \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless 0 \Leftrightarrow \frac{\partial\hat{\alpha}}{\partial X} \gtreqless 0$. It is interesting to note that the last double implication is false if "$r' < 0$" is replaced by "$\varphi''' > 0$",[5] a necessary but not sufficient condition for $r' < 0$ to hold. Thus Arrow's hypothesis does indeed seem to provide an incisive characterization of the behavior one expects from a decision maker generally adverse to risk.

The following lemma is needed for the proof of Theorem 8.

Lemma 8. Assume φ''' monotonic and $E\varphi'(X+t)$, $E\varphi''(X+t)$ and $E\varphi'''(X+t)$ finite $\forall t$. Then $r' < 0 \Rightarrow T' < 0$, where $T(t) \equiv -E\varphi''(X+t)/E\varphi'(X+t)$.

Proof: Under the above assumptions, $T'(t) = -[E\varphi'(X+t)\, E\varphi'''(X+t) - (E\varphi''(X+t))^2]/[E\varphi'(X+t)]^2$. Thus $T' < 0$ if $E\varphi'(X+t)\, E\varphi'''(X+t) - [E\varphi''(X+t)]^2 > 0$.

$r' < 0 \Rightarrow \varphi'(x+t)\, \varphi'''(x+t) > [\varphi''(x+t)]^2 \ \forall x, t$

$\Rightarrow \sqrt{\varphi'(x+t)\, \varphi'''(x+t)} > |\varphi''(x+t)| \ \forall x, t \Rightarrow$

$[E\sqrt{\varphi'(X+t)\, \varphi'''(X+t)}\,]^2 > [E|\varphi''(X+t)|\,]^2$

$= [E\varphi''(X+t)]^2 \ \forall t$. But by Hölder's Inequality

$E\varphi'(X+t)\, E\varphi'''(X+t) \geq [E\sqrt{\varphi'(X+t)\, \varphi'''(X+t)}\,]^2 \ \forall t$.

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[5] For an example where $\varphi''' > 0$, X is independent of Y, $\hat{\alpha} > 0$ and $\frac{\partial\hat{\alpha}}{\partial X} < 0$, take $X \equiv 0$, $P[Y = -1] = 1/3$, $P[Y = 1] = 2/3$, $\varphi' > 0$, $\varphi(0) = 0$, $\lim_{x \to +\infty} \varphi(x) = B > 0$, $\varphi(-1) < -3B$, $\varphi''' > 0$, $\varphi'' < 0$, $\varphi''(-1) = -7$ and $\varphi''(1) = -4$. [It can be shown that $r'(x) > 0$ for some x for such a φ.]

<u>Theorem 8</u>. Let $Z \equiv X - bY$ for some $b \in R$, and suppose $E\varphi'(Z + t)$, $E\varphi''(Z + t)$ and $E\varphi'''(Z + t)$ are finite $\forall t$ and φ''' is monotonic. If Z is independent of Y and $r' < 0$, then

$$EY \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless -b \Leftrightarrow \frac{\partial\hat{\alpha}}{\partial X} \gtreqless 0 \quad .$$

<u>Proof</u>: The first double implication follows immediately from: $\eta'(-b) \equiv EY\,\varphi'(X - bY) = EY\,E\varphi'(Z) \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless -b$.

By Theorem 4, $\frac{\partial\hat{\alpha}}{\partial X}$ and $EY\,\varphi''(X + \hat{\alpha}Y)$ have the same sign. $EY\,\varphi'(X + \hat{\alpha}Y) = 0 \Leftrightarrow EY^-\,\varphi'(X + \hat{\alpha}Y)/EY^+\,\varphi'(X + \hat{\alpha}Y) = 1$, where $Y^- = \max\{-Y, 0\}$ and $Y^+ = \max\{Y, 0\}$. $EY\,\varphi''(X + \hat{\alpha}Y) \gtreqless 0 \Leftrightarrow EY^-[-\varphi''(X + \hat{\alpha}Y)]/EY^+[-\varphi''(X + \hat{\alpha}Y)] \gtreqless 1$; i.e., substituting in the former equality and rearranging terms, $\Leftrightarrow EY^-[-\varphi''(X + \hat{\alpha}Y)]/EY^-\varphi'(X + \hat{\alpha}Y) \gtreqless EY^+[-\varphi''(X + \hat{\alpha}Y)]/EY^+\varphi'(X + \hat{\alpha}Y)$. It will be shown below that this last relationship holds as $\hat{\alpha} \gtreqless -b$.

There exists a null set N such that $\forall\, \omega \notin N$, $\forall B \in \beta$ [β = Borel sets of R], $P(Z + bY \in B|Y)(\omega) = P(Z \in B - bY(\omega))$. For let $B_x \equiv (-\infty, x]$. Then for a.a.ω $P(Z + bY \in B_x|Y)(\omega) = E[I_{B_x}(Z + bY)|Y](\omega) = \int I_{B_x}(Z + bY)\,P(Z \in dz|Y = Y(\omega)) = \int I_{B_x}(Z + bY(\omega))\,P(Z \in dz) = P(Z \in B_x - bY(\omega))$. Hence $\exists$ a null set N such that $\forall\, \omega \in N$, $P(Z + bY \in B_x|Y)(\omega) = P(Z \in B_x - bY(\omega))$ for all rational x . Since $\{B_x : x \text{ rational}\}$ generates β, it holds that $\forall\, \omega \notin N$, $\forall\, B \in \beta$, $P(Z + bY \in B|Y)(\omega) = P(Z \in B - bY(\omega))$. [Carathéodory Extension Theorem.]

Thus

$$E[\varphi''(X + \hat{\alpha}Y)\,|\,Y = y] = E[\varphi''(X - bY + [b + \hat{\alpha}]Y)\,|\,Y = y]$$

$$= \int \varphi''(x - by + [b + \hat{\alpha}]y)\ P(Z + bY \in dx\,|\,Y = y) \quad \text{a.s.}$$

$$= \int \varphi''(x - by + [b + \hat{\alpha}]y)\ P(Z \in d[x - by]) \quad \text{a.s.}$$

$$= \int \varphi''(u + [b + \hat{\alpha}]y)\ P(Z \in du)$$

$$= E\varphi''(Z + [b + \hat{\alpha}]y) \quad .$$

Similarly for φ'.

By Lemma 8, $M' < 0$, where

$$M([b + \hat{\alpha}]y) \equiv - E\varphi''(Z + [b + \hat{\alpha}]y)/E\varphi'(Z + [b + \hat{\alpha}]y)$$

$$= - E[\varphi''(X + \hat{\alpha}Y)\,|\,Y = y]/E[\varphi'(X + \hat{\alpha}Y)\,|\,Y = y] \quad \text{a.s.}$$

Hence, letting $\hat{P}(A) = P(Y \in A)$, $\hat{\alpha} \gtreqless - b \Leftrightarrow$

$$\frac{EY^-[-\varphi''(x+\hat{\alpha}Y)]}{EY^-\varphi'(x+\hat{\alpha}Y)} \equiv \frac{\int_{-\infty}^{0}[-y]\,E[-\varphi''(x+\hat{\alpha}Y)\,|\,Y=y]\,\hat{P}(dy)}{\int_{-\infty}^{0}[-y]\,[E\varphi'(x+\hat{\alpha}Y)\,|\,Y=y]\,\hat{P}(dy)}$$

$$= \frac{\int_{-\infty}^{0}[-y]\,E[-\varphi''(Z+[b+\hat{\alpha}]y)]\,\hat{P}(dy)}{\int_{-\infty}^{0}[-y]\,E\varphi'(Z+[b+\hat{\alpha}]y)\,\hat{P}(dy)}$$

$$\gtreqless \frac{\int_{-\infty}^{0}[-y]\,[-E\varphi''(Z)/E\varphi'(Z)\cdot E\varphi'(Z+[b+\hat{\alpha}]y)]\,\hat{P}(dy)}{\int_{-\infty}^{0}[-y]\,E\varphi'(Z+[b+\hat{\alpha}]y)\,\hat{P}(dy)}$$

$$= -\,E\varphi''(Z)/E\varphi'(Z)$$

$$\equiv \frac{\int_{0}^{\infty}[y]\,[-E\varphi''(Z)/E\varphi'(Z)]\,E\varphi'(Z+[b+\hat{\alpha}]y)\,\hat{P}(dy)}{\int_{0}^{\infty} y\,E\varphi'(Z+[b+\hat{\alpha}]y)\,\hat{P}(dy)}$$

$$\gtreqless \frac{\int_{0}^{\infty} y\,[-E\varphi''(Z+[b+\hat{\alpha}]y)]\,\hat{P}(dy)}{\int_{0}^{\infty} y\,E\varphi'(Z+[b+\hat{\alpha}]y)\,\hat{P}(dy)}$$

$$= \frac{\int_{0}^{\infty} y\,[-E[\varphi''(X+\hat{\alpha}Y)\,|\,Y=y]]\,\hat{P}(dy)}{\int_{0}^{\infty} y\,E[\varphi'(X+\hat{\alpha}Y)\,|\,Y=y]\,\hat{P}(dy)}$$

$$= EY^+[-\varphi''(X+\hat{\alpha}Y)]/EY^+\varphi'(X+\hat{\alpha}Y) \quad .$$

Corollary. Let $E\varphi'(X+t)$, $E\varphi''(X+t)$ and $E\varphi''(X+t)$ be finite $\forall t$ and φ''' be monotonic. If $r' < 0$ and X is independent of Y, then

$$EY \gtreqless 0 \Leftrightarrow \hat{\alpha} \gtreqless 0 \Leftrightarrow \frac{\partial\hat{\alpha}}{\partial\bar{X}} \gtreqless 0 \quad .$$

Proof: Set $b = 0$ in Theorem 8.

For the next two theorems it will be convenient to define the following two conditions:

Condition A: $\exists\tilde{X}$ s.t. $Y(\omega) \geq 0 \Leftrightarrow X(\omega) \geq \tilde{X}$ a.a.w.

Condition B: $\exists\tilde{X}$ s.t. $Y(\omega) \geq 0 \Leftrightarrow X(\omega) \leq \tilde{X}$ a.a.w.

Theorem 9. Condition A, $EY \leq 0 \Rightarrow \hat{\alpha} \leq 0$. If either $EY < 0$ or $P(Y \neq 0,\ X \neq \tilde{X}) > 0$ then the strict inequality holds in the conclusion.

Condition B, $EY \geq 0 \Rightarrow \hat{\alpha} \geq 0$. If either $EY > 0$ or $P(Y \neq 0,$ $X \neq \tilde{X}) > 0$ the strict inequality holds in the conclusion.

Proof: Under A, $\eta'(0) = \int_{Y>0} Y\,\varphi'(X) + \int_{Y<0} Y\,\varphi'(X)$

$\leq \int_{Y>0} Y\,\varphi'(\tilde{X}) + \int_{Y<0} Y\,\varphi'(\tilde{X}) = \varphi'(\tilde{X})\,EY$ which agrees in

sign with EY. The inequality is strict if $P(Y \neq 0,$

$X \neq \tilde{X}) > 0$. Under B the inequality is reversed.

Theorem 10. Condition A, $\hat{\alpha} > 0$, r decreasing implies $\frac{\partial\hat{\alpha}}{\partial\bar{X}} \geq 0$. If r is strictly decreasing, $\frac{\partial\hat{\alpha}}{\partial\bar{X}} > 0$.

Condition B, $\hat{\alpha} < 0$, r decreasing implies $\frac{\partial\hat{\alpha}}{\partial\bar{X}} \leq 0$. If r is strictly decreasing, $\frac{\partial\hat{\alpha}}{\partial\bar{X}} < 0$.

<u>Proof</u>: If Condition A on X and Y holds, r is decreasing, and $\hat{\alpha} > 0$, then since sign $Y\varphi'(X + \hat{\alpha}Y)$ agrees with sign Y ,

$$(*) \quad Y\varphi'(X + \hat{\alpha}Y)\ r\ (X + \hat{\alpha}Y) \leq Y\varphi'(X + \hat{\alpha}Y)\ r\ (\bar{x}) \quad \text{a.s.}$$

Thus $$\frac{\partial\hat{\alpha}}{\partial\bar{x}} = - \Delta^{-1}\ EY\varphi''(X + \hat{\alpha}Y) = \Delta^{-1}\ EY\varphi'(X + \hat{\alpha}Y)\ r\ (X + \hat{\alpha}Y)$$

$$\geq \Delta^{-1}\ EY\varphi'(X + \hat{\alpha}Y)\ r\ (\bar{x}) = \Delta^{-1}\ r\ (\bar{x})\ EY\varphi'(X + \hat{\alpha}Y) = 0$$

which proves the first statement. If r is strictly decreasing then the equality in (*) holds only on (Y = 0) and nontriviality of Y justifies the strict inequality. The remainder follows by noting that Condition B and $\hat{\alpha} < 0$ holding for Y implies Condition A and $\hat{\alpha} > 0$ hold for $-Y$.

<u>Corollary</u>. If r decreasing is replaced by r increasing in Theorem 10 the inequalities of the conclusions are reversed.

<u>Proof</u>: This reverses the inequality in (*) .

Theorems 9 and 10 can be interpreted in terms of a tendency also cited in [1]. If a venture tends to reward the decision maker in events that are unfavorable for his current prospect, the venture has a kind of insurance value. Such a venture should typically be worth more to a risk averter than a venture with similar expectation but no insurance value.

Condition B is one way to express a tendency for a particular venture Y to have positive insurance value when considered in conjunction with a particular initial prospect X. Another expression for such a tendency would be to specify that X and Y are negatively correlated. Two additional versions of this tendency were used in [1]. Utility of a venture αY was decomposed into three parts:

$$(1) \qquad \eta(\alpha) = E\varphi(X + \alpha Y) = [E\varphi(X + \alpha\bar{Y})] + [E\varphi(X + \alpha W) - E\varphi(X + \alpha\bar{Y})] + [E\varphi(X + \alpha Y) - E\varphi(X + \alpha W)] = \eta_G(\alpha) + \eta_S(\alpha) + \eta_D(\alpha)$$

where W is a random variable that is independent of X and Y and has the same distribution as Y. The three components were called utility of gain, utility of spread, and utility of dependence respectively. It is easy to verify that $\eta_S'(0) = 0$ so

$$(2) \qquad \eta'(0) = \eta_G'(0) + \eta_D'(0) = \bar{Y}\, E\varphi'(X) + E(Y - W)\ \varphi'(X) \quad .$$

It seems reasonable to say that Y has some insurance value if $\eta_D'(0) > 0$ since this will tend to be true if Y tends to be large when X is small.

The other expression considered in [1] was $E[Y|X = x]$ decreasing. These two were related by Proposition 9, page 23:

$$(3) \qquad E[Y|X = x] \text{ decreasing (increasing)} \Rightarrow \eta_D'(0) \geq 0 \ (\leq 0)$$

with strict inequality in the conclusions if $E[Y|X = x]$ is strictly monotone.

Recognizing from (1) that $\operatorname{sgn}(\eta_G'(0)) = \operatorname{sgn}(\bar{Y})$ leads immediately to

$$(4) \qquad E[Y|X = x] \text{ decreasing (increasing)},\ \bar{Y} \geq 0\ (\leq 0) \Rightarrow \hat{\alpha} \geq 0\ (\leq 0)$$

with strict inequality in the conclusion if $E[Y|X = x]$ strictly decreasing or $\bar{Y} > 0$.

It is possible to further relate these four possible expressions for a tendency toward positive insurance value of a venture as follows:

Theorem 11.

(i) If $\tilde{x}$ satisfies $\varphi'(\tilde{x}) = E\varphi'(X)$ then Condition B $\Rightarrow \eta_D'(0) \geq 0$. If $P(Y \neq 0, \; X \neq \tilde{x}) > 0$ is also assumed, then $\eta_D'(0) > 0$.

(ii) If $\tilde{x} = \bar{X}$, then Condition B $\Rightarrow \rho_{XY} \leq 0$. $\rho_{XY} < 0$ if $P(Y \neq 0, \; X \neq \tilde{x}) > 0$ is also assumed.

(iii) $E[Y|X = x]$ decreasing $\Rightarrow \rho_{XY} \leq 0$. Strict inequality holds if $E[Y|X = x]$ is strictly decreasing.

Proof:

(i) $EY\varphi'(X) = \int_{Y<0} Y\,\varphi'(X) + \int_{Y>0} Y\,\varphi'(X) \geq \int_{Y<0} Y\,\varphi'(\tilde{x}) + \int_{Y>0} Y\,\varphi'(\tilde{x}) = \bar{Y}\,\varphi'(\tilde{x}) = EW\varphi'(X)$, where the inequality follows from Condition B and the final equality from independence of W and X and the assumed condition $\varphi'(\tilde{x}) = E\varphi'(X)$. The inequality is strict if $P(Y \neq 0, \; X \neq \tilde{x}) > 0$.

(ii) $E\,X\,Y = \int_{Y<0} X\,Y + \int_{Y>0} X\,Y \le \int_{Y<0} \tilde{x}\,Y + \int_{Y>0} \tilde{x}\,Y$

$= \tilde{x}\,\bar{Y} = \bar{X}\,\bar{Y}$. Hence $\rho_{XY} = E\,X\,Y - \bar{X}\,\bar{Y} \le 0$,

with strict inequality holding if

$P(Y \neq 0,\ X \neq \tilde{x}) > 0$ is assumed.

(iii) If $E[Y|X = x] = EY\ \forall x$, then $\rho_{XY} = 0$. Assume

$E[Y|X = x] \neq EY\ \forall x$. Then $\exists \tilde{x}$ such that

$[\omega{:}\, X(\omega) \ge \tilde{x}] = [\omega{:}\, E[Y|X] \le \bar{Y}]$. Hence

$EXY = E[X - \tilde{x}]Y + \tilde{x}\bar{Y} = \int_{\tilde{x}}^{\infty} [x - \tilde{x}]\, E\, [Y|X = x]F(dx)$

$+ \int_{-\infty}^{\tilde{x}} [x - \tilde{x}]\, E\, [Y|X = x]\, F(dx) + \tilde{x}\bar{Y} \le \bar{Y}\, [\int_{\tilde{x}}^{\infty} [x - \tilde{x}]F(dx)$

$+ \int_{-\infty}^{\tilde{x}} [x - \tilde{x}]F(dx) + \tilde{x}] = \bar{Y}\,\bar{X}$. Clearly the inequality

is strict if $E[Y|X = x]$ is strictly decreasing.

Substitution of Condition A for Condition B and

$E[Y|X = x]$ increasing for decreasing reverses

inequalities in the conclusions.

The assumption r = constant may sometimes be appropriate. In this case there is no wealth effect.

Theorem 12. If r is constant, then $\frac{\partial\hat{\alpha}}{\partial\bar{X}} = 0$.

Proof: $\frac{\partial\hat{\alpha}}{\partial\bar{X}} = -\,\Delta^{-1}\, EY\varphi'' = \Delta^{-1}\, EYr\varphi' = \Delta^{-1}\, rEY\varphi'(X + \hat{\alpha}Y) = 0$.

Boundedness of the initial prospect may sometimes help determine the sign of $\hat{\alpha}$ as indicated in Theorem 13. Let $Y^+ = \max\{Y, 0\}$ and $Y^- = \max\{-Y, 0\}$. Our assumption that Y is not an almost sure thing implies $EY^+ > 0$ and $EY^- > 0$.

Theorem 13. If $\tilde{x} \le X \le \overset{*}{x}$ for some $\tilde{x} \le \overset{*}{x}$ then $\dfrac{EY^+}{EY^-} \ge \dfrac{\varphi'(\tilde{x})}{\varphi'(\overset{*}{x})} \Rightarrow \hat{\alpha} \ge 0$ and $\dfrac{EY^+}{EY^-} \le \dfrac{\varphi'(\overset{*}{x})}{\varphi'(\tilde{x})} \Rightarrow \hat{\alpha} \le 0$.

Proof: $\eta'(0) = EY\varphi'(X) = EY^+\varphi'(X) - EY^-\varphi'(X) \ge EY^+\varphi'(\overset{*}{x})$ $- EY^-\varphi'(\tilde{x}) = [EY^-\varphi'(\overset{*}{x})]\left[\dfrac{EY^+}{EY^-} - \dfrac{E\varphi'(\tilde{x})}{E\varphi'(\overset{*}{x})}\right]$, which has the sign of the second factor. Proof of the other case is analogous.

3. Choice between a safe asset and a security

To be widely applicable to economic decision making the foregoing model needs to be generalized in several aspects. Multidimensional families of ventures need to be considered. In many contexts wealth is not the only desideratum so more general spaces of possible outcomes must be studied. Since immediate choices affect circumstances in various future periods, explicitly dynamic models should be formulated. Technical and institutional restraints that help define available choices must also be recognized.

Models that incorporate these considerations in various combinations have been studied to some extent in recent economic literature on choice under uncertainty[6] and will be topics of future papers by the present

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[6] See, for example, [4], [5], [6], [7], [8] and [9].

authors. Meanwhile, it seems useful to briefly consider how the model as developed so far can be used to gain some insight into choice in some highly simplified economic circumstances.

Suppose a decision maker has current prospect $V + c$ where c represents cash on hand and V represents prospective earnings, inheritances, liabilities for unforeseen accidents, etc. His cash will be used to purchase a safe asset with known return b (a negative purchase of the safe asset is interpreted as borrowing cash at interest rate $b - 1$) and a security whose possible returns per share are represented by a random variable W and whose price per share is h. He wants to maximize expected utility of his holdings at a specified date in the future when return from both assets will be realized.

Let α be the number of shares of the security purchased (negative α corresponds to selling short) and β be the number of dollars invested in the safe asset. Then

$$\beta = c - \alpha h$$

and if α shares are purchased, expected utility is given by

$$(1) \qquad \pi^*(\alpha, \beta) = E\varphi(V + \alpha W + \beta b)$$

or

$$(2) \qquad \eta(\alpha) = E\varphi(V + cb + \alpha[W - hb])$$

which has the form of the problem discussed in Sections 1 and 2 except that $V + cb$ has replaced X and $W - hb$ has replaced Y. Thus any of the propositions developed for the abstract problem can readily be applied to the more specific two-asset problem. This will be illustrated for a few selected propositions.

By Theorems 3 and 4, a unique optimal $\hat{\alpha}$ exists if there is positive probability that $W - hb > 0$ and positive probability that $W - hb < 0$. $W - hb$ is positive if $\frac{W}{h} > b$, that is, if dollars invested in the security yield more than dollars invested in the safe asset.

Assuming $\hat{\alpha}$ exists, one is interested in relating the sign of $\hat{\alpha}$ to characteristics of the utility function, the current prospect, the security, and to the rate of return on the safe asset. One is also interested in the marginal response of $\hat{\alpha}$ to changes in the decision maker's circumstances.

Write $V = V^0 + \bar{V}$ and $W = W^0 + \bar{W}$, where $EV = \bar{V}$ and $EW = \bar{W}$. As in the preceding section, let $\Delta = \frac{\partial^2 \eta}{\partial \alpha^2}$ evaluated at $\hat{\alpha}$. Then $\Delta < 0$ and

$$(3) \qquad \frac{\partial \hat{\alpha}}{\partial \bar{V}} = - \Delta^{-1} \frac{\partial^2 \eta}{\partial \alpha \partial \bar{V}} = - \Delta^{-1} EY\varphi''(X + \hat{\alpha}Y)$$

$$(4) \qquad \frac{\partial \hat{\alpha}}{\partial \bar{W}} = - \Delta^{-1} [E\varphi'(X + \hat{\alpha}Y) + \hat{\alpha}\, EY\varphi''(X + \hat{\alpha}Y)]$$

$$= \hat{S} + \hat{\alpha} \frac{\partial \hat{\alpha}}{\partial \bar{V}}$$

where $\hat{S} \equiv - \Delta^{-1} E\varphi'(X + \hat{\alpha}Y) > 0$ and the expectations are taken with respect to the joint distribution of V^0 and W^0 .[7] Continuing,

$$(5) \qquad \frac{\partial \hat{\alpha}}{\partial c} = b \frac{\partial \hat{\alpha}}{\partial \bar{V}}$$

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[7] As in Section 2, the simpler notation in terms of X and Y will be retained in expressing arguments of functions.

(6) $$\frac{\partial\hat{\alpha}}{\partial b} = - h\ \hat{S} + [c - \hat{\alpha}h]\ \frac{\partial\hat{\alpha}}{\partial\bar{V}}$$

$$= - h\ \hat{S} + \hat{\beta}\ \frac{\partial\hat{\alpha}}{\partial\bar{V}} = - h\ \frac{\partial\hat{\alpha}}{\partial\bar{W}} + c\ \frac{\partial\hat{\alpha}}{\partial\bar{V}}$$

(7) $$\frac{\partial\hat{\alpha}}{\partial h} = - b\left[\hat{S} + \hat{\alpha}\ \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right] = - b\ \frac{\partial\hat{\alpha}}{\partial\bar{W}} \quad .$$

These marginal responses presuppose that α can be any real number or that $\hat{\alpha}$ is in the interior of the admissible set if there are restrictions. Effects of boundary solutions will be briefly considered after some interpretation of (3) - (7).

$\bar{V}$ will increase if there is the subjectively certain promise of a previously unanticipated windfall gain at the end of the time period under consideration. For example a mysterious uncle appears with a terminal illness and provision for the decision maker in his will.

From (5) we see, as is intuitively reasonable, that the effect on security purchases of an increment of current cash is b times the effect of a subjectively certain prospective increment of equal amount. There are two kinds of wealth - wealth now and wealth at the end of the period. The cumulation factor b converts a unit of present wealth to a subjectively equivalent number of units of prospective wealth.

In (4), an increase in $\bar{W}$ is a special kind of reevaluation of the prospects for the security. Such a reevaluation could occur if a company experienced an unexpected gain that did not effect calculations of remaining prospects and decided to distribute the gain in end of period common stock dividends. The marginal result of such a reevaluation is the sum of a wealth effect $\left(\hat{\alpha}\ \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right)$ proportional to holdings and a pseudosubstitution effect $(\hat{S})$ known to be positive.

Marginal effect of a change in price of the security $\left(- b \frac{\partial\hat{\alpha}}{\partial\bar{W}}\right)$ is opposite in sign to the effect of a change in expected return and is multiplied by the rate of return on the safe asset since price must be paid at the outset and expected return is available at the end of the period (7). As with price of a consumer good, the wealth effect

$$\left(- \hat{\alpha}b \frac{\partial\hat{\alpha}}{\partial\bar{V}} \quad \text{or} \quad - \hat{\alpha} \frac{\partial\hat{\alpha}}{\partial c}\right)$$

is proportional to quantity and the substitution effect $(- b\, \hat{S})$ is known to be negative.

(6) gives a cross effect - the effect of an increase in yield of a safe asset on optimal purchase of an uncertain asset $\left(\frac{\partial\hat{\alpha}}{\partial b}\right)$. This decomposes into a negative pseudosubstitution effect $(- h\, \hat{S})$ and a wealth effect $\left(\hat{\beta} \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right)$ proportional to holdings. An alternative representation is the sum of a wealth effect $\left(c \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right)$ that would prevail if all cash were put in the safe asset less an allowance for cash put in the uncertain asset $\left(- h\, \hat{\alpha} \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right)$ and a pseudosubstitution effect $(- h\, \hat{S})$.

Using $\hat{\beta} = c - h\, \hat{\alpha}$, responses of security purchases can be converted to responses of safe asset holdings. These are listed without interpretation.

$$(3') \qquad \frac{\partial\hat{\beta}}{\partial\bar{V}} = - h \frac{\partial\hat{\alpha}}{\partial\bar{V}}$$

$$(4') \qquad \frac{\partial\hat{\beta}}{\partial\bar{W}} = - h \frac{\partial\hat{\alpha}}{\partial\bar{W}} = - h\, \hat{S} + \hat{\alpha} \frac{\partial\hat{\beta}}{\partial\bar{V}}$$

$$(5') \qquad \frac{\partial\hat{\beta}}{\partial c} = 1 - h \frac{\partial\hat{\alpha}}{\partial c} = 1 + b \frac{\partial\hat{\beta}}{\partial\bar{V}}$$

$$(6') \quad \frac{\partial\hat{\beta}}{\partial b} = -h\frac{\partial\hat{\alpha}}{\partial b} = h^2\hat{S} + \hat{\beta}\frac{\partial\hat{\beta}}{\partial\bar{V}}$$

$$(7') \quad \frac{\partial\hat{\beta}}{\partial h} = -\hat{\alpha} - h\frac{\partial\hat{\alpha}}{\partial h} = -\hat{\alpha} - b\frac{\partial\hat{\beta}}{\partial\bar{W}}$$

$$= -\hat{\alpha} + h\,b\,\hat{S} - \hat{\alpha}\,b\frac{\partial\hat{\beta}}{\partial\bar{V}} \quad .$$

Application of the propositions of Section 2 to the determination of the signs of $\hat{\alpha}$ and the marginal responses under the various conditions stated is straightforward. To consider conditions of all the propositions would be tedious so two possible circumstances have been chosen. First suppose V and W are independent. This might be approximately true if V depends largely on local circumstances and W on broader national and international developments.

If V, W are independent then $X = V + cb$ is independent of $Y = W - hb$ and the Corollary to Theorem 6, page 13, applies. $\hat{\alpha}$ agrees in sign with $EY = E[W - hb] = h\left[\frac{EW}{h} - b\right]$. $\frac{EW}{h}$ is the expected return from a dollar's worth of the security. If this exceeds the known return on the safe asset the decision maker will purchase some of the security. If, in addition, the decision maker's absolute risk aversion decreases with increased wealth, then the Corollary to Theorem 8, page 20, also applies $\left[\text{note that } \frac{\partial\hat{\alpha}}{\partial\bar{X}} = \frac{\partial\hat{\alpha}}{\partial\bar{V}}\right]$ and it is known that his security purchases will increase with increased wealth.

From Theorem 12, if the decision maker had constant risk aversion, increased wealth would leave security purchases unchanged. This and other examples[8] indicate the importance of learning enough about empirical

[8] See [10], [11].

utility functions to know how risk aversion varies with wealth in the population generally, and among various segments within the population where important differences in behavior might be expected.

From equations (3) - (7), pages 27 and 28, it is seen that $\frac{\partial\hat{\alpha}}{\partial\bar{V}} > 0$ implies $\frac{\partial\hat{\alpha}}{\partial c} > 0$. If it is also known that $\hat{\alpha} \geq 0$, then $\frac{\partial\hat{\alpha}}{\partial\bar{W}} > 0$ and $\frac{\partial\hat{\alpha}}{\partial h} < 0$. Determining $\frac{\partial\hat{\alpha}}{\partial b}$ would require more information. If $\frac{\partial\hat{\alpha}}{\partial\bar{V}} > 0$ and $\hat{\beta} < 0$ (the decision maker is borrowing), then $\frac{\partial\hat{\alpha}}{\partial b} < 0$. If borrowing is not permitted, i.e., β is restricted to be nonnegative which implies $\alpha \leq \frac{c}{h}$, then $\hat{\alpha} > \frac{c}{h}$ for the unrestricted problem implies $\alpha = \frac{c}{h}$ optimal under this restriction (see discussion on page 6).

If expected return per dollar's worth of security $\frac{\bar{W}}{h}$ is less than b, then $\hat{\alpha} < 0$ and $\frac{\partial\hat{\alpha}}{\partial\bar{V}} < 0$, the latter assuming $r' < 0$. This means the decision maker should optimally be in a short position, and should further sell short if wealth increases. If short sales are not permitted, then $\alpha = 0$ is optimal under this restriction.

Now consider another possible relation between V and W . Suppose V consists mainly of earnings and W is disability insurance. If disability is the only important potential source of substantial reductions in earnings, then Condition B, page 20, may approximately hold. If $\tilde{v}$ is the level of earnings at which the disability insurance becomes effective, then $[\omega: V(\omega) \geq \tilde{v}] = [\omega: W(\omega) = 0]$ and $[\omega: V(\omega) < \tilde{v}] = [\omega: W(\omega) \geq \tilde{w}]$ where $\tilde{w}$ is the minimum disability benefit if any benefits are paid. Recalling that $X = V + cb$ and $Y = W - hb$, we have

$$(8) \qquad [\omega: X(\omega) \geq \tilde{v} + cb] = [\omega: Y(\omega) = -hb]$$

$$(9) \qquad [\omega: X(\omega) < \tilde{v} + cb] = [\omega: Y(\omega) \geq \tilde{w} - hb] \quad .$$

h is the disability insurance premium and hb is the cumulated premium. If $\tilde{w} - hb > 0$ (minimum benefit exceeds cumulated premium) then (8) and (9) imply Condition B and Theorems 9 and 10 (page 20) apply. In this case, these general theorems permit only rather limited conclusions. Theorem 9 says that insurance should be purchased if its subjective actuarial value is positive.[9] Theorem 10 says that if purchasing insurance is not favorable under present circumstances it will not become favorable as expected earnings increase. Clearly additional theorems involving various forms of dependence between X and Y are needed to better understand interrelations between current prospects and ventures.

4. Expansion of a business

Suppose the decision maker operates a business with prospective net returns Z. In addition he has various personal contingencies (inheritance, accidental liabilities, etc. as before) represented by V, and some cash c above what is required for transactions. If his production function is linear homogeneous and his operations do not affect prices of inputs or outputs, an expanded business would have prospective net returns $[1 + \alpha] Z$ where α is the amount of expansion.

Suppose he intends to use his cash to expand the business and/or to purchase a security with prospective random returns W per share and price h. Let β be the number of shares purchased.

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[9] Subjective actuarial value could be positive because the personal probabilities of the decision maker differ from the company's tables or because the effective loss to the decision maker would be larger than just the reduced earnings. He might have to borrow under unfavorable circumstances, existing loans might be foreclosed, or he might have to forego immediate treatment and complicate a health difficulty.

If g is the cost of duplicating his present business facilities, then

$$c = \alpha g + \beta h \quad \text{or} \quad \beta = \frac{1}{h}[c - \alpha g]$$

and his expected utility is

$$\eta^*(\alpha, \beta) = E\varphi(V + [1 + \alpha]Z + \beta W)$$

or

$$\eta(\alpha) = E\varphi(V + Z + \frac{c}{h} W + \alpha[Z - \frac{g}{h} \bar{W}]) \quad .$$

This has the form of the abstract model with $X = [V + Z + \frac{c}{h} W]$ and $Y = [Z - \frac{g}{h} W]$.

One would need to know quite a bit about V, Z and W to infer useful properties of the joint distribution of X and Y. For the purpose of providing a simpler illustration, let W be a safe asset instead of an uncertain security. If b is the known yield then, in effect, we take $W = b$ and $h = 1$, so $X = [V + Z + cb]$ and $Y = [Z - gb]$. It then seems reasonable to suppose $E[Y|X = x]$ is an increasing function of x . From relation (3), Section 2, page 22, it is then seen that $\eta_D' < 0$, and from relation (2), Section 2, page 22, it is seen that EZ will have to exceed gb by more than $|\eta_D'|$ to justify any expansion of the business. gb is the known return from investing in the safe asset an amount that would be sufficient to duplicate the business.

In some instances it might be reasonable to assume that personal contingencies V are distributed independently of business contingencies Z . Then $X - Y = V + (c + g)\, b$ is independent of $Y = Z - gb$ and

Theorem 8 may be applied. Theorem 8 indicates that the wealth response $\frac{\partial \hat{\alpha}}{\partial \bar{X}}$ agrees in sign with $EY = EZ - gb$ which compares expected rate of return in the business with known rate of return on the safe asset.

A further simplification results from assuming that personal contingencies are sufficiently small relative to business contingencies to be neglected. Then $X = [Z + cb]$, $Y = [Z - gb]$ and

$$Y \gtreqless 0 \Leftrightarrow X \gtreqless [g + c]b \quad .$$

Thus Condition A of Theorems 9 and 10 is satisfied. Theorem 9 says that the business should not be expanded if $EZ \leq gb$, and Theorem 10 says that if absolute risk aversion is decreasing and if the business has been optimally expanded under current circumstances, then it should be expanded still more if the expected return increases. Although the conclusion does not depend on this coincidence, it is interesting to note that $\bar{X}$ and $\bar{Y}$ vary together in this example.

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