AN INTERGENERATIONAL MODEL OF POPULATION GROWTH

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I. Introduction

The problem of population growth has not yet been formulated in an intertemporal context. Therefore, the link between population theory and the economic growth theory is still missing.[1] In this paper we attempt to provide this link by analyzing an intergenerational model of optimum population growth. The model accommodates the idea that in decisions on increases in population consideration should be given to the "quality" or the "standard of life" of the new population. In a static framework this idea is elaborated upon in Becker and Lewis [1], DeTray [3] and Willis [10]. In some other literature on population growth, such as in Votey [9], this consideration is altogether disregarded.[2] More explicitly, we assume in this paper that the utility of each generation is a function of the level of its consumption and the <u>number</u> and <u>utility</u> of the newly born people. We are thus led to consider utility of infinitely many generations. This model differs from Samuelson's [7] intergenerational model of pure-consumption, since in the latter, the utility of each generation of people depends only on their own consumption.

We first outline a general model of optimum population growth. In order to discuss explicitly various economic aspects of population we use in the analysis a simple case.

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[1] For recent surveys of economic approaches to population, see Nerlove [5] and Schultz [8].

[2] See [2] for some critical discussion of the formulation in [9].

The plan of the paper is as follows. In Section II we set up the basic model. In Section III we analyze the short-run equilibrium conditions while Section IV discusses the long-run equilibrium conditions. In Section V we present an explicit solution for a simple case which yields a constant rate of population growth. Section VI explores the effects of public support to investment in children on population growth. In Section VII we analyze the relationship between population growth and values of time. Section VIII considers taste differences among generations, while Section IX deals with uncertain population changes.

II. The Model

Let there be identical individuals; L_t be the number of people of generation t, c_t their per capita lifetime consumption and U_t the utility indicator of their preferences. Each generation lives one period. Thus, L_{t+1}, the number of people of generation $t + 1$ may also be regarded as the number of newly born people at t. Let $\lambda_t = L_{t+1}/L_t$ be the per capita number of newly born people at t, K_t the total amount of physical capital and $k_t = K_t/L_t$ the per capita amount of capital at t. In general, when U_t is a function of c_t, λ_t and U_{t+1} we can write an intergenerational utility indicator $V = U_t(c_t,\lambda_t,U_{t+1}(c_{t+1},\lambda_{t+1},U_{t+2}(\ldots)))$, where $t = 0, 1, 2, \ldots$. We assume that preferences are the same for each generation and can be represented by an <u>additive</u> utility function $U_t(c_t,\lambda_t) + \beta\, U_{t+1}$; this implies

$$V = \sum_{t=0}^{\infty} \beta^t U(c_t,\lambda_t) \tag{1}$$

where β is the subjective factor by which current generation discounts utility of the next generation.[1] Let the capital of the next generation be produced according to a linear homogeneous production function $F(\cdot)$,[2]

$$(2) \qquad K_{t+1} = F(K_t - C_t,\ L_t)$$

where $K_t - C_t$ is total saving. In per capita terms, (2) can be rewritten as:

$$(3) \qquad \lambda_t k_{t+1} = f(k_t - c_t)$$

where $f(k_t - c_t) \equiv F(k_t - c_t,\ 1)$.

Note that the budget constraint in equation (3) implies that for generation t the amount of k_t , which is inherited from previous generations and is given, can be allocated to three ends: consumption c_t , increase in population λ_t , and the amount of resources to be left over to each individual in the next generation k_{t+1} . This is analogous to the budget constraint used in Becker and Lewis [1] where λ is regarded as the number of children in a family and k_{t+1} as the amount invested in their "quality".[3]

The decision problem of the current generation can now be written

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[1] The inclusion of population growth in the social utility function has also an empirical implication for the measurement of welfare improvement. That is, growth of per capita income, by itself, is an inappropriate measure of welfare improvement, and as a measure it is biased against countries with a high rate of population growth.

[2] The reader may refer to Liviatan [4] for an analysis of optimum economic growth which utilizes the discrete-time formulation.

[3] In the model the inherited capital may include both human and non-human capital (when they are perfectly substitutable in production).

as:

$$(4) \qquad V(k_o) = \max \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t, \lambda_t) \right\}$$

$$0 \leq c_t \leq k_t$$

$$0 \leq \lambda_t \leq \bar{\lambda}$$

subject to (3) and a given level of k_o, where $\bar{\lambda}$ is the maximum feasible level of population growth. Marginal utilities are positive and diminishing.

We can provide now the following characterization of an (interior) solution of (4).[1]

$$(5) \qquad \frac{\partial U}{\partial \lambda}(c_t, \lambda_t) = \frac{\beta}{\lambda_t} k_{t+1} \frac{\partial U}{\partial c}(c_{t+1}, \lambda_{t+1})$$

$$(6) \qquad \frac{\partial U}{\partial c}(c_t, \lambda_t) = \frac{\beta}{\lambda_t} \frac{\partial f}{\partial k}(k_t - c_t) \frac{\partial U}{\partial c}(c_{t+1}, \lambda_{t+1}) \quad .$$

Equation (5) may be interpreted as describing the optimum decision with respect to the level of population growth λ_t. On the one hand an extra unit of λ_t will increase welfare by the marginal utility of population growth, the left-hand side of (5). On the other hand, from (3), for given values of k_t and c_t, this increase in λ_t will reduce next generation (per capita) capital k_{t+1} by the amount of k_{t+1}/λ_t, which entails a loss for the next generation of (k_{t+1}/λ_t) times the marginal utility of consumption. This utility loss (discounted by β) appears on the right-hand side of (6). In equilibrium, according to equation (6), the gain and the loss in utility are equated.

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[1] This characterization can be obtained by a straightforward application of the method of dynamic programming.

Equation (6) may be interpreted as describing the optimizing level of consumption c_t . On the one hand, an extra unit of c_t will increase utility by the marginal utility of consumption, the left-hand side of (6). On the other hand, this entails a reduction in next generation capital. This loss is expressed in utility terms by the right-hand side of (6).

One relevant restriction, often imposed by the society's "standards", is the requirement that each child will be given a minimum amount of capital for purposes such as subsistence and education. This can be represented by a constraint $K_t - C_t \geq mL_{t+1}$, where m is the minimum level of investment in a child. In per capita terms this constraint reduces to

$$k_t - c_t \geq m\lambda_t$$

and can be added to the optimization problem (4). The imposition of this constraint when binding (i.e., when individuals would have invested in a child less than in the absence of the constraint), by increasing the cost of raising a child, is likely to reduce the number of children per family.

III. Short-Run

In the short run when the initial value of k is given the system of equations (3), (5) and (6) may be solved for the quantities of the three "goods": current consumption c_o , population growth λ_o and the next-generation capital k_1 (all expressed in per capita terms). Since k_1 does not appear directly in the utility function and thus is not an ordinary good, we must use its imputed value. Since at equilibrium an extra unit of capital can be used either for consumption or for saving with equal marginal gain (equation (6)) we have:[1]

$$\frac{\partial V(k_t)}{\partial k_t} = \frac{\partial U(c_t, \lambda_t)}{\partial c_t} \qquad (7)$$

Equation (7) specialized for $t = 1$ will yield the imputed value of the next-generation capital k_1 .

Figures 1, 2, and 3 represent a short-run interior solution, where marginal rates of substitution and marginal rates of transformation between the three goods are equalized.

In Figure 1, the marginal rate of transformation between c and k_1 (derived from the transformation curve $k_1 = f(k_o - c)/\lambda$) is $f'(k_o - c)/\lambda$. The marginal rate of substitution between c and k_1 is $\frac{\partial U(c, \lambda)}{\partial c} / \beta \frac{\partial V(k_1)}{\partial k_1}$. At point A the two rates are equalized, as can also be verified formally upon substitution of (7) into (6). Similarly, at point B in Figure 2, the marginal rate of substitution between λ and k_1 , $\frac{\partial U(c, \lambda)}{\partial \lambda} / \beta \frac{\partial V(k_1)}{\partial k}$, is equal to marginal rate of transformation between λ and k_1, k_1/λ . This condition can be derived by substitution of (7) into (5). As shown in Figure 2, a necessary condition for an interior solution for λ and k_1 is that the slope of the indifference curve diminishes faster than the slope of the transformation curve as λ increases. This implies the

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[1] It can be shown that $V(k)$ is concave, i.e., $\partial^2 V/\partial k^2 < 0$.

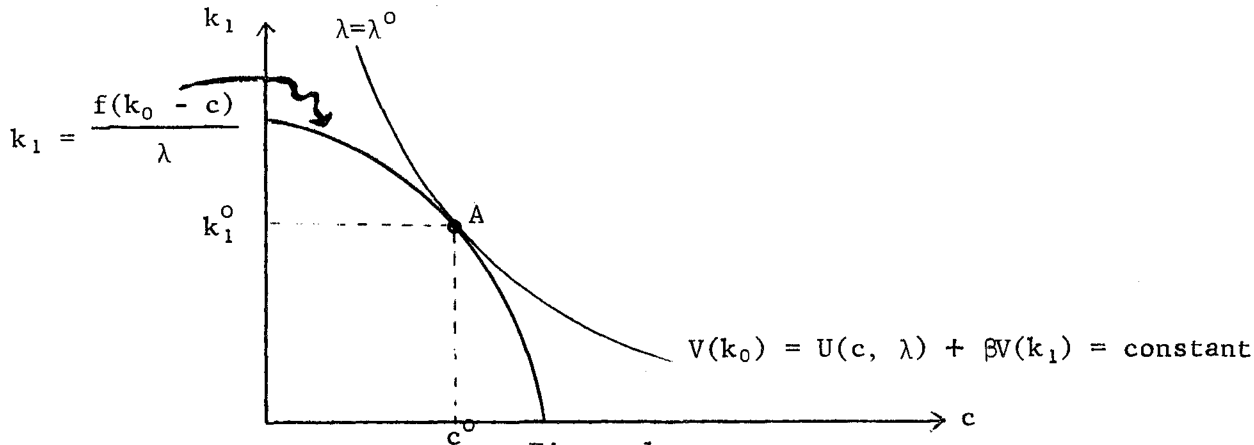

Figure 1

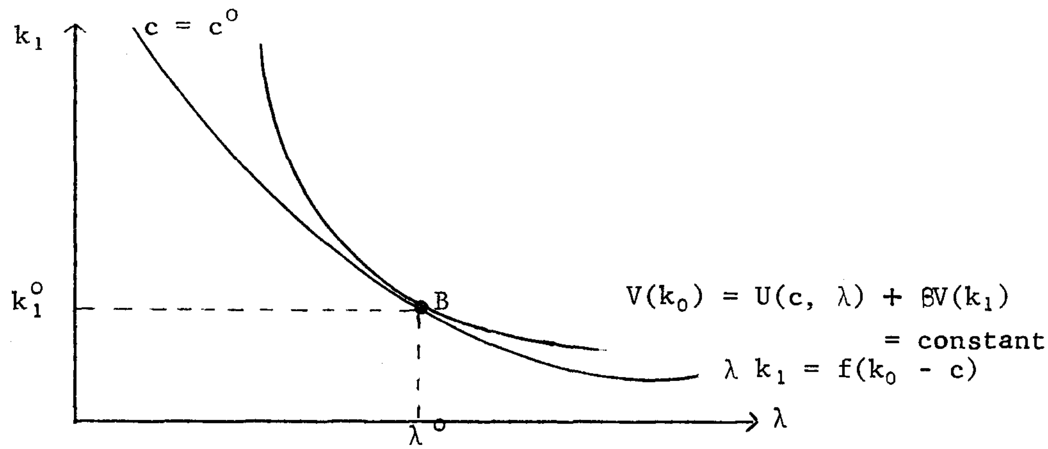

Figure 2

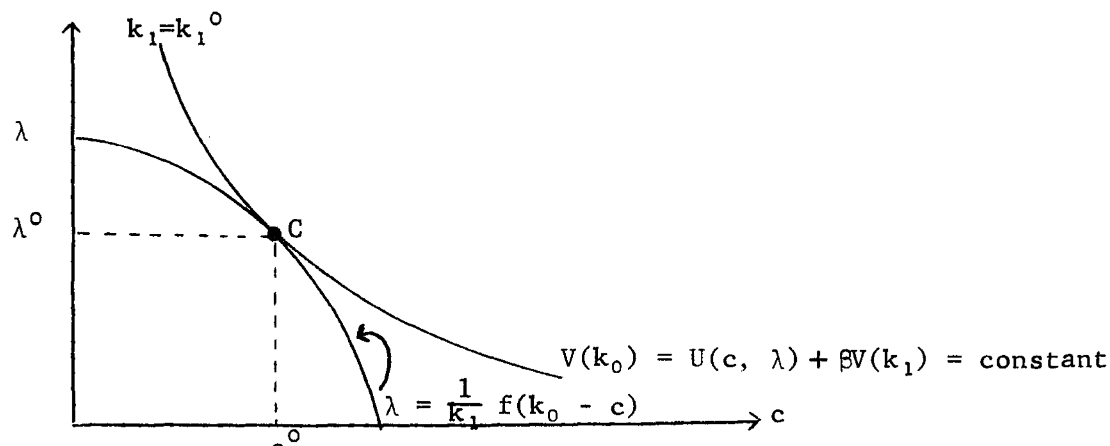

Figure 3

following condition.[1]

$$(8) \qquad \left(\frac{\lambda U_{\lambda\lambda}\ (c,\ \lambda)}{U_{\lambda}\ (c,\ \lambda)} + 1\right) + \left(\frac{k_1 V_{kk}\ (k_1)}{V_k(k_1)} + 1\right) < 0$$

where U_{λ}, V_k, $U_{\lambda\lambda}$ and V_{kk} are first and second order partial derivatives.

At the point C in Figure 3 the marginal rates of substitution and transformation between c and λ , given respectively by $\frac{\partial U\ (c,\ \lambda)}{\partial c} / \frac{\partial U\ (c,\ \lambda)}{\partial \lambda}$ and $\frac{1}{k_1} f'\ (k_o - c)$, are equalized. This condition can be derived from equations (5) and (6). In general, not all the three "goods", c, λ and k_1 are normal. In terms of Figures 1, 2 and 3, when k_o is increased there are shifts in both indifference and transformation curves so that the increase in k_o may result in a decline in the quantity of one or two of these "goods". This point was elaborated upon in Becker and Lewis [1] in the context of their model.

IV. Long-Run

At steady states (when exist) the system of equations (3), (5) and (6) reduces to:

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[1]Note that one of the cases in which an interior equilibrium is not achieved is when the utility function is logarithmic, i.e., $U(c, \lambda) = a \log c + b \log \lambda$. In this case the marginal rate of substitution between k_1 and λ is $\frac{a\lambda}{\beta b k_1}$ while the marginal rate of transformation is λ/k_1 . For example, equality is achieved if $a = \beta b$, but then the indifference curve coincides with the transformation curve, leaving the values k_1 and λ indeterminate. Otherwise, i.e., when $(a \neq b\beta)$ solution may not exist. To see this consider a two-period model with $V = a \log c_o + b \log \lambda_o + \beta \log k_1$. We can rewrite $V = a \log c_o + \beta \log (\lambda_o k_1) + (b - \beta) \log \lambda_o$. Since the product $\lambda_o k_1$ as such appears explicitly in the budget constraint $\lambda_o k_1 = f(k_o - c_o)$, when $b - \beta < 0$, λ_o may be regarded as a free good with a negative marginal utility. However, $\lambda_o = 0$ is not a solution. Therefore, a solution does not exist.

$$(9) \qquad \lambda k - f(k - c) = 0$$

$$(10) \qquad \lambda U_\lambda(c, \lambda) - \beta k U_c(c, \lambda) = 0$$

$$(11) \qquad \lambda/\beta - f'(k - c) = 0$$

where $U_\lambda = \partial U/\partial\lambda$, $U_c = \partial U/\partial c$.

Equation (11) may be interpreted as the "Modified Golden Rule" where the marginal productivity of capital, $f' - 1$, is equal to the rate of population growth, n, plus the subjective rate of discount, δ.[1]

One remarkable property of the steady state is that it is invariant to changes in utility which keep constant the marginal rate of substitution between population growth and consumption U_λ/U_c. Therefore, the steady state equilibrium (but not the path converging to it) results from an ordinal utility.

V. Constant Rate of Population Growth

Since it is difficult to further characterize the general solution it may be instructive to consider a special case. Let the utility function be given by:

$$(12) \qquad U(c, \lambda) = a \log c + v(\lambda)$$

where a is a positive number and $v(\lambda)$ is utility arising from population increase with positive and diminishing marginal utility. In order to guarantee an interior solution, we assume that the elasticity of the marginal utility of population increase is greater than one, (i.e., $-\lambda v_{\lambda\lambda}/v_\lambda > 1$).

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[1] Note that $\lambda = 1 + n$, $\beta = 1/(1 + \delta)$. Therefore, $f' - 1 = (\lambda/\beta) - 1 = (1 + n)(1 + \delta) - 1 \approx \delta + n$.

Let the production function be of a Cobb-Douglas form

$$(13) \qquad f(k - c) = r(k - c)^{\alpha}$$

where the constants r and α are positive and $\alpha \leq 1$. Substituting (12) and (13) into (6) we find a solution

$$(14) \qquad c_t = (1 - \alpha\beta)\, k_t \quad .$$

Substitution of (14) into (3) and (5) yields

$$(15) \qquad \lambda_t k_{t+1} = r(\alpha\beta)^{\alpha}\, k_t^{\alpha}$$

$$(16) \qquad \lambda_t v_{\lambda}(\lambda_t) = \frac{a\beta}{1-\alpha\beta} \rightarrow \lambda = \lambda^* \; .$$

We, therefore obtained consumption function with a unitary-income elasticity (equation (14)) and a constant rate of population growth, λ^* (equation (16)). The next-generation capital stock will increase as current resources increase (equation (15)).

Solving the difference equation (15) for $\log\ k_t$ using (14) and (16) we get

$$(17) \qquad \log k_t = \log\,[r(\alpha\beta)^{\alpha}/\lambda^*]\left[\frac{1-\alpha^{t+1}}{1-\alpha}\right] + \alpha^t \log k_o \quad .$$

Equations (14), (16) and (17) yield a complete short-run solution for our problem.

Turning to the long-run, for values of α below unity we have a stable equilibrium k^* (as $t \rightarrow \infty$),

$$(18) \qquad k^* = [r(\alpha\beta)^{\alpha}/\lambda^*]^{\frac{1}{1-\alpha}} \quad .$$

The specific model which we have presented enables us to illustrate some of the characteristics of the model with regard to changes in the parameters.

(a) A constraint of minimum capital per child

As shown earlier (in Section II) an effective constraint on minimum capital per child can be written as

$$k_t - c_t = \lambda_t m \quad .$$

It can be shown that (14) will not be effected by the constraint and we can therefore write the constraint as

$$k_t - (1 - \alpha\beta)\, k_t = \lambda_t m$$

or

$$\lambda_t = \frac{k_t \cdot \alpha\beta}{m}$$

which indicates that the number of children varies inversely with the required expenditure per child. Also, as long as the constraint is effective, λ is positively related to the stock of capital. However, whenever the constraint is not binding, λ_t is constant.

(b) Changes in the discount factor

Taking the derivative of (16) we can get

$$\frac{d\lambda}{d\beta} = \frac{\alpha}{(1 - \alpha\beta)^2} \cdot \frac{1}{(\lambda \cdot v_{\lambda\lambda} + v_\lambda)}$$

where for an interior solution the second term must be negative. This implies $\frac{d\lambda_0}{d\beta} < 0$. Similarly, from (14) and (17) we can simply see that

$$\frac{dc_0}{d\beta} < 0 \quad \text{and} \quad \frac{dk_1}{d\beta} > 0 \quad .$$

In summary, an increase in the discount rate of utility of future generation will increase the present consumption and the number of children which are undiscounted in the utility of the present generation and will reduce the capital per child which affects the utility function only indirectly through its effect on the utility of the future generation.

VI Public Support to Investment in Children and Population Growth

One characteristic of the modern state is the existence of subsidies for education of the young generation paid by taxes imposed on the parent-generation. From the individual parent's point of view there is no necessary relationship between the subsidies, which are received on a per-child basis, and tax payments, which are paid on the basis of their income. We analyze the effect of this policy on population growth by using the model of Section V.

Subsidies to the investment in children born at period t can be represented by a proportional increase in the amount of capital invested in a child, k_{t+1}. Equation (3) can therefore be rewritten as

$$\lambda_t(1 + s)k_{t+1} = f(k_t - c_t) \tag{19}$$

where s is the rate of the subsidy.

Assume that the public subsidy is financed by a proportional tax τ, so that disposable wealth at period t is $k_t(1 - \tau)$. In order for the government's budget to be balanced over the infinite horizon and for government spending not to exceed its revenue at any point in time we need

$$\sum_{t=0}^{T} \lambda_t \, k_{t+1} \, s \leqq \sum_{t=0}^{T} \tau \, k_t \ , \quad \text{for} \quad T = 1, 2, \ldots \ . \tag{20}$$

We assume that the tax-cum-subsidy policy, when enacted, is expected by individuals to last forever. What is the effect of (once and for all) changes in the rate of subsidy, with a balanced government budget, on population growth?

In the presence of the tax-cum-subsidy policy equations (5) and (6) are modified as follows.

$$U_\lambda(c_t, \lambda_t) = \frac{\beta}{\lambda_t} (1 - \tau) \, k_{t+1} \, U_c(c_{t+1}, \lambda_{t+1}) \tag{5a}$$

$$U_c(c_t, \lambda_t) = \frac{\beta}{\lambda_t} f'((1 - \tau) \, k_t - c_t) \frac{1}{(1+s)} U_c(c_{t+1}, \lambda_{t+1}) \ . \tag{6a}$$

In the special case considered in Section V these equations can be solved explicitly to get

$$c_t = (1 - \tau - \alpha\beta) \, k_t \tag{21}$$

$$\lambda_t \, v_\lambda \, (\lambda_t) = \frac{(1 - \tau) a\beta}{1 - \tau - \alpha\beta} \rightarrow \lambda_t = \lambda^*(\tau) \ . \tag{22}$$

$$\log k_t = \log \left[\frac{r}{(1+s)} (\alpha\beta)^\alpha / \lambda^*(\tau)\right] \left[\frac{1-\alpha^{t+1}}{1-\alpha}\right] + \alpha^t \log (1-\tau) \, k_o \tag{23}$$

where s and τ are chosen as to satisfy (20).

From equation (21) it can be inferred that consumption is negatively related to the rate of the income tax, τ, whereas, it is unaffected by the subsidy. Since the elasticity of the marginal utility of population increase is assumed to exceed one,[1] and the term on the right-hand side of (22) is positively related to τ, an increase in the tax rate τ lowers

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[1]See the discussion in Section V.

the rate of population growth λ, whereas the rate of population growth is not affected by the subsidy. The model thus implies that an increase in the subsidy to the next generation's capital financed by an increase in the rate of income tax will lead, as expected, to lower levels of consumption and population growth and to an increase in the rate of capital accumulation.

VII Population Growth and the Value of Time

The model of the preceding sections did not capture one important aspect of the population problem, namely, the effect of changes in the value of time on the optimal policy with regard to population increase. In this section we incorporate into the model decisions of society with regard to the amount of time devoted to raising children and the amount of time devoted to productive purposes.

Let g denote the <u>fraction</u> of the life time of the parents' generation which is devoted to raising children.[1] Let the number of children at period t, L_{t+1} be an increasing function $\Phi()$ of the <u>total</u> time which the parents' generation devotes to raising children $g\,L_t$,

$$L_{t+1} = \Phi(g_t \cdot L_t)\ , \quad \Phi'(\,) > 0 \quad . \tag{24}$$

For simplicity assume that the function $\Phi(\,)$ is homogeneous of degree one in L_t, then

$$\lambda_t = \varphi(g_t)\ , \quad \varphi'(\,) > 0 \tag{24a}$$

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[1] Investment in the human capital of children can be introduced in a similar way.

where $\varphi(g_t) \equiv \Phi(g_t, 1)$.

The total time devoted by the parents' generation to productive activities is $(1 - g_t) L_t$. This new element will require the following modifications in the equations of Section II.

$$\lambda_t \, k_{t+1} = (1 - g_t) \, f\left(\frac{k_t - c_t}{1 - g_t}\right) \tag{25}$$

$$U_\lambda(c_t, \lambda_t) = \beta \frac{k_{t+1}}{\lambda_t} \, U_c(c_{t+1}, \lambda_{t+1}) + \frac{\beta \, F_L}{\varphi'(g_t)\lambda_t} U_c(c_{t+1}\lambda_{t+1}) \tag{26}$$

where $F_L = f - \left(\frac{k-c}{1-g}\right) f'$ is the marginal productivity of labor.

Equation (26) differs from equation (5) by having an additional term on its right-hand side. This term corresponds to the cost of time associated with the population increase. One extra unit of λ requires an increase of a magnitude $1/\varphi'(g)$ in g (see (24a). The resulting decrease in the labor force participation will lower production of (per capita) capital of the next generation according to the marginal productivity of labor divided by the rate of population growth F_L/λ . The utility value of the loss, viewed by the present generation, is $\beta \, F_L \, U_c/\lambda \, \Phi'$. It is instructive to solve this model explicitly in the special case considered in Section V. The modification of the solution in Section V is in the equation which determines the rate of population growth λ while the solution for c is unaffected. We get

$$\varphi(g_t) \, v_\lambda \, (\varphi(g_t)) = \frac{a\beta}{1-\alpha\beta} + a\beta \, \frac{1-\alpha}{1-\alpha\beta} \, \frac{\varphi(g_t)}{\varphi'(g_t)} \rightarrow g_t = g^* \tag{27}$$

Equation (27) implies a <u>constant</u> fraction of time g^* to be devoted to raising children. This implies, in view of (24a), a constant rate of population growth.

This model suggests that in relation to population growth, it is misleading to measure the value of time by the marginal productivity of labor as such. The marginal productivity of labor varies, over time, together with the capital stock (except at a steady state). The thing which does not vary with the capital stock in this special case (and hence does not vary over time) is the utility value of time: $U_c F_L$. This is so since a proportional increase in the capital stock will lead to an equiproportionate increase in consumption of the next generation and, as a result, to an equiproportionate decrease in its marginal utility. Also, the marginal productivity of labor goes up by the same proportion. This leaves both the value of time $U_c F_L$ and the rate of population growth, λ , unchanged. In general, however, the value of time $U_c F_L$, which determines, among other things, the rate of population growth, might move either together with or in an opposite direction to the capital stock.[1] In particular, development of the economy over time, associated with the accumulation of capital, need not imply an increasing time-trend for the value of time.

VIII. Taste Differences Among Generations

It may be of interest to explore, in the context of an intergenerational decision making model, the effect of taste differences among generations on decisions made by the parents' generation. This may be regarded as an initial step in incorporating into the model situations of uncertainty on the part of the parents' generation with respect to preferences of future generations.[2]

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[1]Correspondingly, its partial effect on population growth is negative or positive.

[2]The analysis of uncertainty regarding preferences is not pursued here.

Let the utility of the parents' generation be denoted by U and that of every future generation by W. Intertemporal utility, in equation (1), is modified accordingly to get

$$U(c_o, \lambda_o) + \sum_{t=1}^{\infty} \beta^t W(c_t, \lambda_t) \quad . \tag{28}$$

Necessary conditions for an interior maximum then become

$$U_\lambda(c_o, \lambda_o) = \frac{\beta}{\lambda_o} k_t W_c(c_1, \lambda_1) \tag{29}$$

$$U_c(c_o, \lambda_o) = \frac{\beta}{\lambda_o} f'(k_o - c_o) W_c(c_1, \lambda_1) \tag{30}$$

where equations (5) - (6) hold for $t \geq 1$ with W substituting for U.

The difference between (5) - (6) (for $t = 0$) and (29) - (30) is that in the right-hand side of the latter marginal utility of the next generation is $W_c(c_1, \lambda_1)$ rather than $U_c(c_1, \lambda_1)$ as a result of the changes in tastes.

We analyze now, using the model of Section V, the effects of differences in tastes among generations with respect to population increase on the current rate of population growth. Specifically, let

$$U(c_o, \lambda_o) = a \log c_o + v(\lambda_o) \tag{31}$$

$$W(c_t, \lambda_t) = a \log c_t + v(\lambda_o, \theta) \ , \quad t \geq 1$$

where θ is a parameter in the function v which distinguishes the utility of future generations, derived from population increase, from that of the parents' generation. Let the production function be as in equation (13). Substituting (13) and (31) into (29) - (30) we can solve for c_o, λ_o and k_1. This solution for <u>every</u> θ is <u>identical</u> to the one in equations

(14) - (16) obtained in the case of unchanged tastes.[1] In this (special) case future generations' preferences with respect to population increase do not affect at all the decisions of parents' generations regarding the level of their consumption, the number of their children and the amount of capital left over for the next generation. Therefore, the introduction of uncertainty on the part of the parents' generation with respect to preferences of future generations regarding population increase will not affect current decisions either.

This may be regarded as a dividing-line case between general class of cases where a decreasing trend in the preferences of future generation with regard to population increases leads to a decrease in the current number of children and another general class of cases where this trend leads to an opposite result.

IX. Uncertain Population Changes

Suppose that population changes are subject to some random events. Among the reasons for uncertainty are random factors determining birth and death rates. What will be the effect of this kind of uncertainty on the population growth policy? To be specific, let

$$\lambda_t = \mu\, h_t \tag{32}$$

where μ is a random variable which is independently distributed over time and h_t is a variable by which population change is controlled. The value of problem (4) then becomes a random variable. Using the expected utility hypothesis the problem is reformulated as:

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[1] Note, however, that when the parameter a in (31) varies over generations all decisions of the parents' generation are affected.

$$(33) \qquad V(k_o) = \max_{\substack{0 \le c_t \le k_t \\ 0 \le h_t \le \bar{h}}} E\left\{\sum_{t=0}^{\infty} \beta^t U(c_t, \lambda_t)\right\}$$

subject to the <u>stochastic</u> constraint (3), where E is expectation operator.

This is a problem in Dynamic Programming that can be characterized by equations like (5) - (6) in expectation.

$$(34) \qquad E[\mu U_\lambda(c_t, \lambda_t)] = \beta\, E[k_{t+1}\, U_c(c_{t+1}, \lambda_{t+1})/h_t]$$

$$(35) \qquad E[U_c(c_t, \lambda_t)] = \beta\, E[f'(k_t - c_t) U_c(c_{t+1}, \lambda_{t+1})/\lambda_t] \quad .$$

To consider the effect of increasing risk in μ on the population growth policy h we analyze again the model in Section V given by equations (9) and (10). The solution of (34) - (35), in this case, is given by (14) and by

$$(36) \qquad h_t\, E[\mu v_\lambda\, (\theta g_t)] = \frac{a\beta}{(1-\alpha\beta)} \quad .$$

Equation (36) implies a <u>constant</u> policy regarding the growth of the population, although the actual growth is random. To obtain an unambiguous result pertaining to the effect of a mean preserving increase in risk on h, we specify

$$(37) \qquad v(\lambda) = \frac{1}{1-\gamma}\, \lambda^{1-\gamma} \quad ,$$

where $\gamma > 1$ (otherwise, no interior solution exists).

Following Rothschild and Stiglitz [6] a mean preserving increase in risk in μ will reduce the level of h if λv_λ is a decreasing function of h and is a convex function of μ. These properties of the function λv_λ are easily verifiable for the case given by (37). Thus, an increase in uncertainty will lower the planned level of population growth.

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