

A MODEL OF INVESTMENT UNDER  
INTEREST RATE UNCERTAINTY

by

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## I. INTRODUCTION

In an interesting sequence of articles, Professors Wright [4], Arrow and Levhari [1] and Flemming and Wright [2] have demonstrated for respectively more general cases that the maximized present value is a monotonic decreasing function of the discount rate. When this is the case, the internal rate of return is unique.

While all the above mentioned contributions dealt with discounting in a deterministic setting, this paper analyzes the effect of uncertainty in the interest rate on the level of investment. It is shown that by using a formal approach closely related to that of Flemming and Wright [2], definite results can be obtained. Interestingly enough it turns out that the level of investment increases under interest rate uncertainty as compared to a world of perfect certainty with the mean interest rate. This somewhat surprising result is due to the fact that, contrary to what seems to be the common view, the kind of uncertainty considered here is shown to imply lighter rather than heavier discounting of the future as compared to certainty.

## II. DISCOUNTING UNDER INTEREST RATE UNCERTAINTY

Consider the present value of a standard project available to a firm:

$$(1) \quad \int_0^T e^{-\bar{r}t} B(t) dt$$

where

$B(t)$  : net benefits at time  $t$   
 $\bar{r}$  : market interest rate  
 $T$  : time horizon of the project

Suppose that the interest rate is not known with certainty and that there is a probability distribution  $\pi(r)$  ( $\pi(r) \geq 0$ ,  $\int_{r_1}^{r_2} \pi(r) dr = 1$ ) associated with it. Possible reasons for uncertainty in the rate of discount are for example an uncertain rate of inflation or uncertain developments in the world capital markets. To simplify the exposition we carry out the analysis in a continuous-time framework and on the assumption that, while uncertain, the interest rate is expected to remain constant throughout the relevant horizon. While the formulation of the problem is more cumbersome when the random interest rates are free to vary independently over time the results of the paper are not likely to be affected.<sup>1</sup>

Under these conditions and assuming the firm to be interested in the expected present value of the project, namely

$$(2) \quad \int_{r_1}^{r_2} \pi(r) \left[ \int_0^T e^{-rt} B(t) dt \right] dr = \int_{r_1}^{r_2} \int_0^T \pi(r) e^{-rt} B(t) dt dr$$

where  $r(t)$  is the instantaneous (random) interest rate at time  $t$ .

Interchanging the order of integration, (2) can be rewritten as:

$$(3) \quad \int_0^T \left[ \int_{r_1}^{r_2} \pi(r) e^{-rt} dr \right] B(t) dt$$

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<sup>1</sup>In the Appendix we develop this problem with variable rates of interest in a discrete-time framework. We conjecture that these results will carry over into the continuous-time model as well.

Denoting now  $R(t) = \int_{r_1}^{r_2} \pi(r) e^{-rt} dr$ , (3) becomes

$$(4) \quad \int_0^T R(t) B(t) dt$$

We have thus reduced the stochastic formulation (2) to its deterministic equivalent (4) where  $R(t)$  plays the role of the standard, deterministic, discount factor.

Defining now  $r^*(t) = -\frac{\dot{R}(t)}{R(t)}$  (where the dot symbol denotes the time derivative operator),  $r^*(t)$  is the implicit (variable) interest rate in the deterministic version of the stochastic evaluation problem.

The main purpose in this section is to compare  $r^*(t)$  with the mean  $\bar{r}$  of the probability distribution of  $r$  ( $\bar{r} = \int_{r_1}^{r_2} \pi(r)r dr$ ). In what follows,  $\bar{r}$  appearing in equation (1) is thus to be thought of as being the mean of the probability distribution of  $r$ .

We first note that since  $e^{-rt}$  is a strictly convex function of  $r$ , we have

$$(5) \quad R(t) > e^{-\bar{r}t} \quad \text{for } t > 0$$

implying that the discount factor implicit in the deterministic version of the evaluation function (4) is always higher (except at  $t = 0$  where both are obviously identical to unity) than the discount factor obtained by using the mean interest rate  $\bar{r}$ .

Furthermore, we show now that  $r^*(t) < \bar{r}$  for any  $t > 0$  (and  $r^*(0) = \bar{r}$ ).

By the definition of  $r^*$

$$(6) \quad r^*(t) = - \frac{\frac{d}{dt} \int_{r_1}^{r_2} \pi(r) e^{-rt} dr}{\int_{r_1}^{r_2} \pi(r) e^{-rt} dr} = \frac{\int_{r_1}^{r_2} r(t) \pi(r) e^{-rt} dr}{\int_{r_1}^{r_2} \pi(r) e^{-rt} dr}$$

From (6) it is easily seen that

$$(7) \quad r^*(0) = \bar{r}$$

implying that the "first instant" of time is discounted at the mean interest rate.<sup>2</sup>

The negative correlation between the random variables  $e^{-rt}$  and  $r(t)$  implies<sup>3</sup>

$$(8) \quad \int_{r_1}^{r_2} r(t) \pi(r) e^{-rt} dr < \int_{r_1}^{r_2} \pi(r) e^{-rt} dr \int_{r_1}^{r_2} \pi(r) r(t) dr$$

(8) together with (6) imply

$$(9) \quad r^*(t) < \bar{r} \quad \text{for } t > 0$$

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<sup>2</sup>It may be of some interest to note that since  $Ee^{-rt}$  is the moment generating function,  $dr^*(0)/dt = -\text{Var}(r)$ , i.e., the deterministic equivalent interest rate decreases initially at a rate equal to the variance of  $r$ . In general, however,  $r^*(t)$  cannot be shown to be a decreasing function of time.

<sup>3</sup>To show that  $r$  and  $e^{-rt}$  are negatively correlated consider the quantity  $b = e^{-(r-\bar{r})t}$ , where  $b < 1$  as  $r > \bar{r}$ . Multiplying  $b$  by  $(r-\bar{r})$  we have  $(r-\bar{r})e^{-(r-\bar{r})t} < r-\bar{r}$ . Taking expectation and using  $E r = \bar{r}$  we get  $E[(r-\bar{r})e^{-(r-\bar{r})t}] < 0$ .

Since in the next section we wish to explore the effect of uncertainty in the rate of interest on the level of investment we introduce now the concept of increasing risk in  $r$ . Following Rothschild and Stiglitz [3] increasing the risk in a random variable is done by moving probability weights to the tails of its probability distribution while keeping its mean constant. This definition can be applied in particular to our analysis of certainty versus uncertainty. Define then  $\alpha$  as a risk parameter (such that its certainty value is zero and the higher the  $\alpha$  the higher the risk) so that<sup>4</sup>

$$(10) \quad r^* = r^*(t, \alpha) .$$

In this section we have shown that when moving from certainty to uncertainty<sup>5</sup>

$$(11) \quad r_{\alpha}^* = \frac{\partial r^*(t, 0)}{\partial \alpha} < 0 .$$

In the next section, we shall use (11) to show that in the familiar framework of project evaluation, uncertainty in the interest rate will increase the level of investment by the firm.

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<sup>4</sup>The parameter  $\alpha$  generates a systematic variation in the discount rates and plays a role analogous to, albeit more general than, that played by the parameter  $\lambda$  in Flemming and Wright [2].

<sup>5</sup>See the Appendix for a similar result where the analysis is not confined to an initial situation of a complete certainty.

### III. INVESTMENT UNDER INTEREST RATE UNCERTAINTY

Consider now a much used framework for project evaluation in which the stream of net benefits  $\{B(t), 0 \leq t \leq \infty\}$  is still outside the firm's control but in which the firm can choose the terminal date  $T$  of projects.<sup>6</sup> The firm can thus be viewed as engaged in a two-step maximization problem. First, for any project the firm will determine the maximized value of the project by selecting  $T$ . Second, it will choose the optimal set of projects.

In order to analyze the first step, define

$$(12) \quad V = \max_T E_r \int_0^T e^{-\int_0^t r(s, \alpha) ds} B(t) dt$$

where  $E_r$  is the expectation operator applied to the (random) instantaneous interest rate  $r$ .

By interchanging the order of integration we obtain

$$(13) \quad V = \max_T \int_0^T E_r \left[ e^{-\int_0^t r(s, \alpha) ds} \right] B(t) dt = \max_T \int_0^T e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt$$

where  $r^*$  is as before the deterministic equivalent of the instantaneous rate of interest. As shown in the previous section  $r_\alpha^* = \frac{\partial r^*}{\partial \alpha} < 0$ .

In order to characterize the optimal terminal date of the project, we differentiate (13) with respect to  $T$  and set the derivative equal to zero getting

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<sup>6</sup>The reader will note that this is essentially the framework considered by Arrow-Levhari [1] and Flemming and Wright [2]. The reader might be tempted to think that what we do here is to prove that  $V$  below is a convex function of  $r$ . While this property of  $V$  could easily be shown, it has no direct bearing on the problem at hand since the investment decision has to be made prior to the realization of the interest rate.

$$(14) \quad \frac{\partial V}{\partial T} = e^{-\int_0^T r^*(s, \alpha) ds} B(T) = 0$$

implying

$$(15) \quad B(T) = 0 \quad \text{for} \quad 0 < T < \infty .$$

Differentiating (14) with respect to  $T$  gives us the second-order condition

$$(16) \quad \frac{\partial^2 V}{\partial T^2} = e^{-\int_0^T r^*(s, \alpha) ds} [B'(T) - r^*(T, \alpha)B(T)] < 0$$

which using (15) implies

$$(17) \quad B'(T) < 0 \quad \text{for} \quad 0 < T < \infty .$$

In order to prove the assertion made in the previous section that the expected present value of any project is increased by increases in the riskiness of the rate of interest (when the mean rate of interest is unchanged) we show now that  $V$  is a strictly increasing function of  $\alpha$ .

Upon differentiation of (13) with respect to  $\alpha$  we get

$$(18) \quad \frac{dV}{d\alpha} = - \int_0^T e^{-\int_0^t r^*(s, \alpha) ds} \left[ \int_0^t r_{\alpha}^*(s, \alpha) ds \right] B(t) dt + e^{-\int_0^T r^*(s, \alpha) ds} B(T) \frac{dT}{d\alpha} .$$

By making use of (15) and by denoting  $\int_0^t r_{\alpha}^*(s, \alpha) ds = f(t)$  we get

$$(19) \quad \frac{dV}{d\alpha} = - \int_0^T e^{-\int_0^t r^*(s, \alpha) ds} f(t) B(t) dt$$

Equations (15) and (17) imply that  $T$  separates (locally) two time intervals such that  $B(t)$  is positive to the left of  $T$  and is negative to the right. Since investment projects are usually characterized by a sequence of time intervals such that at first  $B(t)$  is negative and is positive for some subsequent time interval, there must be some other point of time, say  $\bar{t} < T$ , such that  $B(\bar{t}) = 0$  and where  $B(t)$  is (locally) negative to the left of  $\bar{t}$  and is positive to the right. In general there will be several such points of time. But consider first a project which has only one  $\bar{t}$  (noting that this necessarily implies that  $B(t) < 0$  for  $0 \leq t < \bar{t}$  and  $B(t) > 0$  for  $\bar{t} < t < T$ ).

Since  $f(t) < 0$  and  $f'(t) < 0$  for  $0 < t \leq T$  we obtain

$$(20) \quad \int_0^{\bar{t}} f(t) e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt < f(\bar{t}) \int_0^{\bar{t}} e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt$$

$$(21) \quad \int_{\bar{t}}^T f(t) e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt < f(\bar{t}) \int_{\bar{t}}^T e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt$$

Since  $V > 0$ , (i.e., the project is profitable) we also have<sup>7</sup>

$$(22) \quad \int_t^T e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt > -\int_0^{\bar{t}} e^{-\int_0^t r^*(s, \alpha) ds} B(t) dt$$

<sup>7</sup> It is easily verified that this proof is not confined only to profitable projects. For example, the reader can repeat the same proof for projects whose values, if undertaken, are smaller than zero by a

number which is smaller than  $A = \int_0^{\bar{t}} \frac{f(\bar{t}) - f(t)}{f(\bar{t})} e^{-\int_0^t r(s, \alpha) ds} dt$  (i.e., for projects with  $V + A > 0$  we have  $dV/d\alpha > 0$ ).

Now, (20) - (22) imply

$$(23) \quad \int_0^T e^{-\int_0^t r^*(s, \alpha) ds} f(t)B(t)dt < 0$$

which together with (19) prove the assertion that  $\frac{dV}{d\alpha} > 0$  for a project with only one  $\bar{t}$ . The reader can easily verify that, by induction on the number of  $\bar{t}$ 's the assertion holds in general.

Our results imply that not only are the deterministic-equivalent interest rates lower than the mean of the probability distribution of the random interest rate (implying lighter rather than heavier discounting under uncertainty) but also that if the lifetime of every project is chosen optimally then its expected present value is actually increased. Thus, some marginal projects which were not profitable under certainty may become so and hence total investment will be increased. In any case, increased riskiness in interest rates (around a given mean) will never lead total investment by the firm to decline.

One further implication of the present model may be noted. It is customary in the literature of project evaluation to take account of uncertainty by adding a risk premium to the interest rate (and thus implicitly assuming that risk as such is evil). When the rate of interest is itself a random variable, our analysis in Section II indicates that this risk premium is negative (and variable over time).

## REFERENCES

- [1] Arrow, K. J., and D. Levhari. "Uniqueness of the Internal Rate of Return with Variable Life Investment." Economic Journal, 79 (September, 1969), 560-566.
- [2] Flemming J. S., and J. F. Wright. "Uniqueness of the Internal Rate of Return: A Generalization." Economic Journal, 81 (June, 1971), 256-263.
- [3] Rothschild, M. and J. Stiglitz. "Increasing Risk: I. A Definition." Journal of Economic Theory, 2 (March, 1970), 225-243.
- [4] Wright, J. F. "The Marginal Efficiency of Capital." Economic Journal, 69 (December, 1959), 813-816.

## APPENDIX

In this appendix we consider the effect of interest rate uncertainty when interest rates, governed by independent probability distribution, are free to vary over time. For a discrete time-discrete probability case we will show the counterpart of equation (9), i.e., that  $r_i^*$ , the implicit interest in the deterministic version of the stochastic problem is smaller than the mean interest rate  $\bar{r}_i$  at any period  $i$  (with the exception of the initial period where they are equal).<sup>8</sup>

Consider a problem with  $I$  discrete periods of time and  $N$  states of the world where the interest rate at period  $i$  if state  $j$  occurs is denoted by  $r_{ij}$ . A probability  $P_{ij}$  is associated with the interest rate  $r_{ij}$ .

Denote the mean rate of interest at period  $i$  by  $\bar{r}_i$  where

$$(A.1) \quad \bar{r}_i = \sum_j^N P_{ij} r_{ij}, \quad \sum_j^N P_{ij} = 1.$$

Assume that the probability distributions are such that the interest rate at period  $i$  is independent of the interest rate at period  $j$  ( $i \neq j$ ). Under these conditions the expected present value of the sequence of (positive or negative) benefits  $(B_1, \dots, B_T)$  is given by

$$(A.2) \quad B_1 \sum_j^N P_{1j} \left( \frac{1}{1+r_{1j}} \right) + B_2 \sum_j^N \sum_k^N P_{1j} P_{2k} \left( \frac{1}{1+r_{1j}} \right) \left( \frac{1}{1+r_{2k}} \right) \\ + B_3 \sum_j^N \sum_k^N \sum_\ell^N P_{1j} P_{2k} P_{3\ell} \left( \frac{1}{1+r_{1j}} \right) \left( \frac{1}{1+r_{2k}} \right) \left( \frac{1}{1+r_{3\ell}} \right) + \dots \\ + B_T \sum_j^N \dots \sum_t^N P_{1j} \dots P_{Nt} \left( \frac{1}{1+r_{1j}} \right) \dots \left( \frac{1}{1+r_{Nt}} \right)$$

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<sup>8</sup>A generalization of the analysis to a discrete time-continuous density case is straightforward.

where the last term has  $I$  summation signs.

Denote the certainty equivalent marginal rate of interest between the end of period  $i - 1$  and the end of period  $i$  by  $r_i^*$ . We wish to prove that

$$(A.3) \quad r_i^* < \bar{r}_i \quad \text{for all } i > 1 .$$

To prove (A.3) consider  $r_2^*/(1+r_2^*)$ , which can be obtained from the first two discounting factors in (A.2) as follows:

$$(A.4) \quad \frac{r_2^*}{1+r_2^*} = \frac{\sum_j^N P_{1j} \left( \frac{1}{1+r_{1j}} \right) - \sum_j^N \sum_k^N P_{1j} P_{2k} \left( \frac{1}{1+r_{1j}} \right) \left( \frac{1}{1+r_{2k}} \right)}{\sum_j^N P_{1j} \left( \frac{1}{1+r_{1j}} \right)}$$

$$= \frac{\sum_j^N P_{1j} \left( \frac{1}{1+r_{1j}} \right) \left[ 1 - \sum_k^N P_{2k} \left( \frac{1}{1+r_{2k}} \right) \right]}{\sum_j^N P_{1j} \left( \frac{1}{1+r_{1j}} \right)}$$

$$= 1 - \sum_k^N P_{2k} \left( \frac{1}{1+r_{2k}} \right) .$$

By the convexity of the function  $\frac{1}{1+x}$  we know that

$$(A.5) \quad \sum_k^N P_{2k} \left( \frac{1}{1+r_{2k}} \right) > \frac{1}{1 + \sum_k^N P_{2k} r_{2k}} = \frac{1}{1+\bar{r}_2} .$$

(A.4) - (A.5) imply that

$$(A.6) \quad \frac{r_2^*}{1+r_2^*} < \frac{\bar{r}_2}{1+\bar{r}_2}$$

which implies  $r_2^* < \bar{r}_2$ . In this way (A.3) can be proven for every  $i > 1$ . (Note that, by definition,  $r_1^* = \bar{r}_1$ ) .

Let the deterministic equivalent of the present value of a project given by (A.2), be

$$(A.7) \quad \sum_{i=1}^I R_i^* B_i$$

where  $R_i^* = \prod_{s=1}^i \frac{1}{1+r_s^*}$  is the discount factor at period  $i$ . For a given vector of  $R_i^*$  there exists a value of  $I$ , denoted  $I^0$  for which the present value of the project is maximized. Truncation of the project before  $I^0$  will reduce the present value. Therefore,

$$(A.8) \quad \sum_{i=t}^{I^0} R_i^* B_i > 0 \quad 1 < t < I^0 .$$

The derivative of the maximized present value with respect to  $r_t^*$  (for an unchanged  $I^0$ ) is<sup>9</sup>

$$(A.9) \quad - \sum_{i=t}^{I^0} R_i^* \left( \frac{1}{1+r_t^*} \right)^2 B_i .$$

(A.3), (A.8) - (A.9) imply that a mean preserving increase in risk in the rate of interest will increase the expected present value of the project. Note that this statement is not confined only to an initial situation of a complete certainty. This is so, since (A.3) implies that if we start from a situation with a given amount of uncertainty, with variable rate of interest  $\bar{r}_i$  representing the certainty equivalent discounting, further uncertainty will lead to still lower rates of interests,  $r_i^*$ .

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<sup>9</sup>The sign of this derivative will be the same even in a situation where  $I^0$  changes with  $r_t^*$ .

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