MONOPOLISTIC COMPETITION, OBJECTIVE DEMAND FUNCTIONS AND THE MARXIAN LABOR VALUE IN THE LEONTIEF SYSTEM

by

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I. Monopolistic Competition and General Equilibrium

I.1 Introduction

It is well-known that the theories of monopolistic or imperfect competition have been well evolved and elaborated through the works of J. Robinson [15], E. H. Chamberlin [3], H. von Stackelberg [16] and others ever since that of A. Cournot [4]. Their works analyze not only the monopolistic or monopolistically competitive behavior of a single firm but also the equilibrium situation of a monopolistically competitive market involving several firms.

Nonetheless, illuminating though the achievements of these authors are, we are still left much dissatisfied with the status of our knowledge about monopolistic competition so far reached. For even though market equilibria are analyzed, the analyses are worked out from so fatally the partial equilibrium theoretic points of view that they pay little attention to the national economy-wide interdependence of firms. In the traditional theories of monopolistic competition there is always concealed part of the national economy as a closed system of interdependent vital factors which is hidden and left intact behind the magnificent performances of competing firms in the foreground. Thus, we are dissatisfied with the traditional theories, especially when we are concerned with the income distribution aspects of a non-competitive national economy, since the lack of attention to the entire economy-wide interdependence prevents them from shedding light on income distribution under monopolistic competition.

It is frequently pointed out that an economy involving monopolistic factors fails to achieve the efficiency of resource allocation. Knowledge of this kind is, however, not quite far beyond common sense, unless it is based on an analysis of the overall working of the entire economy in question.

R. Triffin [17] rightly puts emphasis on the need of a general equilibrium theoretic approach to monopolistic competition by paying attention to the interdependent relations among firms. Nonetheless, what he has in mind as an image of a well-formulated model of a monopolistically competitive entire national economy is not quite clear. At any rate everyone is aware that a general equilibrium theoretic viewpoint is undoubtedly needed in order to get insight into the noncompetitive working of the economy peculiar to the present day, and that much work should be done to build general equilibrium theories of noncompetitive economies. An attempt to carry out such work seems to involve much tremendous difficulty since one need to have a new clear-cut formulation of a noncompetitive entire national economy by starting virtually from nothing available in the literature as to the crucial nature of formulations unlike a work based on the well-posed Walrasian general equilibrium formulation of a competitive national economy.

The first pioneering work along with the lines emphasized above was done by T. Negishi ([10]; [11], Chapter 7) in the early 1960s. This illuminating work of Negishi is based on the incorporation of monopolists' perceived demand functions into a general equilibrium situation in an ingenious fashion. It can also be noted that his result generalizes the Walrasian competitive equilibrium so as to include both perfect competition and monopolistic competition, because perfect competition emerges when the perceived demand functions are of a special form. The crucial line of Negishi's thought to formulate a monopolistically competitive equilibrium situation was further elaborated later by K. J. Arrow [1] and Arrow and F. H. Hahn ([2], Chapter 6, pages 151-168). These results, along with the line of Negishi's thought, are the only works available so far in the literature of general equilibrium theories of monopolistic competition. True, they are undoubtedly very illuminating and shed much light on the working of the economy under monopolistic competition. But perceived demand functions embody only firms' subjective perception of the economic situation and conjectures as to rival firms' behaviors rather than a direct recognition of the interdependence of firms in the objective sense. A monopolist controls prices or output through the interdependent relations among economic agents in the objective sense even if his decision making is based on a profit estimate in terms of his perceived demand function. So we are still not completely satisfied with our knowledge about monopolistic competition in the general equilibrium context.

The purpose of this work is to attempt to shed some light on the interdependence of agents in the general equilibrium context in the objective sense by constructing objective demand functions. In general, however, much difficulty is inherent in such an attempt. In order to overcome it this attempt will be pursued in the simple world of the standard Leontief system in which special advantage can be taken of both the well behavedness of the basic model and the Marxian labor and surplus value concepts. The remaining part of this section will clarify certain crucial points in further detail as to the dissatisfactory status of theories of monopolistic competition in the general equilibrium context on one hand, and the basic ideas, concepts, methods and results in this alternative approach on the other, in the respective bisections I.2 - I.6.

I.2 Interdependence in the objective sense

In the Walrasian general equilibrium theory crucial economic magnitudes are mutually interdependent, even when the situation lacks such externalities as external economies and diseconomies and consumption externalities making economic agents directly interdependent not only through market mechanism. Interdependence through market mechanism is imposed on every economic agent by the structural characteristics of the economy irrespective of whether or not he rightly realizes and recognizes it. In general his perception of this interdependence is more or less imperfect and limited. For example, an atomistic consumer in the Walrasian general equilibrium situation is aware only of the current prices, his taste, endowment and income, but not of the current situation of the market. Nonetheless, interdependent relationships among economic agents, including him, exist gravely, which are clear to the omniscient and embodied in the aggregate market demand and supply or excess demand functions. True, his taste is subjective and peculiar to him, but the aggregate market excess demand functions, which integrate the individual behaviors of all economic agents including his, are given to the market in the objective sense. Interdependence of this kind, which may be referred to as interdependence in the objective sense, must exist even in the general equilibrium situation involving monopolistic competition prior to its more or less imperfect perception by economic agents.

This interdependence in the objective sense is what I intend to establish in a monopolistically competitive general equilibrium situation as one of the purposes in this work. One of the reasons why it is important to establish or realize this interdependence is as follows. An atomistic firm can have an illusion to be powerful enough to influence the market price formation, even though the influence is practically negligible in the objective sense. A firm whose perceived demand function is steeply downward sloping cannot necessarily be a powerful monopolist, unless it is practically influential and capable of forcing the market price formation to function through the interdependence in the objective sense towards the goal which it desires to achieve. A mere Don Quixote need not be a powerful monopolist.

It should also be noted that a firm's behavioral pattern and the interdependence which the former is subjected to are different things in principle though they may be mutually intertwined in actuality. The behavioral pattern influences the market price formation, but, conversely, how influential it is hinges on, and can be assessed only in terms of, the objective framework of interdependence.

I.3 Subjective versus objective demand functions

In the theories of monopolistic competition one talks very often and in ease about the demand function of a firm. He is happy enough to be unconscious of what it is so long as his concern is only with the behavior of the firm, or the monopolistic competition among several firms having respective demand functions. But he comes to realize that he is not so happy as he believed as soon as he is seriously interested in the working of a national economy involving monopolistic competition where all economic agents are mutually interdependent in a completely circular way.

Demand for goods must be effective demand coming from the incomes earned by agents in the national economy. The traditional oligopoly theorist pays little attention to the source of effective demand. He lets a monopolist firm seek a maximum of its profit calculated in terms of its demand function. Suppose the maximum monopoly profit is distributed among certain agents. The distributed profit will be spent and result in the effective demand for goods. Thus, the demand function may have the profit as one of its arguments. How ignorant can he be of the possible inconsistency of the profit as one of the arguments in the demand function with that as the firm's maximand calculated in terms of the function?

There might be an objection to the above discussion with a possibility in mind that the firm's profit in question is not among the arguments of its demand function. Now, suppose that there are n goods in the economy and that the supply of each good is monopolized by a respective firm whose demand function may have profits of the firms as its arguments except its own profit. The oligopoly theorist can hardly ignore inconsistency of the above kind even in this special situation. For if the national economy as a closed system of mutually interdependent magnitudes is taken into consideration, the profit of each firm must eventually enter the demand functions of some other firms, though possibly not its own demand function, as arguments, so that the inconsistency problem cannot be ignored in the economy as a whole.

One might regard a firm's demand function as a partly revealed portion of an objective demand function. But this view is not quite decisive unless the objective demand function as such is unambiguously formulated. It is now a dominant view of general equilibrium inclined authors that the demand function of a firm is a perceived or subjective one rather than an objective one. The firm perceives to what extent it is powerful enough to affect the market price formation. The perception may more or less cover the firm's production capacity and the market situation with which the firm is confronted, including its conjectures as to the rival firms' behaviors. The result of the perception is represented in terms of a demand schedule for its product, which is quite a familiar traditional concept in competitive situations.

As was remarked above, Negishi ([10]; [11], Chapter 7) formulated a general equilibrium situation involving monopolists who seek a maximum profit on the basis of their perceived demand functions, and proved the existence of an equilibrium. There are loose connections between their perceived demand functions and the actual market situation. The monopolists' profit maximization is shown in his paper to become consistent with the actual market situation at equilibrium, so that the inconsistency problem touched on above is taken care of.

The result of Negishi is the first important pioneering work in the general equilibrium theoretic analysis of monopolistic competition. Nevertheless, it is not completely satisfactory for the following reason, even though it is taken for granted that he is concerned only with solutions of the Cournot-Nash equilibrium point type.

O. Lange ([9], chapter VII, page 35) states:

> "The nature of economic equilibrium, as well as of disequilibrium, in a monopolistic or monopsonistic market differs from that in a perfectly competitive market. In the latter, disequilibrium consists in excess demand or excess supply. Monopolistic supply, however, is always equal to the demand for the good in question

> and monopsonistic demand is always equal to supply. A monopolistic or monopsonistic market is in equilibrium when the quantity sold and bought is such that it maximizes the profit of the monopolist or monopsonist."

It is not quite sure whether Lange was clearly conscious of the distinction between perceived demand functions and objective demand functions in the passages quoted above. But his lucid characterization of a monopolistic or monopsonistic market in the passages would be not utterly enlightening, if objective demand or supply functions were not in his mind.

This characterization, when carried over to the general equilibrium situation, suggests that a monopolistically competitive national economy can be so characterized that demand and supply are balanced for every good in any price situation, while the economy is in equilibrium if, and only if, monopolists' maximizations of their profits are realized consistently and simultaneously. The situation considered by Negishi diverges from that characterized above because "disequilibrium consists in excess demand or excess supply" generally, except in equilibrium at which excess demand and excess supply disappear while the profits of the monopolists are maximized consistently and simultaneously. This comment may apply more aptly to the special case of Negishi's result where the supply of each good is monopolized by a distinct monopolist, who dominates the market of the good in Negishi's terminology, but there are no competitive producers in the economy. It is not clear whether the economy is in a process of tâtonnement when "disequilibrium consists in excess demand or excess supply." Nor is it clear how convincingly monopolists can perceive the demand schedules for their products based on the information data of the markets in which "disequilibrium consists in excess demand or excess supply" generally. All these doubtful points come from the lack of objective demand functions in his results.

He also considers a dynamic process of market price formation in which the price of a good whose supply is dominated by a monopolist changes in the direction of the discrepancy between his expected price and the current price. If the market accords with the Lange's characterization of a monopolistic market, the current prices must always equate the demand and supply of the goods in the objective sense. At any rate this dynamic process does not make much sense unless it is formulated in terms of well-defined objective demand functions.

The results of Negishi may, therefore, be thought of as a generalization of competitive equilibrium, rather than as an analysis of monopolists' control of the market price formation directly through the national economy-wide framework of interdependence in the objective sense. The results of Negishi can and will be reconsidered and better reformulated in IV.3 on the basis of objective demand functions, whose construction is one of my purposes in this work.

I.4 Difficulties in the construction of objective demand functions

Now that much attention has been called for the importance of the interdependence in the objective sense in the analysis of the general equilibrium situation of monopolistic competition, especially, objective demand functions as a specific form of its conceptual representation, it is due time to consider difficulties inherent in conceiving and, preferably, constructing them.

First, it should be noted and borne in mind that the very too familiar concept of demand function as such presupposes more or less the presence of competitive atomistic agents who behave as price takers. If no competitive atomistic price taker is involved in the national economy as a closed system, so that it is composed solely of nonatomistic price setters, no demand function can be conceivable. In such a situation the economy will be a battlefield of bare bargaining, negotiating and bluffing among all agents, which could be studied only from a purely game theoretic point of view, but not from the traditional point of economic thought based on the concept of demand and supply functions.

A demand function is the representation of integrated behaviors of competitive atomistic agents taking as parameters beyond their control the prices which are the arguments of the function. In this work I will be not so drastically radical that I intend to eliminate completely any form of demand functions. I will be so conservative that what I have in mind is a situation in which compete several monopolists who are confronted with their respective demand functions for their products. Then, what does distinguish my intention from the traditional lines of thought in the theories of monopolistic competition? There are the integrated behaviors of competitive atomistic agents behind the demand functions of the monopolists as in the traditional theories. But unlike the traditional theories, I am seriously concerned with inquiring into the sources of the demands for their products as effective demand, namely, national income. It is extremely important to realize that the very profits of the monopolists compose directly, or indirectly through their distribution among shareholders, part of the national income in question, while the profits are calculated in terms of the demands and the costs, the latter, too, being in turn a source of the effective demand. There is a circular relation which cannot be ignored in the national economy as a closed system, as was noted in I.3.

When one intends to conceive objective demand functions as I do, he must take this circular relation into fullest account. His objective demand functions must be constructed in such a way that they are always compatible with the circular relation at any possible price situation. A general equilibrium model of monopolistic competition, which accords with Lange's characterization, can be constructed only when objective demand functions compatible with the circular relation are defined in a clear-cut way. In general this is a difficult task. I will therefore try to pursue this task, as the first step toward a final goal, in the Leontief system, which is simple in structure enough to make the pursuit of the task less difficult. The pursuit of the task in the Leontief system may also have some bearing on the study of the real economic world from the empirical point of view, unlike a pursuit of the task in a more abstract model.

Mathematically, the construction of well-defined objective demand functions compatible with the circular relation amounts in this work to the existence and uniqueness of solutions of certain systems of equations. The existence of solutions will be proved by making use of Brouwer's fixed point theorem, whereas their uniqueness will be ensured by virtue of my previous joint results with D. Gale [6]. Economically, the construction of these functions amounts in this work to such a unique determination of sectoral profits at any price situation that the circular relation obtains. These will be done in Section III.

Going back to the basic view that a demand function presupposes the presence of competitive atomistic agents and represents their integrated price-taking behaviors, the objective demand functions which will be constructed in Section III are based on the assumption that the households of recipients of monopoly profits, too, behave as price takers, their behaviors being included in the functions. Otherwise, the monopoly profits could hardly be represented as the amounts by which the total revenues calculated in terms of well-determinate demand functions exceed the costs, respectively.

I.5 The role of the Marxian labor value

Generally speaking, for the contemporary economist the Marxian labor value is something like a will-o'-the-wisp and has been dead and buried. Nevertheless it still has not merely a metaphysical meaning but also a positive operational meaning in the simple world of the Leontief system where labor is a unique primary factor of production and capital stocks as produced equipments are not binding.

First of all, the labor value of goods, which are defined as the indirectly as well as directly necessary amounts of labor to produce one unit of goods, respectively, is a crucial characteristic of the set of all possible final demand vectors, especially its efficient frontier, for a given amount of labor. Then, and therefore, they can serve as natural standard weights to conceive a real aggregate magnitude, a counterpart of the aggregate magnitude in macroeconomic theory. Accordingly it is not utterly groundless for the Marxian economist to talk about the surplus value measured in terms of this magnitude.

The prices of goods diverge in general more or less from their labor values except in the special situation where there are no profits in any sectors so that both coincide. But beneath the interplay of the prices on the surface there is what it essentially means in terms of the labor values. This Marxian flavored view will prove to be an operationally meaningful one useful in considering the general equilibrium situation of monopolistic competition in the Leontief system, rather than a dogma, as will be made clear in the following sections. In this work fullest advantage will be taken of this view not in order to foster intentionally the Marxian philosophy, but just because the very essential character of the Leontief system logically induces crystal-clear representations of certain essential aspects of its working in terms of labor values.

This natural, but not artificial, Marxian flavor will be retained as carefully as possible throughout this work. In particular, in presenting the results in the following sections, I will start at reviewing certain known facts about labor values and then considering surplus value in order to grasp vividly the resource allocative and income distributional implication of pricing for the capitalist class in Section II, rather than at discussing the construction of objective demand functions in accordance with the orthodox order of presentation.

I.6 Pricing modes in monopolistic competition

In the standard Leontief system of n goods and n sectors a set of objective demand functions, one for each good, will be constructed in Section III. Thus, the i th sector is confronted with the well determinate demand function for its product, namely, the i th good. The i th sector's demand function is a function of n variables, viz. the prices of the n goods, so that the demand for the i th good depends not only on the price of the good but also on those of the other goods. Moreover, the prices are connected via the inputs coefficients with the labor cost and profit per unit output in such a way that they satisfy the price determining equations, that is, the equations dual to the output determining equations in the Leontief system. The demand functions are so constructed that a complete circular relation obtains for the national income comprising total wage bill and total profits at any price situation. Put in a greater detail, the profit of each sector as its maximand always coincides with the profit income earned in this sector as source of part of effective demand at any price situation. The inconsistency which was touched on in the foregoing bisections never occurs for the objective demand functions.

Each sector may be composed of a single, several or many firms. But, for the sake of analytical simplicity, breaking down the sector into individual firms in their possible competition is ignored, so that the sector is thought of as a single decision-making unit. This presumption is not quite unrealistic, because in the real economic world of the present day, industries behave often as if they were single decision-making units. The steel industry, auto industry, agriculture and so forth have unified strong voices in the battlefield of a nationwide pricing warfare.

It has by now been made clear that the entire national economy, with which I am concerned in this work, is of such a structure that the n sectors (industries) are interdependent in the objective sense through the well-defined objective demand functions. Part, though not all, of the arguments of the functions are at each sector's disposal and capable of being under its control. Each sector is capable of influencing the working of the economy by pricing its product directly or indirectly via the control of the profit per unit output.

This is the bare fundamental aspect of the economy, irrespective of whether or not the sectors are aware of, and perceive, it rightly. Any possible modes of sectoral pricing are effectuated in this framework. If no sector is deceived by its wrong perception of the situation, each sector's maximand will be the sectoral profit calculated in terms of the objective demand function.

Thus, a vast field emerges where a great many solution concepts in the theories of games, cooperative or noncooperative, as well as in the traditional theories of monopolistic competition may find their applications. More specifically, two special pricing modes are conceivable, namely, joint profit maximization and joint surplus value maximization, which need not be equivalent, as will be shown. On the other hand if each sector has a perception of the situation which differs more or less from the reality, the maximand of each sector will depend on the perception. This alternative situation, too, may admit applications of the solution concepts. In this alternative situation a solution is feasible only when the perception becomes compatible with the actual state of things represented by the objective demand functions. In other words the economy is in equilibrium in the sense of Lange when and only when the corresponding price situation is a solution at the same time. A typical example is the Negishi solution in the modified form where each sector's actual profit is a maximum of its expected profit calculated in terms of a perceived demand function at the same time.

Thus, a solution concept will single out a set of points, possibly a single point, on the objective demand schedules as an equilibrium state of the economy. This is the basic idea of the present work. Several, though not all, modes of pricing based on alternative solution concepts will be considered in Section IV.

II. Surplus Value in the Leontief System

II.1 The Leontief system

In the standard Leontief system of n goods and n sectors let

$A = (a_{ij})$ = the input coefficients matrix, square of the n th order and nonnegative, which will be assumed to be indecomposable whenever necessary;

$c = (c_i)$ = the final demand vector, n-dimensional and nonnegative;

$v = (v_j)$ = the value added per unit output vector, n-dimensional and nonnegative;

$x = (x_j)$ = the output vector, n-dimensional and nonnegative;

$p = (p_i)$ = the price vector, n-dimensional and nonnegative.

Throughout in this work the Leontief system is assumed to be viable enough to produce a positive final demand vector, so that the Hawkins-Simon conditions[2] [7] are satisfied. Thus, all the principal minors of the matrix $I - A$, where I is the identity matrix, are positive. Consequently, $I - A$ is nonsingular and its inverse matrix $(I - A)^{-1}$ is nonnegative. Moreover, the output determining equation

$$(I - A)\, x = c \tag{II.1}$$

is uniquely solvable in the nonnegative unknown output vector x for any given nonnegative final demand vector c. On the dual side of the system the price determining equation

$$p'(I - A) = v' \tag{II.2}$$

is uniquely solvable in the nonnegative unknown price vector p for any given nonnegative value added per unit output vector v. These rudimentary and too familiar facts should always be borne in mind.

Now, further let

$\ell = (\ell_j)$ = the labor input vector, n-dimensional and positive;

$\Pi = (\Pi_j)$ = the profit per unit output vector, n-dimensional and nonnegative, Π_j being the j th sector's profit per unit of output;

w = the rate of wage, scalar and positive.

Then, the price determining equation (II.2) can be rearranged to

$$p'(I - A) = w\,\ell' + \Pi' \qquad \text{(II.3)}$$

by breaking down the value added term to wage and profit.

II.2 The capitalists' final demand possibility set and surplus value

The labor values of goods are by definition the total amounts of labor input directly and indirectly necessary to produce units of them, respectively. They cannot be defined separately, but can only be defined simultaneously because of the basic interindustrial relationships in the Leontief system. Specifically, they are defined as a unique solution of equation (II.3) for $w = 1$, $\Pi = 0$. If the solution is denoted by $\sigma = (\sigma_j)$, it is therefore determined by

$$\sigma'(I - A) = \ell' \qquad \text{(II.4)}$$

and given by the formula[3]

$$\sigma' = \ell'(I - A)^{-1} \quad .$$

σ is a positive vector by virtue of the indispensability of labor, $\ell > 0$,[4] and will play a crucial role in the sequel. For the sake of brevity, $\sigma = (\sigma_j)$ and σ_j will be referred to as the labor value vector and the labor value of the j th good, respectively.

The solution p of equation (II.3) can therefore be represented as

$$p' = w\,\sigma' + \Pi'(I - A)^{-1} \quad . \tag{II.5}$$

Now it is assumed that labor force consists of many atomistic workers who behave as price takers in supplying labor and demanding goods. Moreover, their behavior is assumed to be represented by an aggregate supply function of labor

$$L(p,\ w) \tag{II.6}$$

and an aggregate demand function for goods

$$F(p,\ w) = (F_j(p,\ w)) \quad . \tag{II.7}$$

The following assumptions are imposed on these functions:

[A.1] $L(p, w)$ is a nonnegative, continuous, scalar-valued function of the positive price vector p and the positive rate of wage w.

[A.2] $F(p, w)$ is a nonnegative, continuous, vector-valued function of the positive price vector p and the positive rate of wage w, its j th component being the demand for the j th good.

[A.3] There are no savings from the wage income which is entirely spent for the purchase of goods. That is, the following identity holds

$$p'F(p,\ w) = w\,L(P,\ w) \quad . \tag{II.8}$$

[A.4] For any fixed w the labor supply eventually diminishes to zero, when the prices p_i of all the goods tend to infinity simultaneously. That is,

$$\lim L(p,\ w) = 0 \quad \text{as} \quad p_i \to +\infty \ \text{simultaneously}\ (i = 1, 2, \ldots, n) \quad . \tag{II.9}$$

If there is no money illusion, as is assumed in this work, labor can be taken as a numéraire, so that henceforth

$$w = 1 \quad , \tag{II.10}$$

and equations (II.5) - (II.9) will be considered only for the case (II.10). Needless to say, this arrangement has nothing to do with the theory of labor value. In (II.5) for $w = 1$, that is,

$$p' = \sigma' + \Pi' (I - A)^{-1} \quad , \tag{II.11}$$

the sectoral profits are in general nonnegative and possibly positive, so that the prices of goods diverge more or less upward from their labor values.

Finally, an additional assumption is made on a relation between the tastes of workers and the productivity of the underlying technology, namely,

[A.5] The labor force is willing to work at the special price situation $p = \sigma$ and $w = 1$, so that

$$L(\sigma, 1) > 0 \quad . \tag{II.12}$$

It is now presumed that the economy is of such an institutional structure that each sector is a decision making unit that carries out production for the benefit of capitalists while being confronted with the atomistic workers characterized above. It is also recalled that the prices of goods are linear functions of the sectoral profits per units of output, as represented in (II.11). With these in mind, let the following sets be defined by

$$C(\Pi) = \{c \mid c \geqq 0,\ (I-A)x = F(p, 1) + c,\ L(p, 1) \geqq \ell' x \text{ for some } x \geqq 0\} \tag{II.13}$$

$$\Gamma(\Pi) = \{c \mid c \geqq 0,\ (I-A)x = F(p, 1) + c,\ L(p, 1) = \ell' x \text{ for some } x \geqq 0\} \tag{II.14}$$

Given $\Pi \geqq 0$, then p is determined and so $L(p, 1)$ and $F(p, 1)$. Thus, the sets $C(\Pi)$ and $\Gamma(\Pi)$ are determinate.

$C(\Pi)$ is the set of all possible final demand vectors available to the capitalist class, whereas $\Gamma(\Pi)$ is the efficient frontier of $C(\Pi)$. As will be made clear, these two sets are always nonempty and of a very simple structure for any given $\Pi \geqq 0$.

Now, for the sake of economic and mathematical clarity, define

$$M(\Pi) = L(p, 1) - \sigma' F(p, 1) \tag{II.15}$$

as a function of Π . Then, mathematically, the following theorem, which is readily proved, clarifies the crucial properties of $C(\Pi)$ and $\Gamma(\Pi)$:

<u>Theorem 1</u>. (i) $C(\Pi) = \{c \mid c \geqq 0,\ M(\Pi) \geqq \sigma' c\}$, $\Gamma(\Pi) = \{c \mid c \geqq 0,\ M(\Pi) = \sigma' c\}$. (ii) $C(\Pi^1) \supset C(\Pi^2)$ <u>if, and only if,</u> $M(\Pi^1) \geqq M(\Pi^2)$.

<u>Proof.</u> (i) If $c \in C(\Pi)$, then from the definitional equation follows

$$x = (I - A)^{-1} \{F(p, 1) + c\} \quad . \tag{II.16}$$

Premultiplying (II.16) by ℓ' one can reduce (II.16) to

$$\ell' x = \sigma' F(p, 1) + \sigma' c \quad . \tag{II.17}$$

(II.17) combined with

$$L(p, 1) \geqq \ell' x \quad , \tag{II.18}$$

gives

$$M(p, 1) \geqq \sigma' c \tag{II.19}$$

because of (II.15). Conversely, if a nonnegative c satisfies (II.19), then $F(p, 1) + c$ is also nonnegative, so that the x defined by (II.16) is again nonnegative. (II.16), which is equivalent to the definitional equation of $C(\Pi)$, can induce (II.17). Finally, (II.17) and (II.19) imply (II.18) because of (II.15). The proof of the assertion for $\Gamma(\Pi)$ is essentially the same.

(ii) The assertion is obviously true in the light of the characterization of $C(\Pi)$ in (i). Q. E. D.

Geometrically, $C(\Pi)$ and $\Gamma(\Pi)$ are of a much simpler structure. $C(\Pi)$ is a simplex bounded by the hyperplane $M(\Pi) = \sigma' c$ and the n coordinate hyperplanes $c_i = 0$ $(i = 1, 2, \ldots, n)$, whereas $\Gamma(\Pi)$ is a simplex generated as the intersection of the hyperplane $M(\Pi) = \sigma' c$ with the nonnegative orthant $R_+^n = \{c \mid c \geqq 0\}$. $C(\Pi)$ is n-dimensional, and $\Gamma(\Pi)$ is $(n - 1)$ - dimensional and a face of $C(\Pi)$ when $M(\Pi) > 0$. $C(\Pi) = \Gamma(\Pi) = \{0\}$ when $M(\Pi) = 0$. Therefore, they are always nonempty (see Figures 1, 2 and 3), as $M(\Pi) \geqq 0$ for $\Pi \geqq 0$ from (II.23) below.

Suppose that a profit per unit output vector Π is given or possibly chosen by the capitalist class. Then, there will be a supply of $L(p, 1)$ units of labor service, the employment of which leaves any final demand vector c in $C(\Pi)$, especially in $\Gamma(\Pi)$, at the disposal of the capitalist class after $F_j(p, 1)$ units of the j th good $(j = 1, 2, \ldots, n)$ are paid out to the labor force as wages. If the capitalist class is so disposed that the enlargement of the possibility set $C(\Pi)$ is a vital problem, as may be likely to be the case, the greater $M(\Pi)$ the better.

Now, if everything is measured in terms of labor value,

$$L(p, 1) = \text{net national product}$$

$$\sigma' F(p, 1) = \text{the wage bill}$$

$$M(\Pi) = \text{surplus value (Mehrwert).}$$

Moreover, in the Marxian scheme of reproduction, the following equation

$$\sigma' x = \sigma' A x + \sigma' F(p, 1) + M(\Pi)$$

where

$$x = (I - A)^{-1}\{F(p, 1) + c\}$$

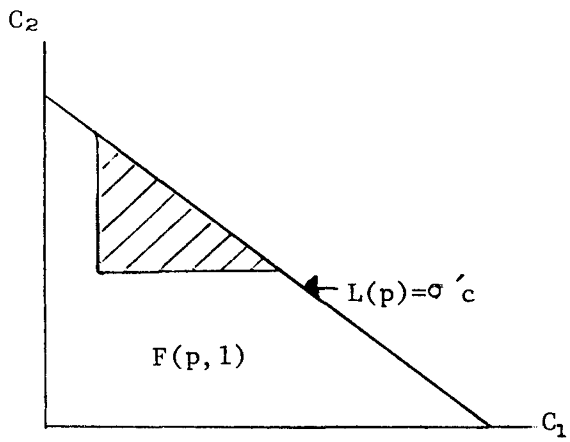

Figure 1

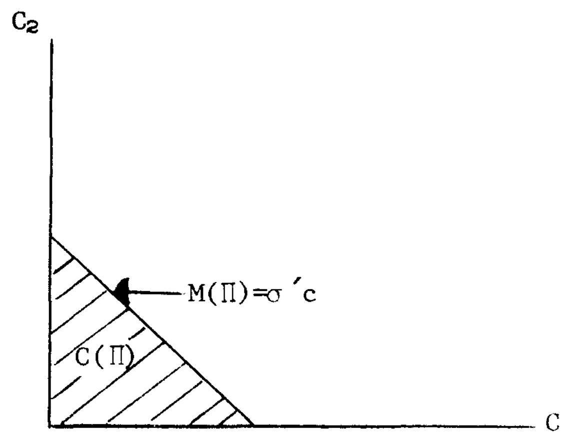

Figure 2

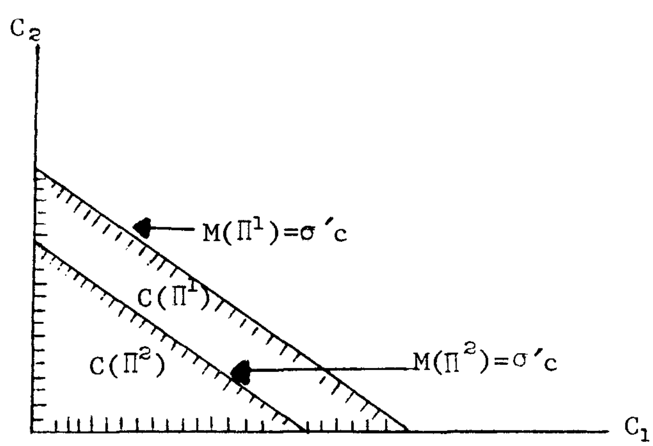

Figure 3

admits the interpretation

$$\sigma' x = \text{gross national product}$$

$$\sigma' Ax = \text{constant capital}$$

$$\sigma' F(p, 1) = \text{variable capital}$$

$$M(\Pi) = \text{surplus value.}$$

From what has been established above, it is now crystal-clear that making surplus value $M(\Pi)$, a scalar, larger is completely equivalent to the enlargement of $C(\Pi)$, the generally n-dimensional final demand possibility set of the capitalist class. In the simple world of the Leontief system this is a fundamental fact which results from the operational function of the labor values of goods in measuring them in real terms beyond any ideological views.

II.3 Maximization of surplus value

In this economy everything emerges in terms of prices on the surface. Nevertheless, it is essentially of such a structure that its working is effectuated by the capitalist's direct or indirect control of the sectoral profits per unit output Π_j, bringing about thereby the allocation of labor in the amount $L(p, 1)$ and the distribution of the net national product $L(p, 1)$ in terms of labor value to the labor force and the capitalist class in the amounts $\sigma' F(p, 1)$ and $M(\Pi)$, respectively (by commodity breakdown, $F(p, 1)$ and a final demand vector c from among $C(\Pi)$).

This by no means asserts that there is a unique mode of the capitalist's control of Π_j. On the contrary capitalists themselves are likely to be deceived by price phenomena on the surface, so that alternative modes of controlling Π_j may be conceivable according to the market structure as well as the capitalist's perception of what is going on beneath the surface. It will be the purpose of Section IV to consider these alternative modes after the construction of objective demand functions has been completed in Section III.

Here, a special conceivable mode of the capitalist's control of Π_j , namely, joint maximization of surplus value will be considered as a polar case. This mode has a normative implication from the capitalist's ethical point of view. For a maximum surplus value, if it ever exists, provides them with the largest possibility set $C(\Pi)$ of final demand vectors. If an allotment out from a final demand vector c in a $C(\Pi)$ is made to capitalist households which are so disposed that more of goods induces greater satisfaction, the allotment can be a Pareto optimum among their households only when Π maximizes surplus value. The following theorem ensures the existence of a maximum surplus value.

<u>Theorem 2</u>. <u>The surplus value</u> $M(\Pi)$ <u>is maximized at a</u> Π <u>over all</u> $\Pi \geqq 0$, <u>if the indecomposability of the input coefficients matrix</u> A <u>is explicitly assumed</u>. <u>The corresponding maximum surplus value is positive</u>.

<u>Proof</u>. By virtue of the indecomposability of A , the inverse of $I - A$ is a positive matrix,[5] as is well-known. Hence, all the components of p in equation (II.11) tend to infinity simultaneously, when even a single component of Π tends to infinity, irrespective of whatever behaviors the other components may have. Therefore, by (II.9),

$$\lim_{\Pi_j \to +\infty} L(\sigma + (I - A')^{-1} \Pi, 1) = 0 \quad (\text{for each } j) \quad . \tag{II.20}$$

Define now the set

$$\Pi(\varepsilon) = \{\Pi \mid \Pi \geqq 0, \ L(\sigma + (I - A')^{-1} \Pi, 1) \geqq \varepsilon\} \quad . \tag{II.21}$$

Then, the set is bounded for any positive number $\varepsilon > 0$. For otherwise it would include such a sequence $\{\Pi^{\nu}\}$ that

$$\lim_{\nu \to +\infty} \Pi_{j\nu} = +\infty \quad \text{(for some } j) \quad . \tag{II.22}$$

But (II.22) contradicts the inclusion of $\{\Pi^{\nu}\}$ in $\Pi(\varepsilon)$ because of (II.20).

Now, in the light of (II.8) for $w = 1$ and the definition (II.15), the surplus value can be expressed as

$$M(\Pi) = (p - \sigma)' F(p, 1) \quad . \tag{II.23}$$

Then, recalling (II.12) in [A.5], one is sure of the positivity of $L(p, 1)$ for a small positive Π by continuity. This implies the positivity of some component of $F(p, 1)$ again by (II.8) for $w = 1$, and therefore the positivity of $M(\Pi)$ in (II.23) for a small positive Π, because $p > \sigma$, $F(p, 1) \geq 0$ then.

Next take such a Π that $M(\Pi)$ is positive, say, Π^{o}, and let

$$\varepsilon = M(\Pi^{o}) \quad . \tag{II.24}$$

Then, the set $\Pi(\varepsilon)$ contains Π^{o} and is therefore nonempty, since $L(\sigma + (I - A')^{-1} \Pi^{o}, 1) \geqq M(\Pi^{o})$. On the other hand, if $\Pi \notin \Pi(\varepsilon)$, then

$$M(\Pi) \leqq L(\sigma + (I - A')^{-1} \Pi, 1) < \varepsilon = M(\Pi^{o}) \quad .$$

Whence, a maximum of $M(\Pi)$ over all nonnegative Π's occurs only in $\Pi(\varepsilon)$, if any. Likewise a maximum of $M(\Pi)$ on $\Pi(\varepsilon)$ is its maximum over all nonnegative Π's. As was shown, $\Pi(\varepsilon)$ is bounded. Moreover, $\Pi(\varepsilon)$ is a closed set by virtue of the continuity of $L(\sigma + (I - A')^{-1}\Pi, 1)$ in Π. Hence $\Pi(\varepsilon)$ is a compact set on which the continuous function $M(\Pi)$ takes on a maximum, which is not less than ε, a positive number. Q.E.D.

Let Π be a solution of this maximization problem. Then, Π must satisfy the Kuhn-Tucker condition, if the differentiability of the relevant functions is further assumed, and one has

$$\frac{\partial M(\Pi)}{\partial \Pi_k} = \sum_{i=1}^{n} b_{ki} \left\{ F_i(p,1) + \sum_{j=1}^{n} (p_j - \sigma_j) \frac{\partial F_j}{\partial p_i} \right\} \leqq 0 \quad (k = 1, 2, \ldots, n) \tag{II.25}$$

with equality holding if $\Pi_k > 0$, where

$$\begin{aligned} b_{ki} &= \text{the } (k, i) \text{ element of } (I - A)^{-1} \\ p_j &= \sigma_j + \sum_{k=1}^{n} b_{kj} \Pi_k \quad (j = 1, 2, \ldots, n) \\ \Pi_k &\geqq 0 \qquad\qquad (k = 1, 2, \ldots, n) \quad . \end{aligned}$$

II.4 The full employment presumption

In the foregoing bisections it is presumed that $L(p, 1)$ which is supplied at the current price-wage situation is fully employed. Even if the capitalist class picks up from among $C(\Pi)$ a final demand vector c which is off the efficient frontier $\Gamma(\Pi)$, so that less labor than $L(p, 1)$ is virtually needed, $F_j(p, 1)$ units of goods $(j = 1, 2, \ldots, n)$ are fully paid out as wages to labor force. Thus, one might imagine if there is any room for the capitalist class to have a larger final demand possibility set than $C(\Pi)$ by carrying out production with less employment of labor and the corresponding less wages than those in the full employment level. This bisection is a digression to see that this does not occur.

To this end, suppose that if μ percent of the labor supply $L(p, 1)$ is employed, the wages paid out are μ percent of the full employment wages $F_j(p, 1) (j = 1, 2, \ldots, n)$. Under this generalized mode of employment, the possibility set $C^*(\Pi)$ of all final demand vectors available to the capitalist class is the set of all $c \geqq 0$ satisfying

$$x = Ax + \{\ell' x / L(p, 1)\} F(p, 1) + c \tag{II.26}$$

$$L(p, 1) \geqq \ell' x \tag{II.27}$$

$$\text{for some} \quad x \geqq 0$$

if

$$L(p,\ 1) > 0 \quad .$$

Otherwise $C^*(\Pi) = \{0\}$.

(II.26) can easily be rearranged to

$$\left\{I - \left(A + \frac{F(p,\ 1)}{L(p,\ 1)}\,\ell'\right)\right\}x = c \quad . \tag{II.28}$$

(II.26) and (II.27) coincide with the definitional equation of $\Gamma(\Pi)$ when equality holds in (II.27). Whence $C^*(\Pi)$ includes $\Gamma(\Pi)$. In the light of (i) in Theorem 1, $\Gamma(\Pi)$ contains positive vectors when $M(\Pi) > 0$, and hence, so does $C^*(\Pi)$. This implies that the left-hand side of (II.28) is a positive vector for some $x \geqq 0$. Since the matrix

$$A + \frac{F(p,\ 1)}{L(p,\ 1)}\,\ell'$$

is nonnegative, it follows from what has been shown above that the coefficients matrix of (II.28) is nonsingular and has its inverse nonnegative.[6] Therefore, when $M(\Pi) > 0$, $C^*(\Pi)$ is nothing but the set of all nonnegative c satisfying

$$L(p,\ 1) \geqq \ell' \left\{I - \left(A + \frac{F(p,\ 1)}{L(p,\ 1)}\,\ell'\right)\right\}^{-1} c \quad . \tag{II.29}$$

It is now obvious by the positivity of

$$\ell' \left\{I - \left(A + \frac{F(p,\ 1)}{L(p,\ 1)}\,\ell'\right)\right\}^{-1}$$

that the efficient frontier of $C^*(\Pi)$ is composed of all nonnegative c satisfying (II.29) with equality and coincides with $\Gamma(\Pi)$. This proves $C^*(\Pi) = C(\Pi)$.

It remains to consider the case $M(\Pi) = 0$, which implies that

$$C(\Pi) = \{0\} \tag{II.30}$$

by (i) in Theorem 1, and that

$$L(p, 1) = \sigma' F(p, 1) \tag{II.31}$$

by (II.15). Then, premultiplying the coefficients matrix of (II.28) by $\sigma' = \ell'(I - A)^{-1}$, one gets, in view of (II.31),

$$\sigma' \left\{ I - \left(A + \frac{F(p, 1)}{L(p, 1)} \ell' \right) \right\} = 0' \quad .$$

Whence

$$\sigma' c = 0 \tag{II.32}$$

for any nonnegative c expressible by (II.28). Since σ is a positive vector, this entails $c = 0$. Thus, $C^*(\Pi) = \{0\}$ and coincides with $C(\Pi)$ in (II.30).

The discussion above may justify the full employment presumption in this work. The reserve army of labor force will therefore not explicitly show up here.

III. Objective Demand Functions

III.1 Capitalists' choice of final demand

The foregoing analysis which is worked out in terms of the labor and surplus values in Section II by no means intends to give a description of what is going on as economic phenomena in the orthodox sense. It simply gives a description of what is going on in terms of the labor value behind the scenes of economic phenomena. The behaviors of economic agents such as capitalists and workers are very likely to be based on something on the surface of the interplay of prices, rather than on the labor value concept beneath it. Nonetheless, whatever behavioral principles of them may bring about a final economic equilibrium, the resource allocative significance of the equilibrium can be assessed in terms of the labor value in an operationally meaningful way, free from any ideological dogma. For this reason the presentation of the results is given intentionally in a reverse order, starting at the consideration of what is going on in the realm of the labor value, then proceeding to the analysis of the factual working of the economy, rather than starting at a setup of behavioral principles of economic agents, then discussing the working of the economy brought about by them and finally interpreting it possibly in terms of the labor value. Before proceeding to the main issue in this section, this peculiar character of the presentation is emphasized again in order to prevent the author's intention in this work from being misunderstood.

It has been made clear in the preceding section that if a nonnegative profit per unit output vector Π is given, then the price vector is determined by equation (II.11), and the possibility set of final demand vectors $C(\Pi)$ and its efficient frontier $\Gamma(\Pi)$ are available to the capitalist class. True, the determination of Π is effected in the factual working of the economy through a market mechanism which is either competitive, monopolistically competitive or of yet another nature. But, whatever pricing principle may determine Π, a given Π provides the capitalist class with the possibility set $C(\Pi)$, from among which the capitalist class can pick up any final demand vector c. As a matter of fact, the production activities of sectors represented by the gross output vector x are influenced, and determined through equation (II.16), by the choice of c from among $C(\Pi)$. Therefore different c's in $C(\Pi)$ are supplied by virtue of different gross output vectors. At any rate the capitalist class can choose any c from among $C(\Pi)$ by virtue of the accomplishment of the corresponding production activities.

How then a final demand vector c is chosen by the capitalist class from among $C(\Pi)$ naturally hinges on what motivates capitalists as consumers, or more precisely, as demanders for goods for their own use. It is noted that the c is the remainder when the workers' demand $F(p, 1)$ and the derived demand Ax are deducted from the gross output x. The capitalists might directly negotiate about the bill of goods to be produced and the distribution of the product among themselves, not by way of a market mechanism. But in a capitalist economy as considered here both the choice of a final demand vector from among $C(\Pi)$ and its distribution are most likely to be worked out through a market mechanism. The capitalists themselves are imprisoned in a market mechanism. They, as producers and entrepreneurs, can possibly influence price formation in the markets to a certain extent, depending on the market structures. But as demanders for goods for their own use they, too, are subjected to the price system which is more or less controlled by themselves. The profits earned are distributed among the households of capitalists. The households spend their profit income to purchase goods. Notwithstanding the possible control of prices by the capitalists as entrepreneurs, their households are very likely to take the prices as given parameters and to behave as competitive consumers in spending their profit income for the purchase of goods that they demand for their own use.

Now, if the above presumption as to the behaviors of the households of capitalists is taken for granted, there will be effective demand originating from the profit income for the product which is supplied by the choice of a final demand vector c from among $C(\Pi)$. The factual choice of a final demand vector and its distribution among the capitalists are thereby worked out when the demand equals the supply. It is the purpose of this section to see how these are worked out.

To this end an aggregate demand function of the capitalists' households for goods will be introduced in parallel with the workers' demand function $F(p, w)$ and on the basis of the above presumption. Let s_i $(i = 1, 2, \ldots, n)$ be the total profit of the i th sector. The aggregate demand function of the capitalists' households

$$G(p, s_1, s_2, \ldots, s_n) = (G_j(p, s_1, s_2, \ldots, s_n)) \tag{III.1}$$

is required to satisfy the following assumptions:

[B.1] $G(p, s_1, s_2, \ldots, s_n)$ is a nonnegative, continuous vector-valued function of the positive price vector p and the nonnegative sectoral profits $s_i (i = 1, 2, \ldots, n)$, its j th component being the demand for the j th good.

[B.2] $G(p, s_1, s_2, \ldots, s_n)$ has either of the following two spending patterns:

J. B. Say's case: The profit income is entirely spent for the purchase of goods. That is, the following identity holds

$$p'G(p, s_1, s_2, \ldots, s_n) = \sum_{j=1}^{n} s_j \quad . \tag{III.2}$$

Keynesian case: There are savings from the profit income. More specifically, there exists a constant[7] θ fulfilling

$$1 > \theta > 0 \tag{III.3}$$

such that the following inequality holds identically

$$p'G(p,\ s_1,\ s_2,\ \ldots,\ s_n) \leqq \theta \sum_{j=1}^{n} s_j \quad . \tag{III.4}$$

It is noted that in Say's case (III.2) need not imply that the spending of the profit income is exclusively for the purchase of consumption goods. The demand function can possibly include demand for goods for investment. However, it is in better accordance with the spirit of Keynesian income analysis to assume the Keynesian case above.

Before proceeding to the main discussion, it should be recalled again and borne in mind that, although the demand function includes p as an argument, p, which need not be a surplus value maximizing price vector, is assumed to be given and kept constant for a while. Suppose that a final demand vector c is chosen from among $C(\Pi)$, in particular, its efficient frontier $\Gamma(\Pi)$. As was remarked, the supply of c is effected by the production activities in the corresponding gross output $x(c)$ determined by (II.16). The profit per unit output vector Π being already given, the profit of the j th sector is

$$s_j(c) = \Pi_j\ x_j(c) \qquad (j = 1,\ 2,\ \ldots,\ n) \quad . \tag{III.5}$$

Then, there will be effective demand for goods originating from these profit incomes (III.5) at the price vector p. However, the effective demand represented by the demand function need not equal the supply c. Does there exist any special c in $C(\Pi)$ such that demand equals supply in the above sense? The consistent working of the economy is effectuated by and only by choosing such a special c from among $C(\Pi)$, if any. For convenience the choice of this special final demand vector c equating demand to supply will henceforth be referred to as a competitive choice. The next step is to prove the existence of a competitive choice in both Say's and Keynesian cases.

III.2 Existence of a competitive choice

It is clear from the definition of a competitive choice that there is a complete circulation of national income flow brought about by this choice. All the wages and profits earned induce effective demand that exactly equals the product supplied, while the very equality of demand to supply makes the profit determinate ex post in addition to the predetermined wages.

There is a crucial relationship between the circulation of national income flow, and the feasibility and efficiency expressed in terms of the labor value of the supply of a final demand vector c. This relationship, which holds true regardless of the introduction of the demand function $G(p, s_1, s_2, \ldots, s_n)$, is of independent interest and of much relevance to the subsequent discussion at the same time. This relationship will be given for easier reference in the sequel in a theorem.

Theorem 3. For any final demand vector c, which need not belong to $C(\Pi)$, and the corresponding gross output vector $x(c)$ determined by (II.16), namely,

$$x(c) = (I - A)^{-1} \{F(p, 1) + c\} \tag{III.6}$$

the relationship

$$\Pi' x(c) = M(\Pi) - \sigma' c + p' c \tag{III.7}$$

holds. In particular, the ex ante profit $\Pi' x(c)$ equals monetary value of supply $p'c$,

$$\Pi' x(c) = p' c \tag{III.8}$$

if and only if c belongs to the efficient frontier $\Gamma(\Pi)$ of $C(\Pi)$.

Proof. The proof is easy and immediate. In fact, premultiplying (III.6) by Π' and taking (II.11) into account, one has

$$\Pi' x(c) = (p - \sigma)' \{F(p, 1) + c\} \quad . \tag{III.9}$$

Then, the relationship (III.7) follows from (III.9), if one recalls a representation of the surplus value $M(\Pi)$ in (II.23). Q.E.D.

It is further noted that the relationship (III.7) translates the relation of the ex ante profit $\Pi' x(c)$ to the monetary value of supply $p'c$ into its resource allocative significance in terms of the labor value. In fact, if c is feasible, $c \in C(\Pi)$, but not efficient, $c \notin \Gamma(\Pi)$, then the ex ante profit income exceeds the monetary value of supply,

$$\Pi' x(c) > p'c \quad . \tag{III.10}$$

On the other hand, if c is infeasible, $c \notin C(\Pi)$, then the ex ante profit income is insufficient to purchase the proposed supply. The equality of the ex ante profit income to the ex ante supply value obtains if and only if a final demand vector c in the efficient frontier $\Gamma(\Pi)$ is supplied. It goes without saying, however, that (III.8) does not necessarily imply equality of demand to supply by commodity breakdown, which can be brought about only by a competitive choice of c, as will be seen below.

Theorem 4. (Existence of a competitive choice in Say's case.) *There exists a final demand vector* c *in the efficient frontier* $\Gamma(\Pi)$ *such that (III.6) and*

$$c = G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c)) \tag{III.11}$$

holds.

Proof. If $M(\Pi) = 0$, then $\Gamma(\Pi)$ contains only one vector, $c = 0$. On the other hand, $\Pi' x(0) = 0$ from (III.7) for $c = 0$ and $M(\Pi) = 0$. Then, in view of (III.5), one sees that the right-hand side is vanishing in (III.2). Hence

$$p'G(p, \Pi_1 x_1(0), \Pi_2 x_2(0), \ldots, \Pi_n x_n(0)) = 0 .$$

But $G(p, s_1, s_2, \ldots, s_n)$ is nonnegative valued by assumption [B.1] and p is positive, so that the right-hand side of (III.11) is the zero vector for $c = 0$. This proves that (III.11) obtains for the only possible $c = 0$ in $\Gamma(\Pi)$ in case $M(\Pi) = 0$.

Next, suppose $M(\Pi) > 0$. As was noted, $\Gamma(\Pi)$ is an (n-1)-dimensional simplex. Define a mapping $\phi : \Gamma(\Pi) \to \Gamma(\Pi)$ by the formula which assigns with any $c \in \Gamma(\Pi)$ the image

$$\phi(c) = \frac{M(\Pi)\, G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c))}{\sigma' G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c))} . \qquad \text{(III.12)}$$

It will be remarked that the construction of the mapping can be worked out in a consistent way. In fact, it follows from $\sigma' c = M(\Pi) > 0$ that the nonnegative c has at least one component positive. Hence, $p'c > 0$ by the positivity of p. Therefore $\Pi' x(c) > 0$ by (III.8). One sees, in view of (III.2) and (III.5), that $p' G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c)) > 0$. This implies that the nonnegative $G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c))$ has at least one component positive. Thus, the denominator, which is the inner product of the positive vector σ and $G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c))$, must be positive for any c in $\Gamma(\Pi)$. On the other hand, $\phi(c)$ is always nonnegative and $\sigma'\phi(c) = M(\Pi)$ by construction. Therefore, ϕ is a well-defined mapping from $\Gamma(\Pi)$ into $\Gamma(\Pi)$. Finally, ϕ is continuous because of the continuity of $G(p, \Pi_1 x_1(c), \Pi_2 x_2(c), \ldots, \Pi_n x_n(c))$.

Therefore, by virtue of the well-known Brouwer fixed point theorem, ϕ has a fixed point $\hat{c}$ such that

$$\hat{c} = \frac{M(\Pi)\, G(p, \Pi_1 x_1(\hat{c}), \Pi_2 x_2(\hat{c}), \ldots, \Pi_n x_n(\hat{c}))}{\sigma' G(p, \Pi_1 x_1(\hat{c}), \Pi_2 x_2(\hat{c}), \ldots, \Pi_n x_n(\hat{c}))} . \qquad \text{(III.13)}$$

Premultiplication of (III.13) by p' gives, in the light of (III.2), (III.5) and (III.8),

$$p'\hat{c} = \frac{M(\Pi)\ p'\hat{c}}{\sigma'\ G\ (p,\ \Pi_1\ x_1\ (\hat{c}),\ \Pi_2\ x_2\ (\hat{c}),\ \ldots,\ \Pi_n\ x_n\ (\hat{c}))}\ , \tag{III.14}$$

which implies

$$M(\Pi) = \sigma'\ G\ (p,\ \Pi_1\ x_1\ (\hat{c}),\ \Pi_2\ x_2\ (\hat{c}),\ \ldots,\ \Pi_n\ x_n\ (\hat{c}))$$

because $p'\hat{c} > 0$. Whence (III.13) reduces to

$$\hat{c} = G\ (p,\ \Pi_1\ x_1\ (\hat{c}),\ \Pi_2\ x_2\ (\hat{c}),\ \ldots,\ \Pi_n\ x_n\ (\hat{c}))\ ,$$

and the choice of this $\hat{c}$ is a competitive choice, as was to be shown. Q.E.D.

Theorem 5. (Existence of a competitive choice in Keynesian case.) Let $d \geq 0$ be a given investment composition vector which is constant.[8] Then, there exist a final demand vector c and a nonnegative scalar ω such that

$$\omega\ d + c\ \varepsilon\ \Gamma(\Pi) \tag{III.15}$$

$$x = (I - A)^{-1}\ \{F(p,\ 1) + \omega\ d + c\} \tag{III.16}$$

$$c = G\ (p,\ \Pi_1\ x_1, \Pi_2\ x_2,\ \ldots,\ \Pi_n\ x_n)\ . \tag{III.17}$$

The above ω is positive if and only if $M(\Pi) > 0$.

Proof. Before proceeding to the proof, it is noted that the capitalist class picks up $\omega\ d + c$ from among $C(\Pi)$ because of the presence of demand for investment goods $\omega\ d$, rather than a mere c . However, Theorem 3 is still valid provided the c in the statement of the theorem is replaced by $\omega\ d + c$.

The proof will be worked out by obtaining a solution as a fixed point of a mapping, which is constructed in a different way. The mapping to be constructed here is defined for all nonnegative vector c , which need not belong to $C(\Pi)$, unlike the mapping in the preceding theorem. First of all, a nonnegative numerical function $\omega(c)$ is defined for all $c \geqq 0$ by the formula

$$\omega(c) = \begin{cases} \dfrac{M(\Pi) - \sigma' c}{\sigma' d} & \text{if } M(p) > \sigma' c \\ 0 & \text{otherwise} \end{cases} \quad . \tag{III.18}$$

The function $\omega(c)$ is continuous on the set of all $c \geqq 0$.

Next let

$$x(c) = (I - A)^{-1} \{F(p, 1) + \omega(c)\ d + c\} \quad . \tag{III.19}$$

Then with respect to (III.19) one obtains as a counterpart equation to (III.7)

$$\Pi' x(c) = M(\Pi) - \sigma'(\omega(c)d + c) + p'(\omega(c)d + c) \quad . \tag{III.20}$$

Now, denoting the set of all nonnegative vectors c in the n-dimensional Euclidean space R^n , the so-called nonnegative orthant, by R^n_+ , define a mapping $\psi: R^n_+ \rightarrow R^n_+$ by the formula

$$\psi(c) = G\ (p,\ \Pi_1\ x_1\ (c),\ \Pi_2\ x_2\ (c),\ \ldots,\ \Pi_n\ x_n\ (c)) \quad . \tag{III.21}$$

It is intended that a solution may be found as a fixed point of the mapping ψ . But the Brouwer fixed point theorem is not directly applicable to the mapping, since its original domain on which it is defined is the entire R^n_+ , an unbounded set. To overcome this difficulty, the mapping will be considered within a suitably chosen bounded subset of the entire domain.

To this end, the following inequality will be established:

$$p' \psi(c) \leqq p' c \tag{III.22}$$

for any c satisfying

$$c \geqq 0 \ , \qquad p' c \geqq \frac{\theta\ p' d\ M(\Pi)}{(1 - \theta)\sigma' d} \tag{III.23}$$

where θ is the θ given in the Keynesian case of [B.2]. In fact, in view of (III.21), (III.4) and (III.20), one sees for any $c \geqq 0$

$$p'\psi(c) = p'G\ (p,\ \Pi_1\ x_1\ (c),\ \Pi_2\ x_2\ (c),\ \ldots,\ \Pi_n\ x_n\ (c))$$

$$\leqq \theta\ \Pi' x(c) \tag{III.24}$$

$$= \theta\ \{M(\Pi) - \sigma'(\omega(c)d + c) + p'(\omega(c)d + c)\} \quad .$$

It is obvious from the definition of $\omega(c)$ that

$$M(\Pi) \leqq \sigma'(\omega(c)d + c) \tag{III.25}$$

$$\omega(c) \leqq \frac{M(\Pi)}{\sigma' d} \quad . \tag{III.26}$$

By taking (III.25) and (III.26) into account, one can reduce (III.24) to

$$p'\psi(c) \leqq \frac{\theta\ p'd\ M(\Pi)}{\sigma' d} + \theta\ p'c \quad . \tag{III.27}$$

The right-hand side of (III.27) is not greater for any c satisfying (III.23) than $p'c$. Whence (III.22) follows.

Now consider the set

$$\Omega = \{c \mid c \geqq 0\ ,\quad \frac{\theta\ p'd\ M(\Pi)}{(1 - \theta)\sigma' d} \geqq p'c\} \quad . \tag{III.28}$$

Ω is a compact set, since p is a positive vector. Therefore the image $\psi(\Omega)$ of Ω under the mapping ψ is also compact because of the continuity of ψ . Hence $\psi(\Omega)$ is bounded, so that it can be included in a simplex

$$\Delta = \{c \mid c \geqq 0,\quad \delta \geqq p'c\}$$

for a sufficiently large positive δ . That is

$$\Delta \supset \psi\ (\Omega) \quad . \tag{III.29}$$

It will be shown that ψ maps each c in Δ into Δ . In fact, let c be a point in Δ . If $c \in \Omega$, then $\psi(c) \in \psi\ (\Omega) \subset \Delta$ by (III.29) .

If $c \notin \Omega$, then

$$\frac{\theta\, p'd\, M(\Pi)}{(1-\theta)\sigma'd} < p'c \quad ,$$

so that (III.22) holds and hence $p'\psi(c) \leqq p'c \leqq \delta$, implying $\psi(c) \in \Delta$.

Now, confining the domain of ψ to Δ, a compact and convex set, one can apply the Brouwer fixed point theorem to the continuous mapping $\psi: \Delta \to \Delta$. Thus, ψ has a fixed point $\hat{c}$,

$$\hat{c} = \psi(\hat{c}) \quad . \tag{III.30}$$

It is obvious that this $\hat{c}$ and the corresponding $\omega = \omega(\hat{c})$ satisfy equations (III.16) and (III.17). It remains to show that they satisfy (III.15). To show this it suffices to see that $\omega(\hat{c})d + \hat{c}$ is feasible, i.e., $\omega(\hat{c})d + \hat{c} \in C(\Pi)$. For, if it is feasible, then $M(\Pi) \geqq \sigma'(\omega(\hat{c})d + \hat{c})$ by definition, which turns out to be $M(\Pi) = \sigma'(\omega(\hat{c})d + \hat{c})$ by (III.25).

With this in mind, suppose that $\omega(\hat{c})d + \hat{c}$ is infeasible, so that strict inequality holds in (III.25). Then,

$$\omega(\hat{c}) = 0 \quad , \tag{III.31}$$

and inequality (III.24) for $c = \hat{c}$ reduces under (III.30) and (III.31) to

$$p'\hat{c} \leqq \theta\, p'\hat{c} \quad . \tag{III.32}$$

But (III.32) is impossible, since $1 > \theta > 0$ and $p'\hat{c} > 0$. Here, $p'\hat{c} > 0$ is seen in the following way. In fact, $\sigma'\hat{c} > 0$ is implied by (III.25) in a strict inequality form and (III.31). This shows that the nonnegative $\hat{c}$ has at least one component positive. Hence $p'\hat{c} > 0$ because $p > 0$.

Finally, the supplementary assertion as to the nonvanishingness of ω in a solution will be considered. In fact, if $M(\Pi) = 0$, then $\omega\, d + c = 0$ by feasibility (III.15), since $\Gamma(\Pi) = \{0\}$ in this case. This clearly implies

$\omega = 0$. Conversely, if $\omega = 0$, then $c \in \Gamma(\Pi)$, and hence $M(\Pi) = \sigma' c$. Therefore, inequality (III.24) reduces to $p'c \leqq \theta p'c$, which implies $p'\hat{c} = 0$ under $1 > \theta > 0$. Thus $c = 0$ because of the nonnegativity of c and the positivity of p . This proves $M(\Pi) = \sigma' c = 0$. It is thereby shown that ω is positive if and only if $M(\Pi) > 0$. The proof is now complete. Q.E.D.

In Theorems 4 and 5 the competitive choices are formulated in terms of certain equations in the unknowns c and ω . But they can be reformulated in terms of equations in the gross output x and the above ω as unknowns.

In fact, the competitive choice in Say's case can be reformulated as the determination of the gross output vector x by virtue of the equation[9]

$$x = (I - A)^{-1}\{F(p, 1) + G\ (P, \Pi_1\ x_1, \Pi_2\ x_2, \ldots, \Pi_n\ x_n)\}\ . \qquad \text{(III.33)}$$

On the other hand, the competitive choice in the Keynesian case can be reformulated as the determination of the gross output vector x and the scale of investment ω by virtue of the equations

$$\omega\ d + G\ (p, \Pi_1\ x_1, \Pi_2\ x_2, \ldots, \Pi_n\ x_n) \in \Gamma\ (\Pi) \qquad \text{(III.34)}$$

$$x = (I - A)^{-1}\{F(p, 1) + \omega\ d + G\ (p, \Pi_1\ x_1, \Pi_2\ x_2, ,,,, \Pi_n\ x_n)\}\ . \qquad \text{(III.35)}$$

Thus, the competitive choice discussed above is actually the determination of national income of a Keynesian type at the fixed price system not only in the Keynesian case but also even in the Say's case.[10]

III.3 Uniqueness of competitive choice

In general the competitive choice whose existence has been proved in Theorems 4 and 5, is not necessarily unique. It is therefore of interest and even important to see under what conditions its uniqueness obtains, all the more as uniqueness is required in order to construct single-valued objective demand functions.

There might be possible alternative conditions for uniqueness. Here certain sufficient conditions for uniqueness will be discussed. These conditions, which slightly specify the spending patterns of the capitalist households as embodied in the demand function $G(P, s_1, s_2, \ldots, s_n)$, ensure the uniqueness of competitive choice under differentiability. Under the differentiability assumption, the relevant concepts and discussion pertaining to uniqueness become transparent. Thus the analysis will be worked out in this bisection as well as elsewhere below under differentiability

[B.3] $G(p, s_1, s_2, \ldots, s_n)$ is differentiable[11] in the rectangular region $s_i \geqq 0 \ (i = 1, 2, \ldots, n)$.

The principal additional assumption on the spending patterns of the capitalist households, which ensures the uniqueness of competitive choice, is the hypothesis of the absence of inferior goods. The hypothesis means that the demand function $G(P, s_1, s_2, \ldots, s_n)$ is monotonically nondecreasing with respect to the arguments $s_1, s_2, \ldots, s_n$ for each fixed p. Under differentiability the hypothesis is expressed as the nonnegativity of the partial derivatives

$$\frac{\partial G_i}{\partial s_j} = G_{ij}(p, s_1, s_2, \ldots, s_n) \geqq 0 \quad (i, j = 1, 2, \ldots, n) \quad . \qquad \text{(III.36)}$$

The hypothesis of no inferior goods suffices to ensure the uniqueness of competitive choice in the Say's case.

In order to ensure the uniqueness of competitive choice in the Keynesian case, however, the hypothesis need be reinforced by yet another additional assumption on the spending patterns, namely, the hypothesis of no-greater-than-one marginal propensities to spend, which means

$$\sum_{i=1}^{n} p_i \frac{\partial G_i}{\partial s_j} \leqq 1 \quad (j = 1, 2, \ldots, n) \quad . \tag{III.37}$$

(III.37) is prevalent, especially when the average propensity to spend

$$\theta\ (p, s_1, s_2, \ldots, s_n) = \frac{p'G(p, s_1, s_2, \ldots, s_n)}{\sum_{K=1}^{n} s_K} \tag{III.38}$$

is nonincreasing with respect to the arguments $s_1, s_2, \ldots, s_n$ in the Keynesian case. For upon differentiation of (III.38), one sees

$$\sum_{i=1}^{n} p_i \frac{\partial G_i}{\partial s_j} = \theta\ (p, s_1, s_2, \ldots, s_n) + \left(\sum_{K=1}^{n} s_K\right) \frac{\partial}{\partial s_j} \theta\ (p, s_1, s_2, \ldots, s_n)$$
$$< 1 \qquad\qquad (j = 1, 2, \ldots, n) \quad ,$$

because of $\theta\ (p, s_1, s_2, \ldots, s_n) < 1$ implied by (III.3) and (III.4), and the nonincreasing average propensity to spend

$$\frac{\partial}{\partial s_j} \theta\ (p, s_1, s_2, \ldots, s_n) \leqq 0 \quad (j, = 1, 2, \ldots, n) \quad .$$

Now, how uniqueness is ensured in a natural way by the above additional assumptions of economic significance will be seen below. All the basic assumptions such as [B.1], [B.2] and so forth are still to be premised and will be used without explicit references thereto.

Theorem 6. (Uniqueness of competitive choice in Say's case.) Under differentiability, if there are no inferior goods for the capitalist households, then the competitive choice is unique. That is, the solution c of equation (III.11), with $x(c)$ defined by (III.6), is unique.

Proof. Let equation (III.11) be rearranged as

$$H_i(c) = c_i - G_i\ (p,\ \Pi_1\ x_1\ (c),\ \Pi_2\ x_2\ (c),\ \ldots,\ \Pi_n\ x_n\ (c)) = 0 \qquad \text{(III.39)}$$

$$(i = 1,\ 2,\ \ldots,\ n) \quad .$$

It is recalled that a solution of (III.39) is obtained as a fixed point of a mapping from $\Gamma(\Pi)$ into $\Gamma(\Pi)$. Moreover, any solution of (III.39) must automatically lie on $\Gamma(\Pi)$. It must satisfy $p'c = \Pi' x(c)$ by (III.2) and (III.39), so that $c \in \Gamma(\Pi)$ by Theorem 3. Nonetheless, equation (III.39) is to be considered on the entire set of all nonnegative c 's, namely, the nonnegative orthant R^n_+ , when it comes to the uniqueness of solution.

With this remark in mind, the Jacobian matrix

$$J_H(c) = \left(\frac{\partial H_i}{\partial c_j}\right) \qquad \text{(III.40)}$$

of the mapping $H: R^n_+ \to R^n$, where $H(c) = (H_i(c))$, will be evaluated. By performing differentiation in view of (III.6), one obtains

$$J_H(c) = I - \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ & & \ddots \\ 0 & & & \Pi_n \end{pmatrix} (I - A)^{-1} \qquad \text{(III.41)}$$

where I is the identity matrix of the n th order. The matrix $(\partial G_i/\partial s_j)$, which is a function of c , is a nonnegative matrix by virtue of the hypothesis of no inferior goods. The matrix between $(\partial G_i/\partial s_j)$ and $(I - A)^{-1}$ in (III.41) is a diagonal matrix with Π_i 's in its principal diagonal. Hence the matrix

$$\left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ & & \ddots \\ 0 & & & \Pi_n \end{pmatrix} (I - A)^{-1} \qquad \text{(III.42)}$$

is a nonnegative matrix.

Next, differentiating (III.2) with respect to s_j , one sees that

$$\sum_{i=1}^{n} p_i \frac{\partial G_i}{\partial s_j} = 1 \qquad (j = 1,\ 2,\ \ldots,\ n) \quad , \qquad \text{(III.43)}$$

holds, or in a matrix form

$$p'\left(\frac{\partial G_i}{\partial s_j}\right) = (1, 1, \ldots, 1) \quad . \tag{III.44}$$

Whence follows

$$p'\left(\frac{\partial G_i}{\partial s_j}\right)\begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix} = \Pi' \quad . \tag{III.45}$$

Therefore, in the light of (III.45) and (II.11), one can evaluate

$$\begin{aligned} p'J_H(c) &= p' - p'\left(\frac{\partial G_i}{\partial s_j}\right)\begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix}(I - A)^{-1} \\ &= p' - \Pi'(I - A)^{-1} \\ &= p' - (p' - \sigma') \\ &= \sigma' > 0' \quad , \end{aligned} \tag{III.46}$$

which gives an important result

$$\begin{cases} p'J_H(c) = \sigma' > 0' \\ p > 0 \quad , \quad \sigma > 0 \quad . \end{cases} \tag{III.47}$$

Since $J_H(c)$ is the identity matrix minus a nonnegative matrix, (III.47) implies by virtue of the well-known Hawkins-Simon's result [7] that all the principal minors of $J_H(c)$ are positive on the domain of the mapping R^n_+, a rectangular region. Therefore, the solution of equation (III.39) is unique by virtue of a univalence theorem on a mapping having a P-Jacobian matrix due to D. Gale and myself ([6], Theorem 4; Nikaido [13], Theorem 20.4, page 370). Q.E.D.

<u>Theorem 7.</u> (Uniqueness of competitive choice in Keynesian case.) <u>Under differentiability, if there are no inferior goods for the capitalist households, and if their marginal propensities to spend never exceed one, then the competitive choice is unique. That is, the solution</u> $\{c, x, \omega\}$ <u>of equations</u> (III.15), (III.16) <u>and</u> (III.17) <u>is unique</u>.

<u>Proof.</u> It is convenient to work out the proof by establishing the uniqueness of solution $\{x, \omega\}$ of equations (III.34) and (III.35), which are obtained by eliminating c from (III.15), (III.16) and (III.17). The domain on which the equations are to be considered is naturally the set of all nonnegative vectors x and scalars ω.

To this end, first define a mapping $K:R_+^n \to R^n$ by

$$K(x) = x - (I - A)^{-1} G(P, \Pi_1 x_1, \Pi_2 x_2, \ldots, \Pi_n x_n) \quad , \tag{III.48}$$

then rearrange equation (III.35) as

$$K(x) = (I - A)^{-1} \{F(p, 1) + \omega d\} \quad . \tag{III.49}$$

The Jacobian matrix $J_K(x)$ of the mapping K can be similarly evaluated as in the preceding theorem, obtaining

$$J_K(x) = I - (I - A)^{-1} \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix} \tag{III.50}$$

where I is the identity matrix of the n th order. $J_K(x)$ is the identity matrix minus a nonnegative matrix at every x of R_+^n, since $(\partial G_i/\partial s_j)$ is nonnegative by virtue of the hypothesis of no inferior goods.

Next, in the light of (III.37) ensured by the hypothesis of no-greater-than-one marginal propensities to spend, one has

$$p' \left(\frac{\partial G_i}{\partial s_j}\right) \leqq (1, 1, \ldots, 1) \quad , \tag{III.51}$$

$$p' \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix} \leqq \Pi' \tag{III.52}$$

instead of (III.44) and (III.45) in the preceding theorem. Therefore, by taking the definition of σ in (II.4) and (II.11) as well as (III.52) into account, one can see

$$\begin{aligned}
(\ell + \Pi)' J_K(x) &= \ell' + \Pi' - (\ell + \Pi)'(I - A)^{-1} \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix} \\
&= \ell' + \Pi' - p' \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & 0 \\ & \Pi_2 & \\ 0 & & \ddots \; \Pi_n \end{pmatrix} \\
&\geqq \ell' + \Pi' - \Pi' \\
&= \ell' > 0' \quad ,
\end{aligned}$$

that is,

$$\begin{cases} (\ell + \Pi)' J_K(x) \geqq \ell' > 0 \\ \ell + \Pi > 0 \quad , \end{cases} \tag{III.53}$$

a counterpart of (III.47).

Since $J_K(x)$ has all its offdiagonal elements nonpositive, as was seen above, (III.53) implies by virtue of the Hawkins-Simon's result [7] that all the principal minors of $J_K(x)$ are positive on R_+^n, a rectangular region. Therefore, the mapping K is univalent on R_+^n again by the theorem, already referred to in the proof of Theorem 6, due to Gale and myself. Therefore, the mapping has an inverse mapping $K^{-1}:K(R_+^n) \rightarrow R_+^n$.

Moreover, this inverse mapping is monotonic increasing by virtue of another related theorem on a mapping with its Jacobian matrix having all offdiagonal elements nonpositive due to Gale and myself ([6], Theorem 5; Nikaido [13], Theorem 20.6, page 373). Here the monotonic increasingness means that

$$K^{-1}(y^1) \geq K^{-1}(y^2) \tag{III.54}$$

for y^1, $y^2 \varepsilon K(R^n_+)$, $y^1 \geq y^2$.

Now, suppose that there are two distinct solutions $\{x^1, \omega^1\}$ and $\{x^2, \omega^2\}$ of equations (III.34) and (III.35). They satisfy equation (III.49), so that

$$K(x^t) = (I - A)^{-1} \{F(p, 1) + \omega^t d\} \quad (t = 1, 2) \quad . \tag{III.55}$$

Then, ω^1 and ω^2 must be distinct. For otherwise, $x^1 = x^2$ by (III.55) and the univalence of K, contradicting $\{x^1, \omega^1\} \neq \{x^2, \omega^2\}$. Hence, without loss of generality, it may be assumed that

$$\omega^1 > \omega^2 \quad . \tag{III.56}$$

(III.56) implies

$$(I - A)^{-1} \{F(p, 1) + \omega^1 d\} \geq (I - A)^{-1} \{F(p, 1) + \omega^2 d\} \quad , \tag{III.57}$$

which entails by virtue of the monotonic increasingness of K^{-1}

$$x^1 = K^{-1}(y^1) \geq K^{-1}(y^2) = x^2 \tag{III.58}$$

where

$$y^t = (I - A)^{-1} \{F(p, 1) + \omega^t d\} \quad (t = 1, 2) \quad . \tag{III.59}$$

Then, (III.56), (III.58) and the hypothesis of no inferior goods imply, because of the positivity of σ,

$$\begin{aligned} &\sigma' \{\omega^1 d + G(p, \Pi_1 x^1_1, \Pi_2 x^1_2, \ldots, \Pi_n x^1_n)\} \\ &> \sigma' \{\omega^2 d + G(p, \Pi_1 x^2_1, \Pi_2 x^2_2, \ldots, \Pi_n x^2_n)\} \quad . \end{aligned} \tag{III.60}$$

But (III.60) obviously contradicts that $\{x^t, \omega^t\}$ $(t = 1, 2)$ satisfy equation (III.34). Q.E.D.

III.4 Objective demand functions

The construction of objective demand functions has already been worked out substantially in the three foregoing bisections in Section III. Thus, in order to construct these functions I have only to put the preceding results in a restated form.

The construction can be done in similar ways for both Say's and Keynesian cases formulated in [B.2] in III.1. Therefore, I will discuss only the Say's case, while I will merely make additional remarks on the Keynesian case.

First of all it should be recalled that the prices p_i and the profits per unit output Π_i are always related to each other by the basic equation (II.11). Moreover, only such prices have been and will be considered in this work that the profits per unit output Π_i are nonnegative. The relevant independent variables are Π_i, rather than the prices, which are functions of the former through equation (II.11). The price vectors fill out a cone P with σ as its vertex, which is in general smaller than the cone obtained by translating the nonnegative orthant by σ. Therefore the independent variables of the objective demand functions to be constructed are the profits per unit output Π_i, though the functions can be thought of as being defined on P.

Now, the behaviors of workers are embodied in the supply function of labor $L(p, 1)$ and the demand function for goods $F(p, 1)$ as set forth in (II.6) and (II.7). Likewise those of capitalists' households are embodied in the demand function for goods $G(p, s_1, s_2, \ldots, s_n)$ formulated in (III.1).

Given an arbitrary profit per unit output vector Π , which is naturally nonnegative, then the price vector p is determined by equation (II.11), and there are supply of labor $L(p, 1)$ and workers' demand for goods $F(p, 1)$. After $F(P, 1)$, which is to be paid out as wages, is deducted, a possibility set of final demand vectors $C(\Pi)$ and its efficient frontier $\Gamma(\Pi)$ are left open to the capitalist class. There exists at least one competitive choice of a final demand vector c from among $\Gamma(\Pi)$ and the corresponding gross output vector x by virtue of the results in III.2. Moreover, they are uniquely determinate by virtue of the results in III.3. Whence they can be thought of as single-valued functions $c(\Pi)$ and $x(\Pi)$ of the independent variable vector Π . As was suggested, they may also be regarded as functions of the price vector defined on P , and $c(\Pi)$ and $x(\Pi)$ may alternatively be denoted by $c(p)$ and $x(p)$, respectively, if desirable.

This $x(\Pi)$ is an objective gross demand function which has been sought for. The term gross means that the demand includes all derived demand $A\,x(\Pi)$. $x(\Pi)$ is consistent with the complete circular flow of national income by commodity breakdown, as should be for an objective demand function.

The consistency immediately follows from the way in which $x(\Pi)$ is constructed. In fact, there are sectoral profit incomes $\Pi_i\, x_i(\Pi)$ $(i = 1, 2, \ldots, n)$ at the price situation. These incomes induce effective demand for goods

$$G\,(p,\ \Pi_1\, x_1\,(\Pi),\ \Pi_2\, x_2\,(\Pi),\ \ldots,\ \Pi_n\, x_n\,(\Pi)) \quad , \tag{III.61}$$

which also equal by construction $c(\Pi)$. At the same time, there holds

$$x(\Pi) = (I - A)^{-1}\{F(p,\ 1) + G(p,\ \Pi_1\, x_1\,(\Pi),\ \Pi_2\, x_2\,(\Pi),\ \ldots,\ \Pi_n\, x_n\,(\Pi))\}\ , \tag{III.62}$$

which shows a complete circular flow of national income guaranteed when production is carried out to meet the gross demand for goods $x(\Pi)$.

There is a related consistency problem which was considered in I.3. In the above construction, the profit of the j th sector is $\Pi_j x_j(\Pi)$. Is this expression of profit equal to the total revenue less the total cost in the sector? The answer is in the affirmative and immediate. In fact, in view of the basic connection with the prices and the profits per unit output in equation (II.11), one has

$$
\begin{aligned}
& p_j x_j(\Pi) - \sum_{i=1}^{n} a_{ij} p_i x_j(\Pi) - \ell_j x_j(\Pi) \\
&= (p_j - \sum_{i=1}^{n} a_{ij} p_i - \ell_j) x_j(\Pi) \qquad \text{(III.63)} \\
&= \Pi_j x_j(\Pi) \qquad (j = 1, 2, \ldots, n) \quad ,
\end{aligned}
$$

where

$$
\begin{aligned}
p_j x_j(\Pi) &= \text{total revenue} \\
\sum_{i=1}^{n} a_{ij} p_i x_j(\Pi) &= \text{material cost} \\
\ell_j x_j(\Pi) &= \text{wages} \qquad (j = 1, 2, \ldots, n) \quad .
\end{aligned}
$$

Thus all the consistency criteria are met by $x(\Pi)$. It is noted that the equality of (III.61) to $c(\Pi)$, (III.62) and (III.63) hold true identically at any Π and the corresponding price vector p . Therefore, I should think that $x(\Pi)$ can be regarded as an objective demand function. Incidentally, as is clear, the corresponding objective <u>net</u> demand function is

$$F(P, 1) + G(P, \Pi_1 x_1(\Pi), \Pi_2 x_2(\Pi), \ldots, \Pi_n x_n(\Pi)) \quad . \qquad \text{(III.64)}$$

The economy is always in equilibrium at any Π and the corresponding price situation in the sense that the market is cleared for each good. On the other hand, the price formation is effected by capitalists' control through more or less monopolistically competitive market structures, as will be considered in the following Section IV.

Supplementary remarks on the Keynesian case are now in order. In this case, an investment composition vector d is given. d is a nonnegative vector having at least one component positive, and is either a constant or possibly a function of the price variables in general. Then, the essentially same reasoning as in the Say's case can apply to the Keynesian case, provided the scale of investment ω is regarded as a function $\omega(\Pi)$ of Π in addition to $c(\Pi)$ and $x(\Pi)$.

III.5 Objective demand functions -- an example

In order to visualize objective demand functions it is useful to have their specific shapes in an extremely simple, special situation. The construction of the functions is very easy. Nonetheless it is worthwhile noting that even in this simple situation their functional shapes are much different from those which the traditional oligopoly theorist has in mind.

The following example of objective demand functions will be given in the Leontief system of two goods and two sectors. Let the capitalist households' demand functions for goods in the Say's case be

$$G_i\ (p,\ s_1,\ s_2) = \frac{\beta_i(s_1 + s_2)}{p_i} \quad (i = 1,\ 2) \quad , \qquad \text{(III.65)}$$

where β_i's are constants such that

$$\beta_1 + \beta_2 = 1 \ , \qquad \beta_i > 0 \qquad (i = 1,\ 2) \quad .$$

It is well-known that (III.65) is derived from maximizing an additive logarithmic utility function subject to the budget constraint. It is well-known that (III.65) satisfies gross substitutability

$$\frac{\partial G_i}{\partial p_j} \geqq 0 \qquad (i \neq j) \quad .$$

On the other hand, (III.65) satisfies all the assumptions set forth in the preceding bisections, including the hypothesis of no inferior goods.

The uniquely determinate final demand vector $c(\Pi) = (c_i(\Pi))$ in the competitive choice is the solution of the system of linear equations

$$c_i(\Pi) = \frac{\beta_i \{p_1 c_1(\Pi) + p_2 c_2(\Pi)\}}{p_i} \qquad (i = 1,\ 2) \tag{III.66}$$

$$\sigma_1 \ c_1 \ (\Pi) + \sigma_2 \ c_2 \ (\Pi) = M \ (\Pi) \tag{III.67}$$

since the competitive choice must satisfy

$$\Pi_1 \ x_1 \ (\Pi) + \Pi_2 \ x_2 \ (\Pi) = p_1 \ c_1 \ (\Pi) + p_2 \ c_2 \ (\Pi) \ . \tag{III.68}$$

The two equations in (III.66) are essentially identical, so that the system of equations determining $c_i(\Pi)$ reduces to

$$\left\{ \begin{aligned} \beta_2 \ p_1 \ c_1 \ (\Pi) - \beta_1 \ p_2 \ c_2 \ (\Pi) &= 0 \\ \sigma_1 \ c_1 \ (\Pi) + \sigma_2 \ c_2 \ (\Pi) &= M \ (\Pi) \ . \end{aligned} \right. \tag{III.69}$$

The solution of (III.69) is

$$\left\{ \begin{aligned} c_1 \ (\Pi) &= \frac{\beta_1 \ M \ (\Pi)}{\beta_1 \ \sigma_1 + \beta_2 \ \sigma_2 \ (p_1/p_2)} \\ c_2 \ (\Pi) &= \frac{\beta_2 \ M \ (\Pi)}{\beta_1 \ \sigma_1 \ (p_2/p_1) + \beta_2 \ \sigma_2} \end{aligned} \right. \tag{III.70}$$

The total profit can be evaluated from (III.68) and (III.70) without evaluating $x_i(\Pi)$. They are

$$\Pi_1 \ x_1 + \Pi_2 \ x_2 = \frac{M\ (\Pi)}{\dfrac{\beta_1 \sigma_1}{p_1} + \dfrac{\beta_2 \sigma_2}{p_2}} \quad . \tag{III.71}$$

Let specific forms of the workers' demand functions be given by

$$F_i \ (p,\ 1) = \frac{\alpha_i \ L\ (P,\ 1)}{p_i} \qquad (i = 1,\ 2) \quad , \tag{III.72}$$

where α_i's are constants such that

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_i > 0 \qquad (i = 1,\ 2)$$

and $L\ (P,\ 1)$ is the supply function of labor.

Then, in view of the definition of surplus value (II.15), one has

$$M\ (\Pi) = \left\{ 1 - \left(\frac{\alpha_1 \sigma_1}{p_1} + \frac{\alpha_2 \sigma_2}{p_2} \right) \right\} L\ (p,\ 1) \quad . \tag{III.73}$$

From (III.70), (III.72) and (III.73) it follows that the objective net demand functions are

$$\begin{cases} F_1\ (p,\ 1) + c_1\ (\Pi) = \dfrac{L(p,1)\{\beta_1 p_2 + \sigma_2(\alpha_1 \beta_2 - \alpha_2 \beta_1)\}}{\beta_2 \sigma_2 p_1 + \beta_1 \sigma_1 p_2} \\[2ex] F_2\ (p,\ 1) + c_2\ (\Pi) = \dfrac{L(p,1)\{\beta_2 p_1 + \sigma_1(\alpha_2 \beta_1 - \alpha_1 \beta_2)\}}{\beta_2 \sigma_2 p_1 + \beta_1 \sigma_1 p_2} \end{cases} \tag{III.74}$$

As was noted, Π_1 and Π_2 are independent variables and p_1 and p_2 are linear functions of Π_1 and Π_2 by equation (II.11) in (III.74). The set of all price vectors is a cone P with the labor value vector σ as its vertex, and a price vector p in P corresponding to a positive Π is an interior point of P . In a small neighborhood of such an interior point the objective net demand functions (III.74) can be regarded as functions of independent variables p_1 and p_2 and need not and are most unlikely to satisfy gross substitutability provided $L(p,\ 1)$ is decreasing with respect to both variables p_1 and p_2 , notwithstanding the gross substitutability of $F(p, 1)$ and $G(p, s_1, s_2, \ldots, s_n)$ in p_1 and p_2. The same remark still applies to a more specific situation where $\alpha_i = \beta_i$ $(i = 1, 2)$. The objective gross demand functions, which are linear functions of the objective net demand functions are therefore most unlikely to satisfy gross substitutability. Moreover they need not be downward sloping even with respect to the price of the good in question. Accordingly, their functional behaviors diverge from what the traditional oligopoly theorist has in mind as to the shapes of demand functions.

III.6 Tâtonnement process for income formation

As was noted at the end of III.2, the determination of a competitive choice either by equation (III.33) or by equations (III.34) and (III.35) is the determination of the sectoral profit incomes $\Pi_i \; x_i$ $(i = 1, 2, \ldots, n)$ of a Keynesian type at the rigid fixed price system. In the situation formulated by these equations there are just x_i units of the gross demands for good i induced by the expected ex ante national income when the i th sector supplies x_i units of the gross output of good i in anticipation of the expected sectoral profit $\Pi_i \; x_i$ $(i = 1, 2, \ldots, n)$. If this mutual consistency of all the ex ante magnitudes is not instantaneously reached, but takes time, the income formation by these equations can be dynamized in the same way as in the usual multiplier process. For instance, the income formation by equation (III.33) in the Say's case can be dynamized to a dynamic process formulated by the system of differential equations

$$\frac{dx_i}{dt} = \lambda_i \; Q_i \; (x) \qquad (i = 1, 2, \ldots, n) \quad , \tag{III.75}$$

where

$Q_i\ (x)$ = the i th component of (III.76)

$$(I - A)^{-1}\{F(p,\ 1) + G(p,\ \Pi_1\ x_1,\ \Pi_2\ x_2,\ \ldots,\ \Pi_n\ x_n)\} - x$$
$$(i = 1,\ 2,\ \ldots,\ n)$$

λ_i = positive constant representing the speed of adjustment
$$(i = 1,\ 2,\ \ldots,\ n) \quad .$$

The system of differential equations generates a solution $x(t)$, starting at an arbitrary gross output vector $x(0) = x^o$, which depicts the variation of the gross output vector over time under a rigid price system. This dynamic process could be thought of as a multiplier process. However, here it might rather be regarded as a tâtonnement towards the situation brought about by the competitive choice, since the full employment relationship $\ell' x(t) = L(p,\ 1)$, or what amounts to the same thing, $\sigma' G (p,\ \Pi_1\ x_1\ (t),\ \ldots,\ \Pi_n\ x_n\ (t)) = M\ (\Pi)$, does not necessarily persist during the working out of the process over time.

The process (III.75) is automatically endowed with a much stronger stabilizing propensity under the basic assumptions premised in III.1 - 3. Explicitly,

Theorem 8. (Global stability in Say's case.) For a given fixed non-negative Π, the system of differential equations (III.75) generates a solution $x(t)$, starting at an arbitrarily given gross output vector $x\ (o) = x^o$. Any such solution converges to the unique gross output vector $\hat{x}$ in the competitive choice as the time t tends to infinity.

Proof. First of all, it is noted, without going into the detail, that the basic assumptions in III.1 - 3 enable the system (III.75) to satisfy certain regularity conditions such that it has a solution starting at an arbitrary x^{0} . It is also important in this connection that the solution can be continued indefinitely for all nonnegative time points, while retaining the nonnegativity of $x(t)$ over time. For $Q_i(x) \geqq 0$ by construction whenever $x_i = 0$.

Now define a mapping $Q:R_+^n \to R^n$ by the formula

$$Q(x) = (Q_i(x)) \tag{III.77}$$

on the basis of (III.76). Then, the Jacobian matrix $J_Q(x)$ of the mapping is given by

$$J_Q(x) = (I - A)^{-1} \left(\frac{\partial G_i}{\partial s_j}\right) \begin{pmatrix} \Pi_1 & & & \\ & \Pi_2 & & \\ & & \ddots & \\ 0 & & & \Pi_n \end{pmatrix} - I \tag{III.78}$$

where I is the identity matrix. $J_Q(x)$ is nothing but $J_K(x)$ in (III.50) multiplied by -1 . The result (III.53) was proved for the Keynesian case, and the proof was based on (III.52) which came from (III.37). Here one has (III.43), a sharpened version of (III.37), in Say's case. In the light of the above remark, one obtains

$$\begin{cases} (\ell + \Pi)' J_Q(x) = -\ell' < 0 \\ \ell + \Pi > 0 \quad , \end{cases} \tag{III.79}$$

a counterpart of (III.53). Since $-J_Q(x)$ has all its offdiagonal elements nonpositive, (III.79) implies that $J_Q(x)$ has all its diagonal elements negative and a dominant diagonal with respect to weighted column sums with constant positive weights $\ell_i + \Pi_i$ $(I = 1, 2, \ldots, n)$. Thus by virtue of a theorem[12] on global stability due to S. Karlin ([8], Chapter 9, Theorem 9.5.1), the value

$$v(x(t)) \tag{III.80}$$

of a Lyapunov function

$$V(x) = \sum_{i=1}^{n} \left(\frac{\Pi_i + \ell_i}{\lambda_i}\right) \left|\lambda_i Q_i(x)\right| \tag{III.81}$$

evaluated at $x = x(t)$ admits a right-hand side derivative with respect to time which is negative unless $x(t)$ satisfies

$$Q_i(x) = 0 \quad (i = 1, 2, \ldots, n) \quad . \tag{III.82}$$

Whence

$$\lim_{t \to +\infty} V(x(t)) = 0 \quad . \tag{III.83}$$

But, since there is just one unique solution of the system of equations (III.82) by virtue of Theorems 4 and 6 in III.2 and 3, formula (III.83) implies that $x(t)$ converges to the unique gross output vector $\hat{x}$ in the competitive choice because of the continuity of the Lyapunov function (III.81). This completes the proof.[13] Q.E.D.

Analogous results seem to be true in the Keynesian case, too, but have not yet been given sufficient justification. Main complication comes from the adjustment of ω, the scale of investment.

IV. Monopolistically Competitive Pricing Modes and the Objective Demand Functions

IV.1 Capitalists' behaviors as entrepreneurs

It is important to recall that the working of the economy, if its bare aspects are boldly viewed, heavily relies upon such a function of the market price mechanism as to regulate the supply of labor $L(p, 1)$ and the real wage bill by commodity breakdown $F(p, 1)$ necessary for the full employment thereof. Once the price system is determined in one way or another, the capitalist class can choose a final demand vector c from among the possibility set $C(\Pi)$, whether or not the choice is made through the market price mechanism. Therefore the market price mechanism is indispensable for the economy to achieve at least the allocation of labor and the distribution of the resulting output between the working and capitalist classes.

If the choice of a final demand vector is made by the capitalist class through the market price mechanism, there will be uniquely determinate objective demand functions for goods, as is the case when the capitalists' households take the prices as given parameters and behave like competitive demanders for goods. As was discussed in the foregoing two sections, the price system determines the surplus value $M(\Pi)$, and the competitive choice of a final demand vector c from among $C(\Pi) = \{c \mid c \geqq 0 , \sigma' c = M(\Pi)\}$ merely allocates the same surplus value to the n goods.

However, even if the capitalists' households behave as price takers, the capitalists as entrepreneurs do behave more or less as price setters. The price system, while regulating the supply of labor and the demand for goods, is determined by their price-setting behaviors and the interactions thereof subject to certain constraints, including the objective demand functions. Along the objective demand schedules the markets of all the goods are always cleared. The more or less monopolistically competitive market structure will single out a point on the objective demand schedules.

The basic general view in this section has already been accounted for in I.6. The n sectors in this economy are assumed to be single entrepreneurial decision-making units, respectively. Each of these sectors is confronted with the determinate objective demand function $x_j(\Pi)$ for its product. If the sector is not deceived by any wrong perception but can see the bare situation that confronts it, its well determinate profit is $\Pi_j \ x_j(\Pi)$, which is identically equal to the total revenue minus cost representation of the profit $p_j x_j(\Pi) - \sum_{i,j=1}^{n} a_{ij} p_i x_j(\Pi) - \ell_j x_j(\Pi)$, as was checked in III.4. Expressed in the terminology of game theory, here is a stage for an n-person game, with the profit $\Pi_j x_j(\Pi)$ as the payoff function of the j th player and the profit per unit output Π_j as his strategy. Much of the interdependence of the n sectors is embodied in this formulation. Thus a solution in game theory will narrow down the values of Π_j $(j = 1, 2, \ldots, n)$ to a set, possibly to a single n-tuple, depending on the solution concept. Expressed in the terminology of the traditional theories of monopolistic competition, here is an oligopolistic market. So a typical solution concept such as the oligopolistic market equilibrium of the Cournot type will be applied to it in order to single out specific values of Π_j's.

It should be noted, however, that here the game situation or the oligopolistic market is formulated in terms of the objective demand functions, which are not so nice-shaped as game theorists and oligopoly theorists usually expect in their theories. In fact, the objective gross demand function for the j th good in the example in III.5 need not be downward sloping. Moreover, the profit $\Pi_j x_j(\Pi)$ is not necessarily a concave function. Therefore a theory of monopolistic competition in terms of objective demand functions has to challenge much of the arbitrary hypothesis setting in the traditional theories.

There might be various possible modes of capitalists' pricing, depending on the monopolistically competitive market structure. An extremely polar case is a joint maximization of the surplus value such as was discussed in II.3. Maximization of the surplus value singles out specific values of Π_j's and hence p_j's, which bring about a situation of monopolistically competitive market equilibrium on the objective demand schedules. A maximum surplus value provides the capitalist class with the largest possibility set of final demand vectors $C(\Pi)$, from whose efficient frontier a competitive choice of final demand vector is made at the corresponding price system (see II and III). Therefore joint surplus value maximization should be a virtue for the capitalist class, as far as the provision of goods in real terms is concerned. In the economy which is much cursed with the price mechanism, however, even the capitalist class is most likely to be unaware of the evaluation in terms of the labor value that prevails beneath the interplay of prices. Capitalists are concerned with profits, rather than surplus value. Therefore another more likely joint optimization may be joint profit maximization, which will be discussed in the following bisection.

In a joint optimization capitalists form a special coalition, namely, that consisting of all of them. More generally, coalitions of some of them are conceivable, and a game theory may be developed in terms of the core concept based on the coalition formation by sectors. But one must face difficulties to be overcome in this line of theory-building. For there is much of pecuniary externalities in connection with a coalition unlike the well-known core story about allocations brought about by competitive equilibria. A coalition in the story can enjoy an autarky by taking advantage of an additive effect of the total resources of its participants, without being influenced by any externalities outside it, just because the total resources of the economy are decomposable. This is not the case with the present situation in question. For the sectoral profit $\Pi_j x_j(\Pi)$ depends not only on Π_j but also on the other Π_j's. If the first and second sectors form a coalition, the sum of their profits $\Pi_1 x_1(\Pi) + \Pi_2 x_2(\Pi)$ is at the mercy of the capitalists outside the coalition. The formation of a coalition by some, but not all, of sectors seems to be impossible. But this is only a rough impression and needs a further careful study, which is beyond the present scope of this work.

If sectors' pricing behaviors are based on perceived demand schedules, yet another type of monopolistically competitive equilibrium, which may be called the Cournot-Negishi solution, is conceivable. The Cournot-Negishi solution is a specific n-tuple of Π_j's such that at the corresponding points of the objective demand schedules the sectoral profits calculated in terms of the perceived demand schedules are maximized, respectively. The Cournot-Negishi solution will be discussed in IV.3.

IV.2 Joint profit maximization

In this monopolistically competitive economy in which everything is interdependent profit maximization need not be a virtue, as far as satisfaction in real terms is concerned. The capitalist class could be better off by making surplus value larger, rather than profits, as was pointed out before. Nonetheless capitalists almost always stick to profit maximization, either cooperatively or noncooperatively.

There is a possibility, however, that profits can be made indefinitely larger by charging higher profits per units of output and hence higher prices, while satisfaction resulting from the final demand vector chosen is getting lesser. Capitalists are likely to seek a nominally larger profit at the cost of having a smaller possibility set of final demand vectors $C(\Pi)$. The values of sectoral profits $\Pi_j x_j(\Pi)$ may vary, as Π_j's vary over all the nonnegative values. It ultimately hinges on the behavior of the supply function of labor $L(p, 1)$ whether the sectoral profits are bounded or not. For $x_j(\Pi)$'s must satisfy by construction the equation $\sum \ell_j x_j(\Pi) = L(p, 1)$, so that $x_j(\Pi)$'s, which are nonnegative, are bounded and eventually tend to zero with $L(p, 1)$ as any of Π_j's tends to infinity. Therefore the boundedness in question depends on how rapidly $x_j(\Pi)$'s approach zero in relation to the growing Π_j's.

The above remark suggests that maximum profits, sectoral or aggregate, can possibly be infinite under circumstances. It is recalled that a maximum surplus value exists under fairly weak conditions on $L(p, 1)$, as was seen in Theorem 2. But finite maximum profits may exist only when $L(p, 1)$ is required to satisfy a more stringent condition. One such additional condition may be given in terms of the elasticity of the supply function of labor.

[A.6] $L(p, 1)$ is differentiable and its elasticity satisfies

$$\frac{\sum_{j=1}^{n} \frac{\partial L}{\partial p_j} p_j}{L} \leqq -\gamma < -1 \quad \text{(identically)} . \tag{IV.1}$$

If $L = 0$, formula (IV.1) does not make sense. Therefore (IV.1) can be given in the rearranged form

$$\begin{cases} \sum_{j=1}^{n} \frac{\partial L}{\partial p_j} p_j \leqq -\gamma L & \text{(identically)} \\ \gamma > 1 . \end{cases} \tag{IV.2}$$

Now, if [A.6] is added to the set of assumptions [A.1] - [A.5], one has

<u>Theorem 9</u>. <u>If</u> p <u>and</u> Π <u>are related by equation (II.11) as before, then</u>

$$\lim_{\Pi_j \to +\infty} \Pi_j \, L(p, 1) = 0 \qquad (j = 1, 2, \ldots, n) \quad , \tag{IV.3}$$

irrespective of whatever behaviors the other Π_i's $(i \neq j)$ may have.

Proof. First, it will be shown that

$$L(t\,p, 1) \leqq t^{-\gamma} L(p, 1) \tag{IV.4}$$

for any scalar $t \geqq 1$. To this end, define

$$f\,(t) = L\,(t\,p, 1) \quad .$$

Then, upon differentiation and using (IV.2), one sees

$$f'(t) \leqq - \frac{\gamma\, f\,(t)}{t} \quad . \tag{IV.5}$$

First consider the case where $f(1) > 0$. Letting

$$g\,(t) = \frac{f\,(t)}{t^{-\gamma}\, f\,(1)} \tag{IV.6}$$

and using (IV.5), one obtains

$$g'(t) \leqq 0 \quad . \tag{IV.7}$$

(IV.7) implies that $g\,(t)$ is nonincreasing, so that

$$g\,(t) \leqq g(1) = 1 \qquad (t \geqq 1) \quad . \tag{IV.8}$$

In the light of the definitions of $f\,(t)$ and $g\,(t)$, inequality (IV.8) is nothing but (IV.4).

Now, (IV.5) implies that $f'(t)$ is nonpositive because $f(t) \geqq 0$. Hence $f\,(t)$ is nonincreasing. Thus, if $f\,(1) = 0$, then

$$f\,(t) \leqq f\,(1) = 0 = t^{-\gamma} f\,(1) \qquad (t \geqq 1) \quad . \tag{IV.9}$$

(IV.9) is again nothing but (IV.4).

Now, as Π varies in such a way that $\Pi_j \to +\infty$ for any fixed j , one sees for $\Pi_j \geqq 1$

$$\Pi_j \; L\,(p,\;1) = \Pi_j \; L\,(\Pi_j\;(p/\Pi_j),\;1) \qquad \text{(IV.10)}$$

$$\leqq \Pi_j^{\,1-\gamma} \; L\,(p/\Pi_j,\;1) \quad .$$

The right-hand side of (IV.10) tends to zero as Π_j tends to infinity, because of $\gamma > 1$ and the known boundedness of $L\,(p,\,1)$. This completes the proof. Q.E.D.

Theorem 9 immediately implies

<u>Theorem 10</u>. <u>If the input coefficients matrix</u> A <u>is explicitly assumed to be indecomposable, then</u>

$$\lim \Pi_j \; x_j \; (\Pi) = 0 \qquad (j = 1,\;2,\;\ldots,\;n)$$

<u>as at least any one</u> Π_k <u>of the</u> Π_j's <u>tends to infinity</u>.

<u>Proof</u>. It is noted that the objective gross demand functions $x_j\;(\Pi)$ $(j = 1,\;2,\;\ldots,\;n)$ satisfy

$$\sum_{j=1}^{n} \ell_j \; x_j \; (\Pi) = L\,(p,\;1) \qquad \text{(IV.11)}$$

identically for all Π . Then, by (IV.11) and the nonnegativity of $\ell_j \; x_j \; (\Pi)$, one obtains

$$\sum_{j=1}^{n} \Pi_j \; x_j \; (\Pi) = \sum_{j=1}^{n} (\Pi_j/\ell_j)\,\ell_j \; x_j \; (\Pi) \qquad \text{(IV.12)}$$

$$\leqq \sum_{j=1}^{n} (\Pi_j/\ell_j) \; L\,(p,\;1) \quad .$$

Now, $L\,(p,\;1)$ tends to zero as the behavior of Π stated above proceeds, because of the indecomposability of A and the basic assumptions on $L\,(p,\;1)$.[14] Hence the j th term on the right-hand side of (IV.12)

$$(\Pi_j/\ell_j) \; L\,(p,\;1)$$

tends to zero by the convergence of $L\,(p,\;1)$ to zero for bounded Π_j's , and by Theorem 9 for unbounded Π_j's. Therefore, the both sides of (IV.12) tend to zero. This proves the theorem, since $\Pi_j\ x_j\ (\Pi)$ are nonnegative.

Q. E. D.

Theorem 11. There exists a positive maximum aggregate profit subject to the objective demand functions.

Proof. By virtue of Theorem 10 the aggregate profit tends to zero, if any of Π_j's tends to infinity. On the other hand $x\ (0) \geq 0$ from (II.12) in assumption [A.5], so that $\Pi'\ x\ (\Pi) > 0$ for a small positive Π by continuity. Thus the existence of a positive maximum of $\Pi'\ x\ (\Pi)$ over all Π's can be proved in the same way as in the proof of Theorem 2.

Q. E. D.

If the profit of the j th sector is measured along the j th axis in the n-dimensional Euclidean space, the point

$$(\Pi_1\ x_1\ (\Pi),\ \Pi_2\ x_2\ (\Pi),\ \ldots,\ \Pi_n\ x_n\ (\Pi)) \tag{IV.13}$$

fills out a set, as the profit per unit output vector Π varies over all nonnegative vectors. The set may be termed the profit set, and its optimal frontier is defined to be the set of all points of the form (IV.13) at which the profit of any sector can not be increased without some other sector's profit being decreased.

The profit set is bounded, since it lies in the nonnegative orthant and the sum of the coordinates of any point in the set does not exceed the maximum aggregate profit. The optimal frontier is naturally bounded as a subset of the profit set. It is nonempty, because it contains special optimal points at which the aggregate profit is maximized.

The Π's whose corresponding points (IV.13) belong to the optimal frontier form a bounded set in the Π space. Otherwise, there is a sequence

$\{\Pi^{\nu}\}$ such that the point (IV.13) for $\Pi = \Pi^{\nu}$ belongs to the optimal frontier, while $\Pi_{j\nu}$ tends to infinity with ν for at least one j. Then, $\Pi_{j\nu} x_j (\Pi^{\nu})$ converges to zero for all $j = 1, 2, \ldots, n$ by Theorem 10. As was noted in the proof of Theorem 11, $x(0) \gtrsim 0$. Moreover, one has even $x(0) > 0$, provided the indecomposability of A is taken into explicit account.[15] Therefore $\Pi_j^o x_j (\Pi^o) > 0$ at the same time for $j = 1, 2, \ldots, n$ for a small positive Π^o. Consequently,

$$\Pi_{j\nu} x_j (\Pi^{\nu}) < \Pi_j^o x_j (\Pi^o) \qquad (j = 1, 2, \ldots, n) \tag{IV.14}$$

for sufficiently large ν. But (IV.14) contradicts the inclusion of the points (IV.13) in the optimal frontier for $\Pi = \Pi^{\nu}$ with large ν. The set of such Π's must therefore be bounded.

IV.3 The Cournot-Negishi solution

The importance of the Negishi solution in the general equilibrium theory of monopolistic competition has been discussed in I.1 and I.3. His theory is a noncooperative game theoretic analysis of entrepreneurial behaviors, and therefore of a Cournot type. In [4] Cournot originated the by now standard noncooperative oligopoly theory. In the market he considered each oligopolist tries to maximize his profit calculated in terms of the demand function for the product by solely controlling his output on the assumption that the outputs of the rivals are given. The Cournot solution is the determination of the output and price by the simultaneous realization of the noncooperative profit-maximizing behaviors of all the oligopolists. The demand function in Cournot's theory is allegedly an objective demand function, which entails no problem just because the theory is of a partial equilibrium nature.

Negishi [10] (and [11], Chapter 7) is rightly conscious of the subjective character of firms' perceived demand functions (more exactly, inverse demand functions). In his world of monopolistic competition each monopolist tries to maximize the profit calculated in terms of a perceived inverse demand function. The perception is not completely arbitrary, but depends more or less on the current state of the economy. The Negishi solution is a general equilibrium situation where both the noncooperative profit-maximizing behaviors of all firms and the market equilibrium in the sense of equality of demand and supply for all goods are simultaneously realized. It goes without saying that in the Negishi solution the firms' expected maximum profits evaluated in terms of the perceived inverse demand functions coincide with the actual realized profits.

Recall, however, Lange's characterization of a monopolistically competitive market in distinction from a perfectly competitive market, which is given in the passages quoted in I.3. The Negishi solution is not completely in accordance with the characterization, in that "disequilibrium consists in excess demand or excess supply" in the economy except at the solution. This is due to the lack of objective demand functions, which represent the current objective state of the markets always and even when there is disequilibrium in such a sense that the simultaneous realization of profit maximizations is not achieved.

Now that the well determinate objective demand functions have been constructed, it is possible to reconsider the Negishi solution in the presence of these functions in more concordance with the characterization of Lange.

Suppose that each sector has a perceived inverse demand function, by which it perceives an inverse demand schedule representing the price of the product of the sector as a function of its output, depending on the current state of the market. Explicitly, let

$$q_j\ (p,\ x,\ y_j) \qquad (j = 1,\ 2,\ \ldots,\ n) \tag{IV.15}$$

be the perceived inverse demand function of the j th sector, representing the expected price of the j th good, where

p = current price vector

x = current gross output vector

y_j = planned gross output in the j th sector .

The perception is so compatible with the current state of the market that (IV.15) satisfies

$$p_j = q_j\ (p,\ x,\ x_j) \quad , \tag{IV.16}$$

where p_j and x_j are the j th components of p and x , respectively. For simplicity's sake, the functions (IV.15) are assumed to be linear in y_j , that is

$$q_j\ (p,\ x,\ y_j) = p_j - \eta_j\ (p,\ x)(y_j - x_j) \qquad (j = 1,\ 2,\ \ldots,\ n) \quad , \tag{IV.17}$$

where $\eta_j\ (p,\ x)$ are functions defined for $p > 0$ and $x \geqq 0$. In general, they are downward sloping, so that

$$\eta_j\ (p,\ x) > 0 \qquad (j = 1,\ 2,\ \ldots,\ n) \quad . \tag{IV.18}$$

However, when perfect competition prevails, (IV.18) may be replaced by

$$\eta_j\ (p,\ x) = 0 \qquad (j = 1,\ 2,\ \ldots,\ n) \quad . \tag{IV.19}$$

Now, in the presence of the objective gross demand function $x(\Pi) = (x_j(\Pi))$, the current state of the economy may be represented by $(p, x(\Pi))$. This means that currently the gross output which is exactly equal to the current demand $x(\Pi)$ is supplied. Whence the markets are always currently cleared.

The perceived inverse demand functions corresponding to the actual current state of the economy takes the form

$$q_j(p, x(\Pi), y_j) \qquad (j = 1, 2, \ldots, n) \quad . \tag{IV.20}$$

Taking p and $x(\Pi)$ as given data, the j th sector maximizes its expected profit

$$\{(1 - a_{jj})\, q_j(p, x(\Pi), y_j) - \sum_{i \neq j} a_{ij}\, p_i - \ell_j\}\, y_j \tag{IV.21}$$

$$(j = 1, 2, \ldots, n)$$

by controlling the planned output y_j.

It is noted that (IV.21) equals the actual profit $\Pi_j\, x_j(\Pi)$ when y_j is set equal to $x_j(\Pi)$. Then, the Negishi solution is defined as such a situation that the j th sector's expected profit (IV.21) is a maximum over all nonnegative y_j at $y_j = x_j(\Pi)$. Thereby the Negishi solution singles out specific values of Π and the corresponding $x(\Pi)$ and p as a monopolistically competitive equilibrium.

<u>Theorem 12</u>. <u>If</u> $\eta_j(p, x)$ <u>are continuous and</u> $\eta_j(p, x(\Pi))$ <u>are bounded</u> $(j = 1, 2, \ldots, n)$, <u>there exists a Negishi solution</u>.

<u>Proof</u>. For each j the function (IV.21) is quadratic in y_j, since (IV.17) and (IV.18) are assumed. The coefficient of the quadratic term in it is negative. (IV.21) vanishes at $y_j = 0$. On the other hand, the expression in the parentheses in (IV.21) equals Π_j at $y_j = x_j(\Pi)$. If

$\Pi_j = x_j(\Pi) = 0$, then $y_j = 0$ is maximizing (IV.21) with the vanishing derivative of (IV.21). Otherwise the expression in the parentheses in (IV.21) vanishes at a positive value of y_j . Hence (IV.21) takes on an interior maximum at y_j = a half of the positive value with its vanishing derivative. In conclusion, for any given $\Pi \geqq 0$ (IV.21) always takes a maximum at some nonnegative value of y_j , with the vanishing derivative.

Differentiating (IV.21) with respect to y_j and setting the derivative equal to zero, one gets

$$(1-a_{jj})\, q_j\, (p,\ x(\Pi),\ y_j) - \sum_{i \neq j} a_{ij}\, p_i = \ell_j + (a_{jj} - 1)\, y_j \frac{\partial}{\partial y_j} q_j\, (p,\ x(p),\ y_j)$$

$$(j = 1,\ 2,\ \ldots,\ n) \quad . \tag{IV.22}$$

Therefore the Negishi solution is determined by the system of equations obtained by substituting $x_j(\Pi)$ for y_j in (IV.22) and taking (IV.16) into account. That is

$$p_j - \sum_{j=1}^{n} a_{ij}\, p_i = \ell_j + (a_{jj} - 1) x_j(\Pi) \frac{\partial}{\partial y_j} q_j\, (p,\ x(\Pi), x_j(\Pi))$$

$$(j = 1,\ 2,\ \ldots,\ n) \quad . \tag{IV.23}$$

But (IV.23) reduces by the definition of Π_j to

$$\Pi_j = (a_{jj} - 1)\, x_j\, (\Pi) \frac{\partial}{\partial y_j} q_j\, (p,\ x(\Pi),\ x_j(\Pi))$$

$$(j = 1,\ 2,\ \ldots,\ n) \quad , \tag{IV.24}$$

which, in the light of (IV.17), becomes

$$\Pi_j = (1 - a_{jj})\, \eta\, (p,\ x(\Pi))\, x_j\, (\Pi)$$

$$(j = 1,\ 2,\ \ldots,\ n) \quad . \tag{IV.25}$$

(IV.25) is the final form of the system of equations determining the Negishi solution, and the proof of existence of the solution will now be taken care of by the Brouwer fixed point theorem again. To this end define the mapping $\chi: R_+^n \to R_+^n$ by the formula

$$\chi(\Pi) = (\chi_j(\Pi))$$
$$\chi_j(\Pi) = \text{the right-hand side of (IV.25)} \quad . \qquad \text{(IV.26)}$$

where naturally p is determined by Π through equation (II.11).

$x_j(\Pi)$ is bounded, as was noted. $\eta(p, x(\Pi))$ is also bounded by assumption. Whence the image $\chi(R_+^n)$ of R_+^n under the mapping is bounded. Therefore one can enclose $\chi(R_+^n)$ in a sufficiently large compact convex subset Λ of R_+^n, be it a cube or a simplex. One can thereby obtain a continuous mapping $\chi: \Lambda \to \Lambda$. By virtue of the Brouwer fixed point theorem there is a fixed point $\hat{\Pi} = \chi(\hat{\Pi})$, and $\hat{\Pi}$ is a solution of (IV.25). This completes the proof. Q.E.D.

On the other hand, if perfect competition prevails so that (IV.19) holds instead of (IV.18), the expected profit can be maximized over all y_j only for $\Pi_j = 0$ and $p_j = \sigma_j$ $(j = 1, 2, \ldots, n)$. This special situation of the Negishi solution can also be characterized by equation (IV.25).

Finally, it is just noted that a dynamic process of monopolitically competitive price formation will be

$$\frac{d\Pi_j}{dt} = \xi_j\{(1 - a_{jj})\, \eta(p, x(\Pi))\, x_j(\Pi) - \Pi_j\}$$
$$\xi_j = \text{positive constant representing the speed of adjustment} \qquad \text{(IV.27)}$$
$$(j = 1, 2, \ldots, n).$$

This process (IV.27) is not a tâtonnement, but proceeds with the current market transactions subject to the objective demand functions. The process is set in motion by the expected profit-maximizing entrepreneurial behaviors. It can be reformulated in terms of a system of difference equations for discrete time. However, the stability property of the process (IV.27) has not yet been explored.

IV.4 Provisional epilogue

The present work is only a prelude to a theory of monopolistic competition in terms of objective demand functions, and should be followed by a further study in the line set forth here.

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FOOTNOTES

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2. Also see Nikaido ([13], page 90, Theorem 6.1; [14], page 14, Theorem 3.1).

3. See Dorfman, Samuelson and Solow ([5], Chapter 10, 10.2).

4. For two n-dimensional vectors $x = (x_i)$ and $y = (y_i)$, $x \geqq y$ means $x_i \geqq y_i$ $(i = 1, 2, \ldots, n)$, $x \geq y$ means $x \geqq y$ but $x \neq y$, and $x > y$ means $x_i > y_i$ $(i = 1, 2, \ldots, n)$.

5. For example, see Nikaido ([13], page 107, Theorem 7.4; [14], page 137, Theorem 20.2).

6. For example, see Nikaido ([13], page 95, Theorem 6.3; [14] page 14, Theorem 3.1 and page 113, Theorem 15.3).

7. θ need not be a constant, but can be a function of p in the reasoning below.

8. d need not be a constant vector, but can be a vector-valued function of p and the sectoral profits in the argument below.

9. This equation is a counterpart of equations studied by the author in connection with the determination and propagation of multisectoral incomes in Nikaido ([12]; [13], Chapter III, Section 11). The main difference is that no homogeneity hypothesis is assumed here as in the latter.

10. Reference to J. B. Say is made on account of the fact that supply creates demand for it in its aggregate value in this case.

11. It is meant by differentiability that each component function G_i $(p, s_1, s_2, \ldots, s_n)$ has a total differential

$$dG_i = \sum_{j=1}^{n} (\partial G_i/\partial s_j)\, ds_j$$

such that the difference between G_i $(p, s_1 + ds_1, s_2 + ds_2, \ldots, s_n + ds_n)$ and G_i $(p, s_1, s_2, \ldots, s_n) + dG_i$ is an infinitesimal magnitude of order higher than

$$\sqrt{\sum_{j=1}^{n} (ds_j)^2}\ .$$

See Gale and Nikaido [6] and Nikaido ([13], Chapter I, Section 5.3)

12. The author is indebted to H. Atsumi for letting his attention to this result of Karlin.

13. It is noted that the stability theorem of Karlin alone can not imply the convergence of $x(t)$ to a single gross output vector in a competitive choice. The stability theorem, in conjunction with the existence (Theorem 4) and uniqueness (Theorem 6) of competitive choice, can establish this theorem.

14. See (II.20).

15. $(I - A)^{-1} > 0$ by indecomposability (for example, see Nikaido ([13], page 107, Theorem 7.4; [14], page 137, Theorem 20.2)). Hence $x(0) = (I - A)^{-1}\, F(\sigma, 1) > 0$.

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