On the local structure of the set of steady-state solutions to the 2D Euler equations

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ABSTRACT

The main result of this thesis is based on the interpretation of Euler's flow of an incompressible fluid as a geodesic flow on the infinite-dimensional Lie group of volume-preserving diffeomorphisms of the region occupied by the fluid equipped with a one-sided invariant metric. In finite dimensions, the dynamics on the cotangent bundle of a Lie group equipped with a one-sided invariant metric can be reduced to a family of Hamiltonian systems on the co-adjoint orbits in the dual Lie algebra. Thus, non-degenerate stationary points are in a (local) one-to-one correspondence with the co-adjoint orbits. We prove that this holds for the most part for two-dimensional Euler's equations of hydrodynamics. Here, the co-adjoint orbits are the sets of isovorticed flows, i.e. sets of vorticity functions obtained by composition with volume-preserving diffeomorphisms, and these are invariant under the vorticity equation. (The latter statement is equivalent to Kelvin-Helmholtz' theorem on conservation of vorticity.) This result is valid for annulus domains in two dimensions, in the category of smooth functions, and in a neighborhood of fairly general steady-states. The co-adjoint orbits are not smooth manifolds if one works in the usual Banach spaces and therefore the proof is based on an application of the Nash-Moser inverse function theorem.

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Dedication

à Teresa,

à ma famille,

à Stéphane.

Contents

Chapter 1

Introduction

Ideal (inviscid) incompressible fluids display many of the features of fluid flows [7], and this in spite of their idealized nature: the volume of any region of the fluid is preserved along its motion, and the only force acting on the fluid is exerted via the pressure, enforcing the incompressibility constraint. That is, the fluid experiences no friction.

In the mathematical study of fluids in general, one has the choice between two descriptions of the flow. Either one fixes a location in space x and observes the evolution of the velocity field $u(x, t)$ of fluid particles going through the said location over time t; or one follows the evolution of the position $\eta(x, t)$ over time t of a fluid particle initially at x. The former is called the Eulerian description of the flow and the latter its Lagrangian description. In the Eulerian description, the velocity field is governed by the Euler equations, written in the usual coordinates as

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \text{div } u = 0. \tag{1.1}$$

Therefore, many problems concerning Euler's equations have been solved using standard techniques of PDE [4], [15], [16], [17], [20], [21], [33], but many questions remain unanswered, and not the least one being whether blow up in finite time is possible for three-dimensional flows.

On the other hand, the Lagrangian description of a fluid in the particular case when it is perfect incompressible offers a very nice geometric interpretation. The flows generated by the solutions $u(x, t)$ represent geodesics in the manifold of volume-preserving diffeomorphisms, with the metric given by the L^2-product in the tangent spaces (see [2], [3],

[27] and references therein). This viewpoint has seen tremendous interest particularly after the work of V. I. Arnold in the 1960s. The application of results from differential geometry to this infinite-dimensional setting (sometimes somewhat formally) has led to new insights [11], [28]. For example, the Arnold stability criterion for steady-state solutions, which is a counterpart to Euler's theorem in classical mechanics on the stability of the rotation about the longest and shortest axes of the inertia ellipsoid, remains one of the strongest results on non-linear stability of steady-state solutions. Of course, many analytical (and functional analytical) difficulties must be overcome, and much has been established rigorously [1], [3], [9], [10], [26], [29], [30]. Yet again, a number of important problems remain open. For example, the structure of the set of steady-state solutions of the equations is not really well understood, even in two dimensions. Of related interest is the result of Arnold for the unit disk that minimizers of the kinetic energy over certain orbits are radially symmetric with a profile that can be determined by the orbit (see [3]). Also noteworthy are the results [5] where Burton shows existence of uncountably many steady-states among vorticities which are rearrangements of one another.

It seems that the geometric point of view can be very useful to gain a deeper understanding of the set of steady-state solutions. Indeed, invariance of the kinetic energy under relabeling of fluid particles lends the space of vorticities ($\omega = \operatorname{curl} u$) a natural foliation whose leaves carry each a symplectic structure, a consequence of symplectic reduction [25], [24]. The symplectic leaves consist of the sets of isovorticed flows, namely the sets of vorticities which differ only by a volume-preserving diffeomorphism,

$$\mathcal{O}_{\omega_0} = \{\omega_0 \circ \eta \ : \ \eta \in \mathcal{D}_{\mathrm{vol}}\}. \tag{1.2}$$

These symplectic leaves also coincide with the co-adjoint orbits of the space of vorticities. The space of vorticities is an example of Poisson manifolds, the structure of which is fairly well understood in finite-dimensions [32]. The Kelvin-Helmholtz theorem in two dimensions states that the vorticity $\omega(x,t)$ is conserved along particle trajectories $\eta(x,t)$,

$$\omega(\eta(x,t),t) = \omega(x,0). \tag{1.3}$$

In the geometric interpretation of ideal incompressible flows, this is simply a restatement of the invariance of the co-adjoint orbits under the Euler flow. The Euler equations describe a Hamiltonian system on each co-adjoint orbit. It is thus natural to conjecture that, under some non-degeneracy assumptions, the steady-state solutions are (locally) in one-to-one correspondence with the co-adjoint orbits.

In a finite-dimensional setting a corresponding statement is easy to prove by the usual implicit function theorem. Suppose that a family of vector fields $v(\cdot, y)\colon \mathbb{R}^n \to \mathbb{R}^n$ is parametrized by $y \in \mathbb{R}^m$. The classical implicit function theorem says that if $v(x_0, y_0) = 0$ and the differential $D_x v(x_0, y_0)$ is not singular, then for y close to y_0 there exists a unique solution $x(y)$ of $v(x, y) = 0$ which is close to x_0.

The geometric interpretation of Euler's equations suggests that a similar statement should be true if we interpret $v(x, y) = 0$ as the Euler steady-state equation, y as a "parameter" for the orbits and x as a suitable "coordinate" on the orbits. In other words, the steady-state solutions should be (in some suitable local setting) in one-to-one correspondence with the orbits. This statement can be verified for special classes of solutions where everything can be computed explicitly, such as simple shear flows or simple flows with rotational symmetry [3]. However, little was known beyond these simple examples.

The main result of this thesis (Theorem 1, Section 1.3) is a rigorous proof that, in a neighborhood of non-degenerate steady flows, each orbit contains a steady flow. This is done under reasonable non-degeneracy assumptions but otherwise in fairly general regimes. Numerous analytical difficulties must be overcome in order to establich this analogy. The first difficulty is that the orbits (1.2) are most likely not submanifolds in the usual Banach-spaces of functions encountered in the theory of PDEs, but they are probably Fréchet-manifolds in the space of smooth functions [14], [19]. This difficulty is overcome by working with an indirect characterization of the orbits in terms of the distribution function $A_\omega(\lambda) = |\{\omega < \lambda\}|$ of the vorticity. The proof is based on the Nash-Moser inverse function theorem [14]. The use of the Nash-Moser theorem seems to be a natural option here, because of the difficulty with the function spaces mentioned above, but also because linearized operators relevant for the problem suffer from loss of derivatives.

The next Sections summarize the content of this thesis.

1.1 Euler's equations and steady flows in 2D

(Chapter 2 contains a more detailed exposition of the material for this section.)

The velocity field u of an ideal incompressible fluid is governed by Euler's equations subject to the boundary condition that the flow be tangent to the boundary of a domain Ω filled by the fluid. In the usual coordinates, we have

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \nabla p &= 0 \qquad \text{in} \quad \Omega \\ \operatorname{div}\, u &= 0 \qquad \text{in} \quad \Omega \\ \langle u, N\rangle &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{1.4}$$

where N denotes the unit outer normal to $\partial\Omega$ and $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. (We will always assume that Ω is bounded with smooth boundary.) In two dimensions, the vorticity $\omega = \operatorname{curl}\, u = u_x^2 - u_y^1$ of the velocity field $u = (u^1, u^2)$ satisfies the transport equation

$$\partial_t\omega + u\cdot\nabla\omega = 0 \tag{1.5}$$

which is equivalent to Euler's equations (1.4) since u is completely determined by ω under suitable boundary conditions. In principle, several choices for these boundary conditions are possible, but Section 3.2 will show that the natural one for the problem at hand is

$$\int_{\Gamma_i}\langle u, dl\rangle = \gamma_i, \qquad i = 1, 2, \dots \tag{1.6}$$

where Γ_i, $i = 1, 2, \dots$ are the inner boundary components (assumed to be in finite number, which we do not specify; Ω need not be simply connected), and the constants γ_i are fixed. Therefore, (1.5) is an equation in ω only. Integrating along particle trajectories, $\eta(x,t)$, the famous conservation law of vorticity (Kelvin-Helmholtz) is immediate:

$$\omega(\eta(x,t),t) = \omega(x,0), \qquad \text{or} \qquad \omega_t = \omega_0\circ\eta_t^{-1} \tag{1.7}$$

where $\omega_t = \omega(\cdot, t)$. In conclusion, (1.5) is a dynamical system on the orbit $\mathcal{O}_{\omega_0}$ defined in (1.2).

Alternatively, the velocity field u of an incompressible fluid is conveniently represented by its stream function ψ according to

$$u = \nabla^{\perp}\psi = \begin{bmatrix} -\psi_y \\ \psi_x \end{bmatrix}, \qquad \psi_{|\Gamma_0} = 0 \tag{1.8}$$

where Γ_0 is the outer component of $\partial\Omega$. In particular, the particle trajectories solve a Hamiltonian system with (time-dependent) Hamiltonian $-\psi$. In terms of the stream function, the vorticity equation (1.5) is written as

$$\partial_t \omega = -\{\psi, \omega\} \tag{1.9}$$

where $\{f, g\} := f_x g_y - f_y g_x = \langle \nabla^{\perp} f, \nabla g\rangle = \det(\nabla f, \nabla g)$ is called the **Poisson bracket** of f and g on Ω. The stream function is completely determined by the following elliptic system

$$\Delta\psi = \omega, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \dots \tag{1.10}$$

where the γ_i's are fixed constants (here $\frac{\partial}{\partial N}$ is the outer normal derivative on the boundary). This system and related elliptic equations are studied in detail in Chapter 2.

The steady-state equation in 2D reduces then to

$$\{\psi, \omega\} = 0. \tag{1.11}$$

The particle trajectories of a steady flow lie on the level sets of ψ and hence are periodic, except at critical points of ψ. (See [17] for a discussion on instability of steady flows.) If the domain Ω is simply connected (i.e. diffeomorphic to a disk) and ψ has a single critical point, then the trajectories are concentric curves or reduce to the fixed point at the critical point of ψ. If the domain is doubly connected (i.e. diffeomorphic to an annulus) and ψ has no critical points, then the trajectories are concentric curves (ψ is locally constant on the boundary as it is a stream function).

The steady-state equation (1.11) offers the following (local) characterization of steady flows. Since $\{\psi, \omega\} = \det(\nabla\psi, \nabla\omega) = 0$, ψ and ω share the same level sets and, locally, away from critical points of ψ and ω, there exists a function F such that

$$\omega = F(\psi). \tag{1.12}$$

Certain restrictions must be imposed in order for F to be global and to be used as a "local coordinate" for steady flows. If Ω is diffeomorphic to a disk and ψ has a single critical point, or if Ω is diffeomorphic to an annulus and ψ has no critical points, then F is global. Therefore the function F parametrizes a family of 2D Euler steady-states via

$$\Delta\psi = F(\psi), \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots. \tag{1.13}$$

Thus, we have arrived at a rather simple (local) description of the steady-states. However, to gain a deeper understanding of the structure of the Euler flow, the viewpoint that the flows generated by the solutions to Euler's equations are geodesics on the manifold of volume-preserving diffeomorphisms equipped with the L^2-metric proves extremely useful. The next section, with a short review of Lagrangian mechanics, will usher into the Lie theoretic interpretation of Euler's equations.

1.2 The Euler flow as a family of Hamiltonian systems

(Chapter 3 contains a more detailed exposition of the material of this section.)

The motion of a mechanical system with configuration space M ($\dim M = n$) is often described by a **Lagrangian** $L\colon TM \to \mathbb{R}$ and the 'principle of least action', which says that the actual motion corresponds to a critical point of the action integral $\int_{t_1}^{t_2} L(q, \dot{q})dt$. In coordinates, this is equivalent to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} = 0, \qquad j = 1, 2, \ldots, n. \tag{1.14}$$

This describes a second order differential equation on the configuration space M, or equivalently a first order differential equation on the **velocity phase space** (or tangent bundle) TM. Alternatively, the motion can be described as a **Hamiltonian system** on the **phase space** T^*M, which carries a natural symplectic structure. The Hamiltonian, in canonical coordinates, is given by (summing over repeated indices)

$$H(q, p) = p_j \dot{q}^j - L(q, \dot{q}) \tag{1.15}$$

where $(q,p) \in T^*M$ is associated to $(q,\dot{q}) \in TM$ via the **Legendre transform**

$$p_j = \frac{\partial L}{\partial \dot{q}^j}, \qquad j = 1, 2, \ldots, n. \tag{1.16}$$

Indeed, the Euler-Lagrange equations (1.14) are equivalent to the Hamiltonian equations

$$\dot{q}^j = \frac{\partial H}{\partial p^j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad j = 1, 2, \ldots, n. \tag{1.17}$$

There is yet another equivalent formulation of this Hamiltonian system. Define the **Poisson bracket** of two functions F, G on T^*M by

$$\{F, G\} = \sum_{j=1}^{n} (F_{q^j} G_{p_j} - F_{p_j} G_{q^j}). \tag{1.18}$$

Then, (1.17) is equivalent to

$$\dot{F} = \{F, H\}, \qquad F \in C^\infty(T^*M). \tag{1.19}$$

Suppose now that a Lie group G acts on M. This induces an action on T^*M and a natural map

$$\pi \colon T^*M \to T^*M/G \tag{1.20}$$

onto the space of orbits of the action. Provided this action satisfies reasonable properties, the quotient space is also a manifold and the identity

$$\{f, g\}^\sim \circ \pi = \{f \circ \pi, g \circ \pi\}, \qquad f, g \in C^\infty(T^*M/G) \tag{1.21}$$

defines a Poisson bracket on the quotient space ($\{\cdot, \cdot\}$ satisfies Jacobi's identity).

The quotient space is no longer a symplectic manifold in general, but the bracket gives it the more general structure of a **Poisson manifold**, which can be described as the disjoint union of immersed submanifolds each carrying a natural symplectic structure. (These immersed submanifolds are in fact embedded into the ambient manifold if, for example, G is compact, which we will assume for purpose of illustration.) These submanifolds are thus called the **symplectic leaves** of the ambient Poisson manifold. The symplectic structure on each leaf can be computed by restricting the Poisson bracket on the ambient manifold to the leaf.

As seen for the case of T^*M, the Poisson bracket on a symplectic manifold gives an alternative characterization of Hamiltonian vector fields. More generally, (1.19) defines a **Hamiltonian system** on a Poisson manifold via its Poisson bracket and such a Hamiltonian system is in fact a collection of Hamiltonian systems on the symplectic leaves of the Poisson manifold. In particular, this description holds for T^*M/G.

If now the Hamiltonian H on T^*M is invariant under the action of G on M, then there exists $h \in C^\infty(T^*M/G)$ such that

$$H = h \circ \pi \tag{1.22}$$

and the system (1.19) reduces to

$$\dot{f} = \{f, h\}^\sim, \qquad f \in C^\infty(T^*M/G). \tag{1.23}$$

This is, essentially, the content of **symplectic reduction** [24]. In other words, the symmetries of H allow to view the original dynamical system on the symplectic manifold T^*M as a Hamiltonian system on the Poisson manifold T^*M/G, and in turn as a family of Hamiltonian systems on the symplectic leaves of T^*M/G. Therefore, given a reference steady-state on a certain leaf, it is reasonable to ask whether the neighboring leaves also have a steady-state (at least in a local setting).

This local one-to-one correspondence between stationary points and leaves of the foliation is easy to establish at a non-degenerate point of the foliation. The proof is a standard application of the implicit function theorem, as we now show. Denote by U a vector field tangent to a foliation, and choose local coordinates (x, y) trivializing the foliation: x is a coordinate on the leaves and y parametrizes the leaves. In these coordinates the vector field U splits as

$$U = (U^{\parallel}, U^{\nparallel}) \tag{1.24}$$

where $U^{\parallel}$ is the component parallel to the leaves, and $U^{\nparallel}$ is transverse to the leaves. The condition that U is tangent to the leaves is simply that $U^{\nparallel}(x, y) = 0$. The stationary points are thus the solutions to

$$U^{\parallel}(x, y) = 0. \tag{1.25}$$

Here, y plays the rôle of a parameter labeling the leaves.

Fix now $(\overline{x}, \overline{y})$ a reference stationary point. Then, if the non-degeneracy condition

$$\det \nabla_x U^{\parallel}(\overline{x}, \overline{y}) \neq 0 \tag{1.26}$$

is satisfied, the classical implicit function theorem allows to solve (1.25) (locally) for x as a function of y. In other terms, each nearby leaf contains a stationary point. If furthermore $\nabla_y U^{\|}(\overline{x}, \overline{y})$ is a square matrix and non-singular, then we have a (local) one-to-one correspondence between the stationary points and the leaves of the foliation.

Finally, suppose that the configuration space M is a Lie group G and it acts on itself by left-multiplication. Then, the quotient space can be identified with the dual $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$,

$$T^*G/G \approx \mathfrak{g}^*; \tag{1.27}$$

the symplectic leaves are the co-adjoint orbits

$$\mathcal{O}_\nu := \{\mathrm{Ad}_g \nu \ : \ g \in G\} \tag{1.28}$$

induced by the action of G on itself by conjugation; and the bracket is given by

$$\{f, g\}^\sim(\mu) = (\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu, \frac{\delta f}{\delta \mu}), \qquad \mu \in \mathfrak{g}^*, \qquad f, g \in C^\infty(\mathfrak{g}^*). \tag{1.29}$$

and is called the **Lie-Poisson bracket**. Here $\frac{\delta f}{\delta \mu}$ denotes the element in $\mathfrak{g}$ identified with the derivative of f in the double dual. The reduced system on $\mathfrak{g}^*$ is given by

$$\dot{\mu} = \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu \tag{1.30}$$

and is called the Euler (or Euler-Poincaré) equation.

1.3 Geometric correspondence for 2D Euler and main theorem

(Chapter 3 contains a more detailed exposition of the background material of this section. The proof of the main Theorem is carried out in Chapter 5, while some preliminary results are established in Chapter 4.)

It is natural to wonder whether the previous local correspondence holds in an infinite-dimensional setting. The main result of this thesis is a step toward establishing an analogue for two-dimensional Euler equations under reasonable assumptions. In this case, the configuration space is the infinite-dimensional Lie group of volume-preserving diffeomorphisms of the domain Ω and the action is compostion from the right. That this is a right-action (while the finite-dimensional analogue presented in Section 1.2 assumed a left-action) will have the effect of introducing a negative sign in the analogue of the Euler-Poincaré equation (1.30) (see (1.34)). This geometric interpretation is somewhat formal, as numerous functional analytic difficulties arise in the infinite-dimensional case. For instance, certain classical theorems for finite-dimensional Lie groups do not have analogues for Banach-Lie groups, although the exponential map is a local diffeomorphism [1]. Furthermore, groups of diffeomorphisms with finite regularity are not even Banach-Lie groups since multiplication in these groups (i.e. composition of diffeomorphisms) is a continuous operation, but it is not differentiable, and in fact the exponential map is not surjective [10], [14]. This suggests that one work in the category of smooth diffeomorphisms - and thus one steps out of the category of Banach-manifolds.

The work of Arnold in particular has made it widely known that Euler's equations of fluid dynamics follow the geometric framework sketched in Section 1.2 (see Section (3.2) for details.) Regard an ideal incompressible fluid filling a domain Ω as a mechanical system with configuration space

$$\mathcal{D}_{\text{vol}} := \{\eta \colon \overline{\Omega} \to \overline{\Omega} \mid \det \operatorname{Jac}(\eta) \equiv 1\}. \tag{1.31}$$

The Lagrangian for this system is given by the kinetic energy and is invariant under right-action by volume-preserving diffeomorphisms:

$$L(\eta, \dot\eta) := \int_\Omega \frac{1}{2}|\dot\eta|^2, \qquad L(\eta \circ \zeta, \dot\eta \circ \zeta) = L(\eta, \dot\eta), \qquad \zeta \in \mathcal{D}_{\text{vol}}. \tag{1.32}$$

In two dimensions, the dual Lie algebra can essentially be identified with the space of vorticity functions, [1] the Hamiltonian is given by the kinetic energy

$$\mathcal{E}(\omega) = \frac{1}{2}\int_\Omega |u|^2 \tag{1.33}$$

[1] See Section 3.2 for a more careful account. Rather than the dual, we will work with the regular dual, which is isomorphic to the Lie algebra. Its elements are actually of the form $(\omega; \gamma_i, i = 1, 2, \dots)$ where ω is a vorticity function and the γ_i's are constants. Ultimately we will work in an annulus domain with a single inner boundary component, and the single constant γ_i will be fixed.

where the velocity field u corresponds to the vorticity ω under boundary conditions (1.6), and the Euler-Poincaré equation is none other than the vorticity equation

$$\dot{\omega} = -\{\psi, \omega\} \tag{1.34}$$

where $\{\psi, \omega\} := \psi_x \omega_y - \psi_x \omega_y$ is the Poisson bracket for functions on Ω. The co-adjoint orbits are the sets

$$\mathcal{O}_\omega = \{\omega \circ \eta \ : \ \eta \in \mathcal{D}_{\text{vol}}\}. \tag{1.35}$$

The steady flows correspond to the conditional critical points of the energy $\mathcal{E}(\omega)$, i.e. the critical points of the energy restricted to each orbit $\mathcal{O}_\omega$.

The main result of this thesis is that some parts of the finite-dimensional picture are valid for 2D Euler. A naive approach would be to mimic the finite-dimensional situation and use an implicit function theorem. However, one immediately runs into difficulties if one works in the usual Banach-spaces of PDE (as already hinted at the beginning of this Section). It seems that the orbits are not Banach-manifolds. To illustrate this phenomenon, consider the space $C^0_{(\mathbb{S}^1,\mathbb{R})}$ of continuous functions on the circle. Clearly, composition with rotations is not a differentiable operator on $C^0_{(\mathbb{S}^1,\mathbb{R})}$ since $\frac{\partial}{\partial\epsilon}_{|\epsilon=0} f(x-\epsilon) = -f'(x)$. At a more fundamental level, it is known [8], [10] that composition of functions is continuous but not differentiable in the usual Banach-spaces. For finite dimensional Lie groups, this is satisfied by definition, and the smooth structure on the orbits is obtained using this smoothness. See [24].

One possible remedy may be to work in the C^∞-category and show that the orbits are smooth tame Fréchet-submanifolds, see Appendix or [14] for terminology. One may speculate that this is the case, but this thesis was not engaged in that direction. Instead, the proof consists in finding appropriate "coordinates" for the steady-states and the orbits, and show a correspondence between these coordinates, as is explained now. We have seen earlier that a large class of Euler steady-states consists of those parametrized by functions F of one variable according to

$$\Delta\psi = F(\psi) \tag{1.36}$$

under the boundary conditions given in (1.13). As for the orbits $\mathcal{O}_\omega$, it turns out that a good "coordinate" is the distribution function

$$A_\omega(\lambda) := \{x \ : \ \omega(x) < \lambda\} \tag{1.37}$$

since, for $\eta \in \mathcal{D}_{\text{vol}}$, we have

$$A_{\omega\circ\eta} = A_\omega. \tag{1.38}$$

Further evidence that A_ω is a suitable parametrization for the orbits is the following partial converse. Let ω_ϵ be a 1-parameter family of steady-states with same distribution function:

$$A_{\omega_\epsilon} = A_\omega, \qquad \forall\ \epsilon. \tag{1.39}$$

Then (see Lemma 26, Section 4.4, for the precise assumptions) $\nu = \frac{d\omega_\epsilon}{d\epsilon}_{|\epsilon=0}$ is tangent to the orbit $\mathcal{O}_\omega$.

Finally, for entirely technical reasons, the correspondence will be established between the functions F and the inverse A_ω^{-1} of the distribution function A_ω (the range of ω may vary with ω).

We now state the main theorem of this thesis. A discussion of the precise statement, its hypotheses, and the proof follows.

Theorem 1 *Let Ω be a domain diffeomorphic to an annulus with single inner boundary component Γ_i and denote N the unit outer normal to $\partial\Omega$. Subject the stream functions to the boundary conditions $\psi_{|\Gamma_o} = 0$ and $\int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i$ where γ_i is a fixed constant. Let a steady-state solution to Euler's equation have stream function $\overline{\psi} \in C^\infty(\overline{\Omega})$ and vorticity function $\overline{\omega}$ without critical points, and let $\overline{F}$ be such that*

$$\Delta\overline{\psi} = \overline{F}(\overline{\psi}). \tag{1.40}$$

If $\overline{F}' > 0$, then there is a locally well-defined solution operator $\psi = S(F)$ to $\Delta\psi = F(\psi)$, and $\psi = S(F)$ is smooth tame. Furthermore, the following non-degeneracy condition automatically holds for F in a neighborhood of $\overline{F}$:

(ND) *There is no nontrivial solution to the linearization to $\Delta\psi = F(\psi)$ at ψ,*

$$\nu = \Delta\phi = F'(\psi)\phi + f(\psi) \tag{1.41}$$

which is also tangent to the orbit $\mathcal{O}_\omega$.

In turn, the map $T(F) = A_\omega^{-1}$ (where $\omega = \Delta\psi$) is locally surjective with a smooth tame right-inverse.

Remark 2

- **The non-degeneracy condition (ND)** is the analogue of the usual non-degeneracy condition for the (classical) implicit function theorem (e.g. (1.26)). It expresses the fact that the set of steady-states intersects the orbits transversally. As the theorem is stated in spaces of C^∞ functions, which are not Banach-spaces but Fréchet-spaces, the Nash-Moser inverse function theorem will be used. An important difference between this theorem and the classical version for Banach-spaces is that the usual non-degeneracy condition to be satisfied must hold not only at the reference point $\overline{F}$, but in an entire neighborhood. The property that $F' > 0$ is an open condition: if it holds for $\overline{F}$, then it holds for F in a (C^1-)neighborhood. The condition that $F' > 0$ implies two things: (1) local uniqueness of the solution to $\Delta\psi = F(\psi)$ (by the maximum principle), and (2) that condition (ND) holds automatically (see Proposition 19, Section 4.2).

  It can in fact be shown that condition (ND) is open for functions such that $F' < 0$: if condition (ND) holds for $\overline{F}$ such that $\overline{F}' < 0$, then it holds for F in a neighborhood of $\overline{F}$. However, this will not be addressed in this thesis, and in the proof of Theorem 1 we will assume that $\overline{F}' > 0$ and restrict to neighborhoods where
  $$F' > 0. \tag{1.42}$$

- **The condition that $\overline{\psi}$ and $\overline{\omega}$ have no critical points** can be interpreted as ensuring the orbits $\mathcal{O}_\omega$ with sufficient smoothness so that the heuristics suggested by the finite-dimensional situation carries over to 2D Euler.

  For a smooth ω (up to the boundary), $A'_\omega(\lambda)$ is discontinuous at absolute and local minima and maxima. Indeed, when λ increases until it reaches, say, a minimum (local or absolute), the area around such minimum makes a sudden contribution to A_ω. If ω has exactly one absolute maximum and one absolute minimum (and no other local extrema), then $A'_\omega(\lambda)$ has discontinuities at these two extrema only, and A_ω^{-1} as a function on $[0, |\Omega|]$ is as smooth as ω is. On the other hand, if ω is the vorticity of a steady-state, then from $\{\psi, \omega\} = 0$ it must be locally constant on the boundary (ψ is, as a stream function). The situation where ω has at most one critical point is only possible if either Ω is simply connected and ω has a unique critical point, or if the domain is diffeomorphic to an annulus and ω has no critical points. (The domain can be any orientable manifold, with or without boundary, such as a torus or a cylinder/annulus.)

- **The C^∞-category** The fact that the orbits are not smooth manifolds if one works in the usual Banach-spaces is an indication here again that we must work in the category of C^∞ functions. The proof hinges on the Nash-Moser theorem, which is an inverse function theorem valid in the special category of smooth tame maps between tame Fréchet-manifolds. (See Section 1.4 for a brief discussion of this category. Also, more extensive background material and an outline of the proof of the Nash-Moser theorem are provided in the Appendix.) The reason one works in the C^∞-category is loss of derivatives. Indeed, the steady-state equation $\Delta\psi = F(\psi)$ shows that $\Delta\psi$ is as smooth as F is by elliptic regularity. However, from the linearized equation $\Delta\phi = F'(\psi)\phi + f(\psi)$, one cannot expect the differentiability of the variation $\Delta\phi$ to be better than that of F', thus incurring a loss of one derivative.

- **The choice of the domain** Up until this point in these remarks we have restricted the domain Ω to be simply- or doubly-connected, and ω, ψ to have a single, or no critical points, respectively. The Nash-Moser inverse function theorem is very sensitive to the removal of any of its assumptions. As it is, the proof for ω without critical point is rather difficult, and it is likely that in the presence of a critical point for ω (and ψ), certain linearized operators will no longer be tame (i.e. one steps out of the category to which the Nash-Moser theorem applies; see the Appendix). To isolate the difficulty, consider the example (see [14], Example II.1.2.2(5), p. 135) of the map $L\colon C^\infty_{[0,1]} \to C^{\infty,\text{even}}_{[-1,1]}$, into the space of even functions, defined by $Lf(x) = f(x^2)$. While L itself satisfies tame estimates $\|Lf\|_n \le C\|f\|_n$ (see Section A.2.2 of the Appendix), its inverse only satisfies estimates of the form $\|L^{-1}g\|_n \le C\|g\|_{2n}$ and hence is not tame. This is enough to derail the inverse function theorem (see [14], Counterexample I.5.5.4, p. 127).

  In conclusion, we consider an annulus-like domain and we denote its boundary components as follows

$$\Gamma_o \ :\ \text{the outer component}, \qquad \Gamma_i \ :\ \text{the (single) inner component}. \tag{1.43}$$

- **In the radial case** the result of the theorem is immediate. It is standard that radial flows $\omega = \omega(r)$ are steady-states of Euler's equations. With r_0 the radius of the inner boundary of the annulus, and assuming $\omega'(r) > 0$, one clearly has $A_\omega(\lambda) = \pi r^2 - \pi r_0^2$ where r is such that $\omega(r) = \lambda$, and so $A_\omega^{-1}(\mu) = \omega\left(\sqrt{\frac{\mu - \pi r_0^2}{\pi}}\right)$.

  In fact, the result of the theorem holds in the case of the disk, which is simply connected, and with ω having a critical point at the origin. Indeed one then easily finds $A_\omega^{-1}(\mu) = \omega\left(\sqrt{\frac{\mu}{\pi}}\right)$. From the previous item in these remarks, the presence of the square root $\sqrt{\frac{\mu}{\pi}}$ is an indication that, for simply connected domains and in the presence of (even) a single critical point, the method of proof of this thesis fails. Nevertheless, the radial case suggests that the statement of the theorem should be true under more general conditions, say with one elliptic (non-degenerate) critical point.

- Only **surjectivity of the map** T can be hoped for, as it can certainly not be injective: it is clear that if F is perturbed outside the range of the corresponding solution ψ to $\Delta\psi = F(\psi)$, then ψ remains a solution to the perturbed steady-state equation. (It can however be shown that, locally, injectivity of T fails only because of this trivial reason. If $T(F_1) = T(F_2)$ for F_1 and F_2 in a small neighborhood, then the corresponding stream functions are equal, $\psi_1 = \psi_2$, i.e. the corresponding flows are the same. This thesis does not address this issue.)

  We will use the surjective part of the Nash-Moser inverse function theorem, and for short we will simply refer to it as the Nash-Moser inverse function theorem.

- **The existence of the solution operator** $\psi = S(F)$ does not actually necessitate the assumption that $\overline{F}' > 0$. The proof in Chapter 2 only assumes that the linearized operator $\Delta - \overline{F}'(\overline{\psi})$ is generic in the sense that it is invertible. This is of course an open condition: it holds for F in a neighborhood of $\overline{F}$ if it does at $\overline{F}$.

1.4 Idea of proof

The proof of the Theorem 1, Section 1.3, is carried out in Chapter 5.

1.4.1 The Nash-Moser inverse function theorem

The Nash-Moser inverse function theorem is a version of the inverse function theorem valid in the category of smooth tame maps in tame Fréchet-spaces. This section highlights the essential features of this category. A more detailed account is provided in the appendix.

Typical proofs of the inverse function theorems involve Picard iterations which require minimal regularity on the part of the function to be inverted (it must be "close to the identity"; see [31]). However, when smoothness of the map is not an issue, one can use Newton's iteration scheme which converges much faster, and thus has greater tolerance for errors. Suppose that a map P defined on some open set satisfies $P(0) = 0$ and one wants to solve $P(f) = g$ for g "small". Following [14], the "continuous" (as opposed to discrete) version of Newton's method amounts to solving a differential equation in f_t such that $P(f_t)$ goes in a straight line from 0 to g. With VP the inverse of the derivative DP, this ODE is

$$\dot{f}_t = cVP(f_t) \cdot (g - P(f_t)) \tag{1.44}$$

where c is a constant which one is free to choose as one wishes. Then, the error $k_t = g - P(f_t)$ satisfies

$$\dot{k}_t + ck_t = 0, \qquad k_t = e^{-ct}g \tag{1.45}$$

and f_t converges to a solution of $P(f) = g$ as $t \to \infty$.

When loss of derivatives occurs, one may hope to restore smoothness via smoothing operators S_t, provided the error incurred is within the tolerance of the scheme. In this case the ODE (1.44) is replaced by an ODE involving these smoothing operators and let's write it in the form

$$\dot{f}_t = V(t, f, g). \tag{1.46}$$

The smoothing operators have the property that

$$S_t \to \mathrm{Id}, \qquad t \to \infty \tag{1.47}$$

and as a consequence the right-hand side of (1.46) behaves "asymptotically" like that of (1.44) so that the solution of this ODE again converges to a solution of $P(f) = g$. In order for this to work, a compromise must be met between the tolerance afforded by Newton's scheme and the error introduced by the smoothing operators. This compromise defines a class of functions to be inverted for which the modified algorithm works, by imposing certain restrictions on how badly these functions are allowed to behave. In the category of smooth tame maps on tame Fréchet-spaces this trade-off is quantified as follows. The existence of the smoothing operators precisely defines **tame Fréchet-spaces** as those Fréchet-spaces with semi-norms $\|\cdot\|_n$ satisfying **interpolation inequalities** of the form

$$\|x\|_m^{n-l} \leq C\|x\|_n^{m-l}\|x\|_l^{n-m}, \qquad l \leq m \leq n. \tag{1.48}$$

This limits how wildly maps may behave, and these (along with their derivatives) must satisfy **tame estimates**, i.e. estimates of the form

$$\|P(F)\|_n \leq C_n \cdot (\|F\|_{n+r} + 1) \tag{1.49}$$

where r is independent of n. A smooth tame map is one which is continuous and tame along with all its derivatives. (See the Appendix for definitions and basic properties.)

We are now in a position to state the (surjective part of the) Nash-Moser inverse function theorem.

Theorem 3 (Part III.1, [14]) *Let $\mathcal{F}$ and $\mathcal{G}$ be tame Fréchet-spaces, and $P\colon \mathcal{U} \subset \mathcal{F} \to \mathcal{G}$ be a smooth tame map. Suppose that the first derivative DP is surjective with a smooth tame family of right-inverses VP. Then T is locally surjective. Moreover, in a neighborhood of any point, P has a smooth tame right-inverse.*

Observe that, in contrast to the inverse function theorem for Banach-spaces, it is required of the derivative $DP(f)$ to be surjective, not only at a reference point f_0, but in a whole neighborhood.

1.4.2 Outline of proof of main theorem

We will apply Theorem 3 to the map

$$T\colon F \in (\mathcal{V} \subset C_I^\infty) \mapsto A_\omega^{-1} \in C^\infty_{[0,|\Omega|]} \tag{1.50}$$

where I is a sufficiently large interval containing range($\overline{\psi}$) and $\mathcal{V}$ is a sufficiently small open neighborhood of $\overline{F}$ so that all corresponding solutions ψ for $F \in \mathcal{V}$ have range contained in I. A crucial point is to show that $DT(F)$ is surjective for F in a neighborhood of $\overline{F}$. In order to do this, we write the first derivative as

$$DT(F) \cdot f = B(F) \cdot f + \tilde{K}(F) \cdot f \tag{1.51}$$

which is morally a "compact perturbation of an isomorphism" (where $B(F)$ plays the rôle of the isomorphism, and $\tilde{K}(F)$ that of the compact perturbation). This decomposition is obtained by writing

$$T(F) = A_{\omega}^{-1} = F \circ A_{\psi}^{-1} \tag{1.52}$$

(see proof in Section 4.3) for then

$$B(F) \cdot f = f \circ A_{\psi}^{-1}. \tag{1.53}$$

(Naively, the inverse of $B(F) \cdot f$ is composition from the right by A_ψ, but as ψ has varying range this causes difficulties. Nevertheless this gives the right intuition.) In the category of Banach-spaces, the Fredholm alternative says that a compact perturbation of an isomorphism is injective if and only if it is surjective. In the present case we are working in spaces of smooth functions, which are not Banach-spaces, but rather Fréchet-spaces as the intersection of infinitely (countably) many Banach-spaces. Thus the Fredholm alternative is used for each Banach-space in the intersection. Another issue is that the map $B(F)$ (for F in a neighborhood of $\overline{F}$) is surjective only. To show that $DT(F)$ is surjective, we show that

$$DT(F) \cdot VB(F) = \mathrm{Id} + K(F) \tag{1.54}$$

is invertible, where $VB(F)$ is a right-inverse for $B(F)$, and to show this it is enough to show (via repeated applications of the Fredholm alternative for Banach-spaces) that $DT(F) \cdot VB(F)$ is injective. This is where the non-degeneracy condition (ND) of Theorem 1 is used.

Remark 4 A remark concerning the gradings is in order. Continuity (and differentiability) of maps of Fréchet-spaces can be established using any grading which are in the same equivalence class. Therefore, we will prove continuity (and differentiability) of maps using the grading that is most convenient depending on the situation. In Chapter 2, where we show that certain maps related to elliptic equations are smooth, the availability of Schauder-type estimates will make it more convenient to use the $C^{n,\alpha}$-grading (of functions whose n-th derivatives are Hölder-continuous). Therefore we introduce the spaces of stream functions with finite regularity in the $C^{n,\alpha}$-grading:

$$\mathcal{U}^{n,\alpha} := \{\psi \in C^{n,\alpha}_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\}, \tag{1.55}$$

$$\mathcal{U}^{n,\alpha}_{\gamma_i} := \{\psi \in \mathcal{U}^{n,\alpha} \mid \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, i = 1, 2, \dots\}. \tag{1.56}$$

On the other hand, the proof that A_ω^{-1} is smooth in Chapter 5 will be easier in the C^n-grading (of functions which are n-times continuously differentiable) and thus we introduce the spaces of stream functions with such differentiability:

$$\mathcal{U}^{n} := \{\psi \in C^{n}_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\}, \tag{1.57}$$

$$\mathcal{U}^{n}_{\gamma_i} := \{\psi \in \mathcal{U}^{n} \mid \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, i = 1, 2, \dots\}, \tag{1.58}$$

In contrast,

all tame estimates will be derived in the $C^{n,\alpha}$-grading

as it is crucial to determine exactly the degree of the map to be inverted.

Finally, the space of smooth stream functions will be denoted

$$\mathcal{U} := \{\psi \in C^{\infty}_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\} \tag{1.59}$$

and we will use also the (affine) subspace

$$\mathcal{U}_{\gamma_i} := \{\psi \in \mathcal{U} \mid \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, i = 1, 2, \dots\}. \tag{1.60}$$

The spaces $\mathcal{U}_{\gamma_i}, \mathcal{U}^n_{\gamma_i}, \mathcal{U}^{n,\alpha}_{\gamma_i}$ are, in general, affine subspaces of $\mathcal{U}, \mathcal{U}^n, \mathcal{U}^{n,\alpha}$ respectively. $\mathcal{U}_0$, $\mathcal{U}^n_0$, and $\mathcal{U}^{n,\alpha}_0$ are the *linear* subspaces subtending the affine spaces $\mathcal{U}_{\gamma_i}$, $\mathcal{U}^n_{\gamma_i}$, and $\mathcal{U}^{n,\alpha}_{\gamma_i}$ respectively. ■

1.5 Summary of notation

We list below certain notational conventions. These apply in particular to Chapters 2, 4, and 5.

- Ω denotes a smooth bounded domain in $\mathbb{R}^2$. It is connected, but not necessarily simply-connected. The outer boundary component is denoted Γ_0, and the inner boundary components are denoted Γ_i, $i = 1, 2, \ldots$ (the precise number will not be specified, but is finite). In Chapters 4 and 5, we will consider an annulus domain Ω, with outer boundary component Γ_o and (single) inner boundary component Γ_i.

- ψ (almost always) denotes a stream function.

- ω always denotes a vorticity function.

- The reference steady-state solution has quantities denoted $\overline{\psi}$, $\overline{\omega}$, and $\overline{F}$. That is, $\overline{\omega} = \Delta\psi = \overline{F}(\overline{\psi})$.

- F generally (i.e. except in the Appendix) denotes the function such that $\Delta\psi = F(\psi)$ is a steady-state solution.

- N generally denotes the unit outer normal to the boundary. If Ω is diffeomorphic to an annulus and if ψ is a function without critical points, then we will denote by $N = \frac{\nabla\psi}{|\nabla\psi|}$ a vector field normal to the level sets of ψ.

- $\langle \cdot, \cdot \rangle$ denotes inner product in $\mathbb{R}^2$.

- γ_i corresponds to the boundary condition $\int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i$, $i = 1, 2, \ldots$

- The boundary conditions for the elliptic equation $\Delta\psi = \omega$ will be that $\psi_{|\Gamma_0} = 0$, $\psi_{|\Gamma_i}$ =constant, $\int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i$, $i = 1, 2, \ldots$.

- We will observe the summation convention. In particular, when Ω is allowed to have several inner boundary components Γ_i, $i = 1, 2, \ldots$, then $\psi_{|\Gamma_i}\gamma = \sum_i \psi_{|\gamma_i}\gamma_i$ (since $\psi_{|\Gamma_0} = 0$, it is not necessary to specify the range for i in the sum).

- $\psi = S(F)$ is the solution operator sending F to (a uniquely determined) ψ such that $\Delta\psi = F(\psi)$ under the boundary conditions specified above.

- The distribution function of ψ is $A_\psi(\lambda) = |\{\psi < \lambda\}|$, with inverse $A_\psi^{-1}(\mu)$. Likewise for ω.

- $T(F) = A_\omega^{-1}$ where $\omega = \Delta\psi = F(\psi)$.

- If ψ and u are functions on $\overline{\Omega}$, and the level set $\{x \;:\; \psi(x) = \lambda\}$ is a closed curve, we set

$$J_\psi u(\lambda) = \int\limits_{x \;:\; \psi(x)=\lambda} u(x) dl(x). \tag{1.61}$$

- $\{f, g\} = f_x g_y - f_y g_x$ is the Poisson bracket of two functions f, g of x and y.

- If $\mathcal{K}$ is the closure of a smooth, bounded region in Euclidean space (most often we will take $\mathcal{K} = \overline{\Omega}$), then $C^n_{\mathcal{K}}$ denotes the space of n-times continuously differentiable functions on $\mathcal{K}$. The norm is given by

$$\|f\|_n := \|f\|_{C^n_{\mathcal{K}}} := \sup_{0\le j\le n} \sup_{\mathcal{K}} |\nabla^j f|. \tag{1.62}$$

- $C^{n,\alpha}_{\mathcal{K}}$ denotes the subspace of functions in $C^n_{\mathcal{K}}$ whose n-th derivatives are Hölder continuous with exponent α. Throughout, α will be a fixed constant in $(0, 1)$. The norm is denoted

$$\|f\|_{n,\alpha} := \|f\|_{C^{n,\alpha}_{\mathcal{K}}} := \|f\|_n + [\nabla^n f]_\alpha \qquad [f]_\alpha := \sup_{x\neq y\in\mathcal{K}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{1.63}$$

- $C^\infty_{\mathcal{K}}$ denotes the space of smooth functions on $\mathcal{K}$.

- $\mathcal{U}$, $\mathcal{U}^n$, and $\mathcal{U}^{n,\alpha}$ denote the (linear) spaces of stream functions of class C^∞, C^n, and $C^{n,\alpha}$ respectively. The (affine) subspaces $\mathcal{U}_{\gamma_i}$, $\mathcal{U}^n_{\gamma_i}$, and $\mathcal{U}^{n,\alpha}_{\gamma_i}$ take into account the boundary conditions (see (1.55)-(1.60)).

- $\mathcal{V}$ will generally denote an open set of some Fréchet-space $C^\infty_{\mathcal{K}}$, or $\mathcal{U}$, etc.

- The relevant elliptic operator is denoted $E(c)\cdot\phi := \Delta\phi + c\phi$. The solution operator of $E(c)\cdot\phi$ with boundary conditions $\gamma_i = 0$ ($i = 1, 2, \dots$) is denoted $\phi = VE(c)\cdot k$.

- More generally, when a map of two variables F and g in Fréchet-spaces is linear in g, it is denoted $L(F)\cdot g$, and if $L(F)$ has an inverse, it is denoted $VL(F)\cdot g$.

1.6 Organization of thesis

The rest of the document is organized as follows.

The main goal of Chapter 2 is to construct the solution operator $\psi = S(F)$ of Theorem 1 (Section 1.3). Existence, uniqueness, and estimates for linear and semi-linear elliptic equations arising in incompressible hydrodynamics are established. The results are the standard ones, but the boundary conditions are not the usual Dirichlet or Neumann boundary conditions, and therefore a detailed account is included for the sake of completeness. It is hoped that it could be of some use to have it all recorded somewhere with this level of details. On the other hand, tame estimates for the solution operator $\phi = VE(c) \cdot k$ are far from obvious, as the constants in the estimates depend on c. This must be done with care.

Chapter 3 begins with the necessary review of classical mechanics and symplectic reduction in order to arrive at the Euler-Poincaré equation in the dual Lie algebra. Then, this is applied to the infinite-dimensional setting of two-dimensional ideal incompressible fluids.

Chapter 4 establishes that the non-degeneracy condition (ND) holds whenever $F' > 0$. It also contains useful results on distribution functions for functions on $\overline{\Omega}$ and on the co-adjoint orbits (i.e. the sets of isovorticed flows).

Chapter 5 contains the bulk of the proof of Theorem 1, Section 1.3.

Finally, an Appendix contains the necessary definitions and properties related to smooth tame maps of tame Fréchet-spaces and the Nash-Moser inverse function theorem as it is used in this thesis. The proof in [14] is carried out for the abstract case of tame Fréchet spaces. The appendix shows that it indeed applies to the situation of this thesis, namely that of the map T defined in (1.50) between spaces of smooth functions on closed intervals.

Chapter 2

Elliptic systems arising in hydrodynamics

References for this Chapter are [3], [5], [6], [22], [23].

The ultimate goal of this section is to construct a smooth tame map $\psi = S(F)$ returning the solution to the nonlinear elliptic equation on $\Omega \subset \mathbb{R}^2$

$$\Delta\psi = F(\psi), \quad \psi_{|\Gamma_0} = 0, \quad \psi_{|\Gamma_i} = \text{constant}, \quad \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \quad i = 1, 2, \ldots \tag{2.1}$$

corresponding to a nonlinearity F and fixed constants γ_i (N denotes the outer unit normal and $\frac{\partial}{\partial N}$ denotes the outer normal derivative). These boundary conditions are justified in Section 3.2. The domain Ω is assumed bounded, smooth, and connected, but not necessarily simply connected. We will denote its (single) outer and (finitely many) inner boundary components

$$\Gamma_0, \qquad \Gamma_i, \qquad i = 1, 2, \ldots \tag{2.2}$$

respectively. (We will not need to specify exactly how many inner boundary components Ω has.) The results of this Chapter are valid with any number of inner boundary components. However, we will eventually (in Chapters 4 and 5) consider the case of a single inner boundary component, and then the 'i' in Γ_i will stand for "inner boundary component". Also, it is *not* assumed that $F' > 0$, but rather that F is generic in the sense that $\Delta - F'(\psi)$ is invertible.

The boundary conditions on ψ are

$$\psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.3}$$

where the γ_i's are fixed constants. The first two identities mean that ψ is the stream function of some velocity field u,

$$u = \nabla^{\perp} \psi, \qquad \langle u, N \rangle = 0 \tag{2.4}$$

where N is the unit outer normal to the boundary and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and the third identity means that the circulation of u is prescribed along each inner boundary component:

$$\int_{\Gamma_i} \langle u, dl \rangle = \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots . \tag{2.5}$$

In order to construct the map $\psi = S(F)$ we will first need to derive estimates for the linear elliptic equation

$$\Delta \psi = \omega, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots . \tag{2.6}$$

These boundary conditions are different from the more usual Dirichlet or Neumann boundary conditions, and we present a detailed discussion of this elliptic problem. In order to study the linearized equation

$$\Delta \phi = F'(\psi)\phi + f(\psi), \qquad \phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, \qquad i = 1, 2, \ldots \tag{2.7}$$

we will need estimates for the linear equation

$$\Delta \phi + c\phi = k, \qquad \phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, \qquad i = 1, 2, \ldots . \tag{2.8}$$

The availability of Schauder-type estimates makes the choice of Hölder spaces $C^{n,\alpha}$ the most convenient one to establish smoothness and tameness of maps associated with these elliptic equations.

2.1 Incompressible flows

In this section we discuss relations between velocity fields, stream functions and vorticity functions (for two-dimensional flows), and particle trajectories. In particular, we show in which sense they are equivalent.

A velocity field u is a divergence-free vector field

$$\text{div } u = 0 \qquad \text{in} \quad \Omega \tag{2.9}$$

which is tangent to the boundary

$$\langle u, N \rangle = 0 \qquad \text{on} \quad \partial\Omega. \tag{2.10}$$

The individual particle trajectories are then the integral curves of the (time-dependent) vector field u: the solution $\eta(x,t)$ to

$$\dot{X}(t) = u(X,t), \qquad X(0) = x \tag{2.11}$$

gives the position at time t of a particle initially at position x. Conversely, if $\eta(x,t)$ is the flow of an incompressible fluid, then the velocity field is

$$u(x,t) = \dot{\eta}(\eta^{-1}(x,t),t), \qquad u = \dot{\eta} \circ \eta^{-1}. \tag{2.12}$$

(The divergence-free condition on u is the linearized version of the condition that $\text{Jac}(\eta) \equiv 1$, and by definition it is clear that u is tangent to the boundary.)

In two dimensions, the vorticity of the flow $u = (u^1, u^2)$ is a scalar function given by

$$\omega = \text{curl } u = u^2_x - u^1_y. \tag{2.13}$$

Given ω, u satisfies the following Cauchy-Riemann system of equations

$$\text{curl } u = \omega \tag{2.14}$$

$$\text{div } u = 0 \tag{2.15}$$

which is an elliptic system. The velocity field is completely determined from the vorticity if the circulation is specified along each inner boundary component:

$$\int_{\Gamma_i} \langle u, dl \rangle = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.16}$$

where γ_i are fixed constants. (The quantity $\int_{\Gamma_0}\langle u, dl\rangle$ should *not* be specified as it is determined by the other integrals since by the divergence theorem we have $\int_{\Gamma_0}\langle u, dl\rangle + \sum_{i\geq 1}\int_{\Gamma_i}\langle u, dl\rangle = \int_\Omega \text{div } u = 0$.)

In two dimensions, the velocity field $u = (u^1, u^2)$ can be represented by its stream function ψ

$$u = \nabla^\perp \psi = \begin{bmatrix} -\psi_y \\ \psi_x \end{bmatrix}. \tag{2.17}$$

Specifically, we have a one-to-one correspondence between (smooth) velocity fields and stream functions:

Proposition 5 *Let u be a smooth velocity field on $\overline{\Omega}$ such that*

$$\int_{\Gamma_i}\langle u, dl\rangle = \gamma_i, \qquad i = 1, 2, \ldots. \tag{2.18}$$

Then there exists a unique smooth function ψ on $\overline{\Omega}$ such that

$$u = \nabla^\perp \psi, \qquad \psi_{|\Gamma_0} = 0. \tag{2.19}$$

In addition, this stream function satisfies

$$\psi_{|\Gamma_i} = \mathit{constant}, \qquad \int_{\Gamma_i}\frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.20}$$

by which we mean that ψ equals a constant, but unspecified, value on each of the inner boundary components Γ_i, $i = 1, 2, \ldots$, and the γ_i are fixed (known) values. (Here N denotes the normal to $\partial\Omega$ pointing outside of Ω, and Γ_i, $i = 1, 2, \ldots$ has the conventional orientation, i.e. it is oriented clockwise.)

Proof If u has a stream function ψ (defined as satisfying (2.17)), then it is easilty seen that $\psi_{|\partial\Omega}$ = locally constant and one may impose without loss of generality $\psi_{|\Gamma_0} = 0$. Furthermore, the conditions (2.16) on u translate into the conditions on ψ that $\int_{\Gamma_i}\frac{\partial\psi}{\partial N} = \gamma_i$, $i = 1, 2, \ldots$.

Conversely, if a function ψ satisfies the boundary conditions in (2.19) and (2.20), fix $x_0 = (x_0^1, x_0^2)$ any point on the boundary (say). For any point $x = (x^1, x^2) \in \Omega$, let $\mathcal{C}$ denote a path joining x_0 to x, and set

$$\psi(x) := -\int_{\mathcal{C}}\langle u^\perp, dl\rangle \tag{2.21}$$

where

$$u^{\perp} = \begin{bmatrix} -u^2 \\ u^1 \end{bmatrix}. \tag{2.22}$$

Provided this is well-defined, one verifies at once that $\nabla\psi = -u^{\perp}$. That ψ is well-defined is a consequence of the divergence-free condition and the boundary condition on u. Indeed, let first $\mathcal{C}'$ be another path connecting x_0 to x and homotopic to $\mathcal{C}$. Then by the Gauss-Green theorem, and denoting D the region enclosed by $\mathcal{C}$ and $\mathcal{C}'$ (taking into account orientations), and N the outer normal to D, we have

$$\int_{\mathcal{C}} \langle u^{\perp}, dl\rangle = \int_{\mathcal{C}} \langle u, N\rangle dl = \int_{\mathcal{C}'} \langle u, N\rangle dl + \int_D \text{div } u \; dx = \int_{\mathcal{C}'} \langle u^{\perp}, dl\rangle \tag{2.23}$$

Next, suppose that $\mathcal{C}'$ differs from $\mathcal{C}$ in path(s) going a multiple number of times around one or several of the inner boundaries. For $i = 1, 2, \ldots$

$$\int_{\Gamma_i} \langle u^{\perp}, dl\rangle = \int_{\Gamma_i} \langle u, N\rangle dl = 0 \tag{2.24}$$

by the tangency of u on $\partial\Omega$. $\psi(x_0)$ is well-defined.

Finally, the integrals in (2.20) coincide with the circulations of u in (2.16):

$$\int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \int_{\Gamma_i} \langle u, dl\rangle, \qquad i = 1, 2, \ldots. \tag{2.25}$$

■

Remark 6

- Leaving functional analytical issues for now, the Lie algebra of the group of volume-preserving diffeomorphisms is the space of divergence-free vector fields tangent to the boundary. From Proposition 5, it is identified with the **space of stream functions**

$$\mathcal{U} = \{\psi \in C^{\infty}_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \ldots\}. \tag{2.26}$$

  Later we will work in (linear and affine) subspaces of stream functions with finite regularity. (See the notation in Section 1.5.)

- It is also interesting to note that the flow of a 2D incompressible fluid is a time-dependent Hamiltonian system with Hamiltonian $-\psi$. From this observation follow a number of conclusions on the topology of the trajectories, particularly useful for steady-state solutions. See Proposition 18, Section 4.1, and Section 5.1.

$\blacksquare$

The stream function is related to the vorticity according to $\Delta\psi = \omega$. A consequence of Proposition 5 is that the Cauchy-Riemann system

$$\text{curl } u = \omega, \qquad \text{div } u = 0, \qquad \langle u, N\rangle = 0, \qquad \int_{\Gamma_i} \langle u, dl\rangle = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.27}$$

is equivalent to the elliptic equation

$$\Delta\psi = \omega, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \langle u, dl\rangle = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.28}$$

In the next Proposition we show that there exists a unique smooth solution u to (2.27) (and equivalently a unique solution ψ to (2.28)).

Proposition 7 *Given a smooth function ω and constants γ_i, $i = 1, 2, \ldots$, there exists a unique smooth solution u to*

$$\textit{curl } u = \omega, \qquad \textit{div } u = 0, \qquad \int_{\Gamma_i} \langle u, dl\rangle = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.29}$$

Proof

$\boxed{1}$ **Existence** The solution u will be constructed as the sum

$$u := \nabla^{\perp}\psi_1 + u_0 + \nabla\phi \tag{2.30}$$

where

- ψ_1 takes care of the right-hand side ω,
- u_0 takes care of the circulations γ_i, and
- ϕ takes care of the boundary condition $\langle u, N\rangle = 0$.

Let ψ_1 be the unique solution to the Poisson problem

$$\Delta\psi_1 = \omega, \qquad \psi_{1|\partial\Omega} = 0. \tag{2.31}$$

Set

$$u_0(x) = \sum_{i\geq 1} \gamma_i' K(x, x_i), \qquad \gamma_i' := \gamma_i - \int_{\Gamma_i} \frac{\partial\psi_1}{\partial N}, \quad i = 1, 2, \ldots \tag{2.32}$$

where each x_i is an arbitrary, fixed point selected inside the region bounded by Γ_i, and

$$K(x, y) := -\frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2}. \tag{2.33}$$

Note that

$$\mathrm{curl}_x K(x, y) = 0, \qquad \mathrm{div}_x K(x, y) = 0, \qquad x \neq y \tag{2.34}$$

and thus by Stokes' theorem, for any curve $\mathcal{C}(y)$ encircling y exactly once, for r sufficiently small, and denoting D the region bounded by $\mathcal{C}(y)$ and the circle centered at y with radius r (note that the orientations have to be switched), we have

$$\begin{aligned}
& \int_{\mathcal{C}(y)} \langle K(x, y), dl(x)\rangle & (2.35)\\
= & -\frac{1}{2\pi}\int_{\mathcal{C}(y)} \langle \frac{(x-y)^\perp}{|x-y|^2}, dl(x)\rangle & (2.36)\\
= & +\frac{1}{2\pi}\int_{\mathcal{C}(y)} \langle \frac{x-y}{|x-y|^2}, N\rangle dl(x) & (2.37)\\
= & \frac{1}{2\pi}\int_{|x-y|=r} \langle \frac{x-y}{|x-y|^2}, N\rangle dl(x) - \int_D \mathrm{div}_x\left(-\frac{1}{2\pi}\frac{x-y}{|x-y|^2}\right) dx & (2.38)\\
= & \frac{1}{2\pi}\int_{|x-y|=r} \langle \frac{x-y}{|x-y|^2}, N\rangle dl(x) + \int_D \mathrm{curl}_x\left(-\frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2}\right) dx & (2.39)\\
= & \frac{1}{2\pi}\int_{|x-y|=r} \langle \frac{x-y}{|x-y|^2}, N\rangle dl(x) & (2.40)\\
= & +\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{d\theta}{r} & (2.41)\\
= & 1 & (2.42)
\end{aligned}$$

since $\mathrm{curl}_x K(x, y) = 0$. Finally, let ϕ solve

$$\Delta\phi = 0, \qquad \frac{\partial\phi}{\partial N} = -\langle u_0 + \nabla^\perp\psi_1, N\rangle \tag{2.43}$$

(uniquely determined up to a constant). Define now u as $\nabla^\perp\psi_1 + u_0 + \nabla\phi$. We verify:

$$\text{curl } u = \text{curl } \nabla^\perp\psi_1 + \text{curl } u_0 + \text{curl } \nabla\phi = \omega, \tag{2.44}$$

$$\text{div } u = \text{div } \nabla^\perp\psi_1 + \text{div } u_0 + \text{div } \nabla\phi = 0, \tag{2.45}$$

$$\langle u, N\rangle = \langle \nabla^\perp\psi + u_0, N\rangle + \frac{\partial\phi}{\partial N} = 0, \tag{2.46}$$

and

$$\int_{\Gamma_i}\langle u, dl\rangle = \int_{\Gamma_i}\frac{\partial\psi_1}{\partial N} + \int_{\Gamma_i}\langle u_0, dl\rangle + \int_{\Gamma_i}\langle\nabla\phi, dl\rangle = \int_{\Gamma_i}\frac{\partial\psi_1}{\partial N} + \gamma_i' + 0 = \gamma_i \tag{2.47}$$

(the term in ϕ vanishes since it is the integral of a gradient round a closed curve). □

$\boxed{2}$ **Uniqueness** It is equivalent to show that, if ψ solves

$$\Delta\psi = 0, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{const.}, \qquad \int_{\Gamma_i}\frac{\partial\psi}{\partial N} = 0, \qquad i = 1, 2, \ldots \tag{2.48}$$

then $\psi \equiv 0$. By the maximum principle, ψ reaches its maximum and minimum values on $\partial\Omega$. Suppose $\psi > 0$ somewhere in Ω. Then its maximum is attained on one of the Γ_i's, where ψ is constant, and thus $\frac{\partial\psi}{\partial N} \geq 0$ on all of this Γ_i. But, $\int_{\Gamma_i}\frac{\partial\psi}{\partial N} = 0$ implies that $\frac{\partial\psi}{\partial N} = 0$ on Γ_i, which contradicts the Hopf boundary-point lemma. Likewise one shows that ψ cannot take on negative values. ■

2.2 Estimates for $\Delta\psi = \omega$ in Hölder spaces

We denote $C^{n,\alpha}(\overline{\Omega})$ the space of functions on some compact $\overline{\Omega}$ with the following norm

$$\|f\|_{n,\alpha} := \|f\|_{C^{n,\alpha}_{\overline{\Omega}}} := \|f\|_{C^n_{\overline{\Omega}}} + [\nabla^n f]_\alpha \tag{2.49}$$

where

$$\|f\|_n := \|f\|_{C^n_{\overline{\Omega}}} = \sup_{0\leq j\leq n}\sup_{\overline{\Omega}}|\nabla^j f|, \qquad [f]_\alpha := \sup_{x\neq y\in\overline{\Omega}}\frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{2.50}$$

Proposition 8 *Let* $\omega \in C^{n,\alpha}_{\overline{\Omega}}$ *(*$n = 0, 1, 2, \dots$*). Then there exists a unique* $\psi \in C^{n+2,\alpha}_{\overline{\Omega}}$ *such that*

$$\Delta\psi = \omega, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \textit{constant}, \qquad \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \dots \tag{2.51}$$

and it satisfies

$$\|\psi\|_{n+2,\alpha} \leq C(\Omega, n) \cdot (\|\omega\|_{n,\alpha} + \sum_i |\gamma_i|), \qquad n \geq 0. \tag{2.52}$$

That is, the solution operator associated to $\Delta\psi = \omega$ *is tame with degree* -2 *and base* 2*:*

$$\|\psi\|_{n,\alpha} \leq C(\Omega, n) \cdot (\|\omega\|_{n-2,\alpha} + \sum_i |\gamma_i|), \qquad n \geq 2. \tag{2.53}$$

Before proving Proposition 8, we first establish estimates which will be useful on numerous instances. Let

$$X := \{\psi \in H^1(\Omega) \mid \psi_{|\Gamma_0} = 0,\ \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\}. \tag{2.54}$$

Lemma 9 *There exist constants (independent of* ψ*) such that*

$$|\psi_{|\Gamma_i}| \leq C_1(\Omega)\|\nabla\psi\|_{L^2(\Omega)}, \qquad \psi \in X, \tag{2.55}$$

$$\|\psi\|_{L^2(\Omega)} \leq C_2(\Omega)\|\nabla\psi\|_{L^2(\Omega)}, \qquad \psi \in X. \tag{2.56}$$

Furthermore, if $\psi \rightharpoonup \overline{\psi}$ *(weakly) in* X*, then* $\psi_{|\Gamma_i} \to \overline{\psi}_{|\Gamma_i}$, $i = 1, 2, \dots$*. Finally,* X *is a Banach space with respect to the norm*

$$\|\psi\|_{H^1(\Omega)} := \|\nabla\psi\|_{L^2(\Omega)}. \tag{2.57}$$

Proof of Lemma 9 Estimate (2.56) is just the Poincaré inequality, valid in X. The estimate (2.55) is a consequence of the trace theorem

$$\|\psi\|_{L^2(\partial\Omega)} \leq C \cdot \|\nabla\psi\|_{L^2(\Omega)} \qquad \psi \in H^1(\Omega). \tag{2.58}$$

since then $\|\psi\|^2_{L^2(\partial\Omega)}$ is a weighted sum of the $|\gamma_i|$, $i = 1, 2, \dots$.

The claim on convergence follows since the trace theorem $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is in fact compact.

As for completeness of X, observe first that it is indeed a linear subspace of $H^1(\Omega)$. If now $\psi^n \in X$ converges to $\overline{\psi}$ in $H^1(\Omega)$, then by (2.55) the $\psi^n_{|\Gamma_i}$ converge as well. ■

Proof of Proposition 8 We first derive a priori estimates for smooth functions. Given $\omega \in C^\infty_{\overline{\Omega}}$ we know from Proposition 5 and Proposition 7 of Section 2.1 that there exists a unique $\psi \in C^\infty_{\overline{\Omega}}$ solving the problem. Let g_i be a fixed smooth function on $\overline{\Omega}$, equal to 1 on Γ_i and equal to 0 on the other Γ_j's. Note that once the g_i's are fixed, then

$$\|g_i\|_{n,\alpha} \tag{2.59}$$

are fixed constants (depending on n, i). The function

$$u = \psi - \psi_{|\Gamma_i} g_i \tag{2.60}$$

is a solution to the Dirichlet problem

$$\Delta u = \omega - \psi_{|\Gamma_i} \Delta g_i, \qquad u_{|\partial\Omega} = 0 \tag{2.61}$$

and for which we have the Schauder estimates (with constants *depending* on n)

$$\|u\|_{n+2,\alpha} \leq C \cdot (\|\Delta u\|_{n,\alpha} + \|u\|_{0,\alpha}). \tag{2.62}$$

Then,

$$\begin{aligned} \|\psi\|_{n+2,\alpha} &\leq \|u\|_{n+2,\alpha} + C \sum_i |\psi_{|\Gamma_i}| && (2.63) \\ &\leq C \cdot (\|\Delta u\|_{n,\alpha} + \|u\|_{0,\alpha} + \sum_i |\psi_{|\Gamma_i}|) && (2.64) \\ &\leq C \cdot (\|\omega\|_{n,\alpha} + \|\psi\|_{0,\alpha} + \sum_i |\psi_{|\Gamma_i}|). && (2.65) \end{aligned}$$

We are done with the estimates once we have the bound

$$\|\psi\|_{0,\alpha} \leq C(\|\omega\|_{0,\alpha} + \sum_i |\psi_{|\Gamma_i}|). \tag{2.66}$$

First note that we have in a similar way the estimate

$$\|\psi\|_{H^2(\Omega)} \leq C(\|\omega\|_{L^2(\Omega)} + \sum_i |\psi_{|\Gamma_i}|). \tag{2.67}$$

The embeddings $C^{0,\alpha}_{\overline{\Omega}} \hookrightarrow H^2_\Omega$ (Ω is two-dimensional) and $C^{0,\alpha}_{\overline{\Omega}} \hookrightarrow L^2_\Omega$ give

$$\|\psi\|_{0,\alpha} \leq C\|\psi\|_{H^2} \leq C(\|\omega\|_{L^2} + \sum_i |\psi_{|\Gamma_i}|) \leq C(\|\omega\|_{0,\alpha} + \sum_i |\psi_{|\Gamma_i}|). \tag{2.68}$$

Let now ω_j be as sequence of smooth functions converging to $\omega \in C^{n,\alpha}_{\overline{\Omega}}$ in the $C^{n,\alpha}$-norm. Then, the corresponding ψ_j's form a Cauchy sequence in $C^{n+2,\alpha}$, and $\Delta(\psi_j - \psi_l) = \omega_j - \omega_l$, with $\int_{\Gamma_i} \frac{\partial(\psi_j - \psi_l)}{\partial N} = 0$, $i = 1, 2, \ldots$. Thus, the limit $\psi \in C^{n+2,\alpha}_{\overline{\Omega}}$ solves $\Delta\psi = \omega$ with $\int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i$, $i = 1, 2, \ldots$. Taking limits (as we may), it satisfies the desired estimates. ■

2.3 Estimates for $\Delta\phi + c\phi = k$ in Hölder spaces

Given $c \in C^\infty_{\overline{\Omega}}$, we consider the linear equation

$$\Delta\phi + c\phi = k, \qquad \phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, \qquad i = 1, 2, \ldots. \tag{2.69}$$

In this section we establish that the solution operator for $\Delta\phi + c\phi = k$, when it exists, is a smooth tame family of linear maps. (See Appendix A.)

If $\overline{c}$ is such that $\Delta + \overline{c}$ is invertible, then by the open mapping theorem there exists a constant $\overline{C}_n$ depending on $\overline{c}$ (and n) such that

$$\|\phi\|_{n,\alpha} \leq \overline{C}_n \|\Delta\phi + \overline{c}\phi\|_{n-2,\alpha}. \tag{2.70}$$

For c in a neighborhood of $\overline{c}$, $\Delta + c$ is also invertible and thus also satisfies estimates of the form

$$\|\phi\|_{n,\alpha} \leq C_n \|\Delta\phi + c\phi\|_{n-2,\alpha} \tag{2.71}$$

but the constant C_n depends on c, and it is not at all clear if one can choose a neighborhood of $\overline{c}$ in which these constants can be made independent of c. This serves as indication that tame estimates for elliptic equations pose subtle issues. In addition, the boundary conditions for our problem are not the usual ones (Dirichlet, Neumann, ...) and thus we must handle this other issue with care.

In order to obtain tame estimates for the inverse of $\Delta\phi + c\phi = k$, we will start from those for $\Delta u + cu = f$ with zero Dirichlet boundary conditions, established in

Theorem II.3.3.5, p. 161, [14], and then write $\phi = u - \phi_{|\Gamma_i} g_i$ where the g_i's are as in the proof of Proposition 8, Section 2.2.

Introduce the following linear spaces of stream functions

$$\begin{aligned} \mathcal{U} &= \{\psi \in C^{\infty}_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\} && (2.72) \\ \mathcal{U}_0 &= \{\phi \in \mathcal{U} \mid \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, i = 1, 2, \dots\}. && (2.73) \end{aligned}$$

The mapping

$$E\colon C^{\infty}_{\overline{\Omega}} \times \mathcal{U}_0 \to C^{\infty}_{\overline{\Omega}}, \qquad E(c)\cdot\phi := \Delta\phi + c\phi = k \tag{2.74}$$

in the $C^{n,\alpha}$-grading, has degree 0 in c, 2 in ϕ and base 0 (Lemma 47 and Lemma 48 in Section A.2.3). We also define the spaces of stream functions of class $C^{n,\alpha}$, $n \geq 1$:

$$\begin{aligned} \mathcal{U}^{n,\alpha} &:= \{\psi \in C^{n,\alpha}_{\overline{\Omega}} \mid \psi_{\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \dots\} && (2.75) \\ \mathcal{U}^{n,\alpha}_0 &:= \{\phi \in \mathcal{U}^{n,\alpha} \mid \int_{\partial\Omega} \frac{\partial \phi}{\partial N} = 0, i = 1, 2, \dots\} && (2.76) \end{aligned}$$

The estimates in $C^{n,\alpha}$-spaces for the elliptic problem

$$\Delta\phi + c\phi = k, \qquad \phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, \qquad i = 1, 2, \dots \tag{2.77}$$

will be derived using those for the Dirichlet problem. This is a special case of Theorem II.3.3.5, p. 161, [14].

Proposition 10 (Theorem II.3.3.5, p. 161, [14]) *Consider the elliptic problem with zero Dirichlet boundary conditions*

$$L(c)\cdot u := \Delta u + cu = f, \qquad u_{|\partial\Omega} = 0. \tag{2.78}$$

Suppose that $\overline{c}$ is such that $L(\overline{c})\cdot u = f$ has a unique solution u for each f. Then, there exists a neighborhood

$$\mathcal{V}_*(\overline{c}) := \{c \mid \|c - \overline{c}\|_{0,\alpha} < \epsilon_*\} \tag{2.79}$$

such that, if $c \in \mathcal{V}_*(\overline{c})$, then $L(c)\cdot u = f$ has a unique solution u for each f. Furthermore, the solution operator

$$VL(c)\cdot f = u, \qquad \Delta u + cu = f, \qquad u_{|\partial\Omega} = 0 \tag{2.80}$$

is a smooth tame map, and satisfies the tame estimates

$$\|u\|_{n,\alpha} \leq C\left(\|f\|_{n-2,\alpha} + \|c\|_{n-2,\alpha}\|f\|_{0,\alpha}\right), \qquad n \geq 2 \tag{2.81}$$

for c in the $C^{0,\alpha}$-neighborhood $\mathcal{V}_*(\overline{c})$, and all f without restriction.

For emphasis, we repeat that the constant in (2.81) depends on n, but not on $c \in \mathcal{V}_*(\overline{c})$.

For our boundary conditions, we have the following

Lemma 11 *Let $\overline{c}$ and $\mathcal{V}_*(\overline{c})$ be as in Proposition 10. Then, there exists a neighborhood of $\overline{c}$ of the form*

$$\mathcal{V}_H(\overline{c}) = \{c \in C^{\infty}_{\overline{\Omega}} \mid \|c - \overline{c}\|_{C^{0,\alpha}_{\overline{\Omega}}} < \epsilon\} \subset \mathcal{V}_*(\overline{c}) \tag{2.82}$$

such that for any $c \in \mathcal{V}_H(\overline{c})$, $\phi \in \mathcal{U}_0$, and $n = 0, 1, 2, \ldots$ we have

$$\|\phi\|_{n+2,\alpha} \leq C \cdot \left(\|\Delta\phi + c\phi\|_{n,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\|c\|_{n,\alpha}\right) \tag{2.83}$$

and $\Delta\phi + c\phi = k$ is invertible for such c.

Proof

$\boxed{1}$ **Estimates in $C^{2,\alpha}_{\overline{\Omega}}$** We first establish estimates in the $C^{2,\alpha}$-norm,

$$\|\phi\|_{2,\alpha} \leq C \cdot \|\Delta\phi + c\phi\|_{0,\alpha}, \tag{2.84}$$

along with the useful inequalities

$$|\phi_{|\Gamma_i}|, \|\phi\|_{0,\alpha} \leq C\|\Delta\phi + c\phi\|_{0,\alpha}. \tag{2.85}$$

These will be proved valid for c in some $C^{0,\alpha}$-neighborhood of $\overline{c}$.

From Propostion 8, we know that $\Delta\colon \mathcal{U}^{2,\alpha}_0 \mapsto C^{0,\alpha}_{\overline{\Omega}}$ is an isomorphism. Since $c\phi$ is a compact perturbation, $\Delta + c\colon \mathcal{U}^{2,\alpha}_0 \mapsto C^{0,\alpha}_{\overline{\Omega}}$ is injective if and only if it is bijective, and by the open mapping theorem, it is an isomorphism. In particular, $\Delta + \overline{c}\colon \mathcal{U}^{2,\alpha}_0 \to C^{0,\alpha}_{\overline{\Omega}}$ is an isomorphism, and there exists a constant depending on $\overline{c}$ such that

$$\|\phi\|_{2,\alpha} \leq C(\overline{c}) \cdot \|\Delta\phi + \overline{c}\phi\|_{0,\alpha}. \tag{2.86}$$

In turn

$$\begin{aligned} \|\phi\|_{2,\alpha} &\leq C(\overline{c}) \cdot (\|\Delta\phi + c\phi\|_{0,\alpha} + \|(c-\overline{c})\phi\|_{0,\alpha}) && (2.87) \\ &\leq C(\overline{c}) \cdot (\|\Delta\phi + c\phi\|_{0,\alpha} + \|c-\overline{c}\|_{0,\alpha}\|\phi\|_{0,\alpha}). && (2.88) \end{aligned}$$

Choose ϵ_{**} sufficiently small (depending on the constant $C(\overline{c})$) so that if c is in the neighborhood

$$\mathcal{V}_{**}(\overline{c}) := \{c \mid \|c-\overline{c}\|_{0,\alpha} < \epsilon_{**}\} \tag{2.89}$$

then the last term on the right-hand side of the estimate on $\|\phi\|_{2,\alpha}$ can be incorporated into the left-hand side and so we obtain

$$\|\phi\|_{2,\alpha} \leq C \cdot \|\Delta\phi + c\phi\|_{0,\alpha}, \qquad c \in \mathcal{V}_H(\overline{c}), \quad \phi \in \mathcal{U}_0. \tag{2.90}$$

Note that the constant is independent of c taken in the neighborhood $\mathcal{V}_{**}(\overline{c})$.

As for the inequalities (2.85), we have $\sum_i |\phi_{|\Gamma_i}| \leq C\|\nabla\phi\|_{L^2}$ (Lemma 9, Section 2.2) and with (2.90), valid for any c in $\mathcal{V}_H(\overline{c})$, we arrive at

$$\sum_i |\phi_{|\Gamma_i}|, \|\phi\|_{0,\alpha} \leq C\|\phi\|_{2,\alpha} \leq C\|\Delta\phi + c\phi\|_{0,\alpha}. \tag{2.91}$$

□

$\boxed{2}$ **Higher-order estimates** We apply Proposition 10 to

$$u = \phi - \phi_{|\Gamma_i} g_i \tag{2.92}$$

where the g_i are as in the proof of Proposition 8 in Section 2.2 and set

$$\mathcal{V}_H(\overline{c}) := \mathcal{V}_*(\overline{c}) \cap \mathcal{V}_{**}(\overline{c}) \tag{2.93}$$

where $\mathcal{V}_*(\overline{c})$ is from Proposition 10.

Now,

$$\|\phi\|_{n,\alpha} \leq \|u\|_{n,\alpha} + C\sum_i |\phi_{|\Gamma_i}|. \tag{2.94}$$

Next, recall the tame estimates for the Dirichlet problem (2.78)

$$\|u\|_{n,\alpha} \leq C\Big(\|f\|_{n-2,\alpha} + \|c\|_{n-2,\alpha}\|f\|_{0,\alpha}\Big), \qquad n \geq 2. \tag{2.95}$$

By (2.85),

$$\|\phi\|_{n,\alpha} \leq \|u\|_{n,\alpha} + C \sum_i |\phi_{|\Gamma_i}| \leq \|u\|_{n,\alpha} + C\|\Delta\phi + c\phi\|_{0,\alpha}. \tag{2.96}$$

Also,

$$f = \Delta u + cu = \Delta\phi + c\phi - \phi_{|\Gamma_i}\Delta g_i - c\phi_{|\Gamma_i} g_i \tag{2.97}$$

satisfies then

$$\begin{aligned} \|f\|_{0,\alpha} &\leq \|\Delta\phi + c\phi\|_{0,\alpha} + C\sum_i |\phi_{|\Gamma_i}| + C\sum_i |\phi_{|\Gamma_i}| \|c\|_{0,\alpha} && (2.98)\\ &\leq C\|\Delta\phi + c\phi\|_{0,\alpha}(1 + \|c\|_{0,\alpha}) && (2.99)\\ &\leq C\|\Delta\phi + c\phi\|_{0,\alpha} && (2.100) \end{aligned}$$

since $\|c\|_{0,\alpha}$ is bounded for $c \in \mathcal{V}_H(\overline{c})$. Finally, more generally for $m \geq 0$,

$$\begin{aligned} \|f\|_{m,\alpha} &\leq \|\Delta\phi + c\phi\|_{m,\alpha} + C\sum_i |\phi_{|\Gamma_i}| + C\sum_i |\phi_{|\Gamma_i}| \|c\|_{m,\alpha} && (2.101)\\ &\leq C\|\Delta\phi + c\phi\|_{m,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\|c\|_{m,\alpha}. && (2.102) \end{aligned}$$

Putting all the above estimates together, we conclude with the desired estimate on ϕ:

$$\begin{aligned} \|\phi\|_{n,\alpha} &\leq \|u\|_{n-2,\alpha} + C\|\Delta\phi + c\phi\|_{0,\alpha} && (2.103)\\ &\leq C\left(\|f\|_{n-2,\alpha} + \|c\|_{n-2,\alpha}\|f\|_{0,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\right) && (2.104)\\ &\leq C\Big(\|\Delta\phi + c\phi\|_{n-2,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\|c\|_{n-2,\alpha} && (2.105)\\ &\quad + \|c\|_{n-2,\alpha}\|\Delta\phi + c\phi\|_{0,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\Big) && (2.106)\\ &\leq C\Big(\|\Delta\phi + c\phi\|_{n-2,\alpha} + \|\Delta\phi + c\phi\|_{0,\alpha}\|c\|_{n-2,\alpha}). && (2.107) \end{aligned}$$

■

Corollary 12 *The solution operator*

$$VE\colon \mathcal{V}_H(\overline{c}) \times C^\infty_{\overline{\Omega}} \to \mathcal{U}_0 \tag{2.108}$$

defined by $VE(c) \cdot k = \phi$ *such that*

$$\Delta\phi + c\phi = k, \quad \phi_{|\Gamma_0} = 0, \quad \phi_{|\Gamma_i} = \textit{constant}, \quad \int_{\Gamma_i} \frac{\partial\phi}{\partial N} = 0, \quad i = 1, 2, \ldots \tag{2.109}$$

is a smooth tame family of linear maps. $VE(c)\cdot k$ *has degree* -2 *in both* c *and* k*, and base 2:*

$$\|VE(c)\cdot k\|_{n,\alpha} \le C\cdot(\|k\|_{n-2,\alpha} + \|c\|_{n-2,\alpha}\|k\|_{0,\alpha}), \qquad n\ge 2 \tag{2.110}$$

for c *in the* $C^{0,\alpha}$*-neighborhood* $\mathcal{V}_H(\overline{c})$ *defined in Lemma 11 and any* $k\in C^\infty_{\overline{\Omega}}$ *(without restriction). In particular,*

$$\|VE(c)\cdot k\|_{n,\alpha} \le C\cdot(\|c\|_{n-2,\alpha}+1)\|k\|_{n-2,\alpha}, \qquad n\ge 2. \tag{2.111}$$

The first derivative with respect to c *is given by*

$$DVE(c)\cdot(k,\chi) = VE(c)\cdot(-\chi\phi). \tag{2.112}$$

Proof

$\boxed{1}$ **Continuity in c and k** Let $c,\tilde{c}\in\mathcal{V}_H(\overline{c})$ and $k,\tilde{k}\in C^\infty_{\overline{\Omega}}$, and set

$$\phi := VE(c)\cdot k, \qquad \Delta\phi + c\phi = k, \tag{2.113}$$

$$\tilde{\phi} := VE(\tilde{c})\cdot\tilde{k}, \qquad \Delta\tilde{\phi} + \tilde{c}\tilde{\phi} = \tilde{k}. \tag{2.114}$$

Then, $\phi-\tilde{\phi}$ solves

$$(\Delta+\tilde{c})\cdot(\phi-\tilde{\phi}) = (k-\tilde{k}) - (c-\tilde{c})\phi \tag{2.115}$$

and thus estimates of Lemma 11 and tame estimates for a product of functions (see Lemma 48 in Section A.2.3 of the Appendix) give

$$\begin{aligned}
\|\phi-\tilde{\phi}\|_{n+2,\alpha} &\le C\cdot\Big(\|k-\tilde{k}\|_{n,\alpha} + \|(c-\tilde{c})\phi\|_{n,\alpha} && (2.116)\\
&\quad +\|\tilde{c}\|_{n,\alpha}(\|k-\tilde{k}\|_{0,\alpha} + \|(c-\tilde{c})\phi\|_{0,\alpha}\Big) && (2.117)\\
&\le C_{(\|\tilde{c}\|_{n,\alpha})}\Big(\|k-\tilde{k}\|_{n,\alpha} + \|c-\tilde{c}\|_{n,\alpha}\|\phi\|_{n,\alpha}\Big) && (2.118)
\end{aligned}$$

where the constant depends on $\|\tilde{c}\|_{n,\alpha}$. Now with $\tilde{c}$ and $\tilde{k}$ fixed, if $\|k-\tilde{k}\|_{n,\alpha}$ and $\|c-\tilde{c}\|_{n,\alpha}$ are sufficiently small, so are $\|k-\tilde{k}\|_{n-2,\alpha}$ and $\|c-\tilde{c}\|_{n-2,\alpha}$, and thus $\|k\|_{n-2,\alpha}$ and $\|c\|_{n-2,\alpha}$ remain bounded, as well as $\|\phi\|_{n,\alpha}$ in view of linear estimates from Lemma 11. In turn, $\|\phi-\tilde{\phi}\|_{n+2,\alpha}$ can be made arbitrarily small for $\|k-\tilde{k}\|_{n,\alpha}$ and $\|c-\tilde{c}\|_{n,\alpha}$ sufficiently small. Since $n\ge 2$ is arbitrary, this shows that $VE(c)\cdot k$ is continuous as a map of

Fréchet spaces. $\square$

$\boxed{\mathbf{2}}$ **First derivative of VE** Since $VE(c) \cdot k$ is linear in k, the derivative in k is $VE(c) \cdot k$ itself, and we only need to compute the derivative in c. (See Proposition 38 in Section A.1.)

Fix then $\chi \in C^\infty_{\overline{\Omega}}$ and let $c_t = c + t\chi$ which is in $\mathcal{V}_H(\overline{c})$ for sufficiently small t. Denote $\phi_t = VE(c_t) \cdot k$, $\phi = \phi_0$:

$$\Delta\phi_t + c_t\phi_t = k, \qquad \Delta\phi + c\phi = k. \tag{2.119}$$

Then,

$$\Delta\frac{\phi_t - \phi}{t} + c\frac{\phi_t - \phi}{t} = -\chi\phi_t \tag{2.120}$$

or

$$\frac{\phi_t - \phi}{t} = VE(c) \cdot (-\chi\phi_t). \tag{2.121}$$

Since VE is continuous as a map of Fréchet spaces, the limit on the right-hand side exists in the C^∞-topology, i.e. the first derivative of $VE(c) \cdot k$ in c exists and is given by

$$DVE(c) \cdot (k, \chi) = D_c VE(c) \cdot (k, \chi) = VE(c) \cdot (-\chi\phi). \tag{2.122}$$

Since VE is (continuous and) tame, as well as multiplication of functions (Lemma 48, Section A.2.3), VE is continuously differentiable and tame. $\square$

$\boxed{\mathbf{3}}$ **VE is smooth tame** $E(c) \cdot \phi = \Delta\phi + c\phi$ is a smooth tame map as the sum of a linear differential operator with constant coefficients (Lemma 47, Section A.2.3), and multiplication of functions (Lemma 48, Section A.2.3), so it is smooth tame. We have just established the the inverse $VE(c) \cdot k$ is tame and continuously differentiable with tame first derivative. By Theorem I.5.3.1, p. 102, and Theorem II.3.1.1, p. 150 of [14], this implies that $VE(c) \cdot k$ is a smooth tame map. (This is the analogue of the one-dimensional case where, if f is of class C^n and has continuously differentiable inverse $g = f^{-1}$, then the inverse is of class C^n. This is obtained by induction and the formula $g' = \frac{1}{f' \circ g}$ and the fundamental theorem of calculus.) $\blacksquare$

2.4 A class of solutions for $\Delta\psi = F(\psi)$

In this section we construct solutions to $\Delta\psi = F(\psi)$ for a large class of nonlinearity F.

Let $G \in C^\infty(\mathbb{R})$ satisfy the growth conditions

$$|G(\psi)| < C \cdot (1 + |\psi|^{2-\delta}), \qquad |G''(\psi)| \leq C, \qquad \psi \in \mathbb{R}, \tag{2.123}$$

for some small $\delta > 0$. Letting

$$F = G' \tag{2.124}$$

a solution to the semilinear elliptic problem

$$\Delta\psi = F(\psi), \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots \tag{2.125}$$

where $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots$ are fixed constants is obtained by minimizing the functional

$$\mathcal{J}(\psi) := \int_\Omega [\frac{1}{2}|\nabla\psi|^2 + G(\psi)] - \psi_{|\Gamma_i}\gamma_i \tag{2.126}$$

over the space X introduced in (2.54):

$$X := \{\psi \in H^1(\Omega) \mid \psi_{|\Gamma_0} = 0,\ \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \ldots\}. \tag{2.127}$$

The proof is the usual direct method in the calculus of variations. Observe that in case $F' > 0$, $\mathcal{J}$ is strictly convex and the solution will be unique.

Heuristics The first derivative of $\mathcal{J}$ at ψ in the direction $\phi \in X$ is given by

$$\mathcal{J}'(\psi) \cdot \phi = \int_\Omega [\langle\nabla\psi, \nabla\phi\rangle + F(\psi)\phi] - \phi_{|\Gamma_i}\gamma_i \tag{2.128}$$

where we observe the usual convention of summing over repeated indices. If $\psi \in X$ is a smooth critical point of $\mathcal{J}$ then integrating by parts one finds

$$\int_\Omega (-\Delta\psi + F(\psi))\phi + \phi_{|\Gamma_i}\left(\int_{\Gamma_i} \frac{\partial\psi}{\partial N} - \gamma_i\right) = 0, \qquad \forall\phi \in X. \tag{2.129}$$

In particular, since we may choose $\phi \in H_0^1(\Omega)$ arbitrarily,

$$\Delta\psi = F(\psi) \qquad \text{in } \Omega, \tag{2.130}$$

and selecting $\phi = \phi_i$ equal to 1 in a neighborhood of Γ_i and letting its support shrink,

$$\int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots. \tag{2.131}$$

Proposition 13 *If F is such that the growth conditions (2.123) are satisfied, then $\mathcal{J}$ has a minimizer in X. If, in addition, $F' > 0$, then this minimizer is unique.*

Proof The direct method works as in the more usual examples, as we now verify. From the inequality (2.55) and the growth condition (2.123), $\mathcal{J}$ is continuously differentiable on X. We first show that

$$m := \inf_X \mathcal{J} > -\infty. \tag{2.132}$$

By Young's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$ and $(2-\delta)q = 2$,

$$\begin{aligned}
\int_\Omega |G(\psi)| &\leq C|\Omega| + C\int_\Omega |\psi|^{2-\delta} && (2.133)\\
&\leq C|\Omega| + C\left[\int_\Omega \frac{\left(\frac{1}{\delta_1}\right)^p}{p} + \frac{\left(\delta_1|\psi|^{2-\delta}\right)^q}{q}\right] && (2.134)\\
&\leq C(\Omega) + \frac{|\Omega|}{p\delta_1^p} + \frac{\delta_1^q}{q}\int_\Omega |\psi|^2 && (2.135)
\end{aligned}$$

so that, setting $\epsilon_1 = \frac{\delta_1^q}{q}$ and using estimates of Lemma 9 in Section 2.2.

$$\begin{aligned}
\mathcal{J}(\psi) &\geq \frac{1}{2}\|\nabla\psi\|^2_{L^2(\Omega)} - C(\Omega,\epsilon_1) - \epsilon_1\int_\Omega |\psi|^2 && (2.136)\\
&\quad -\sum_i\left[\frac{1}{4\epsilon_2}|\gamma_i|^2 + \epsilon_2|\psi_{|\Gamma_i}|^2\right] && (2.137)\\
&\geq (\frac{1}{2} - C_1(\Omega)^2\epsilon_1 - \Sigma_i\epsilon_2 C_1(\Omega)^2)\|\nabla\psi\|^2_{L^2(\Omega)} - C(\Omega,\epsilon_1) && (2.138)\\
&\quad -\sum_i \frac{1}{4\epsilon_2}|\gamma_i|^2 && (2.139)
\end{aligned}$$

Thus, choosing ϵ_1, $\epsilon_2 > 0$ sufficiently small, $\mathcal{J}$ is coercive:

$$\mathcal{J}(\psi) \geq \frac{1}{4}\|\nabla\psi\|^2_{L^2(\Omega)} + \kappa, \qquad \psi \in X \tag{2.140}$$

where $\kappa = \kappa(\Omega, \delta, \gamma_i)$. In particular, $\mathcal{J}(\psi)$ is bounded from below for $\psi \in X$.

Let now $\psi^n \in X$ be a minimizing sequence:

$$\mathcal{J}(\psi^n) \to_n m. \tag{2.141}$$

In particular, from (2.140) we see that ψ^n is bounded in H^1, hence by weak compactness, extracting a subsequence if necessary, ψ^n converges weakly in $H^1(\Omega)$ and strongly in

$L^2(\Omega)$ to some $\psi \in H^1(\Omega)$. Now

$$\begin{aligned}
\mathcal{J}(\psi^n) &= \int \left[\frac{1}{2}|\nabla\psi^n|^2 + G(\psi^n)\right] - \psi^n_{|\Gamma_i}\gamma_i && (2.142)\\
&\geq \int \left[\frac{1}{2}|\nabla\psi|^2 + G(\psi)\right] - \psi_{|\Gamma_i}\gamma_i && (2.143)\\
&\quad + \int_\Omega [\langle\nabla\psi, \nabla\psi^n - \nabla\psi\rangle + G(\psi^n) - G(\psi)] + (\psi_{|\Gamma_i} - \psi^n_{|\Gamma_i})\gamma_i && (2.144)\\
&\geq \mathcal{J}(\psi) && (2.145)\\
&\quad + \int_\Omega [\langle\nabla\psi, \nabla\psi^n - \nabla\psi\rangle + G(\psi^n) - G(\psi)] + (\psi_{|\Gamma_i} - \psi^n_{|\Gamma_i})\gamma_i && (2.146)
\end{aligned}$$

and by Taylor's expansion,

$$G(\psi^n) - G(\psi) = G'(\psi)(\psi^n - \psi) + \int_0^1 (1-\theta)G''(\psi + \theta(\psi^n - \psi))(\psi^n - \psi)^2 d\theta. \quad (2.147)$$

Since ψ^n converges to ψ in L^2 and $G'(\psi) \in L^2(\Omega)$ by the growth conditions (2.123) and the fact that $\psi \in L^2(\Omega)$, we have that $\int_\Omega G'(\psi)(\psi^n - \psi) \to_n 0$. Also by the growth conditions, the last term in Taylor's expansion is bounded by $|\psi^n - \psi|^2$ and thus integrating over Ω and letting $n \to \infty$, this term vanishes as well. In other words,

$$m = \lim_n \mathcal{J}(\psi^n) \geq \mathcal{J}(\psi) \quad (2.148)$$

hence we have equality by definition of m as an infimum. □

In case $F' > 0$, the minimizer is unique as follows easily since $|\nabla\psi|^2$ and $G(\psi)$ are strictly convex in ψ, while $\psi_{|\Gamma_i}\gamma_i$ is linear in ψ: suppose $\tilde{\psi} \neq \psi$ is another minimizer, and let $\overline{\psi} = \frac{1}{2}(\psi + \tilde{\psi})$ be their midpoint, so that

$$\begin{aligned}
m &\leq \mathcal{J}(\overline{\psi}) && (2.149)\\
&= \int \left(\frac{1}{2}|\nabla\overline{\psi}|^2 + G(\overline{\psi})\right) - \overline{\psi}_{|\Gamma_i}\gamma_i && (2.150)\\
&< \frac{1}{2}\left\{\int \left(\frac{1}{2}|\nabla\psi|^2 + G(\psi)\right) - \psi_{|\Gamma_i}\gamma_i\right\} && (2.151)\\
&\quad + \frac{1}{2}\left\{\int \left(\frac{1}{2}|\nabla\tilde{\psi}|^2 + G(\tilde{\psi})\right) - \tilde{\psi}_{|\Gamma_i}\gamma_i\right\} && (2.152)\\
&= m && (2.153)
\end{aligned}$$

which is absurd. ■

2.5 The solution operator $\psi = S(F)$ for $\Delta\psi = F(\psi)$

We conclude this chapter by defining a solution operator, which to each F in a certain neighborhood of a generic $\overline{F}$ (see below on this) sends a uniquely determined solution ψ to $\Delta\psi = F(\psi)$, and we show that such an operator is a smooth tame map.

A first technicality must be dealt with: let $\overline{F} \in C^\infty_{\mathbb{R}}$ and $\overline{\psi} \in C^\infty_{\overline{\Omega}}$ be such that

$$\Delta\overline{\psi} = \overline{F}(\overline{\psi}), \qquad \overline{\psi}_{|\Gamma_o} = 0, \qquad \overline{\psi}_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial\overline{\psi}}{\partial N} = \gamma_i, \qquad i = 1, 2, \ldots. \tag{2.154}$$

$\overline{F}$ need only be defined on range$(\overline{\psi})$, that is, if $\overline{F}$ is modified outside of range$(\overline{\psi})$, $\overline{\psi}$ remains a solution of the modified equation. On the other hand, for F "near" $\overline{F}$, the corresponding stream function ψ may have different range. Thus, we would like to control the range of the solution with F itself.

The following Lemma shows that we can choose an interval I containing the range of $\overline{\psi}$, and a neighborhood of $\overline{F}$ in C^∞_I in which each F has at least one solution ψ whose range is contained in I. Note that this does not imply uniqueness, and therefore this does not yet define a solution operator. This is done next in Proposition 15.

Lemma 14 *Let $\overline{\psi} \in C^\infty_{\overline{\Omega}}$ solve $\Delta\overline{\psi} = \overline{F}(\overline{\psi})$, where $\overline{F} \in C^\infty_{\mathbb{R}}$, subject to $\overline{\psi}_{|\Gamma_0} = 0$, $\overline{\psi}_{|\Gamma_i}$ =constant, $\int_{\Gamma_i} \frac{\partial\overline{\psi}}{\partial N} = \gamma_i$, $i = 1, 2, \ldots$. There exists an interval I containing range$(\overline{\psi})$ and a neighborhood of the restriction of $\overline{F}$ to I (still denoted $\overline{F}$) of the form*

$$\mathcal{V}_1(\overline{F}) = \{F \in C^\infty_I \mid \|F - \overline{F}\|_{C^0_I} < \epsilon'\} \tag{2.155}$$

such that for each $F \in \mathcal{V}_1(\overline{F})$, there exists at least one solution $\psi \in C^\infty_I$ to $\Delta\psi = F(\psi)$ with the same boundary conditions, and whose range is contained in I.

Proof The solution ψ corresponding to F will be a minimizer of a functional $\mathcal{J}$ as in (2.126) where $F = G'$. The L^2-estimate for the Dirichlet problem gives for ψ solving $\Delta\psi = \omega$ the estimate

$$\|\psi\|_{H^2(\Omega)} \le C(\Omega) \cdot (\|\omega\|_{L^2(\Omega)} + \sum_i |\gamma_i|) \tag{2.156}$$

by writing $u = \psi - \psi_{|\Gamma_i} g_i$ where the g_i's are introduced in Lemma 11 of Section 2.3, From the embedding $H^2(\Omega) \hookrightarrow C^0(\Omega)$, there exists a constant $C_3 = C_3(\overline{\Omega})$ independent of ψ such that

$$\|\overline{\psi}\|_{C^0_{\overline{\Omega}}} \leq C_3 \cdot (\sup_{\text{range }(\overline{\psi})} |\overline{F}| + \sum_i |\gamma_i|). \tag{2.157}$$

Set $K := \sup_{\text{range }\overline{\psi}} |\overline{F}| + \sum_i |\gamma_i| + 1$ and let $M > C_3 \cdot (K+1)$. Modify now $\overline{F}$ outside of range$(\overline{\psi})$ to a new function $\tilde{F} \in C^\infty_{\mathbb{R}}$ so that

$$\tilde{F}(\zeta) = -K \quad \text{for} \quad \zeta \leq -M-1, \qquad \tilde{F}(\zeta) = K \quad \text{for} \quad \zeta \geq M+1. \tag{2.158}$$

For any $F \in C^\infty_{\mathbb{R}}$ such that $|F - \tilde{F}|$ where $0 < \epsilon' < 1$ *and* for which $G = \int F$ satisfies the growth conditions (2.123) then a solution ψ to $\Delta\psi = F(\psi)$ exists obtained by minimizing a functional $\mathcal{J}$ depending on F, and this solution satisfies

$$\sup_{\Omega} |\psi| \leq C_3 \cdot (K + \epsilon') \leq M. \tag{2.159}$$

Set

$$I := [-M, M]. \tag{2.160}$$

Let now $F \in C^\infty_I$ such that $|F - \overline{F}| < \epsilon'$ on I. Extend F to a smooth function on $\mathbb{R}$ in such a way that $|F| < K$ and such that the growth conditions (2.123) are satisfied. Then, a solution to $\Delta\psi = F(\psi)$ exists by Proposition 13 of Section 2.4 and range$(\psi) \subset I$ so that it is independent of the extension of F. ■

The previous lemma guarantees existence of solutions, but not uniqueness. even restricting to neighborhoods of $\overline{F}$ and $\overline{\psi}$. For this we need to consider a **generic solution** in the sense that the linearized operator $\Delta - \overline{F}'(\overline{\psi})$ is invertible (and thus $\Delta - F'(\psi)$ is invertible for F in a neighborhood of $\overline{F}$). Note that when $F' > 0$, this is automatically the case by the maximum principle. (See also Section 4.2.)

Proposition 15 *Suppose that $\overline{F}$ is generic in the sense that $\Delta - \overline{F}'(\overline{\psi})$ is invertible. Then there exists a $C^{2,\alpha}$-neighborhood*

$$\mathcal{V}_2(\overline{F}) \subset \mathcal{V}_1(\overline{F}) \tag{2.161}$$

of $\overline{F}$ in C^∞_I, a neighborhood $\mathcal{V}(\overline{\psi})$ of $\overline{\psi}$ in $\mathcal{U}_{\gamma_i}$, and a smooth tame map

$$S\colon \mathcal{V}_2(\overline{F}) \to C^\infty_{\overline{\Omega}} \tag{2.162}$$

returning the unique solution to $\Delta\psi = F(\psi)$ *in* $\mathcal{V}(\overline{\psi})$*. The map* $\psi = S(F)$ *is tame with degree* -2 *in* F*:*

$$\|\psi\|_{n,\alpha} \leq C(\|F\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{2.163}$$

for $F \in \mathcal{V}_2(\overline{F})$*. The first derivative is given by*

$$\phi = DS(F) \cdot f = VE(-F'(\psi)) \cdot (f \circ \psi). \tag{2.164}$$

If Ω *is diffeomorphic to an annulus, and* $\overline{\psi}$ *has no critical points, then we make* $\mathcal{V}_2(\overline{F})$ *smaller so that* $\psi = S(F)$ *for* $F \in \mathcal{V}_2(\overline{F})$ *has no critical points either.*

Proof $\mathcal{V}_1(\overline{F})$ is defined in Lemma 14. Note that by elliptic regularity (Proposition 8, Section 2.2), smoothness of $\overline{F}$ implies smoothness of $\overline{\psi}$. Recall that the spaces $\mathcal{U}_0^{n,\alpha}$ and $\mathcal{U}_{\gamma_i}^{n,\alpha}$ of stream functions of class $C^{n,\alpha}$ are defined in (2.75) and (2.76) and that the spaces $\mathcal{U}_0$ and $\mathcal{U}_{\gamma_i}$ of smooth stream functions are defined in (2.72) and (2.73).

$\boxed{1}$ **Existence of the solution map** S Consider the continuously differentiable map

$$H\colon C^{2,\alpha}(I) \times \mathcal{U}_{\gamma_i}^{2,\alpha} \to C^{0,\alpha}_{\overline{\Omega}}, \qquad H(F,\psi) := \Delta\psi - F(\psi). \tag{2.165}$$

Since

$$D_\psi H(\overline{F}, \overline{\psi}) \cdot \phi = \Delta\phi - \overline{F}'(\overline{\psi})\phi \tag{2.166}$$

is an isomorphism of Banach spaces $\mathcal{U}_0^{2,\alpha} \to C^{0,\alpha}_{\overline{\Omega}}$ ($\overline{F}$ is assumed to be generic), the (classical) implicit function theorem guarantees existence of neighborhoods $V_2(\overline{F}) \subset C^{2,\alpha}(I)$ of $\overline{F}$ and $V(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^{2,\alpha}$ of $\overline{\psi}$, and a continuously differentiable map

$$\tilde{S}\colon V_2(\overline{F}) \to \mathcal{U}_{\gamma_i}^{2,\alpha} \tag{2.167}$$

returning the unique solution $\psi \in \mathcal{U}_{\gamma_i}^{2,\alpha}$ to $\Delta\psi = F(\psi)$ in $V(\overline{\psi})$. If $F \in V_2(\overline{F}) \cap C^{l,\alpha}(I)$ ($l \geq 2$), then by bootstrapping $\psi = \tilde{S}(F) \in \mathcal{U}_{\gamma_i}^{l+2,\alpha}$, and if $F \in V_2(\overline{F}) \cap C^\infty(I)$, then the solution is smooth.

Let now $\epsilon'' > 0$ be sufficiently small so that if $F \in C^\infty$ and $\|F - \overline{F}\|_{C^{2,\alpha}_I} < \epsilon''$, then on the one hand $F \in \mathcal{V}_1(\overline{F})$ (as defined in (2.155) of Lemma 14) and on the other, by continuity of $\tilde{S}$, $F'(\psi) \in \mathcal{V}_H(\overline{F}'(\overline{\psi}))$ (taking $\overline{c} = \overline{F}'(\overline{\psi})$ in the definition of $\mathcal{V}_H(\overline{c})$ in (2.82) of Lemma 11 from Section 2.3). Setting

$$\mathcal{V}_2(\overline{F}) \;\; := \;\; V_2(\overline{F}) \cap \{F \in C^\infty_I \mid \|F - \overline{F}\|_{C^2_I} < \epsilon''\}, \tag{2.168}$$

$\tilde{S}$ restricts to a map

$$S\colon \mathcal{V}_2(\overline{F}) \to \mathcal{U}_{\gamma_i}, \qquad \psi = S(F), \qquad \Delta\psi = F(\psi) \tag{2.169}$$

with image contained in $\mathcal{U}_{\gamma_i} \cap V(\overline{\psi})$. Also, for all $F \in \mathcal{V}_2(\overline{F})$, $\Delta - F'(\psi)$ is invertible. □

$\boxed{2}$ **Continuity of S as a map of Fréchet spaces** Let F_j converge to F in the $C^{n,\alpha}$-norm for some $n \geq 2$. Then, by continuity of $\tilde{S}$, $\psi_j = S(F_j)$ converges to $\psi = S(F)$ in the $C^{2,\alpha}$-norm (at least), and $\psi_j - \psi$ solves the elliptic equation

$$\Delta(\psi_j - \psi) = F_j(\psi_j) - F(\psi). \tag{2.170}$$

Now composition $C^{l+1,\alpha} \times C^{l,\alpha}$ is continuous (Lemma 50, Section A.2.3) so that the right-hand side goes to 0 in $C^{1,\alpha}_{\overline{\Omega}}$, and therefore $\psi_j \to_j \psi$ in $C^{3,\alpha}_{\overline{\Omega}}$ by Proposition 8 of Section 2.2. Repeating, $\psi_j \to_j \psi$ in $C^{n+1,\alpha}_{\overline{\Omega}}$. In other words, $\tilde{S}$ restricts to continuous maps

$$C^{n,\alpha}(I) \to \mathcal{U}^{n+1,\alpha}_{\gamma_i}, \qquad n \geq 2. \tag{2.171}$$

□

$\boxed{3}$ **Tame estimates** From estimates of Proposition 8 in Section 2.2 and tame estimates on composition of functions (see Lemma 50 of Section A.2.3 of the Appendix), we obtain

$$\begin{aligned} \|\psi\|_{n+2,\alpha} &\leq C \cdot (\|F \circ \psi\|_{n,\alpha} + \sum_i |\gamma_i|), \qquad n \geq 0 && (2.172)\\ &\leq C \cdot (\|F\|_{n,\alpha} + \|\psi\|_{n,\alpha} + 1), \qquad n \geq 0 && (2.173)\end{aligned}$$

for F in a $C^{3,\alpha}$-neighborhood and as long as ψ remains in a $C^{1,\alpha}$-neighborhood, which is the case from (2.171). In turn, interpolation inequality $\|\psi\|_{n,\alpha} \leq \epsilon_1 \|\psi\|_{C^{n+1,\alpha}} + C(\epsilon_1)\|\psi\|_{0,\alpha}$ with ϵ_1 sufficiently small (though depending on n) gives

$$\|\psi\|_{n+2,\alpha} \leq C \cdot (\|F\|_{n,\alpha} + \|\psi\|_{0,\alpha} + 1), \qquad n \geq 0. \tag{2.174}$$

In order to get rid of the term in ψ on the right-hand side use

$$\|\psi\|_{0,\alpha} \leq C\|\psi\|_{H^2} \leq C(\|F(\psi)\|_{L^2} + \sum_i |\gamma_i|) \leq C \cdot (\|F\|_{C^0_I} + 1) \tag{2.175}$$

and we conclude with

$$\|\psi\|_{n+2,\alpha} \leq C \cdot (\|F\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{2.176}$$

for F in a $C^{1,\alpha}_{\overline{\Omega}}$-neighborhood, and in particular for $F \in \mathcal{V}_2(\overline{F})$ if one makes it smaller. Relabelling indices, we obtain the desired tame estimates. □

$\boxed{4}$ **First derivative** Fix $F \in \mathcal{V}_2(\overline{F})$ and set $\psi = S(F)$.

Let $f \in C^{3,\alpha}(I)$ and set $F_t := F + tf \in V_2(\overline{F})$ for small t, $\psi_t := \tilde{S}(F_t)$, $\psi := \tilde{S}(F)$ both in $V(\overline{\psi})$. Since $\tilde{S}$ is continuously differentiable, denote by

$$\phi := C^{2,\alpha} - \lim_{t} \frac{\psi_t - \psi}{t}. \tag{2.177}$$

Write

$$\Delta \frac{\psi_t - \psi}{t} = f(\psi_t) + \frac{F(\psi_t) - F(\psi)}{t}. \tag{2.178}$$

Since $f \in C^{3,\alpha}_I$, by continuity of S, (2.171), we have that $\psi_t \to \psi$ in $C^{1,\alpha}_{\overline{\Omega}}$ (at least) and so $f(\psi_t) \to f(\psi)$ in $C^{1,\alpha}_{\overline{\Omega}}$ (see Lemma 50, Section A.2.3 on composition of functions). In addition, F is smooth so that $\frac{F(\psi_t)-F(\psi)}{t} \to F'(\psi)\phi$ in $C^{1,\alpha}_{\overline{\Omega}}$. In conclusion, the right-hand side converges in $C^{1,\alpha}_{\overline{\Omega}}$ and therefore by elliptic regularity (Proposition 8, Section 2.2) $\frac{\psi_t - \psi}{t} \to \phi$ in $C^{3,\alpha}$ and ϕ satisfies

$$\Delta\phi = f(\psi) + F'(\psi)\phi \tag{2.179}$$

in $C^{1,\alpha}_{\overline{\Omega}}$. Repeating with $f \in C^{4,\alpha}_I$, $C^{5,\alpha}_I$, $\ldots$, we conclude that if $f \in C^{l,\alpha}$, then

$$\phi = C^{l+1,\alpha} - \lim_{t \to 0} \frac{\psi_t - \psi}{t} \tag{2.180}$$

and ϕ satisfies (2.179) in $C^{l-1,\alpha}_{\overline{\Omega}}$. This implies that S is differentiable at $F \in \mathcal{V}_2(\overline{F})$, with derivative

$$\phi = DS(F) \cdot f = VE(-F'(\psi)) \cdot (f \circ \psi) \tag{2.181}$$

where we use notation from Corollary 12 of Section 2.3:

$$VE(c) \cdot k = \phi, \qquad \Delta\phi + c\phi = k \tag{2.182}$$

and ϕ satisfies the boundary condition

$$\phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial \phi}{\partial N} = 0, \qquad i = 1, 2, \ldots. \tag{2.183}$$

Since $\psi = S(F)$ and F' are tame maps of F, that composition of functions is tame (Lemma 50, Section A.2.3) and $VE(c) \cdot k$ is tame as well (Corollary 12 of Section 2.3), we conclude that $\psi = S(F)$ is continously differentiable and its derivative is tame. $\square$

$\boxed{5}$ **Higher derivatives** The first derivative ϕ_1 of $\psi = S(F)$ in the direction f_1 has been obtained by differentiating $\Delta\psi = F(\psi)$:

$$\Delta\phi_1 = F'(\psi)\phi_1 + f_1(\psi). \tag{2.184}$$

We see that $\phi_1 = VE(-F' \circ \psi) \cdot f_1(\psi)$ is a tame map of F and f_1, and continuously differentiable. Thus the second derivative ϕ_{12} of ϕ_1 with respect to F in the direction f_2 can be computed by differentiating (2.184):

$$\Delta\phi_{12} - F'(\psi)\phi_{12} = f_2'(\psi)\phi_1 + F''(\psi)\phi_1\phi_2 + f_1'(\psi)\phi_2 \tag{2.185}$$

where ϕ_2 is the first derivative of ψ in the direction f_2. Thus ϕ_{12} is a tame map of F, f_1 and f_2, and continuously differentiable in F. More generally, denote $\phi_{i_1 \ldots i_k}$, $i_1 < i_2 < \cdots < i_k$, the k-th derivative of ϕ in the direction $(f_{i_1}, \ldots, f_{i_k})$. By induction, the n-th derivative $\phi_{1\ldots n} = D^n(S(F)) \cdot (f_1, \ldots, f_n)$ is such that $\Delta\phi_{1\ldots n} - F'(\psi)\phi_{1\ldots n}$ is the sum of products of $f_1^{(n-1)}(\psi), \ldots, f_n^{(n-1)}(\psi)$ or $F^{(n)}(\psi)$ with a number of factors of the form $\phi_{i_1 \ldots i_k}$ where $1 \leq k \leq n-1$ and $i_k \leq n-1$. Thus, ψ has derivatives of all orders, and the n-th derivative is a tame map of $F, f_1, \ldots, f_n$. $\blacksquare$

Chapter 3

The Euler flow as a family of Hamiltonian systems

In this Chapter we make precise the sense in which the Euler flow of hydrodynamics can be viewed as a family of Hamiltonian systems. We begin with a brief summary of classical mechanics. References for this Section are [3], [24].

3.1 Lagrangian mechanics, Hamiltonian formalism, and symplectic reduction

Let an n-dimensional manifold M be the configuration space of a mechanical system $\mathcal{S}$, and let its motion be dictated by a Lagrangian

$$L\colon TM \to \mathbb{R}. \tag{3.1}$$

For instance, L can be the kinetic energy minus the potential energy,

$$L(q, \dot{q}) = E(q, \dot{q}) - V(q). \tag{3.2}$$

Namely, the actual motion $q(t)$ between two moments t_1 and t_2 corresponds to a critical point of the action

$$\int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt. \tag{3.3}$$

In other words, $(q, \dot{q})$ satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} = 0, \qquad j = 1, \ldots, n. \tag{3.4}$$

(Under certain conditions of convexity, these equations are sufficient to determine the motion $(q, \dot{q})$.) In the case where the Lagrangian reduces to the kinetic energy,

$$L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2, \tag{3.5}$$

the latter lends M a metric, and the Euler-Lagrange equations are equivalent to the geodesic equations relative to that metric. In other words, the motion of $\mathcal{S}$ is represented by a geodesic on the configuration space M.

Alternatively, the motion can be described by a Hamiltonian system on the phase space T^*M. The cotangent bundle T^*M has a natural **symplectic structure**, that is, a closed, nondegenerate 2-form given in (local) canonical coordinates $(q^1, \ldots, q^n; p_1, \ldots, p_n)$ by

$$\omega = dq^i \wedge dp_i. \tag{3.6}$$

A **Hamiltonian vector field** X_H associated to the function H on T^*M is determined by

$$\omega(X_H, v) = dH \cdot v, \qquad v \in T\,T^*M. \tag{3.7}$$

In local canonical coordinates,

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \sum_{i=1}^{n} \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{3.8}$$

To the Lagrangian system described at the beginning of this section one may associate the Hamiltonian H according to

$$H(q, p) = (p, \dot{q}) - L(q, \dot{q}) \tag{3.9}$$

where $(\cdot, \cdot)$ denotes the pairing $T^*M \times TM$ and the point $(q, \dot{q}) \in TM$ is related to the point $(q, p) \in T^*M$ via the **Legendre transform**

$$p_j = \frac{\partial L}{\partial \dot{q}^j}, \qquad j = 1, 2, \ldots, n. \tag{3.10}$$

Then, $(q, \dot{q})$ is a solution to the Euler-Lagrange equations (3.4) if and only if (q, p) is a solution to the Hamiltonian system given in canonical coordinates by

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad j = 1, \ldots, n. \tag{3.11}$$

Introducing the Poisson bracket

$$\{F, G\} := \omega(X_F, X_G) = \sum_{j=1}^{n} F_{q^j} G_{p_j} - F_{p_j} G_{q^j}, \tag{3.12}$$

Hamilton's equations (3.11) are equivalent to

$$\dot{F} = \{F, H\}, \qquad F \in C^\infty(T^*M) \tag{3.13}$$

by which we mean that the solution $(q(t), p(t))$ solves Hamilton's equations (3.11) if and only if $\frac{d}{dt}(F(q(t), p(t))) = \{F, H\}(q(t), p(t))$. The Poisson bracket is a bilinear, skew-symmetric operator, and satisfies the Leibniz rule

$$\{FG, H\} = F\{G, H\} + G\{F, H\} \tag{3.14}$$

and the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0. \tag{3.15}$$

This gives T^*M the structure of a **Poisson manifold**, which is a more general structure than that of a symplectic manifold (which T^*M is). On a Poisson manifold, (3.13) defines a **Hamiltonian system**. (When the manifold is symplectic, the two definitions coincide.) In general, a Poisson manifold can loosely be described as the union of symplectic manifolds called the symplectic leaves of the foliation. We will return to this description later.

Suppose now that a group G acts on M on the left. That is, there is a map $G \times M \to M, (g, q) \mapsto g \cdot q$ (the **action**) such that

$$g \cdot (h \cdot q) = (gh) \cdot q, \qquad g, h \in G, \quad q \in M. \tag{3.16}$$

This extends to a left-action on the tangent bundle, denoted $g \cdot \dot{q}$. We say that G is a group of symmetry for the system $\mathcal{S}$ if it leaves the Lagrangian invariant:

$$L(g \cdot (q, \dot{q})) = L(q, \dot{q}). \tag{3.17}$$

Likewise, the left-action $G \times M \to M$ extends to a left-action on the cotangent bundle T^*M, denoted

$$G \times T^*M \to T^*M, \qquad (g \cdot p) \mapsto g \cdot p. \tag{3.18}$$

G-invariance of the Lagrangian L, $L(g \cdot q, g \cdot \dot{q}) = L(q, \dot{q})$, implies G-invariance of the hamiltonian H

$$H(g \cdot q, g \cdot p) = H(q, p) \tag{3.19}$$

since it can be shown that $(g \cdot p, g \cdot \dot{q}) = (p, \dot{q})$ (these parentheses denote the pairing $T^*M \times TM$).

Let

$$\pi \colon T^*M \to T^*M/G \tag{3.20}$$

be the canonical projection onto the quotient space T^*M/G, i.e. the space of orbits of this action. We may identify the (smooth) functions on the quotient space with the (smooth) G-invariant functions on the phase space T^*M. Thus, the identity

$$\{f, k\}^{\sim} \circ \pi = \{f \circ \pi, k \circ \pi\}, \qquad f, k \in C^\infty(T^*M/G) \tag{3.21}$$

allows to define a Poisson bracket on T^*M/G, i.e. it is clearly bilinear and skew-symmetric, and it satisfies Leibniz' rule and Jacobi's identity because the bracket $\{\cdot, \cdot\}$ on T^*M does. By construction of $\{\cdot, \cdot\}^{\sim}$, π is a **Poisson map** with respect to these Poisson brackets, that is, π preserves the Poisson structure.

Let P be a Poisson manifold with Poisson bracket $\{\cdot, \cdot\}$. Given a function H on P, the **Hamiltonian vector field** associated to H is the unique vector field X_H such that

$$\{F, H\} = X_H(F), \qquad F \in C^\infty(P). \tag{3.22}$$

Existence and uniqueness of this Hamiltonian vector field is guaranteed since the bracket satisfies Leibniz' rule.

In particular, since T^*M and T^*M/G are Poisson manifolds, we may define Hamiltonian systems on them. Denote ϕ_t the flow of the Hamiltonian h on T^*M/G, and ψ_t the flow of the Hamiltonian $h \circ \pi$ on T^*M. Then,

$$\phi_t \circ \pi = \pi \circ \psi_t \tag{3.23}$$

since both sides are solutions to the same Hamiltonian system with same initial conditions. In other words, π takes the Hamiltonian flow for $H = h \circ \pi$ on T^*M to the

Hamiltonian flow for h on T^*M/G. We also say that we have **reduced** the Hamiltonian system X_H on T^*M to that of X_h on T^*M/G.

A nice, geometric description of Poisson manifolds is that they are the disjoint union of immersed submanifolds, each of which carries a symplectic structure. These submanifolds are called the **symplectic leaves** of the ambient Poisson manifold. (When they are actually embedded submanifolds, they form a foliation, in some loose sense which will be made clearer later.) Let's now see how they arise. At a point $z \in P$ on a Poisson manifold, $\{F, H\}(z)$ depends on $dF(z)$ and $dH(z)$ only via a tensor (the Poisson tensor) and the vectors $X_{H|z}$ span a subspace of T_zP, the dimension of which depends on the rank of the Poisson tensor. Therefore, from the Poisson bracket we obtain a distribution in the tangent bundle of P, and the integral manifolds are those symplectic leaves. Observe that the rank of the Poisson tensor may vary with z, and therefore a singular Frobenius theorem needs to be invoked. The symplectic structure on a leaf can be calculated by first restricting the Poisson bracket on the whole manifold P to the leaf, and then using this bracket to define the symplectic structure as in (3.12). See the **Symplectic Stratification Theorem**, [24], for a proof of the above. (The term 'stratification' is chosen instead of 'foliation' to reflect the fact that the dimension of the symplectic leaves may vary.)

When P has in fact a symplectic structure determined by the two-form ω, the Poisson bracket is given by

$$\{F, H\} = \omega(X_F, X_H). \tag{3.24}$$

Then, both definitions of Hamiltonian vector fields, namely (3.7), valid more generally for any symplectic manifold with two-form ω, and (3.22), agree. Since ω is non-degenerate, the Hamiltonian vector fields span, at each $z \in P$, the whole tangent space, and therefore P coincides with its single symplectic leaf.

Consider now the important case where $M = G$ acts on itself by left-translation:

$$h \cdot g = hg =: L_h \cdot g. \tag{3.25}$$

This extends to a left-action on TG according to

$$u \in T_gG \qquad \mapsto \qquad h \cdot u = T_gL_h \cdot u \in T_{hg}G \tag{3.26}$$

and to a left-action on T^*G according to

$$\alpha \in T_g^*G \qquad \mapsto \qquad h \cdot \alpha = T_g^* L_{h^{-1}} \cdot \alpha \in T_{hg}^* G. \tag{3.27}$$

The quotient space T^*G/G can be identified with the dual $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ and the projection is given by

$$\pi \colon T^*G \to \mathfrak{g}^*, \qquad \pi(g, \alpha) = T_g^* L_g \cdot \alpha. \tag{3.28}$$

The Poisson bracket $\{\cdot,\cdot\}^\sim$ on $\mathfrak{g}^*$ is obtained from the Poisson bracket $\{\cdot,\cdot\}$ on T^*G and the identity

$$\{f,k\}^\sim \circ \pi = \{f\circ\pi, k\circ\pi\}, \qquad f,k \in C^\infty(\mathfrak{g}^*). \tag{3.29}$$

The functions $F = f\circ\pi$ and $K = k\circ\pi$ are none other then the left-invariant extensions of f and k from $\mathfrak{g}^*$ to T^*G:

$$F(g,\alpha) = f(T_g^* L_g \cdot \alpha), \qquad K(g,\alpha) = k(T_g^* L_g \cdot \alpha), \tag{3.30}$$

since (by definition) they satisfy for all $h \in G$ and $\alpha \in T_g^*G$,

$$F(hg, T_t^* L_{h^{-1}} \cdot \alpha) = F(g, \alpha_g), \qquad F(hg, T_t^* L_{h^{-1}} \cdot \alpha) = F(g, \alpha_g). \tag{3.31}$$

It can be shown that the Poisson bracket on $\mathfrak{g}^*$ so obtained is given by the **Lie-Poisson** bracket

$$\{f,g\}^\sim(\mu) = -(\mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\partial \mu}]), \qquad \mu \in \mathfrak{g}^* \tag{3.32}$$

where $(\cdot,\cdot)$ denotes the pairing $\mathfrak{g}^* \times \mathfrak{g}$, $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{g}$, and $\frac{\delta f}{\delta \mu}$ is the element of $\mathfrak{g}$ identified with the differential df on the double dual of $\mathfrak{g}$. The Lie-Poisson bracket can also be expressed as follows

$$\{f,h\}^\sim(\mu) = -(\mu, [\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}]) = (\mu, [\frac{\delta h}{\delta \mu}, \frac{\delta f}{\delta \mu}]) = (\mu, \mathrm{ad}_{\frac{\delta h}{\delta \mu}} \frac{\delta f}{\delta \mu}) = (\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu, \frac{\delta f}{\delta \mu}) \tag{3.33}$$

where ad is the adjoint operator on $\mathfrak{g}$ simply given by the Lie bracket on $\mathfrak{g}$

$$\mathrm{ad}_\xi \eta = [\xi, \eta], \qquad \xi, \eta \in \mathfrak{g} \tag{3.34}$$

and ad^* is the co-adjoint operator, i.e. the dual to ad:

$$(\mathrm{ad}_\xi^* \nu, \eta) = (\nu, \mathrm{ad}_\xi \eta), \qquad \nu \in \mathfrak{g}^*, \qquad \xi, \eta \in \mathfrak{g}. \tag{3.35}$$

In order to obtain an equation in $\mathfrak{g}^*$ directly, let $\mu(t)$ be a solution to $\dot{f} = \{f, h\}$. On the one hand,

$$\dot{f} = \frac{d}{dt} f(\mu(t)) = df \cdot \dot{\mu} = (\dot{\mu}, \frac{\delta f}{\delta \mu}) \tag{3.36}$$

and on the other

$$\{f, h\}^{\sim}(\mu) = (\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu, \frac{\delta f}{\delta \mu}). \tag{3.37}$$

Since f is arbitrary, we find

$$\dot{\mu} = \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu \tag{3.38}$$

which is also known as **Euler's equation** (or the **Euler-Poincaré equation**).

Now, identifying $T\mathfrak{g}^* \approx \mathfrak{g}^*$, from (3.37) we see that the Hamiltonian vector field associated with $h \in C^\infty(\mathfrak{g}^*)$ is given by

$$X_h(\mu) = \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu, \qquad \mu \in \mathfrak{g}^*. \tag{3.39}$$

Therefore, the symplectic leaves are the **co-adjoint orbits**

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*_g \mu \ : \ g \in G\} \tag{3.40}$$

which are the orbits in $\mathfrak{g}^*$ of the **co-adjoint action**[1]

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \qquad (g, \nu) \mapsto \mathrm{Ad}^*_{g^{-1}} \nu \tag{3.41}$$

of G on $\mathfrak{g}^*$ induced by the action of G on itself by conjugation

$$G \times G \to G, \qquad (g, h) \mapsto ghg^{-1}. \tag{3.42}$$

Euler's equation (3.38) says that the flow is tangent to the symplectic leaves. Also, since the symplectic structure on each symplectic leaf coincides with that given by the Poisson bracket restricted to the leaf, Euler's equation is a Hamiltonian system on this leaf. In other words, the original system on T^*G can be viewed as a *family* of Hamiltonian systems parametrized by the symplectic leaves, and each such Hamiltonian system is a perturbation of its neighbors.

[1] Taking g^{-1} guarantees that this is a *left*-action.

3.2 Geometric correspondence for 2D Euler

Let an ideal incompressible fluid fill a domain $\Omega \subset \mathbb{R}^d$ (where for now $d \geq 2$). It can be viewed as a mechanical system with infinitely many degrees of freedom, composed of fluid particles subject to the constraint of incompressibility. Thus, the configuration space for this system is the set of volume-preserving diffeomorphisms

$$\mathcal{D}_{\mathrm{vol}} := \{\eta \colon \overline{\Omega} \to \overline{\Omega} \mid \det \mathrm{Jac}(\eta) \equiv 1\}. \tag{3.43}$$

Suppose it is set in motion with some initial velocity field, but is otherwise under no external force. (The incompressibility constraint is enforced by the pressure through the term $-\nabla p$, but it is an internal force, already encoded in the configuration space $\mathcal{D}_{\mathrm{vol}}$.) The Lagrangian of this system reduces to the kinetic energy

$$L(\eta, \dot{\eta}) = \int_\Omega \frac{1}{2}|\dot{\eta}|^2 \tag{3.44}$$

which is invariant under composition from the *right*

$$L(\eta \circ \zeta, \dot{\eta} \circ \zeta) = L(\eta, \dot{\eta}), \qquad \zeta \in \mathcal{D}_{\mathrm{vol}} \tag{3.45}$$

In particular, the kinetic energy can be computed from the velocity field $u = \dot{\eta} \circ \eta^{-1}$:

$$L(\eta, \dot{\eta}) = L(\mathrm{id}, u) = \frac{1}{2}\int_\Omega |u|^2 \tag{3.46}$$

This invariance is in contrast to the previous exposition, where the Lagrangian was invariant under a *left*-action. This will have the effect of introducing a negative sign inteh Euler-Poincaré equation.

The Lie algebra is the vector space of divergence-free vector fields tangent to the boundary,

$$\mathcal{U} := \{u \colon \overline{\Omega} \to \mathbb{R}^d \mid \operatorname{div} u = 0, \langle u, N\rangle = 0\} \tag{3.47}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$. The adjoint action of $\mathcal{U}$ on itself is the derivative of the adjoint action of $\mathcal{D}_{\mathrm{vol}}$ on $\mathcal{U}$, which is itself the derivative of the action of $\mathcal{D}_{\mathrm{vol}}$ on itself by conjugation. Namely, taking derivative in ζ of $(\eta, \zeta) \mapsto \eta \circ \zeta \circ \eta^{-1}$, the **adjoint action** of $\mathcal{D}_{\mathrm{vol}}$ on $\mathcal{U}$ is push-forward by diffeomorphisms:

$$(\eta, u) \in \mathcal{D}_{\mathrm{vol}} \times \mathcal{U} \qquad \mapsto \qquad \mathrm{Ad}_\eta u = \eta_* u = d\eta \circ u \circ \eta^{-1} \in \mathcal{U}. \tag{3.48}$$

This is clearly a vector field on Ω. To see that $\eta_* u$ is divergence-free and tangent to the boundary, write the adjoint action as $\frac{d}{d\epsilon}_{|\epsilon=0}(\eta \circ \zeta_\epsilon \circ \eta^{-1})$ where $u = \dot{\zeta}(0)$. The **adjoint action** of $\mathcal{U}$ on itself is determined by the bracket giving $\mathcal{U}$ the structure of a Lie group from the group operation in $\mathcal{D}_{\text{vol}}$. It is *minus* the usual Lie bracket of vector fields on Ω, with components written in cartesian coordinates x^i as

$$(\mathrm{ad}_u v)^i = v^j \frac{\partial u^i}{\partial x^j} - u^j \frac{\partial v^i}{\partial x^j}. \tag{3.49}$$

We will be working in two dimensions. The domain $\Omega \subset \mathbb{R}^2$ is assumed bounded, smooth, and connected, but not necessarily simply connected. We will denote its (single) outer and (finitely many) inner boundary components

$$\Gamma_0, \qquad \Gamma_i, \qquad i = 1, 2, \ldots \tag{3.50}$$

respectively. (We will not need to speficy exactly how many inner boundary components Ω has.) The result of this Section are valid with any number of inner boundary components. In Section 4.4, however, we will restrict to the case of a single inner boundary component. Γ_0 will be changed to Γ_o (for 'outer') and the 'i' in Γ_i will no longer be a numerical index, but will rather stand for 'inner'.

In two dimensions it is convenent to use stream functions instead of velocity fields. Thus, we will identify the Lie algebra with the **space of stream functions**, that is, of smooth functions, locally constant on the boundary, and vanishing on Γ_0 (see Proposition 5, Section 2.1):

$$\mathcal{U} = \{\psi \in C^\infty_{\overline{\Omega}} \mid \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_i} = \text{constant}, i = 1, 2, \ldots\}. \tag{3.51}$$

In terms of stream functions, the **adjoint action** of $\mathcal{D}_{\text{vol}}$ on $\mathcal{U}$ is

$$(\eta, \psi) \in \mathcal{D}_{\text{vol}} \times \mathcal{U} \mapsto \mathrm{Ad}_\eta \psi = \eta_* \psi = \psi \circ \eta^{-1} \in \mathcal{U}. \tag{3.52}$$

That $\eta_*\psi$ is a stream function (i.e. a function belonging to the space $\mathcal{U}$) is immediate in this representation, as η preserves each boundary components. The Lie bracket in $\mathcal{U}$ in terms of stream functions is as follows. By direct computation one verifies that for

$$u = \nabla^\perp \psi = \begin{bmatrix} -\psi_y \\ \psi_x \end{bmatrix}, \qquad v = \nabla^\perp \Phi = \begin{bmatrix} -\Phi_y \\ \Phi_x \end{bmatrix}. \tag{3.53}$$

the adjoint action is

$$\mathrm{ad}_u v = -\nabla^\perp\{\psi, \Phi\} \tag{3.54}$$

where $\{\psi, \Phi\} = \psi_x\Phi_y - \psi_y\Phi_x$ denotes the Poisson bracket of functions on Ω already introduced in (1.34).

Strictly speaking, the dual to the Lie algebra is the space of continuous linear forms on $\mathcal{U}$ and hence is a space of distributions. We will actually work with the regular dual, denoted $\mathcal{U}^*$, identified with a space of smooth functions in duality with $\mathcal{U}$ via the kinetic energy $\frac{1}{2}\int_\Omega |u|^2$ of the flow. Even leaving functional-analytical issues aside, the description of $\mathcal{U}^*$ is subtle and depends on the topology of Ω. To ψ we assign a linear functional on $\mathcal{U}$ defined by

$$l_\psi(\Phi) = \int_\Omega \langle \nabla^\perp\psi, \nabla^\perp\Phi\rangle. \tag{3.55}$$

Integrating by parts (rigid rotations are isometries in the Euclidean space)

$$l_\psi(\Phi) = \int_\Omega \langle \nabla\psi, \nabla\Phi\rangle = -\int_\Omega (\Delta\psi)\Phi + \int_{\partial\Omega} \Phi\frac{\partial\psi}{\partial N} = -\int_\Omega (\Delta\psi)\Phi + \Phi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\psi}{\partial N} \tag{3.56}$$

where we observe the convention of summing over repeated indices (note thate $\Phi_{|\Gamma_0} = 0$ so it does not actually appear in the sum). In other words, the **inertia operator** defined by

$$I\colon \mathcal{U} \to \mathcal{U}^*, \qquad \psi \mapsto \left(\Delta\psi; \int_{\Gamma_i}\frac{\partial\psi}{\partial N}, i = 1, 2, \dots\right) \tag{3.57}$$

identifies the Lie agebra $\mathcal{U}$ with the **regular dual** $\mathcal{U}^* = C^\infty_{\overline{\Omega}} \times \mathbb{R} \times \mathbb{R} \dots$. This is an isomorphism since, by Proposition 5 and Proposition 7 of Section 2.1, for any $\omega \in C^\infty_{\overline{\Omega}}$ and constants γ_i, $i = 1, 2, \dots$, there exists a unique solution to the linear elliptic equation

$$\Delta\psi = \omega, \qquad \psi_{|\Gamma_0} = 0, \qquad \psi_{|\Gamma_i}, \qquad \int_{\Gamma_i}\frac{\partial\psi}{\partial N} = \gamma_i, \qquad i = 1, 2, \dots. \tag{3.58}$$

To alleviate notation, we may in the sequel refer to an element l_ψ in $\mathcal{U}^*$ simply by $\omega = \Delta\psi$, with the tacit understanding that there are constants $\int_{\Gamma_i}\frac{\partial\psi}{\partial N}$, $i = 1, 2, \dots$ associated to it. (This will not cause any confusion since the values of $\int_{\Gamma_i}\frac{\partial\psi}{\partial N} = \gamma_i$ are unchanged by the co-adjoint action, as we see next.) Thus, we say that an element $\omega \in \mathcal{U}^*$ acts on $\Phi \in \mathcal{U}$ according to

$$(\omega, \Phi) := (\omega, \Phi)_{\mathcal{U}^*\times\mathcal{U}} := -\int_\Omega \omega\Phi + \Phi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\psi}{\partial N}. \tag{3.59}$$

The co-adjoint action of $\mathcal{D}_{\rm vol}$ on $\mathcal{U}^*$ is defined as the dual to the adjoint action Ad_η: for any stream functions $\alpha, \Phi \in \mathcal{U}$, and $\omega \in \mathcal{U}^*$,

$$(\mathrm{Ad}^*_\eta l_\psi)(\alpha) := l_\psi(\mathrm{Ad}_\eta \alpha) = -\int_\Omega \omega(\alpha \circ \eta^{-1}) + (\alpha \circ \eta^{-1}) \int_{\partial\Omega} \frac{\partial \psi}{\partial N} = -\int_\Omega \alpha(\omega \circ \eta) + \alpha_{|\Gamma_i} \int_{\Gamma_i} \frac{\partial \psi}{\partial N} \tag{3.60}$$

since $\det(\mathrm{Jac}\eta) \equiv 1$ and η preserves each boundary component. Thus the **co-adjoint action** of $\mathcal{D}_{\rm vol}$ on $\mathcal{U}^*$ is simply composition of ω by η and leaves the constants γ_i, $i = 1, 2, \ldots$ unchanged:

$$\mathrm{Ad}^*_\eta = \eta^*(\omega; \gamma_i, i = 1, 2, \ldots) = (\omega \circ \eta; \gamma_i, i = 1, 2, \ldots). \tag{3.61}$$

The **co-adjoint orbits** are the sets

$$\mathcal{O}_{(\omega;\gamma_i,i=1,2,\ldots)} := \{(\omega \circ \eta; \gamma_i, i = 1, 2, \ldots) \ : \ \eta \in \mathcal{D}_{\rm vol}\}. \tag{3.62}$$

The **co-adoint action** of $\mathcal{U}$ on $\mathcal{U}^*$ is computed as follows. With $\frac{d}{d\epsilon}_{|\epsilon=0} \eta_\epsilon = \nabla^\perp \Phi$, and

$$\frac{d}{d\epsilon}_{|\epsilon=0}(\omega \circ \eta_\epsilon) = \omega_x(-\Phi_y) + \omega_y \Phi_x = \{\Phi, \omega\} \tag{3.63}$$

so that

$$\mathrm{ad}^*_\Phi(\omega; \gamma_i, i = 1, 2, \ldots) = (\{\Phi, \omega\}; 0, 0, \ldots). \tag{3.64}$$

Remark 16 Alternatively, this can be derived as follows. By definition, the co-adjoint action of $\mathcal{U}$ on $\mathcal{U}^*$ is the dual to the adjoint action of $\mathcal{U}$ on itself:

$$(\mathrm{ad}^*_\Phi \omega, \alpha) = (\omega, \mathrm{ad}_\Phi \alpha) = -\int_\Omega \omega\{\alpha, \Phi\} + \{\alpha, \Phi\}_{|\Gamma_i} \int_{\Gamma_i} \frac{\partial \psi}{\partial N} \tag{3.65}$$

Now α, Φ are constant on the boundary, being stream functions, and therefore $\{\alpha, \Phi\} = 0$ there. As for the first term, it is of independent interest to note the following identity, immediate by integration by parts,

$$\int_\Omega f\{g, h\} = \int_\Omega g\{h, f\} - \int_{\partial\Omega} fg\langle \nabla^\perp h, N\rangle. \tag{3.66}$$

With this identity,

$$(\mathrm{ad}^*_\Phi \omega, \alpha) = -\int_\Omega \alpha\{\Phi, \omega\} - \int_{\partial\Omega} \omega\alpha \langle \nabla^\perp \Phi, N\rangle = -\int_\Omega \alpha\{\Phi, \omega\} \tag{3.67}$$

where the boundary term again vanishes since $\nabla\Phi$ is normal to the boundary. ■

When the Lagrangian reduces to the kinetic energy, it coincides with the Hamiltonian:

$$h(\omega) = \mathcal{E}(\omega) := \frac{1}{2}\int_\Omega |u|^2 = -\frac{1}{2}\int_\Omega \omega\psi + \frac{1}{2}\psi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\psi}{\partial N}. \tag{3.68}$$

Let's compute the L^2-gradient $\frac{\delta h}{\partial\omega}$ where with some abuse of notation ω denotes an element of the regular dual. Let ν, $\delta\gamma_i$ be variations of ω and γ_i, respectively, and let ϕ be such that

$$\Delta\phi = \nu, \qquad \phi_{|\Gamma_0} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i}\frac{\partial\phi}{\partial N} = \delta\gamma_i, \quad i = 1, 2, \ldots. \tag{3.69}$$

(Again, we will simply represent the element in $\mathcal{U}^*$ by the variation ν.) Taking derivatives at $\epsilon = 0$ and using Green's identity,

$$\begin{aligned}
& \frac{d}{d\epsilon}_{|\epsilon=0} h(\omega + \epsilon\nu) && (3.70)\\
= \; & \frac{1}{2}\frac{d}{d\epsilon}_{|\epsilon=0}\left(-\int_\Omega(\omega+\epsilon\nu)(\psi+\epsilon\phi) + (\psi+\epsilon\phi)_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial(\psi+\epsilon\phi)}{\partial N}\right) && (3.71)\\
= \; & \frac{1}{2}\left(-\int_\Omega(\omega\phi+\nu\psi) + \psi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\phi}{\partial N} + \phi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\psi}{\partial N}\right) && (3.72)\\
= \; & \frac{1}{2}\left(-\int_\Omega(\psi\Delta\phi+\nu\phi) + \int_{\partial\Omega}(\psi\frac{\partial\phi}{\partial N} - \phi\frac{\partial\psi}{\partial N}) + \psi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\phi}{\partial N} + \phi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\psi}{\partial N}\right) && (3.73)\\
= \; & -\int_\Omega \psi\nu + \psi_{|\Gamma_i}\int_{\Gamma_i}\frac{\partial\phi}{\partial N} && (3.74)\\
= \; & (\nu, \psi). && (3.75)
\end{aligned}$$

Thus,

$$\frac{\delta h}{\delta\omega} = \psi. \tag{3.76}$$

Finally, the Euler equations in this context are given by

$$\dot{\omega} = -\mathrm{ad}^*_{\frac{\delta h}{\partial\omega}}\omega = -\{\psi, \omega\} \tag{3.77}$$

where the negative sign is due to the fact that the Lagrangian is invariant under a *right*-action rather than a left-action. Equation (3.77) is nothing but the vorticity equation.

3.3 Alternative characterization of steady flows

There is an alternative characterization of steady-state solutions, which offers additional motivation for Theorem 1 (Section 1.3). This is due to A. Shnirelman.

Consider a differential equation

$$\dot{z} = f(z) \tag{3.78}$$

on a finite-dimensional manifold with a foliation (not in the strict sense of the term since we allow the dimensions of the leaves to vary). If all leaves near a point $\overline{z}$ are of same dimensions, then $\overline{z}$ is said to be a **regular point** of the foliation. Suppose that the system (3.78) leaves the foliation invariant and has a first integral E, i.e. E remains constant along the integral curves of (3.78).

Proposition 17 (Theorem 3.3, p. 85, [3]) *If the following conditions are satisfied,*

(a). $\overline{z}$ is a regular point of the foliation,

(b). $\overline{z}$ is a critical point of E restricted to its leaf, and

(c). the second differential of E restricted to the leaf of $\overline{z}$ is a non-degenerate quadratic form,

then on every nearby leaf there exists a nearby stationary point.

In other words, the leaves are locally parametrized by the stationary points. As we have just seen in Section 3.1, this situation occurs naturally when the system (3.78) is the Euler-Poincaré equation, the manifold is the dual Lie algebra, and the first integral E is the energy, i.e. the square norm of the velocity field.

Let's now discuss the analogue of Proposition 17 in the infinite-dimensional setting of two-dimensional incompressible flows. Let $\overline{\psi}$ and $\overline{\omega}$ be the stream function and vorticity function of a steady flow related according to $\overline{\omega} = \Delta\overline{\psi} = \overline{F}(\overline{\psi})$.

Condition (a) has analogue, in the infinite-dimensional case of 2D Euler, that the topology of the profile of the vorticity function should be the same for ω in a neighborhood. This is enforced by imposing that ω not have critical points.

Condition (b) says that Euler steady-states can be characterized as critical points of the energy E restricted to their co-adjoint orbits as we now show. Taking first derivatives of E in the direction $\phi \in \mathcal{U}$ corresponding to $(\nu; 0, 0, \dots) \in \mathcal{U}^*$ with $\nu = \{\alpha, \overline{\omega}\}$ for some $\alpha \in \mathcal{U}$ (see (3.64)),

$$DE(\overline{\psi})\cdot\phi = \int_\Omega \langle \nabla\overline{\psi}, \nabla\phi\rangle = \int_{\partial\Omega} \overline{\psi}\frac{\partial\phi}{\partial N} - \int_\Omega \overline{\psi}\Delta\phi = -\int_\Omega \overline{\phi}\nu = -\int_\Omega \overline{\psi}\{\alpha, \overline{\omega}\} = \int_\Omega \{\overline{\psi}, \overline{\omega}\}\alpha. \tag{3.79}$$

(The boundary terms vanish since $\overline{\psi}_{|\Gamma_o} = 0$, $\overline{\psi}_{|\Gamma_i} =$ constant, and $\int_{\Gamma_i} \frac{\partial\phi}{\partial N} = 0$ and the last equality holds by (3.66) and the fact that $\nabla\alpha \perp \partial\Omega$.) Therefore, $\overline{\psi}$ is a critical point of E restricted to its co-adjoint orbit if and only if $\{\overline{\psi}, \overline{\omega}\} = 0$, i.e. if and only if the flow is a steady-state.

Condition (c) corresponds to the non-degeneracy condition for the implicit function theorem to be invoked, and is the analogue of condition (ND) in Theorem 1, Section 1.3. The second variation of energy restricted to the co-adjoint orbit through a steady-state is given by

$$D^2E(\overline{\psi})\cdot(\phi, \phi) = \int_\Omega |\nabla\phi|^2 + \frac{(\Delta\phi)^2}{\overline{F}'(\overline{\psi})}. \tag{3.80}$$

In particular, condition (c) is automatically satisfied when $\overline{F}' > 0$. This will be revisited in Proposition 19 of Section 4.2.

Proof of (3.80) To compute the second variation of the energy, let $(\omega_\epsilon, \overline{\gamma}) = (\overline{\omega} \circ \eta_\epsilon, \overline{\gamma})$ be a smooth curve in the co-adjoint orbit starting at $(\overline{\omega}, \overline{\gamma})$, and denote ψ_ϵ the corresponding curve in $\mathcal{U}$ under the inertia operator. From the first derivative

$$\frac{d}{d\epsilon}E(\psi_\epsilon) = \int_\Omega \langle \nabla\psi_\epsilon, \nabla\frac{\partial}{\partial\epsilon}\psi_\epsilon\rangle \tag{3.81}$$

we find that the second derivative at $\epsilon = 0$ is, posing $\dot{\psi} = \frac{\partial}{\partial\epsilon}_{|\epsilon=0}\psi_\epsilon$ and $\ddot{\psi} = \frac{\partial^2}{\partial^2\epsilon}_{|\epsilon=0}\psi_\epsilon$,

$$\frac{d^2}{d^2\epsilon}_{|\epsilon=0}E(\psi_\epsilon) = \int_\Omega \langle \nabla\dot{\psi}, \nabla\dot{\psi}\rangle + \langle \nabla\overline{\psi}, \nabla\ddot{\psi}\rangle \tag{3.82}$$

Integrating the second term by parts, the boundary terms vanish ($\overline{\psi}$ is locally constant on the boundary, and $\int_{\Gamma_i} \frac{\partial\psi_\epsilon}{\partial N} = \overline{\gamma}$ is independent of ϵ):

$$\int_\Omega \langle \nabla\overline{\psi}, \nabla\ddot{\psi}\rangle = \int_{\partial\Omega} \overline{\psi}\frac{\partial\ddot{\psi}}{\partial N} - \int_\Omega \overline{\psi}\Delta\ddot{\psi} = -\int_\Omega \overline{\psi}\ddot{\omega}. \tag{3.83}$$

To compute $\ddot{\omega} = \frac{d^2}{d\epsilon^2}_{|\epsilon=0}\omega_\epsilon$ it is enough to expand up to second orders. We can take η_ϵ to be the flow corresponding to some velocity field $v = \nabla^\perp\alpha$. For $x \in \Omega$,

$$\eta_\epsilon(x) = x + \epsilon v(x) + \frac{\epsilon^2}{2}\nabla_v v + \cdots \tag{3.84}$$

and the last term is obtained as follows: the flow is the solution to $\dot{x} = v(x)$, thus $\ddot{x} = \nabla_v v(x)$. Now taking second derivatives at $\epsilon = 0$,

$$\begin{aligned}
\frac{d^2}{d\epsilon^2}_{|\epsilon=0}\omega_\epsilon(x) &= \frac{d^2}{d\epsilon^2}_{|\epsilon=0}\overline{\omega}(x + \epsilon v + \frac{\epsilon^2}{2}\nabla_v v + \cdots) && (3.85)\\
&= \langle\nabla\overline{\omega}, \nabla_v v\rangle + \nabla^2\overline{\omega}\cdot(v, v) && (3.86)\\
&= \overline{\omega}_{,k}v_l v_{k,l} + \overline{\omega}_{,kl}v_k v_l && (3.87)\\
&= (\overline{\omega}_{,k}v_k v_l)_{,l} && (3.88)\\
&= \mathrm{div}(\langle\nabla\overline{\omega}, \nabla^\perp\alpha\rangle v\rangle) && (3.89)\\
&= \mathrm{div}(\nu v) && (3.90)
\end{aligned}$$

using that div $v = 0$. In turn, integrating by parts

$$\begin{aligned}
\int_\Omega -\overline{\psi}\ \mathrm{div}(\nu v) &= -\int_\Omega \mathrm{div}(\overline{\psi}\nu v) + \int_\Omega\langle\nabla\overline{\psi}, \nu v\rangle && (3.91)\\
&= -\int_{\partial\Omega}\overline{\psi}\nu\langle v, N\rangle + \int_\Omega\langle\nabla\overline{\psi}, \nu v\rangle && (3.92)\\
&= \int_\Omega \nu\langle\nabla\overline{\psi}, v\rangle. && (3.93)
\end{aligned}$$

Now $\overline{\psi}$ solves $\overline{\omega} = \overline{F}(\overline{\psi})$, so taking gradients, $\nabla\overline{\psi} = \frac{1}{\overline{F}'(\overline{\psi})}\nabla\overline{\omega}$ and

$$\int_\Omega -\overline{\psi}\ \mathrm{div}(\nu v) = \int_\Omega \frac{\nu\langle\nabla\overline{\omega}, v\rangle}{\overline{F}'(\overline{\psi})} = \int_\Omega \frac{\nu^2}{\overline{F}'(\overline{\psi})}. \tag{3.94}$$

This establishes (3.80). ■

Chapter 4

The non-degeneracy condition (ND) and some preliminary results

In this Chapter we show that if $F' > 0$, then the non-degeneracy condition (ND) of Theorem 1, Section 1.3, is satisfied. We also establish a number of preliminary results on the level sets of stream functions, the distribution functions of functions which are locally constant on the boundary and without critical points, and on the co-adjoint orbits. These will be essential in the proof of Theorem 1, Section 1.3, carried out in Chapter 5.

From now on we will restrict to annulus-like domains: the domain Ω has a single outer boundary component and a single inner boundary component denoted respectively

$$\Gamma_o, \qquad \Gamma_i. \tag{4.1}$$

(The subscript 'o' is for 'outer', and the subscript 'i' is no longer an index, but stands for 'inner'.) We retain the notation $\int_{\Gamma_i} \frac{\partial \psi}{\partial N} = \gamma_i$ for the boundary condition on stream functions, where N is the unit outer normal to the boundary, and $\frac{\partial}{\partial N}$ is the outer normal derivative.

4.1 Level sets of ψ for annulus domains

We begin with general properties of the level sets of ψ that will be used throughout, often without reference.

Proposition 18 (Properties of level sets) *Let $\Omega \subset \mathbb{R}^2$ be diffeomorphic to the annulus $B_2 \setminus B_1$ with exterior boundary denoted Γ_o and interior boundary Γ_i. Suppose a function ψ has no critical points, is locally constant on the boundary, and that*

$$\psi_{|\Gamma_o} = 0, \qquad \psi_{|\Gamma_i} = c, \qquad (c < 0). \tag{4.2}$$

Then, the level sets form a 1-parameter family of simple closed curves, and ψ is increasing across the level curves. If ψ is of class C^n, $n \geq 1$, then the level sets are curves of class C^n.

Furthermore, if ψ is the stream function of a solution to the steady-state Euler equation with vorticity ω, then there exists a function F defined on range$(\psi) = [c, 0]$ such that

$$\omega = F(\psi). \tag{4.3}$$

If $F' \neq 0$, then ω has no critical points, it is locally constant on $\partial\Omega$, and the level sets of ω coincide with the level sets of ψ.

Finally, for a sufficiently small $\|\cdot\|_1$-neighborhood $\mathcal{V}_*(\psi)$ of ψ in $\mathcal{U}$, there exists $L > 0$ such that the length $\mathcal{L}(\{\psi = \lambda\})$ of the level set $\{\psi = \lambda\}$ satisfies

$$L^{-1} < \mathcal{L}(\{\psi = \lambda\}) < L, \qquad \lambda \in \mathit{range}(\psi). \tag{4.4}$$

Proof (sketch)
The level set $\Gamma_o = \{\psi = 0\}$ is a simple closed curve since $\nabla\psi \neq 0$. By the implicit function theorem, the nearby level sets are again simple closed curves. By repeated application of the implicit function theorem, one covers the entire domain $\overline{\Omega}$. That ψ is monotone across its level sets is obtained by solving $\dot{x} = \nabla\psi(x)$, $x(0) \in \Gamma_o$, and $\psi(x(t))$ is monotone in t. ∎

4.2 $F' > 0$ implies the non-degeneracy condition (ND)

The stream function $\overline{\psi}$ of the reference steady-state as in Theorem 1 (Section 1.3) is such that $\nabla\overline{\psi} \neq 0$ on $\overline{\Omega}$. Thus its level sets are concentric curves ($\overline{\psi}$ is constant on each component of the boundary), and $\overline{\psi}$ is monotone across its level sets. Since $\overline{\psi}_{|\Gamma_o} = 0$, we may assume without loss of generality that

$$\overline{\psi} \leq 0, \qquad \overline{\psi}_{|\Gamma_o} = 0, \qquad \overline{\psi}_{|\Gamma_i} = \overline{c} \qquad (\overline{c} < 0). \tag{4.5}$$

The solution operator $\psi = S(F)$ constructed in Proposition 15, Section 2.5, is defined in a neighborhood $\mathcal{V}_2(\overline{F})$ of $\overline{F}$ in which each F is generic in the sense that $\Delta - F'(\psi)$ is invertible. If now $\overline{F}$ is monotone, this neighborhood $\mathcal{V}_2(\overline{F})$ can be chosen so that F is also monotone, and thus $\psi \leq 0$. That is, the interval I from Lemma 14, Section 2.5, can be chosen in the form

$$I = [\underline{c}, 0] \qquad (\underline{c} < 0). \tag{4.6}$$

Proposition 19 *Supppose $F \in C^\infty_I$ satisfies $F' > 0$ and is such that $\Delta\psi = F(\psi)$ has a solution. Then, this solution is unique, the linearized operator $\Delta - F'(\psi)$ is invertible, and condition (ND) of Theorem 1, Section 1.3, is automatically satisfied.*

Proof That $\Delta\psi = F(\psi)$ has a unique solution is easily seen from Proposition 13 of Section 2.4 ($\mathcal{J}$ is strictly convex) or simply by the maximum principle. The maximum principle also shows that the solution is generic. Since the boundary conditions are not the more usual Dirichlet boundary conditions, we give a proof of this fact. Suppose then that ϕ satisfies

$$\Delta\phi = \overline{F}'(\overline{\psi})\phi, \qquad \phi_{|\Gamma_o} = 0, \qquad \phi_{|\Gamma_i} = \text{constant}, \qquad \int_{\Gamma_i} \frac{\partial\phi}{\partial N} = 0. \tag{4.7}$$

Then, by the maximum principle ($\overline{F}'(\overline{\psi}) > 0$), ϕ attains its maximum and minimum on $\partial\Omega$. If $\phi > 0$ somewhere, then $\max_{\overline{\Omega}} \phi = \phi_{|\Gamma_i} > 0$, and the strong maximum principle implies that $\frac{\partial\phi}{\partial N} > 0$ everywhere on Γ_i. But this contradicts the boundary condition $\int_{\Gamma_i} \frac{\partial\phi}{\partial N} = 0$, hence $\phi \leq 0$ throughout. Likewise, one shows that $\phi \geq 0$ and thus that $\phi \equiv 0$. In other words, the linearized operator is injective, and by the Fredholm alternative (see also Lemma 11, Section 2.3), it is invertible.

As for condition (ND), let $\nu = \{\alpha, \omega\}$ solve the linearized equation $\nu = \Delta\phi = F'(\psi)\phi + f(\psi)$. Recall the identity

$$\int_\Omega f\{g,h\} = \int_\Omega g\{h,f\} - \int_{\partial\Omega} fg\langle\nabla^\perp h, N\rangle. \tag{4.8}$$

This gives

$$\int_\Omega \frac{f(\psi)}{F'(\psi)}\{\alpha,\omega\} \quad = \quad \int_\Omega \alpha\{\omega, \frac{f(\psi)}{F'(\psi)}\} - \int_{\partial\Omega} \frac{f(\psi)}{F'(\psi)}\alpha\langle\nabla^\perp\omega, N\rangle = 0 \tag{4.9}$$

where the integral on Ω is zero since the bracket $\{\cdot,\cdot\} = \det(\nabla\cdot,\nabla\cdot)$ vanishes identically (both ω and $\frac{f(\psi)}{F'(\psi)}$ are functions of ψ) and the boundary integral is also zero since ω is locally constant on the boundary. On the other hand,

$$\begin{aligned}\int_\Omega \frac{f(\psi)}{F'(\psi)}\{\alpha,\omega\} \quad &= \quad \int_\Omega(-\phi + \frac{\nu}{F'(\psi)})\nu = \int_\Omega \left(|\nabla\phi|^2 + \frac{\nu^2}{F'(\psi)}\right) - \int_{\partial\Omega}\phi\frac{\partial\phi}{\partial N} && (4.10)\\ &= \quad \int_\Omega \left(|\nabla\phi|^2 + \frac{\nu^2}{F'(\psi)}\right) && (4.11)\end{aligned}$$

where the boundary term vanishes because $\phi = 0$ on Γ_0, $\phi_{|\Gamma_i}$ is a constant and $\int_{\Gamma_i}\frac{\partial\phi}{\partial N} = 0$. Thus, since $F' > 0$ we must have $\nu = 0$. ■

4.3 Useful facts about distribution functions

We first prove the expression (1.52) when $F' > 0$,

$$T(F) = A_\omega^{-1} = F \circ A_\psi^{-1}. \tag{4.12}$$

Proof of (4.12) With $F' > 0$ so that

$$A_\psi(\lambda) = |\{\psi < \lambda\}| = |\{\omega < F(\lambda)\}|, = A_\omega(F(\lambda)) \tag{4.13}$$

and setting

$$A_\omega^{-1}(\mu) = \lambda', \qquad A_\psi^{-1}(\mu) = \lambda, \tag{4.14}$$

we find

$$A_\omega(\lambda') = \mu = A_\psi(\lambda) = A_\omega(F(\lambda)). \tag{4.15}$$

By monotonicity of $A_\omega^{-1}(\mu) = \lambda' = F(\lambda) = F(A_\psi^{-1}(\mu))$. ■

Remark 20 When $F' > 0$ (as we assume), it will be shown that $T(F)$ is smooth tame. This is still the case if $F' < 0$: introduce the tame linear operator $F \in C^\infty_{[\underline{c},0]} \mapsto \tilde{F} \in C^\infty_{[0,-\underline{c}]}$, $\tilde{F}(\psi) = F(-\psi)$, noting that then $\tilde{F}' > 0$. Then, $\omega = \tilde{F}(-\psi)$ if $\omega = F(\psi)$ and therefore

$$T(F) = A_\omega^{-1} = \tilde{F} \circ A_{-\psi}^{-1} \tag{4.16}$$

Since $\psi \mapsto A_\psi^{-1}$ and $\psi \mapsto -\psi$ are smooth tame maps (see Section 5.2) $\psi \mapsto A_{-\psi}^{-1}$ is also smooth tame. Thus, $T(F)$ is smooth tame. ■

The results of the remainder of this section are valid for any function ψ which is locally constant on the boundary and has no critical points. We do not assume that ψ vanishes on the outer boundary component. The results apply to stream functions in general, as well as vorticity functions of steady-state solutions (for these must be locally constant on the boundary). We repeat the assumption that the domain Ω is diffeomoprhic to the annulus, with outer and inner boundary components denoted Γ_o and Γ_i respectively.

The distribution function of the sets $\{\psi < \lambda\}$ is given by

$$A_\psi(\lambda) := |\{\psi < \lambda\}| = \int_{\min\psi}^{\lambda} \int_{x:\psi(x)=\lambda} \frac{1}{|\nabla\psi|} dl(x) d\lambda, \qquad \lambda \in [\min\psi, \max\psi]. \tag{4.17}$$

This is easily seen using the coarea formula (see § 3.2 in [12])

$$\int_\Omega u(x)|\nabla\psi(x)|\zeta(\psi(x))\ dx = \int_{\min\psi}^{\max\psi} \zeta(\lambda') \int_{x:\psi(x)=\lambda'} u(x) dl(x)\ d\lambda' \tag{4.18}$$

with $u(x) = \frac{1}{|\nabla\psi(x)|}$ and $\zeta(\lambda')$ the characteristic function over the interval $[\min\psi, \lambda]$. Much of the work will be devoted to the detailed study of

$$J_\psi u(\lambda) := \int_{\psi=\lambda} u := \int_{x:\psi(x)=\lambda} u(x) dl(x), \qquad \lambda \in [\min\psi, \max\psi]. \tag{4.19}$$

(Note that this makes sense for any C^n-function ψ ($n \geq 1$) which is locally constant on the boundary and without critical points.)

Lemma 21 *The derivatives of A_ψ and A_ψ^{-1} are given by*

$$A'_\psi = J_\psi \frac{1}{|\nabla\psi|}, \qquad (A_\psi^{-1})' = \frac{1}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}}. \tag{4.20}$$

Proof Note first that since ψ has no critical points and its level sets are (nondegenerate) concentric curves, their length is bounded below (Proposition 18, Section 4.1) and $J_\psi \frac{1}{|\nabla\psi|}$ is positive. The derivative A'_ψ is obvious from (4.17) while that of A_ψ^{-1} is obtained by differentiating

$$A_\psi^{-1}(A_\psi(\lambda)) = \lambda, \qquad (A_\psi^{-1})'(A_\psi(\lambda)) \times A'_\psi(\lambda) = 1. \tag{4.21}$$

■

Lemma 22 *Setting $N = \frac{\nabla\psi}{|\nabla\psi|}$, the first derivative of $J_\psi u$ is given by*

$$(J_\psi u)'(\lambda) := \frac{d}{d\lambda}(J_\psi u)(\lambda) = J_\psi \left(\frac{div(uN)}{|\nabla\psi|}\right)(\lambda), \qquad \lambda \in [\min\psi, \max\psi] \tag{4.22}$$

Proof This is done using the co-area formula (see § 3.2 in [12]): for any function ζ,

$$\int_\Omega u(x)|\nabla\psi(x)|\zeta(\psi(x))dx = \int_c^0 \zeta(\lambda) J_\psi u(\lambda)\ d\lambda. \tag{4.23}$$

Let ζ be smooth with compact support in $(\min\psi, \max\psi)$. Integrating by parts, using the co-area formula (4.23), the identities $\nabla(\zeta\circ\psi) = \zeta'(\psi)\nabla\psi$ and $\mathrm{div}((\zeta\circ\psi)uN) = (\zeta\circ\psi)\mathrm{div}(uN) + \langle\nabla(\zeta\circ\psi), uN\rangle$, and the coarea formula again,

$$\begin{aligned}
\int_{\min\psi}^{\max\psi} \zeta(\lambda)(J_\psi u)'(\lambda)\ d\lambda &= -\int_{\min\psi}^{\max\psi} \zeta'(\lambda) J_\psi u(\lambda)\ d\lambda && (4.24)\\
&= -\int_\Omega u(x)|\nabla\psi(x)|\zeta'(\psi(x))\ dx && (4.25)\\
&= -\int_\Omega u(x)\langle\nabla(\zeta(\psi(x))), N\rangle\ dx && (4.26)\\
&= -\int_\Omega \langle\nabla(\zeta\circ\psi), uN\rangle\ dx && (4.27)\\
&= \int_\Omega \mathrm{div}(uN)(\zeta\circ\psi)\ dx && (4.28)\\
&= \int_\Omega \frac{\mathrm{div}(uN)}{|\nabla\psi(x)|}|\nabla\psi(x)|\zeta(\psi(x))\ dx && (4.29)\\
&= \int_{\min\psi}^{\max\psi} \zeta(\lambda) J_\psi \left(\frac{\mathrm{div}\ (uN)}{|\nabla\psi|}\right)(\lambda)\ d\lambda. && (4.30)
\end{aligned}$$

This is true for an arbitrary test function ζ with arbitrary support in $(\min\psi, \max\psi)$, so (4.22) holds. ■

Lemma 23 *Let ψ_ϵ be a smooth family of functions without critical points, each one locally constant on the boundary, and let u be a smooth function. Then, with $\phi = \frac{\partial}{\partial\epsilon}_{|\epsilon=0}\psi_\epsilon$,*

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} J_{\psi_\epsilon} u(\lambda) = -J_\psi \left\{ \frac{\phi\ div(uN)}{|\nabla\psi|} \right\}(\lambda) \tag{4.31}$$

Proof This is done by the co-area formula (4.23). Fix $\min\psi < c < d < \max\psi$. If $\epsilon > 0$ is sufficiently small, then $[c,d] \subset\subset (\min\psi_\epsilon, \max\psi_\epsilon)$. For an arbitrary function ζ with compact support in (c,d), differentiating the co-area formula

$$\int_\Omega u(x)|\nabla\psi_\epsilon(x)|\zeta(\psi_\epsilon(x))dx = \int_c^d \zeta(\lambda)J_{\psi_\epsilon}u(\lambda)\ d\lambda \tag{4.32}$$

at $\epsilon = 0$ we obtain

$$\int_\Omega u(x)\left[\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|}\zeta(\psi(x)) + |\nabla\psi|\zeta'(\psi(x))\phi(x)\right]\ dx = \left[\int_c^d \zeta(\lambda)\left(\frac{\partial}{\partial\epsilon}_{|\epsilon=0} J_{\psi_\epsilon}u\right)(\lambda)d\lambda\right]. \tag{4.33}$$

On the one hand, the coarea formula (4.23) and integration by parts give (note that ζ' has support in $[c,d]$)

$$\int_\Omega u(x)\phi(x)|\nabla\psi|\zeta'(\psi(x))dx = \int_c^d \zeta'(\lambda)(J_\psi(u\phi))(\lambda)\ d\lambda = -\int_c^d \zeta(\lambda)(J_\psi(u\phi))'(\lambda)\ d\lambda \tag{4.34}$$

and on the other, using (4.22) and again the coarea formula (4.23),

$$\begin{aligned}
\int_\Omega u(x)\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|}\zeta(\psi(x))\ dx &= \int_\Omega u(x)\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|^2}|\nabla\psi(x)|\zeta(\psi(x))\ dx && (4.35)\\
&= \int_c^d \zeta(\lambda)J_\psi\left(u\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|^2}\right)(\lambda)\ d\lambda && (4.36)
\end{aligned}$$

This being true for any test funtion ζ with support in (c,d) and all $\min\psi < c < d <$

$\max\psi$, we conclude that for all $\lambda \in [\min\psi, \max\psi]$

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0}(J_{\psi_\epsilon}u(\lambda)) = -(J_\psi(u\phi))'(\lambda) + J_\psi\left(u\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|^2}\right)(\lambda) \tag{4.37}$$

$$= J_\psi\left\{-\frac{\mathrm{div}(u\phi N)}{|\nabla\psi|} + u\frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|^2}\right\}(\lambda) \tag{4.38}$$

$$= J_\psi\left\{\frac{-\mathrm{div}(\phi\ uN) + \langle\nabla\phi, uN\rangle}{|\nabla\psi|}\right\}(\lambda) \tag{4.39}$$

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0}(J_{\psi_\epsilon}u(\lambda)) = -J_\psi\left\{\frac{\phi\ \mathrm{div}(uN)}{|\nabla\psi|}\right\}(\lambda) \tag{4.40}$$

where again we have used (4.22). ∎

Corollary 24 *For* $\min\psi < \lambda' < \lambda < \max\psi$,

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0}A_{\psi_\epsilon}(\lambda) - \frac{\partial}{\partial\epsilon}_{|\epsilon=0}A_{\psi_\epsilon}(\lambda') = J_\psi\frac{\phi}{|\nabla\psi|}(\lambda') - J_\psi\frac{\phi}{|\nabla\psi|}(\lambda) \tag{4.41}$$

Proof With $u = \frac{1}{|\nabla\psi|}$ in Lemma 23,

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0}A_{\psi_\epsilon}(\lambda) - \frac{\partial}{\partial\epsilon}_{|\epsilon=0}A_{\psi_\epsilon}(\lambda') \tag{4.42}$$

$$= \int_{\lambda'}^{\lambda}\frac{d}{d\lambda}\frac{\partial}{\partial\epsilon}_{|\epsilon=0}A_{\psi_\epsilon}(l)dl \tag{4.43}$$

$$= \int_{\lambda'}^{\lambda}\frac{\partial}{\partial\epsilon}_{|\epsilon=0}\left(J_{\psi_\epsilon}\frac{1}{|\nabla\psi_\epsilon|}\right)(l)dl \tag{4.44}$$

$$= \int_{\lambda'}^{\lambda}\left[-J_\psi\left\{\frac{\phi\ \mathrm{div}(\frac{N}{|\nabla\psi|})}{|\nabla\psi|}\right\} + J_\psi\left(\frac{\partial}{\partial\epsilon}_{|\epsilon=0}\frac{1}{|\nabla\psi_\epsilon|}\right)\right](l)dl \tag{4.45}$$

$$= \int_{\lambda'}^{\lambda}-J_\psi\left[\frac{\phi\ \mathrm{div}(\frac{N}{|\nabla\psi|})}{|\nabla\psi|} + \frac{\langle\nabla\psi,\nabla\phi\rangle}{|\nabla\psi|^3}\right](l)dl \tag{4.46}$$

$$= \int_{\lambda'}^{\lambda}-J_\psi\left[\frac{\mathrm{div}(\phi\frac{N}{|\nabla\psi|})}{|\nabla\psi|}\right](l)dl \tag{4.47}$$

$$= -\int_{\lambda'}^{\lambda}\frac{d}{d\lambda}J_\psi\left(\frac{\phi}{|\nabla\psi|}\right)(l)dl \tag{4.48}$$

$$= J_\psi\frac{\phi}{|\nabla\psi|}(\lambda') - J_\psi\frac{\phi}{|\nabla\psi|}(\lambda) \tag{4.49}$$

where yet again we have used (4.22). ∎

4.4 Properties of the co-adjoint orbits

We repeat the assumption that the domain Ω is diffeomorphic to the annulus, with outer and inner boundary components denoted Γ_o and Γ_i respectively.

From (3.64) we know that a tangent ν at ω to the orbit $\{\omega \circ \eta \,:\, \eta \in \mathcal{D}_{\mathrm{vol}}\}$ is given by a stream function α according to

$$\nu = \{\alpha, \omega\}, \qquad \alpha \in \mathcal{U}. \tag{4.50}$$

The next Lemma derives another characterization of tangents, when ω verifies certain properties (these properties will be verified for example when ω is a steady-state).

Lemma 25 *Let $\omega \in C^\infty_{\overline{\Omega}}$ be without critical points and suppose that $\omega_{|\partial\Omega}$ = locally constant. Then, for $\nu \in C^\infty_{\overline{\Omega}}$, there exists a stream function $\alpha \in \mathcal{U}$ such that*

$$\nu = \{\alpha, \omega\} \tag{4.51}$$

if and only if

$$\nu_{|\partial\Omega} = loc.\ const. \qquad and \qquad \int_{\omega=\lambda} \frac{\nu}{|\nabla\omega|} = 0, \qquad \lambda \in range(\omega). \tag{4.52}$$

In this case, $\nu_{|\partial\Omega} \equiv 0$.

Proof The level sets of ω are connected, simple, concentric closed curves. We will denote $[m, M]$ the range of ω.

Suppose that $\nu = \{\alpha, \omega\} = -\langle \nabla\alpha, \nabla^\perp\omega\rangle$ for some stream function $\alpha \in \mathcal{U}$. Then, $\nabla\alpha \parallel \nabla\omega$ on the boundary and thus ν vanishes there. On the other hand,

$$\int_{\omega=\lambda} \frac{\nu}{|\nabla\omega|} = -\int_{\omega=\lambda} \langle \nabla\alpha, \frac{\nabla^\perp\omega}{|\nabla\omega|}\rangle = -\int_{\omega=\lambda} \langle \nabla\alpha, dl\rangle = 0 \tag{4.53}$$

since $\{\omega = \lambda\}$ is a closed curve. $\square$

Suppose now that ν is locally constant on the boundary and that the integrals in (4.52) vanish for $\lambda \in \mathrm{range}(\omega)$. In particular, taking $\lambda = m$ and M,

$$\nu_{|\partial\Omega} = 0. \tag{4.54}$$

Therefore, if a smooth function α satisfies $\nu = \{\alpha, \omega\} = -\langle \nabla\alpha, \nabla^\perp\omega\rangle$, then $\nabla\alpha \parallel \nabla\omega$ on $\partial\Omega$ and $\alpha_{|\partial\Omega}$ is locally constant. Without loss of generality we may impose $\alpha_{|\Gamma_0} = 0$ and $\alpha \in \mathcal{U}$. All is left to do is to solve for α. If it exists, then its gradient can be written as

$$\nabla\alpha = a(x)\nabla\omega + b(x)\nabla^\perp\omega \tag{4.55}$$

where $a(x), b(x)$ are smooth functions. The coefficient $b(x)$ is easily solved:

$$\nu = -\langle \nabla\alpha, \nabla^\perp\omega\rangle = -b(x)|\nabla\omega|^2, \qquad b(x) = -\frac{\nu}{|\nabla\omega|^2} \tag{4.56}$$

and is uniquely determined. The vector field $a(x)\nabla\omega + b(x)\nabla^\perp\omega$ is locally a gradient if and only if

$$0 = \mathrm{curl}\Big(a(x)\nabla\omega + b(x)\nabla^\perp\omega\Big) = -\mathrm{div}\Big(a(x)\nabla^\perp\omega - b(x)\nabla\omega\Big) \tag{4.57}$$

(here curl $X = X^2_x - X^1_y = -\mathrm{div}\ X^\perp$ is a scalar). Using the identity $\mathrm{div}(hX) = h\ \mathrm{div}\ X + \langle\nabla h, X\rangle$ for a function h and a vector field X, and that $\mathrm{div}(\nabla^\perp\omega) = 0$, $a(x)$ must then satisfy

$$\langle\nabla a, \nabla^\perp\omega\rangle = \mathrm{div}(b\nabla\omega) = -\mathrm{div}(\frac{\nu}{|\nabla\omega|}N) \tag{4.58}$$

where $N = \frac{\nabla\omega}{|\nabla\omega|}$. This equation is solved by the method of characteristics. It reduces to a collection of ODEs on the level sets of ω, which are concentric closed curves. One therefore must verify that the solution $a(x)$ is single valued. Let $x(t)$, $0 \le t \le T$, parametrize the level set $\{\omega = \lambda\}$, e.g. $x(t)$ is the solution to $\dot{x} = \nabla^\perp\omega(x)$ with $x(T) = x(0)$. Then we must have

$$\begin{aligned}
0 &= a(x(T)) - a(x(0)) && (4.59)\\
&= \int_{t=0}^{T} \frac{d}{dt} a(x(t))dt && (4.60)\\
&= \int_{\omega=\lambda} \langle\nabla a, \nabla^\perp\omega\rangle \frac{dl}{|\nabla\omega|} && (4.61)\\
&= -\int_{\omega=\lambda} \frac{1}{|\nabla\omega|}\mathrm{div}\left(\frac{\nu}{|\nabla\omega|}N\right) dl && (4.62)\\
&= -J_\omega\left(\frac{\mathrm{div}(\frac{\nu}{|\nabla\psi|}N)}{|\nabla\omega|}\right)(\lambda) && (4.63)\\
&= -\frac{d}{d\lambda}\left(J_\omega \frac{\nu}{|\nabla\omega|}\right)(\lambda) && (4.64)
\end{aligned}$$

by Lemma 22, Section 4.3. This is indeed verified using the assumption (4.52). Therefore $a(x)$ is single-valued and $a(x)\nabla\omega + b(x)\nabla^{\perp}\omega$ agrees with the gradient of a locally defined function α. Now, α is globally defined provided that

$$0 = \int_{\Gamma_i} \langle a(x)\nabla\omega + b(x)\nabla^{\perp}\omega, dl\rangle = \int_{\Gamma_i} \langle \nabla\alpha, dl\rangle \tag{4.65}$$

which holds since $\nabla\alpha \perp \partial\Omega$ as observed earlier. ■

As remarked in the Introduction, the distribution function

$$A_\omega(\lambda) = |\{x \in \Omega \mid \omega(x) < \lambda\}| \tag{4.66}$$

of a vorticity function ω is unchanged under the action of volume-preserving diffeomorphisms:

$$A_{\omega\circ\eta} = A_\omega, \qquad \eta \in \mathcal{D}_{\mathrm{vol}}. \tag{4.67}$$

That is, A is constant on the co-adjoint orbits. The next Lemma establishes a partial converse, indicating that A_ω is indeed a "good coordinate" for the orbits. Roughly speaking, if ω_ϵ is a one-parameter family of steady-state vorticities passing through ω with same distribution function, then it is tangent at ω to its orbit.

Lemma 26 *Let ω_ϵ be a smooth curve of functions passing through ω at $\epsilon = 0$ and which are locally constant on the boundary. Suppose that they have no critical points and that*

$$A_{\omega_\epsilon} = A_\omega, \qquad \forall\epsilon. \tag{4.68}$$

Then, $\nu = \frac{d}{d\epsilon}_{|\epsilon=0}\omega_\epsilon$ is tangent to the orbit through ω, i.e. there exists a stream function α such that $\nu = \{\alpha, \omega\}$.

Proof This is a corollary to Lemma 25. It is equivalent to show that (4.52) is satisfied. Suppose without loss of generality that $\omega_{\epsilon|\Gamma_o} = \max\omega_\epsilon$.

By assumption,

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega_\epsilon}(\lambda) = 0, \qquad \lambda \in [\min\omega, \max\omega]. \tag{4.69}$$

On the other hand, by Corollary 24, Section 4.3, we have for $\lambda, \lambda' \in [\min\omega, \max\omega]$

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega_\epsilon}(\lambda) - \frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega\epsilon}(\lambda') \quad = \quad J_\omega \frac{\nu}{|\nabla\omega|}(\lambda') - J_\omega \frac{\nu}{|\nabla\omega|}(\lambda) \tag{4.70}$$

It is then enough to show that $J_\omega \frac{\nu}{|\nabla\omega|}(\max\omega) = -\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega\epsilon}(\max\omega)$. Since $\Gamma_o = \{\omega = \max\omega\}$ we have

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A^{-1}_{\omega_\epsilon}(|\Omega|) = \frac{\partial}{\partial\epsilon}_{|\epsilon=0} \omega_{\epsilon|\Gamma_o} = \nu_{|\Gamma_o} \tag{4.71}$$

while from Lemma 21, Section 4.3,

$$\frac{dA^{-1}_\omega}{d\mu}(|\Omega|) = \frac{1}{\left(J_\omega \frac{1}{|\nabla\omega|}\right)(\max\omega)} = \frac{1}{\int_{\Gamma_o} \frac{1}{|\nabla\omega|}}. \tag{4.72}$$

But differentiating $A^{-1}_{\omega_\epsilon}(A_{\omega_\epsilon(\lambda)}) = \lambda$ at $\epsilon = 0$, we find

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A^{-1}_{\omega_\epsilon}(\mu) + \frac{dA^{-1}_\omega}{d\mu}(\mu) \frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega_\epsilon}(\lambda), \qquad \mu = A_\omega(\lambda) \tag{4.73}$$

and in particular at $\mu = |\Omega|$ we have

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega_\epsilon}(\max\omega) = -\frac{\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A^{-1}_{\omega_\epsilon}(\max\omega)}{\frac{dA^{-1}_\omega}{d\mu}(|\Omega|)} = -\nu_{|\Gamma_o} \int_{\Gamma_o} \frac{1}{|\nabla\omega|} = -J_\omega \frac{\nu}{|\nabla\omega|}(\max\omega) \tag{4.74}$$

as desired. Therefore, we have

$$J_\omega \frac{\nu}{|\nabla\omega|}(\lambda) = -\frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\omega_\epsilon}(\lambda) = 0, \qquad \lambda \in [\min\omega, \max\omega]. \tag{4.75}$$

■

Chapter 5

Proof of Main Theorem

5.1 Notation and assumptions

In this Chapter we prove the main theorem (Theorem 1, Section 1.3). The fluid occupies a domain Ω diffeomorphic to an annulus, i.e. it is smooth, bounded, connected, and doubly-connected. We denote its outer and inner boundary components respectively

$$\Gamma_o, \qquad \Gamma_i. \tag{5.1}$$

We consider a reference steady-state with stream function $\overline{\psi}$ and vorticity function $\overline{\omega}$ such that

$$\overline{\omega} = \Delta\overline{\psi} = \overline{F}(\overline{\psi}), \qquad \overline{F}' > 0. \tag{5.2}$$

There is no loss of generality in assuming that

$$\overline{\psi} \leq 0. \tag{5.3}$$

Therefore, for $F \in \mathcal{V}_2(\overline{F}) \subset C_I^\infty$ (see Proposition 15, Section 2.5), we have $\psi = S(F) \leq 0$ and we may choose the interval I obtained in Lemma 14, Section 2.5, in the form

$$I = [\underline{c}, 0] \qquad (\underline{c} < 0). \tag{5.4}$$

We recall some notation (see Section 1.5): linear and affine spaces of smooth stream functions are denoted

$$\begin{aligned} \mathcal{U} &= \{\psi \in C^\infty_{\overline{\Omega}} \mid \psi_{|\Gamma_o} = 0, \psi_{|\Gamma_i} = \text{constant}\}, & (5.5)\\ \mathcal{U}_{\gamma_i} &= \{\psi \in \mathcal{U} \mid \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i\}, & (5.6)\end{aligned}$$

We will prove smoothness of maps of Fréchet spaces using the C^n-grading and therefore we introduce the spaces of stream functions with finite regularity

$$\begin{aligned} \mathcal{U}^n &= \{\psi \in C^n_{\overline{\Omega}} \mid \psi_{|\Gamma_o} = 0, \psi_{|\Gamma_i} = \text{constant}\}, & (5.7)\\ \mathcal{U}^n_{\gamma_i} &= \{\psi \in \mathcal{U}^n \mid \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i\} & (5.8)\end{aligned}$$

for $n = 1, 2, \ldots$. The C^n-norms will be denoted

$$\|\cdot\|_n := \|\cdot\|_{C^n}, \qquad n = 0, 1, 2, \ldots. \tag{5.9}$$

As for tame estimates, we will derive them in the $C^{n,\alpha}$-grading since we will need tame estimates in Hölder norms of elliptic equations (see Section 2.5) and we recall the notation already in use in Chapter 2:

$$\begin{aligned} \mathcal{U}^{n,\alpha} &= \{\psi \in C^{n,\alpha}_{\overline{\Omega}} \mid \psi_{|\Gamma_o} = 0, \psi_{|\Gamma_i} = \text{constant}\}, & (5.10)\\ \mathcal{U}^{n,\alpha}_{\gamma_i}, &= \{\psi \in \mathcal{U}^{n,\alpha} \mid \int_{\Gamma_i} \frac{\partial\psi}{\partial N} = \gamma_i\}, & (5.11)\\ & & (5.12)\end{aligned}$$

for $n = 1, 2, \ldots$ where $\alpha \in (0, 1)$ is fixed. The $C^{n,\alpha}$-norms will be denoted

$$\|\cdot\|_{n,\alpha} := \|\cdot\|_{C^{n,\alpha}}, \qquad n = 0, 1, 2, \ldots. \tag{5.13}$$

5.2 A_ψ^{-1} is a smooth tame map of ψ

A major step in the proof of Theorem 1 is to show that the operator A_ψ^{-1}, the inverse of the distribution function of a stream function ψ,

$$A_\psi(\lambda) = |\{x \in \Omega \mid \psi(x) < \lambda\}| \tag{5.14}$$

is a smooth tame map of ψ and to establish a tractable expression for its derivative. This will rely heavily on the co-area formula [12] (see also Section 4.3) and expressions involving

$$J_\psi u(\lambda) := \int_{x:\psi(x)=\lambda} u(x)dl(x) \tag{5.15}$$

introduced in Section 4.3. It is not quite an operator in ψ and u since the range of ψ may vary, adding to the technical difficulties.

We state the main results in the following

Proposition 27 ($Q(\psi) = A_\psi^{-1}$ is smooth tame) *Let $\overline{\psi} \in \mathcal{U}_{\gamma_i}$ have no critical points. Then, there exists a neighborhood $\mathcal{V}_*(\overline{\psi})$ such that the operator*

$$Q\colon \psi \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \mapsto A_\psi^{-1} \in C^\infty_{[0,|\Omega|]} \tag{5.16}$$

is a smooth tame map of Fréchet spaces. A_ψ^{-1} is tame with degree 0:

$$\|A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.17}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood. The first derivative is given by

$$DQ(\psi)\cdot\phi = \frac{J_\psi \frac{\phi}{|\nabla\psi|} \circ A_\psi^{-1}}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}}. \tag{5.18}$$

More generally, the operator $(\psi, u) \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \times C^\infty_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^\infty_{[0,|\Omega|]}$ is a smooth tame map of Fréchet spaces. $J_\psi u \circ A_\psi^{-1}$ is tame with degree 0 in u and 1 in ψ:

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha}\|u\|_{1,\alpha} + \|u\|_{n,\alpha}), \qquad n \geq 0 \tag{5.19}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction). In particular,

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha} + 1)\|u\|_{n,\alpha}, \qquad n \geq 1 \tag{5.20}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction).

Alternatively, the tame estimates (5.19) are equivalent (see Lemma 45, Section A.2.2) to

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha} + \|u\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.21}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and all u in a $\|\cdot\|_{1,\alpha}$-neighborhood.

A major technical inconvenience is the fact that $J_\psi u$ is not quite an operator in the spaces of functions which we will be working in since the range of ψ may vary. Thus one must work with both A_ψ^{-1} and $J_\psi u \circ A_\psi^{-1}$ in parallel.

Lemma 28 (Smoothness of $A_\psi^{-1}(\lambda)$) *Let $\psi \in \mathcal{U}_{\gamma_i}^n$ $(n \geq 2)$ have no critical points and denote its range $[c, 0]$. The distribution function of the sets $\{\psi < \lambda\}$ is given by*

$$A_\psi(\lambda) := |\{\psi < \lambda\}| = \int_c^\lambda \left(J_\psi \frac{1}{|\nabla\psi|} \right)(l) dl, \qquad \lambda \in [c, 0]. \tag{5.22}$$

It is of class C^n, and it has an inverse

$$A_\psi^{-1} \in C^n_{[0,|\Omega|]} \qquad \textit{with derivative} \qquad \frac{dA_\psi^{-1}}{d\mu} = \frac{1}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}}. \tag{5.23}$$

More generally, if in addition $u \in C^l_{\overline{\Omega}}$, $l \geq 0$, then

$$J_\psi u \circ A_\psi^{-1} \in C^{\min(n-2,l)}_{[0,|\Omega|]}. \tag{5.24}$$

Proof

Formulas (5.22) and (5.23) have been seen in Section 4.3.

If ψ is of class C^n, then the classical implicit function theorem implies that each level set of ψ is a curve of class C^n. Fix $\lambda_0 \in (\min\psi, \max\psi)$ and a point $(x_1^0, x_2^0) \in \Omega$ on the level set $\{\psi = \lambda_0\}$. Then, there exists a neighborhood W_0 of (x_1^0, x_2^0) and a function $h(s, \lambda)$ defined for s in some interval $[s_0, s_1]$ and for λ on a neighborhood $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, such that the portions of the level sets $\{(x_1, x_2) \in W_0 \mid \psi(x_1, x_2) = \lambda\}$ are given by the graphs of $h(\cdot, \lambda)$. The funcion h is of class C^n by the (classical) implicit function theorem applied to the function $((x, \lambda), y) \mapsto \psi(x, y) - \lambda$ in order to solve for y in terms of x and λ (afer a rotation). Cropping W_0 and reducing the interval $[s_0, s_1]$ and the neighborhood $(\lambda_0 - \delta, \lambda_0 + \delta)$ if necessary, we have for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$

$$|\{(x_1, x_2) \in W_0 \mid \psi(x_1, x_2) < \lambda\}| = \int_{s_0}^{s_1} h(s, \lambda) ds \tag{5.25}$$

which is of class C^n in λ. Covering $\{\psi = \lambda_0\}$ with finitely many open sets such as W_0, we conclude that $A_\psi(\lambda)$ is of class C^n.

Later on, the following remark will be useful. For ψ in a $\|\cdot\|_1$-neighborhood of $\mathcal{U}_{\gamma_i}$ such that $\nabla\psi$ stays away from 0 and the bounds (4.4) hold, the quantity

$$A'_\psi(\lambda) = J_\psi \frac{1}{|\nabla\psi|}(\lambda), \qquad \lambda \in [c, 0], \tag{5.26}$$

stays away from 0. Therefore A_ψ has an inverse $A_\psi^{-1} \in C^n_{[0,|\Omega|]}$ with derivative given by (5.23) for ψ in such a neighborhood. Making this neighborhood even smaller if necessary, there exists a constant C (depending on the bound (4.4)) such that

$$\left\| \frac{dA_\psi^{-1}}{d\mu} \right\|_0 \leq C. \tag{5.27}$$

As for (5.24), it is enough to show that $J_\psi u(\lambda)$ is of class $C^{\min(n-2,l)}$ in $\lambda \in \text{range}(\psi)$. To see this, use a fixed level set $\{\psi = \lambda_0\}$ to parametrize nearby level sets $\{\psi = \lambda\}$, $\lambda \approx \lambda_0$. Denote $X = X(x_0, \lambda)$ the (unique nearby) point on $\{\psi = \lambda\}$ such that $X - x_0$ is parallel to the normal $N(x_0) = \frac{\nabla\psi(x_0)}{|\nabla\psi(x_0)|}$ to the level set $\{\psi = \lambda_0\}$. As noted earlier in this proof, the level sets $\{\psi = \lambda\}$ are locally the graphs of $h(\cdot, \lambda)$ for varying λ, and h is of class C^n in both s and λ. Further, the normal N is a function of class C^{n-1}. In turn $X = X(x_0, \lambda)$ is of class C^{n-1}, and the Jacobian $\rho = \rho(x_0, \lambda)$ of the change of variables $x_0 \to X$ is of class C^{n-2}. Thus

$$J_\psi u(\lambda) = \int_{x_0 : \psi(x_0) = \lambda_0} u(X(x_0, \lambda))\rho(x_0, \lambda) dl(x_0) \tag{5.28}$$

is clearly of class $C^{\min(n-2,l)}$. ∎

The following preliminary result will be used in the subsequent lemmas.

If f, g are two functions of one variable, then the n-th derivative of the composition $f \circ g$ is of the form

$$(f \circ g)^{(n)} = \sum_{k=1}^{n} \sum_{j_1 + \cdots + j_k = n} c_{k;j_1,\ldots,j_k} (f^{(k)} \circ g) g^{(j_1)} \cdots g^{(j_k)}, \qquad n \geq 0 \tag{5.29}$$

where the second sum is taken over $j_1, \ldots, j_k \geq 1$ and $c_{k;j_1,\ldots,j_k}$ are constants. In particular, if $g = f^{-1}$, then $(f \circ g)^{(n)} = 0$ for $n \geq 2$, and in turn $\sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} (f^{(k)} \circ g) g^{(j_1)} \cdots g^{(j_k)} = 0$, $n \geq 2$ or,

$$\sum_{k=1}^{n} \sum_{j_1 + \cdots + j_k = n} c_{k;j_1,\ldots,j_k} f^{(k)} (g^{(j_1)} \circ f) \cdots (g^{(j_k)} \circ f) = 0, \qquad n \geq 2. \tag{5.30}$$

Further, the first term in this sum is $f^{(n)}(g' \circ f)^n = \frac{f^{(n)}}{(f')^n}$ and therefore

$$f^{(n)} = -(f')^n \sum_{k=1}^{n-1} \sum_{j_1 + \cdots + j_k = n} c_{k;j_1,\ldots,j_k} f^{(k)} (g^{(j_1)} \circ f) \cdots (g^{(j_k)} \circ f), \qquad n \geq 2. \tag{5.31}$$

This will be used with $f = A_\psi^{-1}$ and $g = A_\psi$, and also with $f = J_\psi u$ and $g = A_\psi^{-1}$.

Fix now $\overline{\psi} \in \mathcal{U}_{\gamma_i}$ without critical points (this will be our reference steady-state) and let

$$\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i} \tag{5.32}$$

be a neighborhood such as described in Lemma 18 of Section 4.1: $\psi \in \mathcal{V}_*(\overline{\psi})$ has no critical points (hence has range of the form $[c, 0]$), and the bounds (4.4), (5.27) hold. Furthermore, introduce neighborhoods of spaces with finite regularity

$$\mathcal{V}_*^n(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^n, \qquad n = 1, 2, \ldots \tag{5.33}$$

such that $\mathcal{V}_*(\overline{\psi}) \subset \mathcal{V}_*^n(\overline{\psi}) \subset \mathcal{V}_*^1(\overline{\psi})$ for $n \geq 1$, and in which each ψ has no critical points. (For instance, $\mathcal{V}_*^n(\overline{\psi})$ can be the completion of $\mathcal{V}_*(\overline{\psi})$ in the $\|\cdot\|_n$-norm.)

Lemma 29 ($Q(\psi) = A_\psi^{-1}$ is continuous) *The map $Q\colon \psi \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \mapsto A_\psi^{-1} \in C^\infty_{[0,|\Omega|]}$ is a continuous map of Fréchet spaces. Specifically,*

$$\psi \in (\mathcal{V}_*^n(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^n) \mapsto A_\psi^{-1} \in C^n_{[0,|\Omega|]}, \qquad n \geq 2 \tag{5.34}$$

is a continuous map of Banach spaces.

More generally, $(\psi, u) \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \times C^\infty_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^\infty_{[0,|\Omega|]}$ is continuous. Specifically,

$$(\psi, u) \in (\mathcal{V}_*^{n+1}(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^{n+1}) \times C^n_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^n_{[0,|\Omega|]}, \qquad n \geq 1 \tag{5.35}$$

is continuous.

Proof

We begin the proof that Q is continuous as a map of Fréchet spaces by showing that it (or rather its extension) is continuous from a neighborhood in $\mathcal{U}_{\gamma_i}^1$ into $C^0_{[0,|\Omega|]}$.

$\boxed{1}$ **Continuity of $\psi \in (\mathcal{V}_*^1(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^1) \mapsto A_\psi^{-1} \in C^0_{[0,|\Omega|]}$.** Let $\psi \in (\mathcal{V}_*^2(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^2)$ have range $[c, 0]$ and fix $\epsilon > 0$. Let $\mu \in [0, A_\psi(c + \epsilon)]$ so that $|A_\psi^{-1}(\mu) - c| < \epsilon$. If $\|\tilde{\psi} - \psi\|_0$ is small and range$(\tilde{\psi}) = [\tilde{c}, 0]$, then $|A_{\tilde{\psi}}^{-1}(0) - A_\psi^{-1}(0)| = |\tilde{c} - c| < \epsilon$. If $\tilde{\psi}$ is restricted to a $\|\cdot\|_1$-neighborhood of ψ, then by (5.27) $|A_{\tilde{\psi}}^{-1}(\mu) - A_{\tilde{\psi}}^{-1}(0)| \leq C\mu \leq CA_\psi(c + \epsilon)$. Thus, for $\mu \in [0, A_\psi(c + \epsilon)]$ and $\tilde{\psi}$ in such neighborhood,

$$|A_{\tilde{\psi}}^{-1}(\mu) - A_\psi^{-1}(\mu)| < 2\epsilon + CA_\psi(c + \epsilon). \tag{5.36}$$

Let now $\mu \in [A(c+\epsilon), |\Omega|]$, set $\lambda = A_\psi^{-1}(\mu)$ and $\tilde\lambda = A_{\tilde\psi}^{-1}(\mu)$, and $\tilde\mu = A_{\tilde\psi}(\lambda)$. (With $\|\tilde\psi - \psi\|_0$ sufficiently small, then $\lambda \in \text{range}(\tilde\psi)$.) If $\tilde\psi$ is in a sufficiently small $\|\cdot\|_1$-neighborhood of ψ, then by the classical implicit function theorem $\{\psi < \lambda\}$ and $\{\tilde\psi < \lambda\}$ differ in a region with area as small as one desires, and $|\tilde\mu - \mu| = |A_{\tilde\psi}(\lambda) - A_\psi(\lambda)|$ is as small as one desires. In turn, from the bound (5.27), $|\lambda - \tilde\lambda| = |A_\psi^{-1}(\mu) - A_\psi^{-1}(\tilde\mu)| \leq C|\mu - \tilde\mu|$ can be made as small as one desires. We have proved continuity of

$$\psi \in (\mathcal{V}_*^1(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^1) \mapsto A_\psi^{-1} \in C^0_{[0,|\Omega|]}. \tag{5.37}$$

$\square$

$\boxed{2}$ **Continuity of** $(\psi, u) \in (\mathcal{V}_*^2(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^2) \times C^0_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^0_{[0,|\Omega|]}$ First we estimate $J_\psi u(\lambda) - J_{\psi_0} u_0(\lambda_0)$ which we write as a single integral using $\{\psi_0 = \lambda_0\}$ as a common parametrization. Overriding previous notation, we denote $X = X(\psi, x_0, \lambda)$ the (unique nearby) point on $\{\psi = \lambda\}$ such that $X - x_0$ is parallel to the normal $N_0(x_0) = \frac{\partial\psi(x_0)}{|\nabla\psi(x_0)|}$ to the level set $\{\psi_0 = \lambda_0\}$. This is well-defined for ψ in a small $\|\cdot\|_2$-neighborhood of ψ_0, $\lambda \approx \lambda_0$, and $x_0 \in \{\psi_0 = \lambda_0\}$.[1] With ψ of class C^2, $X = X(\psi, x_0, \lambda)$ is of class C^1 and the Jacobian $\rho = \rho(\psi, x_0, \lambda)$ is of class C^0. Observe also that

$$X(\psi_0, x_0, \psi_0(x_0)) = x_0, \qquad \rho(\psi_0, x_0, \psi_0(x_0)) = 1. \tag{5.38}$$

For $\psi \in (\mathcal{V}_*^2(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^2)$ as above, $u, u_0 \in C^0_{\overline{\Omega}}$, and $\lambda \approx \lambda_0$,

$$J_\psi u(\lambda) - J_{\psi_0} u_0(\lambda_0) \tag{5.39}$$

$$= \int\limits_{x_0 : \psi_0(x_0) = \lambda_0} [u(X(\psi, x_0, \lambda))\rho(\psi, x_0, \lambda) - u_0(X(\psi_0, x_0, \lambda_0))]\, dl(x_0) \tag{5.40}$$

If $\|u - u_0\|_0$, $\|\psi - \psi_0\|_2$, and $\lambda - \lambda_0$ are small, then the integrand is small. Finally, by continuity of (5.37), $\lambda - \lambda_0 = A_\psi^{-1}(\mu) - A_{\psi_0}^{-1}(\mu)$ can be made arbitrarily small, and thus $J_\psi u \circ A_\psi^{-1}(\mu) - J_{\psi_0} u_0 \circ A_{\psi_0}^{-1}(\mu)$ is arbitrarily small (uniformly in μ) provided $\|u - u_0\|_0$ and $\|\psi - \psi_0\|_2$ are small. This shows that

$$(\psi, u) \in (\mathcal{V}_*^2(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^2) \times C^0_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^0_{[0,|\Omega|]}. \tag{5.41}$$

[1] There is a technical difficulty when λ_0 is not in the range of ψ, but this can be arranged by extending ψ over some neighborhood of $\overline{\Omega}$. This is not a crucial issue, so we will omit the proof.

is continuous. $\square$

$\boxed{3}$ $\psi \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \mapsto A_\psi^{-1} \in C^1_{[0,|\Omega|]}$ **is continuous** This is immediate since

$$\psi \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \mapsto \frac{dA_\psi^{-1}}{d\mu} = \frac{1}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}} \in C^0_{[0,|\Omega|]} \tag{5.42}$$

is continuous by (5.41), by Lemma 49, Section A.2.3, and continuity of $\psi \in \mathcal{V}^2_*(\overline{\psi}) \mapsto \frac{1}{|\nabla\psi|} \in C^0_{\overline{\Omega}}$. $\square$

Applying (5.31)

$$f^{(n)} = -(f')^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} f^{(k)}(g^{(j_1)} \circ f) \cdots (g^{(j_k)} \circ f), \qquad n \geq 2. \tag{5.43}$$

to $f = A_\psi^{-1}$ and $g = A_\psi$, the n-th derivative ($n \geq 2$) is of the form

$$-\left(\frac{dA_\psi^{-1}}{d\mu}\right)^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} c_{k;j_i} \left(\frac{d^k A_\psi^{-1}}{d\mu^k}\right) \left(\frac{d^{j_1} A_\psi}{d\lambda^{j_1}} \circ A_\psi^{-1}\right) \cdots \left(\frac{d^{j_k} A_\psi}{d\lambda^{j_k}} \circ A_\psi^{-1}\right) \tag{5.44}$$

where the second sum is taken over $j_1, \ldots, j_k \geq 1$. As a preliminary we compute the derivatives (in λ) of $J_\psi u$. $\square$

$\boxed{4}$ **The derivatives $\frac{d^m}{d\lambda^m}(J_\psi u)$** Recall from Lemma 22, Section 4.3,

$$(J_\psi u)'(\lambda) := \frac{d}{d\lambda}(J_\psi u)(\lambda) = J_\psi \left(\frac{\operatorname{div}(uN)}{|\nabla\psi|}\right)(\lambda), \qquad \lambda \in [c, 0] \tag{5.45}$$

Posing

$$u_0 := u, \qquad u_m := \frac{\operatorname{div}(u_{m-1}N)}{|\nabla\psi|}, \qquad m \geq 1 \tag{5.46}$$

we have

$$\frac{d^m J_\psi u}{d\lambda^m} = J_\psi u_m, \qquad m \geq 0. \tag{5.47}$$

$\square$

$\boxed{\mathbf{5}}$ $(\psi, u) \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \times C^1_{\overline{\Omega}} \mapsto J_\psi u \circ A_\psi^{-1} \in C^1_{[0,|\Omega|]}$ **is continuous** We know that

$$(\psi, u) \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \times C^1_{\overline{\Omega}} \mapsto \frac{\operatorname{div}(uN)}{|\nabla\psi|} \in C^0_{\overline{\Omega}} \tag{5.48}$$

is continuous (see Lemma 48 and Lemma 49 of Section A.2.3), and we have seen earlier in this proof that

$$(\psi, v) \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \times C^0_{\overline{\Omega}} \mapsto J_\psi v \circ A_\psi^{-1} \in C^0_{[0,|\Omega|]} \tag{5.49}$$

is also continuous. Thus, with

$$(J_\psi u \circ A_\psi^{-1})' = J_\psi \frac{\operatorname{div}(uN)}{|\nabla\psi|} \circ A_\psi^{-1} \tag{5.50}$$

we immediately find that

$$(\psi, u) \in (\mathcal{V}^2_*(\overline{\psi}) \subset \mathcal{U}^2_{\gamma_i}) \times C^1_{\overline{\Omega}} \mapsto J_\psi \frac{\operatorname{div}(uN)}{|\nabla\psi|} \circ A_\psi^{-1} \in C^0_{[0,|\Omega|]} \tag{5.51}$$

is continuous. □

$\boxed{\mathbf{6}}$ $\psi \in (\mathcal{V}^{n+1}_*(\overline{\psi}) \subset \mathcal{U}^{n+1}_{\gamma_i}) \mapsto A_\psi^{-1} \in C^n_{[0,|\Omega|]}$ **is continuous** $(n \geq 2)$ The n-th derivative of A_ψ^{-1}, given in (5.44), now reads

$$-\left(\frac{dA_\psi^{-1}}{d\mu}\right)^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} c_{k;j_i} \left(\frac{d^k A_\psi^{-1}}{d\mu^k}\right) \Big(J_\psi v_{j_1-1} \circ A_\psi^{-1}\Big) \cdots \Big(J_\psi v_{j_k-1} \circ A_\psi^{-1}\Big) \tag{5.52}$$

where

$$v_0 := \frac{1}{|\nabla\psi|}, \qquad v_m := \frac{\operatorname{div}(v_{m-1}N)}{|\nabla\psi|}, \qquad m \geq 1 \tag{5.53}$$

and recall that the second sum is over $j_1, \ldots, j_k \geq 1$. Multiplication of functions $C^l \times C^l \to C^l$ is a continuous operation (Lemma 48, Section A.2.3). By continuity of (5.42) and of $\psi \in (\mathcal{V}^{m+1}_*(\overline{\psi}) \subset \mathcal{U}^{m+1}_{\gamma_i}) \mapsto v_0 = \frac{1}{|\nabla\psi|} \in C^m_{\overline{\Omega}}$ $(m \geq 0)$, and by a simple induction, it is enough to show that

$$(\psi, u) \in (\mathcal{V}^{m+1}_*(\overline{\psi}) \subset \mathcal{U}^{m+1}_{\gamma_i}) \times C^m_{\overline{\Omega}} \mapsto J_\psi u_m \circ A_\psi^{-1} \in C^0_{[0,|\Omega|]}, \qquad m \geq 1 \tag{5.54}$$

is continuous, where u_m is defined from u in (5.46). This is just (5.51) if $m = 1$, which also shows that in fact it is enough to prove that

$$(\psi, u) \in (\mathcal{V}^{m+1}_*(\overline{\psi}) \subset \mathcal{U}^{m+1}_{\gamma_i}) \times C^m_{\overline{\Omega}} \mapsto u_m \in C^0_{\overline{\Omega}} \tag{5.55}$$

is continuous. To show this, we prove that for all $k \geq 0$ and $m \geq 1$

$$(\psi, u) \in (\mathcal{V}_*^{m+k+1}(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^{m+k+1}) \times C_{\overline{\Omega}}^{m+k} \mapsto u_m \in C_{\overline{\Omega}}^{k} \tag{5.56}$$

is continuous. Taking (first) derivatives is a continuous operator $C^{l+1} \to C^l$ (Lemma 47, Section A.2.3) and multiplication $C^l \times C^l \to C^l$ is continuous (Lemma 48, Section A.2.3) so that

$$(\psi, u) \in (\mathcal{V}_*^{m+k+1}(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}^{m+k+1}) \times C_{\overline{\Omega}}^{m+k+1} \mapsto u_1 \in C_{\overline{\Omega}}^{m+k-1} \tag{5.57}$$

is continuous (recall that $u_1 = \frac{\operatorname{div}(uN)}{|\nabla\psi|}$). Repeating this process proves the claim. □

$\boxed{7}$ **$J_\psi u \circ A_\psi^{-1}$, as a map of Fréchet spaces, is jointly continuous in ψ and u**
Immediate from the expression (5.29)

$$(f \circ g)^{(n)} = \sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} (f^{(k)} \circ g) g^{(j_1)} \cdots g^{(j_k)}, \qquad n \geq 0 \tag{5.58}$$

with $f = J_\psi u$ and $g = A_\psi^{-1}$ and the previous paragraphs since now

$$\frac{d^n}{d\mu^n}\left(J_\psi u \circ A_\psi^{-1}\right) \tag{5.59}$$

$$= \sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} \left(\frac{d^k J_\psi u}{d\lambda^k} \circ A_\psi^{-1}\right) \left(\frac{d^{j_1} A_\psi^{-1}}{d\mu^{j_1}}\right) \cdots \left(\frac{d^{j_k} A_\psi^{-1}}{d\mu^{j_k}}\right) \tag{5.60}$$

$$= \sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} \left(J_\psi u_k \circ A_\psi^{-1}\right) \left(\frac{d^{j_1} A_\psi^{-1}}{d\mu^{j_1}}\right) \cdots \left(\frac{d^{j_k} A_\psi^{-1}}{d\mu^{j_k}}\right). \tag{5.61}$$

■

Lemma 30 ($Q(\psi) = A_\psi^{-1}$ is tame) *The map $A_\psi^{-1} = Q(\psi)$ is tame with degree 0 in ψ:*

$$\|A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.62}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood.

More generally, the map $(\psi, u) \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \times C_{\overline{\Omega}}^{\infty} \mapsto J_\psi u \circ A_\psi^{-1} \in C_{[0,|\Omega|]}^{\infty}$ is tame with degree 0 in u, 1 in ψ:

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha}\|u\|_{1,\alpha} + \|u\|_{n,\alpha}), \qquad n \geq 0 \tag{5.63}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction). In particular,

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha} + 1)\|u\|_{n,\alpha}, \qquad n \geq 1 \tag{5.64}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction).

Proof

First we establish estimates for $J_\psi u \circ A_\psi^{-1}$. □

$\boxed{1}$ **Estimate on $\|J_\psi u \circ A_\psi^{-1}\|_{0,\alpha}$** In this paragrpah we show taht

$$\|J_\psi u \circ A_\psi^{-1}\|_{0,\alpha} \leq C \cdot \|u\|_{0,\alpha} \tag{5.65}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction).

It is clear that there exists a constant depending on the bound (4.4) such that

$$\|J_\psi u \circ A_\psi^{-1}\|_0 \leq C \cdot \|u\|_0. \tag{5.66}$$

for ψ in a $\|\cdot\|_1$-neighborhood and any u (without restriction). To estimate the Hölder-constant $[J_\psi u \circ A_\psi^{-1}]_\alpha$ (see notation in (1.63)), we use as before a (local) change of coordinate $X = X(x, \lambda)$ denoting the (unique nearby) point on $\{\psi = \lambda\}$ such that $X - x$ is parallel to the normal $N(x) = \frac{\nabla\psi(x)}{|\nabla\psi(x)|}$ to $\{\psi = \psi(x)\}$. This is a smooth function and the corresponding Jacobian $\rho(x, \lambda)$ is smooth as well. Observe that $X(x, \psi(x)) = x$ and $\rho(x, \psi(x)) = 1$. Fix $\mu, \mu' \in [0, |\Omega|]$ and set $\lambda = A_\psi^{-1}(\mu)$, $\lambda' = A_\psi^{-1}(\mu')$. Then

$$\begin{aligned}
& J_\psi u(\lambda') - J_\psi u(\lambda) && (5.67)\\
= & \int\limits_{x':\psi(x')=\lambda'} u(x')dl(x') - \int\limits_{x:\psi(x)=\lambda} u(x)dl(x) && (5.68)\\
= & \int\limits_{x:\psi(x)=\lambda} \Big[u(X(x,\lambda'))\rho(x,\lambda') - u(X(x,\lambda))\rho(x,\lambda)\Big]\, dl(x) && (5.69)\\
= & \int\limits_{x:\psi(x)=\lambda} \Big[u(X(x,\lambda'))(\rho(x,\lambda') - \rho(x,\lambda)) && (5.70)\\
& +(u(X(x,\lambda')) - u(X(x,\lambda)))\,\underbrace{\rho(x,\lambda)}_{=1}\Big]\, dl(x) && (5.71)\\
=: & \; I + II && (5.72)
\end{aligned}$$

Taylor's formula gives

$$\begin{aligned} & \rho(x,\lambda') - \rho(x,\lambda) && (5.73)\\ = \;& D_\lambda \rho(x,\lambda)(\lambda'-\lambda) + (\lambda'-\lambda)^2 \int_0^1 (1-t) D_\lambda^2 \rho(x, \lambda + t(\lambda'-\lambda))\ dt && (5.74) \end{aligned}$$

ρ as a C^2-function depends continuously on ψ as a C^4-function. Thus for ψ in a $\|\cdot\|_4$-neighborhood (and hence in a $\|\cdot\|_{4,\alpha}$-neighborhood) there exists a constant C independent of ψ and of $x \in \overline{\Omega}$ such that

$$|\rho(x,\lambda') - \rho(x,\lambda)| \leq C \cdot |\lambda' - \lambda|. \tag{5.75}$$

In turn, for such ψ and any u (without restriction),

$$|I| \leq C \cdot \|u\|_0 |\lambda' - \lambda|. \tag{5.76}$$

Similarly, for II we have by Taylor's formula

$$X(x,\lambda') - X(x,\lambda) = D_\lambda X(x,\lambda) \cdot (\lambda'-\lambda) + (\lambda'-\lambda)^2 \int_0^1 (1-t) D_\lambda^2 X(x, \lambda + t(\lambda'-\lambda)) dt. \tag{5.77}$$

Since X as a C^2-function depends continuously on ψ as a C^3-function, for ψ in a $\|\cdot\|_3$-neighborhood (and hence in a $\|\cdot\|_{4,\alpha}$-neighborhood), we have

$$|X(x,\lambda') - X(x,\lambda)| \leq C \cdot |\lambda' - \lambda| \tag{5.78}$$

hence for such ψ and any u (without restriction),

$$|II| \leq C \cdot [u]_\alpha |\lambda' - \lambda|^\alpha. \tag{5.79}$$

In turn we have $|J_\psi u(\lambda') - J_\psi u(\lambda)| \leq C\left\{\|u\|_0 |\lambda'-\lambda| + [u]_\alpha |\lambda'-\lambda|^\alpha\right\}$ which implies that $J_\psi u$ is α-Hölder continuous on $[c,0]$ and we have the estimate

$$\left| J_\psi u(\lambda') - J_\psi u(\lambda) \right| \leq C \|u\|_{0,\alpha} |\lambda' - \lambda|^\alpha \tag{5.80}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction). Finally, by the bound (5.27) on $(A_\psi^{-1})'$,

$$\left| J_\psi u \circ A_\psi^{-1}(\mu') - J_\psi u \circ A_\psi^{-1}(\mu) \right| \leq C \cdot \|u\|_{0,\alpha} |\mu' - \mu|^\alpha \tag{5.81}$$

for ψ in $\|\cdot\|_{4,\alpha}$-neighborhood. Putting (5.66) and (5.81) together yields

$$\|J_\psi u \circ A_\psi^{-1}\|_{0,\alpha} \leq C \cdot \|u\|_{0,\alpha} \tag{5.82}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and any u (without restriction). □

$\boxed{\mathbf{2}}$ **Estimate on** $\|u_m\|_{0,\alpha}$ We show next that

$$\|u_m\|_{0,\alpha} \leq C \cdot \left(\|\psi\|_{m+1,\alpha}\|u\|_{0,\alpha} + \|u\|_{m,\alpha}\right), \qquad m \geq 0 \tag{5.83}$$

for ψ in a $\|\cdot\|_{2,\alpha}$-neighborhood and any u without restriction. For sake of simplicity, we first consider the following situation. Suppose that u, a are smooth functions of one variable and set

$$u_0 := u, \qquad u_{m+1} := (u_m a)' a, \qquad m \geq 0. \tag{5.84}$$

We show by induction on m that

$$u_m = \sum_{j_0+\cdots+j_m=m} c_{m;j_1,\ldots,j_m} u^{(j_0)} a^{(j_1)} \cdots a^{(j_m)} a^m \tag{5.85}$$

where the sum is taken over $j_1, \ldots, j_m \geq 0$. This is verified for $m = 0$ and $m = 1$:

$$u_0 = u, \qquad u_1 = (u'a + ua')a. \tag{5.86}$$

Suppose this is verified up to some $m \geq 1$. Then, (the constants will change values)

$$
\begin{aligned}
& u_{m+1} && (5.87)\\
= & \sum_{j_0+\cdots+j_m=m} c_{m;j_0,\ldots,j_m}(u^{(j_0)}a^{(j_1)}\cdots a^{(j_m)}a^{m+1})'a && (5.88)\\
= & \sum_{j_0+\cdots+j_m=m} c_{m;j_0,\ldots,j_m}\Big(u^{(j_0+1)}a^{(j_1)}\cdots a^{(j_m)}a^{m+1} && (5.89)\\
& +u^{(j_0)}a^{(j_1+1)}a^{(j_2)}\cdots a^{(j_m)}a^{m+1}+\cdots && (5.90)\\
& +u^{(j_0)}a^{(j_1)}\cdots a^{(j_{m-1})}a^{(j_m+1)}a^{m+1} && (5.91)\\
& +(m+1)u^{(j_0)}a^{(j_1)}\cdots a^{(j_{m-1})}a^{(j_m)}a'a^m\Big)a && (5.92)\\
= & \sum_{j_0+\cdots+j_m=m} c_{m;j_0,\ldots,j_m}\Big(u^{(j_0+1)}a^{(j_1)}\cdots a^{(j_m)}a^{(0)} && (5.93)\\
& +u^{(j_0)}a^{(j_1+1)}a^{(j_2)}\cdots a^{(j_m)}a^{(0)}+\cdots && (5.94)\\
& +u^{(j_0)}a^{(j_1)}\cdots a^{(j_{m-1})}a^{(j_m+1)}a^{(0)} && (5.95)\\
& +(m+1)u^{(j_0)}a^{(j_1)}\cdots a^{(j_{m-1})}a^{(j_m)}a^{(1)}\Big)a^{m+1} && (5.96)\\
= & \sum_{j_0'+\cdots+j_m'+j_{m+1}'=m+1} c_{m+1;j_0',\ldots,j_{m+1}'}u^{(j_0')}a^{(j_1')}\cdots a^{(j_m')}a^{m+1} && (5.97)
\end{aligned}
$$

Let's now estimate $\|u_m\|_{0,\alpha}$ using interpolation inequalities (see Lemma 41 of Section A.2.1) and estimates for products of functions (Lemma 48, Section A.2.3):

$$
\begin{aligned}
& \|u_m\|_{0,\alpha} && (5.98)\\
\leq & C\sum_{j_0+\cdots+j_m=m}\|u\|_{j_0,\alpha}\|a\|_{j_1,\alpha}\cdots\|a\|_{j_m,\alpha}\|a^m\|_{0,\alpha} && (5.99)\\
\leq & C\sum_{j_0+\cdots+j_m=m}\|u\|_{j_0,\alpha}\times && (5.100)\\
& \times\|a\|_{0,\alpha}^{\frac{m-j_0-j_1}{m-j_0}}\|a\|_{m-j_0,\alpha}^{\frac{j_1}{m-j_0}}\cdots\|a\|_{0,\alpha}^{\frac{m-j_0-j_m}{m-j_0}}\|a\|_{m-j_0,\alpha}^{\frac{j_m}{m-j_0}}\times\|a\|_{0,\alpha}^m && (5.101)\\
\leq & C\|a\|_{0,\alpha}^{l_m}\sum_{j_0+\cdots+j_m=m}\|u\|_{j_0,\alpha}\|a\|_{m-j_0,\alpha}^{\frac{j_1}{m-j_0}}\cdots\|a\|_{m-j_0,\alpha}^{\frac{j_m}{m-j_0}} && (5.102)\\
\leq & C\|a\|_{0,\alpha}^{l_m}\sum_{j_0+\cdots+j_m=m}\|u\|_{j_0,\alpha}\|a\|_{m-j_0,\alpha} && (5.103)\\
\leq & C\|a\|_{0,\alpha}^{l_m}\sum_{j_0=0}^{m}(\|u\|_{0,\alpha}\|a\|_{m,\alpha}+\|u\|_{m,\alpha}\|a\|_{m,\alpha}) && (5.104)\\
\leq & C\|a\|_{0,\alpha}^{l_m}(\|u\|_{0,\alpha}\|a\|_{m,\alpha}+\|u\|_{m,\alpha}\|a\|_{m,\alpha}) && (5.105)
\end{aligned}
$$

since $j_1 + \cdots + j_m = m - j_0$, and where l_m is some positive integers depending on m. In turn, we have the estimate

$$\|u_m\|_{0,\alpha} \leq C(\|u\|_{m,\alpha} + \|a\|_{m,\alpha}\|u\|_{0,\alpha}), \qquad m \geq 0 \tag{5.106}$$

for all u without restriction, and a in a $\|\cdot\|_{0,\alpha}$-neighborhood.

The situation with u_m as defined by $u_m = \frac{\operatorname{div}(uN)}{|\nabla\psi|}$ can be dealt with in a similar fashion, only the details are more tedious. Here, a plays the rôle of $\frac{1}{|\nabla\psi|}$ or N, which are both smooth tame functions of $\nabla\psi$. In conclusion,

$$\|u_m\|_{0,\alpha} \leq C(\|u\|_{m,\alpha} + \|\psi\|_{m+1,\alpha}\|u\|_{0,\alpha}), \qquad m \geq 0 \tag{5.107}$$

for all ψ in a $\|\cdot\|_{2,\alpha}$-neighborhood and all u without restriction. □

$\boxed{3}$ **Estimates on $\|J_\psi u_m \circ A_\psi^{-1}\|_{0,\alpha}$** We easily conclude from (5.65) and (5.83) that

$$\left\|\frac{d^m J_\psi u}{d\lambda^m} \circ A_\psi^{-1}\right\|_{0,\alpha} = \left\|J_\psi u_m \circ A_\psi^{-1}\right\|_{0,\alpha} \leq C\cdot(\|u\|_{m,\alpha} + \|\psi\|_{m+1,\alpha}\|u\|_{0,\alpha}), \qquad m \geq 0 \tag{5.108}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, and any u without restriction. Alternatively, since $J_\psi u \circ A_\psi^{-1}$ is linear in u,

$$\left\|\frac{d^m J_\psi u}{d\lambda^m} \circ A_\psi^{-1}\right\|_{0,\alpha} = \left\|J_\psi u_m \circ A_\psi^{-1}\right\|_{0,\alpha} \leq C\cdot(\|u\|_{m,\alpha} + \|\psi\|_{m+1,\alpha} + 1), \qquad m \geq 0 \tag{5.109}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, and u in a $\|\cdot\|_{0,\alpha}$-neighborhood. □

$\boxed{4}$ **Estimate on $\|A_\psi^{-1}\|_{n,\alpha}$** In this section we show that

$$\left\|\frac{d^n A_\psi^{-1}}{d\mu^n}\right\|_{0,\alpha} \leq (\|\psi\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.110}$$

for ψ in a $\|\psi\|_{4,\alpha}$-neighborhood.

To alleviate notation we put

$$f = A_\psi^{-1}, \qquad g = A_\psi, \qquad \text{and} \qquad J = J_\psi \frac{1}{|\nabla\psi|} \tag{5.111}$$

and simply write f', g', J' for their derivatives (with respect to μ or λ accordingly). Note that $g' = J$ by (21), so that (5.31) reads

$$\begin{aligned} f^{(n)} &= -(f')^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} f^{(k)}(g^{(j_1)} \circ f) \cdots (g^{(j_k)} \circ f) && (5.112) \\ &= -(f')^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k} f^{(k)}(J^{(j_1-1)} \circ f) \cdots (J^{(j_k-1)} \circ f) && (5.113) \end{aligned}$$

where the second sum is over $j_1, \ldots, j_k \geq 1$, and by tame estimates for a product of functions (Lemma 48, Section A.2.3)

$$\|f^{(n)}\|_{0,\alpha} \leq C\|f'\|_{0,\alpha}^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} \|f^{(k)}\|_{0,\alpha}\|J^{(j_1-1)} \circ f\|_{0,\alpha} \cdots \|J^{(j_k-1)} \circ f\|_{0,\alpha} \quad (5.114)$$

where the second sum is over $j_1, \ldots, j_k \geq 1$. We prove by induction the estimate (5.62):

$$\|f^{(n)}\|_{0,\alpha} \leq C \cdot (\|\psi\|_{n,\alpha} + 1), \qquad n \geq 0 \quad (5.115)$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood. By continuity of (5.34) with $n = 4$, for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, there exists a constant C such that

$$\|f^{(4)}\|_{0,\alpha} \leq C \quad (5.116)$$

which implies that (5.115) holds up to $n = 4$. Suppose (5.115) holds up to some $n-1 \geq 4$. Using (5.109) with $u = \frac{1}{|\nabla\psi|}$ and $m = j - 1$ ($j \geq 1$), we find

$$\|J^{(j-1)} \circ f\|_{0,\alpha} \leq C \cdot (\|\psi\|_{j,\alpha} + 1), \qquad j \geq 1 \quad (5.117)$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood. This and the induction hypothesis for $n - 1$ give

$$\|f^{(n)}\|_{0,\alpha} \leq C\|f'\|_{0,\alpha}^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_k=n} (\|\psi\|_{k,\alpha} + 1)(\|\psi\|_{j_1,\alpha} + 1) \cdots (\|\psi\|_{j_k,\alpha} + 1) \quad (5.118)$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, and where the sum is over $j_1, \ldots, j_k \geq 1$. The double sum is the sum of 1, $\|\psi\|_{k,\alpha}\|\psi\|_{j_1,\alpha} \cdots \|\psi\|_{j_k,\alpha}$, and products involving fewer factors. Consider first the term $\|\psi\|_{k,\alpha}\|\psi\|_{j_1,\alpha} \cdots \|\psi\|_{j_k,\alpha}$. By multiplicative interpolation inequalities (Lemma 41, Section A.2.1),

$$\|\psi\|_{l,\alpha} \leq C \cdot \|\psi\|_{1,\alpha}^{\frac{n-l}{n-1}} \|\psi\|_{n,\alpha}^{\frac{l-1}{n-1}}, \qquad 1 \leq l \leq n \quad (5.119)$$

(which we can use since $1 \leq k, j_i \leq n-1$) we find

$$
\begin{aligned}
& \|\psi\|_{k,\alpha}\|\psi\|_{j_1,\alpha}\cdots\|\psi\|_{j_k,\alpha} && (5.120)\\
\leq\ & C\cdot\|\psi\|_{1,\alpha}^{\frac{n-k}{n-1}}\|\psi\|_{n,\alpha}^{\frac{k-1}{n-1}}\cdot\|\psi\|_{1,\alpha}^{\frac{n-j_1}{n-1}}\|\psi\|_{n,\alpha}^{\frac{j_1-1}{n-1}}\cdots\|\psi\|_{1,\alpha}^{\frac{n-j_k}{n-1}}\|\psi\|_{n,\alpha}^{\frac{j_k-1}{n-1}} && (5.121)\\
\leq\ & C\cdot\|\psi\|_{n,\alpha}^{\frac{k-1}{n-1}+\sum_{i=1}^{k}\frac{j_i-1}{n-1}} && (5.122)\\
\leq\ & C\cdot\|\psi\|_{n,\alpha} && (5.123)
\end{aligned}
$$

for ψ in a $\|\cdot\|_{1,\alpha}$-neighborhood, hence for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood. In case $\|\psi\|_{k,\alpha}$ and/or some of the $\|\psi\|_{j,\alpha}$'s are missing, replace the remaining ones by $\|\psi\|_{j_i',\alpha}$ where $j_i' \geq j_i$ and such that $j_1' + j_2' + \cdots = n$. (This can only increase the quantity, up to a constant, independent of ψ and u.) Then the proof above gives the same result.

Putting the above together, we have derived (5.115) for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, which is none other than the tame estimates (5.62) on A_ψ^{-1}. □

$\boxed{5}$ $J_\psi u \circ A_\psi^{-1}$ **is tame** In this section we show that

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C\cdot(\|\psi\|_{n+1,\alpha}\|u\|_{1,\alpha} + \|u\|_{n,\alpha}), \qquad n \geq 0 \tag{5.124}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and all u without restriction.

Recall the expression (5.29) for the n-th derivative of $f\circ g$:

$$(f\circ g)^{(n)} = \sum_{k=1}^{n}\sum_{j_1+\cdots+j_k=n} c_{k;j_1,\ldots,j_k}(f^{(k)}\circ g)g^{(j_1)}\cdots g^{(j_k)}, \qquad n\geq 2. \tag{5.125}$$

With $f = J_\psi u$ and $g = A_\psi^{-1}$ and (5.47), this becomes

$$\frac{d^n(J_\psi u\circ A_\psi^{-1})}{d\mu^n} = \sum_{k=1}^{n}\sum_{j_1+\cdots+j_k=n} c_{k;j_i}\left(J_\psi u_k \circ A_\psi^{-1}\right)\left(\frac{d^{j_1}A_\psi^{-1}}{d\mu^{j_1}}\right)\cdots\left(\frac{d^{j_k}A_\psi^{-1}}{d\mu^{j_k}}\right). \tag{5.126}$$

Recall estimates (5.109) and (5.115):

$$
\begin{aligned}
\left\|J_\psi u_k \circ A_\psi^{-1}\right\|_{0,\alpha} &\leq C\cdot(\|u\|_{k,\alpha} + \|\psi\|_{k+1,\alpha} + 1), \qquad k\geq 0, && (5.127)\\
\|A_\psi^{-1}\|_{j,\alpha} &\leq C\cdot(\|\psi\|_{j,\alpha}+1), \qquad j\geq 0 && (5.128)
\end{aligned}
$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, and all u in a $\|\cdot\|_{1,\alpha}$-neighborhood. Then

$$\begin{aligned}
&\left\|\left(J_\psi u_k \circ A_\psi^{-1}\right)\left(\frac{d^{j_1} A_\psi^{-1}}{d\mu^{j_1}}\right)\cdots\left(\frac{d^{j_k} A_\psi^{-1}}{d\mu^{j_k}}\right)\right\|_{0,\alpha} && (5.129)\\
\leq\ & C\cdot(\|u\|_{k,\alpha}+\|\psi\|_{k+1,\alpha}+1)(\|\psi\|_{j_1,\alpha}+1)\cdots(\|\psi\|_{j_k,\alpha}+1) && (5.130)
\end{aligned}$$

for ψ in $\|\cdot\|_{4,\alpha}$-neighborhood and all u in a $\|\cdot\|_{0,\alpha}$-neighborhood. The last equation is the sum of products of $\|\psi\|_{j,\alpha}$'s, and of products of a number of $\|\psi\|_{j,\alpha}$'s with either $\|u\|_{k,\alpha}$ or $\|\psi\|_{k+1,\alpha}$. We will use again multiplicative interpolation inequalities:

$$\|u\|_{k,\alpha} \leq C\|u\|_{1,\alpha}^{\frac{n-k}{n-1}}\|u\|_{n,\alpha}^{\frac{k-1}{n-1}}, \quad 1\leq k\leq n, \tag{5.131}$$

$$\|\psi\|_{j,\alpha} \leq C\|\psi\|_{1,\alpha}^{\frac{n-j}{n-1}}\|\psi\|_{n,\alpha}^{\frac{j-1}{n-1}}, \qquad 1\leq j\leq n. \tag{5.132}$$

Products of $\|\psi\|_{j,\alpha}$'s can be estimated by $\|\psi\|_{n,\alpha}$ in a way similar to that which led to (5.123). As for the estimate of a product with $\|u\|_{k,\alpha}$ as one of the factors, using that $j_1+\cdots+j_k=n$, $\|\psi\|_{1,\alpha}^{\frac{(k-1)n}{n-1}} = \|\psi\|_{1,\alpha}^{k-1}\|\psi\|_{1,\alpha}^{\frac{k-1}{n-1}}$, and the inequality $x^\delta y^{1-\delta}\leq x+y$ $(0\leq\delta\leq 1)$, we find

$$\begin{aligned}
&\|u\|_{k,\alpha}\|\psi\|_{j_1,\alpha}\cdots\|\psi\|_{j_k,\alpha} && (5.133)\\
\leq\ & C\|u\|_{1,\alpha}^{\frac{n-k}{n-1}}\|u\|_{n,\alpha}^{\frac{k-1}{n-1}}\|\psi\|_{1,\alpha}^{\frac{n-j_1}{n-1}}\|\psi\|_{n,\alpha}^{\frac{j_1-1}{n-1}}\cdots\|\psi\|_{1,\alpha}^{\frac{n-j_k}{n-1}}\|\psi\|_{n,\alpha}^{\frac{j_k-1}{n-1}} && (5.134)\\
\leq\ & C\|u\|_{1,\alpha}^{\frac{n-k}{n-1}}\|u\|_{n,\alpha}^{\frac{k-1}{n-1}}\|\psi\|_{1,\alpha}^{\frac{(k-1)n}{n-1}}\|\psi\|_{n,\alpha}^{\frac{n-k}{n-1}} && (5.135)\\
=\ & C\cdot\|\psi\|_{1,\alpha}^{k-1}\left(\|u\|_{1,\alpha}\|\psi\|_{n,\alpha}\right)^{\frac{n-k}{n-1}}\left(\|u\|_{n,\alpha}\|\psi\|_{1,\alpha}\right)^{\frac{k-1}{n-1}} && (5.136)\\
\leq\ & C\cdot\|\psi\|_{1,\alpha}^{k-1}\Big[\|u\|_{1,\alpha}\|\psi\|_{n,\alpha}+\|u\|_{n,\alpha}\|\psi\|_{1,\alpha}\Big] && (5.137)
\end{aligned}$$

so that for ψ in a $\|\cdot\|_{1,\alpha}$-neighborhood and any u (without restriction),

$$\|u\|_{k,\alpha}\|\psi\|_{j_1,\alpha}\cdots\|\psi\|_{j_k,\alpha}\leq C\cdot(\|u\|_{n,\alpha}+\|u\|_{1,\alpha}\|\psi\|_{n,\alpha}). \tag{5.138}$$

Taking $u=\nabla\psi$ one immediately finds

$$\|\psi\|_{k+1,\alpha}\|\psi\|_{j_1,\alpha}\cdots\|\psi\|_{j_k,\alpha}\leq C\cdot\|\psi\|_{n+1,\alpha}, \qquad n\geq 0 \tag{5.139}$$

for ψ in a $\|\cdot\|_{2,\alpha}$-neighborhood. Putting these together, we arrive at

$$\begin{aligned}
&\left\|\left(J_\psi u_k \circ A_\psi^{-1}\right)\left(\frac{d^{j_1} A_\psi^{-1}}{d\mu^{j_1}}\right)\cdots\left(\frac{d^{j_k} A_\psi^{-1}}{d\mu^{j_k}}\right)\right\|_{0,\alpha} && (5.140)\\
\leq\ & C\cdot(1+\|\psi\|_{n+1,\alpha}+\|u\|_{n,\alpha}+\|u\|_{1,\alpha}\|\psi\|_{n,\alpha}), \qquad n\geq 0 && (5.141)
\end{aligned}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood, and any u without restriction. Summing all such terms implies the following tame estimates for $J_\psi u \circ A_\psi^{-1}$ with degree 0 in u and 1 in ψ:

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha} + \|u\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.142}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and all u in a $\|\cdot\|_{1,\alpha}$-neighborhood. Alternatively, since $J_\psi u \circ A_\psi^{-1}$ is linear in u, we may remove the restriction on u (Lemma 45, Section A.2.2):

$$\|J_\psi u \circ A_\psi^{-1}\|_{n,\alpha} \leq C \cdot (\|\psi\|_{n+1,\alpha}\|u\|_{1,\alpha} + \|u\|_{n,\alpha}\|), \qquad n \geq 0 \tag{5.143}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood and all u without restriction. This is (5.63). ■

Lemma 31 ($Q(\psi) = A_\psi^{-1}$ is smooth tame) *The map $Q\colon (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \to C^\infty_{[0,|\Omega|]}$ is a smooth tame map of Fréchet spaces with first derivative*

$$DQ(\psi) \cdot \phi = \frac{J_\psi \frac{\phi}{|\nabla\psi|} \circ A_\psi^{-1}}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}}. \tag{5.144}$$

More generally, the map $(\psi, u) \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \times C^\infty_{0,|\Omega|]} \mapsto J_\psi u \circ A_\psi^{-1} \in C^\infty_{[0,|\Omega|]}$ is a smooth tame map of Fréchet spaces.

Proof

$\boxed{1}$ **Differentiability of $Q\colon (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}) \to C^\infty_{[0,|\Omega|]}$** Let $\psi \in (\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i})$ and $\phi \in \mathcal{U}_0$. For $\epsilon > 0$ sufficiently small, $\psi_\epsilon = \psi + \epsilon\phi$ is in $\mathcal{V}_*(\overline{\psi}) \subset \mathcal{U}_{\gamma_i}$. We first want to show that

$$C^n - \lim_{\epsilon \to 0} \frac{A_{\psi_\epsilon}^{-1} - A_\psi^{-1}}{\epsilon} \tag{5.145}$$

exists for all $n \geq 0$. If this is the case, then the derivative is given pointwise by

$$\left(\frac{d}{d\epsilon}_{|\epsilon=0} A_{\psi_\epsilon}^{-1}\right)(\mu) = \frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\psi_\epsilon}^{-1}(\mu). \tag{5.146}$$

Writing

$$A_{\psi_\epsilon}^{-1}(\mu) = \int_{|\Omega|}^{\mu} \frac{1}{J_{\psi_\epsilon} \frac{1}{|\nabla\psi_\epsilon|} \circ A_{\psi_\epsilon}^{-1}(l)} dl \tag{5.147}$$

we see that $A_{\psi_\epsilon}^{-1}(\mu)$ is a smooth function of (ϵ, μ). Thus, for each $n \geq 0$

$$C^0 - \lim_{\epsilon \to 0} \frac{\frac{d^n}{d\mu^n} A_{\psi_\epsilon}^{-1} - \frac{d^n}{d\mu^n} A_\psi^{-1}}{\epsilon} = \frac{\partial^n}{\partial\mu^n} \frac{\partial}{\partial\epsilon}_{|\epsilon=0} A_{\psi_\epsilon}^{-1}. \tag{5.148}$$

$DQ(\psi) \cdot \phi$ exists and is given pointwise by $\frac{\partial}{\partial \epsilon}_{|\epsilon=0} A^{-1}_{\psi_\epsilon}(\mu)$. $\square$

$\boxed{2}$ **First derivative** $DQ(\psi) \cdot \phi$ For $\mu \in (0, |\Omega|]$, set $\lambda = A^{-1}_\psi(\mu)$ and denote the range of ψ by $[c, 0]$. If $\epsilon > 0$ is sufficiently small, then $\lambda \in$ range (ψ_ϵ) and it makes sense to differentiate $A^{-1}_{\psi_\epsilon}(A_{\psi_\epsilon}(\lambda))$ with respect to ϵ:

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} A^{-1}_{\psi_\epsilon}(\mu) + \frac{dA^{-1}_\psi}{d\mu}(\mu) \frac{\partial}{\partial \epsilon}_{|\epsilon=0} A_{\psi_\epsilon}(\lambda) = 0. \tag{5.149}$$

Corollary 24, Section 4.3, gives, for $c < \lambda' < \lambda \leq 0$,

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} A_{\psi_\epsilon}(\lambda) - \frac{\partial}{\partial \epsilon}_{|\epsilon=0} A_{\psi_\epsilon}(\lambda') \tag{5.150}$$

$$= J_\psi \frac{\phi}{|\nabla \psi|}(\lambda') - J_\psi \frac{\phi}{|\nabla \psi|}(\lambda) \tag{5.151}$$

For all ϵ, $A_{\psi_\epsilon}(0) = |\Omega|$ and $\phi_{|\Gamma_o} \equiv 0$ so that taking $\lambda' = 0$ we find

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} A_{\psi_\epsilon}(\lambda) = 0 = -J_\psi \frac{\phi}{|\nabla \psi|}(\lambda), \qquad \lambda \in [c, 0] \tag{5.152}$$

and in turn

$$\frac{\partial}{\partial \epsilon}_{|\epsilon=0} A^{-1}_{\psi_\epsilon}(\mu) = \frac{dA^{-1}_\psi}{d\mu}(\mu) \ \left(J_\psi \frac{\phi}{|\nabla \psi|} \circ A^{-1}_\psi \right)(\mu). \tag{5.153}$$

We can also write this as

$$DQ(\psi) \cdot \phi \ = \ \frac{J_\psi \frac{\phi}{|\nabla \psi|} \circ A^{-1}_\psi}{J_\psi \frac{1}{|\nabla \psi|} \circ A^{-1}_\psi} = R_1(J_\psi v^1_1 \circ A^{-1}_\psi, J_\psi v^1_2 \circ A^{-1}_\psi) \tag{5.154}$$

where R_1 is a rational function, and $v^1_1 = \frac{\phi}{|\nabla \psi|}$, $v^1_2 = \frac{1}{|\nabla \psi|}$ are smooth tame maps of ϕ and ψ. In particular, $DQ(\psi) \cdot \phi$ is continuous in ψ and ϕ by Lemma 29 and satisfies tame estimates. $\square$

$\boxed{3}$ **First derivative of** $J_\psi u \circ A^{-1}_\psi$ The operator $J_\psi u \circ A^{-1}_\psi$ is linear in u so we only need to worry about differentiability in ψ (Proposition 38, Section A.1). With $\psi_\epsilon = \psi + \epsilon\phi$ as above, $J_{\psi_\epsilon} u \circ A^{-1}_{\psi_\epsilon}(\mu)$ is a smooth function of (μ, ϵ), and a similar argument as for A^{-1}_ψ shows that $\frac{d}{d\epsilon}_{|\epsilon=0}(J_{\psi_\epsilon} u \circ A^{-1}_{\psi_\epsilon})$ exists in the C^∞-topology. It is computed pointwise as follows: for $\mu \in (0, |\Omega|)$, setting $\lambda = A_\psi^{-1}(\mu)$, and using (4.22), (5.18), and (4.31).

$$\begin{aligned}
& \frac{\partial}{\partial \epsilon}_{|\epsilon=0} (J_{\psi_\epsilon} u(A_{\psi_\epsilon}^{-1}(\mu))) && (5.155)\\
= \; & \frac{\partial J_{\psi_\epsilon} u}{\partial \epsilon}_{|\epsilon=0} (A_\psi^{-1}(\mu)) + (J_\psi u)'(\lambda) \left(\frac{\partial A_{\psi_\epsilon}^{-1}}{\partial \epsilon}_{|\epsilon=0} (\mu) \right) && (5.156)\\
= \; & -J_\psi \left(\frac{\phi \mathrm{div}(uN)}{|\nabla \psi|} \right) \circ A_\psi^{-1}(\mu) && (5.157)\\
& + \left(J_\psi \left(\frac{\mathrm{div}(uN)}{|\nabla\psi|} \right) \circ A_\psi^{-1} \right)(\mu) \left(\frac{J_\psi \frac{\phi}{|\nabla\psi|} \circ A_\psi^{-1}}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}} \right)(\mu). && (5.158)
\end{aligned}$$

Thus, the first derivative of $J_\psi u \circ A_\psi^{-1}$ in ψ is in the form of a rational function of tame maps

$$D(J_\psi u \circ A_\psi^{-1}) \cdot \phi = \tilde{R}_1(J_\psi w_1 \circ A_\psi^{-1}, \ldots, \ldots, J_\psi w_4 \circ A_\psi^{-1}) \quad (5.159)$$

where $w_1 = \frac{\mathrm{div}(uN)}{|\nabla\psi|}$, $w_2 = \frac{\phi}{|\nabla\psi|}$, $w_3 = \frac{1}{|\nabla\psi|}$, $w_4 = \frac{\phi \mathrm{div}(uN)}{|\nabla\psi|}$ which are all smooth tame maps of ψ, u, and ϕ. Note also that this expression is bilinear in u and ϕ. □

$\boxed{4}$ **Higher derivatives of Q** The second derivative is of the form

$$D^2Q(\psi) \cdot (\phi_1, \phi_2) = D_\psi(DQ(\psi) \cdot \phi_1) \cdot \phi_2 = R_2(J_\psi v_1^2 \circ A_\psi^{-1}, \ldots, J_\psi v_{l_2}^2 \circ A_\psi^{-1}) \quad (5.160)$$

where $v_1, \ldots, v_{l_2}$ are smooth tame maps of ϕ_1, ϕ_2, ψ, and in particular $D^2Q(\psi) \cdot (\phi_1, \phi_2)$ is tame. By induction, Q has derivatives of all orders satisfying tame estimates. They are of the form

$$D^mQ(\psi) \cdot (\phi_1, \ldots, \phi_m) = R_m(J_\psi v_1^m \circ A_\psi^{-1}, \ldots J_\psi v_{l_m}^m \circ A_\psi^{-1}) \quad (5.161)$$

where R_m is a rational function, and each v_j^m is a smooth (tame) map of $\psi, \phi_1, \ldots, \phi_m$.

By almost identical arguments, one also sees that $J_\psi u \circ A_\psi^{-1}$ has derivatives of all orders, and that these are tame maps: the n-th derivative is a rational function $\tilde{R}_n$ of $J_\psi w_j^n \circ A_\psi^{-1}$'s where the w_j^n's are smooth tame maps of $u, \psi, \phi_1, \ldots, \phi_n$ (by linearity in u, we only need to worry about differentiability in ψ; see Proposition 38, Section A.1). ■

5.3 $T(F)$ is a smooth tame map of F

We recall the underlying assumptions. The constant γ_i is fixed, the stream function $\overline{\psi}$ and vorticity function $\overline{\omega}$ of the reference steady-state have no critical points in $\overline{\Omega}$ and satisfy $\overline{\omega} = \Delta\overline{\psi} = \overline{F}(\overline{\psi})$ where $\overline{F}' > 0$. A solution operator $\psi = S(F)$ is defined for $F \in \mathcal{V}_2(\overline{F})$ (see Proposition 15, Section 2.5), and such that ψ and $\omega = \Delta\psi$ have no critical points in $\overline{\Omega}$. We assume without loss of generality that $\psi \leq 0$ and has range contained in an interval $[\underline{c}, 0]$ where $\underline{c} < \overline{c} = \overline{\psi}_{|\Gamma_i}$.

Proposition 32 *The distribution function $T(F)$ is a smooth tame map of F. $T(F)$ has degree 0 in F:*

$$\|T(F)\|_{n,\alpha} \leq C(\|F\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.162}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood. Its first derivative is given by

$$\begin{aligned} DT(F)\cdot f &= f \circ A_{\psi}^{-1} + \left(\frac{dA_{\omega}^{-1}}{d\mu}\right)\left(J_{\psi}\frac{\phi}{|\nabla\psi|} \circ A_{\psi}^{-1}\right) && (5.163)\\ &= B(F)\cdot f + \tilde{K}(F)\cdot f && (5.164)\end{aligned}$$

where $\phi = DS(F)\cdot f = VE(-F' \circ \psi)\cdot(f \circ \psi)$ (using notation from Corollary 12, Section 2.3); where

$$B(F)\colon f \in C^{\infty}_{[\underline{c},0]} \mapsto f \circ A_{\psi}^{-1} \in C^{\infty}_{[0,|\Omega|]} \tag{5.165}$$

and $B(F)\cdot f$ is a smooth tame map of F and f:

$$\|B(F)\cdot f\|_{n,\alpha} \leq C(\|f\|_{n,\alpha} + \|F\|_{n-2,\alpha}\|f\|_{1,\alpha}), \qquad n \geq 2 \tag{5.166}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood, and all f (without restriction); and where

$$\tilde{K}(F)\colon f \in C^{\infty}_{[\underline{c},0]} \mapsto \left(\frac{dA_{\omega}^{-1}}{d\mu}\right)\left(J_{\psi}\frac{\phi}{|\nabla\psi|} \circ A_{\psi}^{-1}\right) \in C^{\infty}_{[0,|\Omega|]} \tag{5.167}$$

and $\tilde{K}(F)\cdot f$ is a smooth tame map of F and f:

$$\|\tilde{K}(F)\cdot f\|_{n,\alpha} \leq C(\|f\|_{n-2,\alpha} + \|F\|_{n+1,\alpha}\|f\|_{2,\alpha}), \qquad n \geq 2 \tag{5.168}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f without restriction.

Proof

$\boxed{1}$ **Tame estimates on** $T(F)$ We have shown in Section 4.3 that the operator returning the inverse of the distribution function of ω corresponding to $F \in \mathcal{V}_2(\overline{F})$ can be written as

$$T(F) = A_\omega^{-1} = F \circ A_\psi^{-1}. \tag{5.169}$$

Composition is tame with degre 0 and base 1 (Lemma 50, Section A.2.3) so that

$$\|T(F)\|_{n,\alpha} \le C(\|F\|_{n,\alpha} + \|A_\psi^{-1}\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.170}$$

for F in a $\|\cdot\|_{1,\alpha}$-neighborhood, and provided that A_ψ^{-1} remains in a $\|\cdot\|_{1,\alpha}$-neighborhood; A_ψ^{-1} is tame with degree 0:

$$\|A_\psi^{-1}\|_{n,\alpha} \le C(\|\psi\|_{n,\alpha} + 1) \le C(\|\psi\|_{n+2,\alpha} + 1), \qquad n \ge 0 \tag{5.171}$$

for ψ in a $\|\cdot\|_{4,\alpha}$-neighborhood (and thus A_ψ^{-1} remains in a $\|\cdot\|_{1,\alpha}$-neighborhood as needed); the solution operator $\psi = S(F)$ has degree -2:

$$\|\psi\|_{n+2,\alpha} \le C(\|F\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.172}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood. Putting these together, we conclude that $T(F)$ has degree 0 in F:

$$\begin{aligned}
\|T(F)\|_{n,\alpha} &\le C(\|F\|_{n,\alpha} + \|A_\psi^{-1}\|_{n,\alpha} + 1), \qquad n \ge 0 &(5.173)\\
&\le C(\|F\|_{n,\alpha} + \|\psi\|_{n+2,\alpha} + 1), \qquad n \ge 0 &(5.174)\\
&\le C(\|F\|_{n,\alpha} + \|F\|_{n,\alpha} + 1), \qquad n \ge 0 &(5.175)\\
\|T(F)\|_{n,\alpha} &\le C(\|F\|_{n,\alpha} + 1), \qquad n \ge 0 &(5.176)
\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood of $\overline{F}$. □

$\boxed{2}$ **The first derivative** $DT(F) \cdot f$ Writing $T(F) = F \circ A_\psi^{-1} = F \circ (Q(S(F)))$, we use (5.18), (5.23), Lemma 50, Section A.2.3, and that

$$(A_\omega^{-1})' = F'(A_\psi^{-1}) \times (A_\psi^{-1})' = (F' \circ A_\psi^{-1}) \times \frac{1}{J_\psi \frac{1}{|\nabla\psi|} \circ A_\psi^{-1}} \tag{5.177}$$

to find that

$$DT(F)\cdot f = f\circ A_\psi^{-1} + (F'\circ A_\psi^{-1})DQ(\psi)\cdot DS(F)\cdot f \tag{5.178}$$

$$= f\circ A_\psi^{-1} + (F'\circ A_\psi^{-1})\left(\frac{J_\psi\frac{\phi}{|\nabla\psi|}\circ A_\psi^{-1}}{J_\psi\frac{1}{|\nabla\psi|}\circ A_\psi^{-1}}\right) \tag{5.179}$$

$$= f\circ A_\psi^{-1} + \left(\frac{dA_\omega^{-1}}{d\mu}\right)\left(J_\psi\frac{\phi}{|\nabla\psi|}\circ A_\psi^{-1}\right) \tag{5.180}$$

where we have let $\phi = DS(F)\cdot f$. Write this as

$$DT(F)\cdot f = B(F)\cdot f + \tilde{K}(F)\cdot f \tag{5.181}$$

where we have posed, for $F\in\mathcal{V}_2(\overline{F})$,

$$B(F)\colon f\in C^\infty_{[\underline{c},0]}\mapsto f\circ A_\psi^{-1}\in C^\infty_{[0,|\Omega|]} \tag{5.182}$$

and

$$\tilde{K}(F)\colon f\in C^\infty_{[\underline{c},0]}\mapsto \left(\frac{dA_\omega^{-1}}{d\mu}\right)\left(J_\psi\frac{\phi}{|\nabla\psi|}\circ A_\psi^{-1}\right)\in C^\infty_{[0,|\Omega|]}. \tag{5.183}$$

□

$\boxed{3}$ **Tame estimates on** $B(F)\cdot f$ Using tame estimates for composition (Lemma 50, Section A.2.3),

$$\|f\circ A_\psi^{-1}\|_{n,\alpha}\le C(\|f\|_{n,\alpha}+\|A_\psi^{-1}\|_{n,\alpha}+1),\qquad n\ge 0 \tag{5.184}$$

for f and A_ψ^{-1} in $\|\cdot\|_{1,\alpha}$-neighborhoods, and from the derivation of the tame estimates on $T(F)$ above we conclude that $B(F)\cdot f$ is tame with degree 0 in f and -2 in F:

$$\|f\circ A_\psi^{-1}\|_{n,\alpha}\le C(\|f\|_{n,\alpha}+\|F\|_{n-2,\alpha}+1),\qquad n\ge 2 \tag{5.185}$$

for all f in a $\|\cdot\|_{1,\alpha}$-neighborhood and F in a $\|\cdot\|_{2,\alpha}$-neighborhood, and since $B(F)\cdot f$ is linear in f, we may convert this (see Lemma 45, Section A.2.2) into

$$\|f\circ A_\psi^{-1}\|_{n,\alpha}\le C(\|f\|_{n,\alpha}+\|F\|_{n-2,\alpha}\|f\|_{1,\alpha}),\qquad n\ge 2 \tag{5.186}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood, and all f (without restriction). □

$\boxed{4}$ **Tame estimates on $\tilde{K}(F) \cdot f$** Multiplication of functions is tame with degree 0 and base 0 (Lemma 48, Section A.2.3), so

$$\|\tilde{K}(F) \cdot f\|_{n,\alpha} \leq C \cdot \left(\|A_\omega^{-1}\|_{n+1,\alpha} + \left\| J_\psi \frac{\phi}{|\nabla \psi|} \circ A_\psi^{-1} \right\|_{n,\alpha} + 1 \right), \qquad n \geq 0 \tag{5.187}$$

as long as A_ω^{-1} remains in a $\|\cdot\|_{1,\alpha}$-neighborhood and $J_\psi \frac{\phi}{|\nabla\psi|} \circ A_\psi^{-1}$ in a $\|\cdot\|_{0,\alpha}$-neighborhood. But with $F \in \|\cdot\|_{2,\alpha}$, ψ remains in a $\|\cdot\|_{3,\alpha}$-neighborhood and in turn A_ω^{-1} does remain in a $\|\cdot\|_{1,\alpha}$-neighborhood, as well as $J_\psi \frac{\phi}{|\nabla\psi|} \circ A_\psi^{-1}$ provided ϕ remains in a $\|\cdot\|_{1,\alpha}$-neighborhood (Lemma 30, Section 5.2). By tame estimates (5.21) on $J_\psi u \circ A_\psi^{-1}$, and tame estimates on $T(F) = A_\omega^{-1}$ above, this becomes

$$\|\tilde{K}(F) \cdot f\|_{n,\alpha} \leq C \cdot \left(\|F\|_{n+1,\alpha} + \|\psi\|_{n+1,\alpha} + \left\| \frac{\phi}{|\nabla \psi|} \right\|_{n,\alpha} + 1 \right), \qquad n \geq 0 \tag{5.188}$$

for all F in $\|\cdot\|_{2,\alpha}$-neighborhood (which guarantees that ψ remains in a $\|\cdot\|_{3,\alpha}$-neighborhood, as needed) and all ϕ in a $\|\cdot\|_{1,\alpha}$-neighoborhood. We use again that multiplication of functions is tame with degree 0 and base 0 (Lemma 48, Section A.2.3), and tame estimates for Nemitskii operators (Lemma 49, Section A.2.3) to find that

$$\|\tilde{K}(F) \cdot f\|_{n,\alpha} \leq C \cdot \left(\|F\|_{n+1,\alpha} + \|\phi\|_{n,\alpha} + 1 \right), \qquad n \geq 0 \tag{5.189}$$

for all F in $\|\cdot\|_{3,\alpha}$-neighborhood, and all ϕ in a $\|\cdot\|_{1,\alpha}$-neighborhood.

Before concluding we write tame estimates for $\phi = DS(F) \cdot f = VE(-F' \circ \psi) \cdot (f \circ \psi)$ as a function of F and f. Corollary 12, Section 2.5, shows that $VE(c) \cdot k$ has degree -2 in both k and c, and base 2, so that we have

$$\|\phi\|_{n,\alpha} \leq C(\|f \circ \psi\|_{n-2,\alpha} + \| - F' \circ \psi\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.190}$$

as long as $f \circ \psi$ and $-F' \circ \psi$ remain in $\|\cdot\|_{0,\alpha}$-neighoborhoods, which is satisfied for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f in a $\|\cdot\|_{2,\alpha}$-neighborhood. But

$$\|f \circ \psi\|_{m,\alpha} \leq C(\|f\|_{m,\alpha} + \|\psi\|_{m,\alpha} + 1), \qquad m \geq 0 \tag{5.191}$$

for all f and ψ in $\|\cdot\|_{1,\alpha}$-neighborhoods, and

$$\begin{aligned} \| - F' \circ \psi\|_{m,\alpha} &\leq C(\|F\|_{m+1,\alpha} + \|\psi\|_{m,\alpha} + 1), \qquad m \geq 0 &(5.192)\\ &\leq C(\|F\|_{m+1,\alpha} + 1), \qquad m \geq 0 &(5.193) \end{aligned}$$

for all F in $\|\cdot\|_{2,\alpha}$-neighborhood, so that

$$\begin{aligned}\|\phi\|_{n,\alpha} &\leq C\left(\|f\|_{n-2,\alpha}+\|\psi\|_{n-2,\alpha}+\|F\|_{n-1,\alpha}+1\right), \qquad n\geq 2 &(5.194)\\ &\leq C\left(\|f\|_{n-2,\alpha}+\|F\|_{n-1,\alpha}+1\right), \qquad n\geq 2 &(5.195)\end{aligned}$$

for all f in a $\|\cdot\|_{2,\alpha}$-neighborhood and all F in a $\|\cdot\|_{2,\alpha}$-neighborhood.

We may now conclude that $\tilde{K}(F)\cdot f$ is tame with degree 1 in F and -2 in f:

$$\|\tilde{K}(F)\cdot f\|_{n,\alpha}\leq C(\|F\|_{n+1,\alpha}+\|f\|_{n-2,\alpha}+1), \qquad n\geq 2 \tag{5.196}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f in a $\|\cdot\|_{2,\alpha}$-neighborhood. Alternatively, since $\tilde{K}(F)\cdot f$ is linear in f, we may convert these estimates (see Lemma 45, Section A.2.2) into

$$\|\tilde{K}(F)\cdot f\|_{n,\alpha}\leq C(\|F\|_{n+1,\alpha}\|f\|_{2,\alpha}+\|f\|_{n-2,\alpha}), \qquad n\geq 2 \tag{5.197}$$

for all F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f without restriction. ■

Our next objective is to show that $DT(F)$ is surjective for each F in a neighborhood of $\overline{F}$, and has a smooth tame family of right-inverses. To do this we will use a sort of Fredholm alternative, as we now explain.

We have just seen that $B(F)\cdot f$ is tame in F and f, and in particular we have from (5.166) that

$$\|B(F)\cdot f\|_{n,\alpha}\leq C(\|F\|_{n-2,\alpha}+1)\|f\|_{n,\alpha}, \qquad n\geq 2 \tag{5.198}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and all f without restriction. In particular, $B(F)$ extends to continuous maps (still denoted $B(F)$) of Banach spaces

$$B(F)\colon C^{n,\alpha}_{[\underline{c},0]}\to C^{n,\alpha}_{[0,|\Omega|]}, \qquad n\geq 2 \tag{5.199}$$

(here F, hence A_ψ^{-1}, is fixed and smooth). Note that $B(F)$ and its extensions are surjective but cannot be injective since range(ψ) = range$(A_\psi^{-1})\subset[\underline{c},0]$ = domain(f) and is not the entire interval. Nevertheless $B(F)$ admits a smooth tame right-inverse (see Section 5.4).

Similarly, from (5.168), $\tilde{K}(F)\cdot f$ satisfies

$$\|\tilde{K}(F)\cdot f\|_{m,\alpha}\leq C(\|F\|_{m+1,\alpha}+1)\|f\|_{m-2,\alpha}, \qquad m\geq 3 \tag{5.200}$$

hence $\tilde{K}(F)$ extends to continuous maps of Banach spaces

$$\tilde{K}(F)\colon C^{n,\alpha}_{[\underline{c},0]} \to C^{n+2,\alpha}_{[0,|\Omega|]}, \qquad n \geq 1 \tag{5.201}$$

and in fact to compact linear maps

$$\tilde{K}(F)\colon C^{n,\alpha}_{[\underline{c},0]} \to C^{n+1,\alpha}_{[0,|\Omega|]}, \qquad n \geq 1 \tag{5.202}$$

as well as

$$\tilde{K}(F)\colon C^{n,\alpha}_{[0,|\Omega|]} \to C^{n,\alpha}_{[0,|\Omega|]}, \qquad n \geq 1. \tag{5.203}$$

In some sense, modulo a right-inverse, $DT(F)$ is a "compact perturbation" of an isomorphism.

To show that $DT(F)$ has a smooth tame family of right-inverses, we first construct a smooth tame family of right-inverses for $B(F) \cdot f$.

5.4 Right-inverse for $B(F) \cdot f = f \circ A_\psi^{-1}$

Since $B(F)$ is surjective for each F, we know that a right-inverse exists for each F. However, we need to work with smooth tame maps, so we establish the following

Naively, the inverse of $g = f \circ A_\psi^{-1}$ should be "$f = g \circ A_\psi$". However, f is defined on an interval larger than the domain of A_ψ. We first extend the smooth functions A_ψ defined on $[\min \psi, \max \psi]$ to smooth functions on a larger interval, and this is done with tame estimates.

Lemma 33 *The map $B(F) \cdot f$ has a smooth tame family of right-inverses $f = VB(F) \cdot g$ defined on a $\| \cdot \|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$ of $\overline{F}$: for $F \in \mathcal{V}_3(\overline{F})$*

$$VB(F)\colon C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[\underline{c},0]}, \qquad B(F) \cdot VB(F) = Id_{C^{\infty}_{[0,|\Omega|]}}. \tag{5.204}$$

$f = VB(F) \cdot g$ has degree 0 in g and -2 in F:

$$\|f\|_{n,\alpha} \leq C \cdot (\|F\|_{n-2,\alpha} + \|g\|_{n,\alpha} + 1), \qquad n \geq 2 \tag{5.205}$$

for $F \in \mathcal{V}_3(\overline{F})$ and g in $\| \cdot \|_{1,\alpha}$-neighborhood. Equivalently (Lemma 45, Section A.2.2)

$$\|f\|_{n,\alpha} \leq C \cdot (\|F\|_{n-2,\alpha}\|g\|_{1,\alpha} + \|g\|_{n,\alpha}), \qquad n \geq 2 \tag{5.206}$$

for $F \in \mathcal{V}_3(\overline{F})$ and any g (without restriction). In particular,

$$\|f\|_{n,\alpha} \leq C \cdot (\|F\|_{n-2,\alpha} + 1)\|g\|_{n,\alpha}, \qquad n \geq 2 \tag{5.207}$$

for $F \in \mathcal{V}_3(\overline{F})$ and any g (without restriction).

Proof The idea is to extend A_ψ, in a tame fashion, from the (varying) interval $[\min\psi, 0]$ to a larger but fixed interval $[\underline{c}, 0]$. This is done by extending the inverse A_ψ^{-1} to a function with specified values at the endpoints, and then taking the inverse of this extension. Specifically, the right-inverse is constructed in the form

$$VB(F) \cdot g := \left(\mathcal{E}g \circ \left(\mathcal{E}(A_\psi^{-1})\right)^{-1}\right)\Big|_{[\underline{c},0]} \tag{5.208}$$

where $\mathcal{E}g$ is a (linear) extension of g and $\mathcal{E}(A_\psi^{-1})$ is an extension of A_ψ^{-1} (we use the same notation $\mathcal{E}$ for these two different operators in order to avoid proliferation of symbols).

$\boxed{1}$ **The extension $\mathcal{E}g$** The proof of Corollary II.1.3.7, p. 138, [14] guarantees existence of an operator $g \in C^\infty_{[0,|\Omega|]} \mapsto \mathcal{E}g \in C^\infty_{[-D,D]}$ extending functions defined on $[0, |\Omega|]$ to any larger interval $[-D, D]$. This can be made tame linear of degree 0 and base 0:

$$\|\mathcal{E}g\|_{n,\alpha} \leq C \cdot \|g\|_{n,\alpha}, \qquad n \geq 0 \tag{5.209}$$

for all $g \in C^\infty_{[0,|\Omega|]}$. $\square$

$\boxed{2}$ **The extension $\mathcal{E}(A_\psi^{-1})$** We show that there exist constants $\delta > 0$, $c^* < \underline{c}$, $\delta' > 0$, and a smooth tame operator, defined for H in a $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}(A_\psi^{-1})$ where $H' > 0$,

$$\mathcal{E}\colon (\mathcal{V}(A_\psi^{-1}) \subset C^\infty_{[0,|\Omega|]}) \to \{\hat{H} \in C^\infty_{[-\delta,|\Omega|+\delta]} \mid \hat{H}(|\Omega| + \delta) = \delta',\ \tilde{H}(-\delta) = c^*\} \tag{5.210}$$

extending an increasing function H defined on $[0, |\Omega|]$ to a still increasing function defined on $[-\delta, |\Omega| + \delta]$,

$$\mathcal{E}(H) = H \qquad \text{on} \qquad [0, |\Omega|] \tag{5.211}$$

and such that $\mathcal{E}(H)$ has fixed "endpoints": $\mathcal{E}(H)(|\Omega| + \delta) = \delta'$, $\mathcal{E}(H)(-\delta) = c^*$.

Let $\overline{F}$ be fixed, whose corresponding solution $\overline{\psi}$ has range $[\overline{c}, 0]$ and (strictly increasing) distribution function $A_{\overline{\psi}}^{-1}$ defined on $[0, |\Omega|]$. Again from the proof of Corollary II.1.3.7, p. 138, [14], there exists a tame linear map (with degree 0 and base 0) extending $H \in C^{\infty}_{[0,|\Omega|]}$ to a function on $C^{\infty}_{[-\delta,|\Omega|+\delta]}$. Choosing $\delta > 0$ sufficiently small, and restricting to a $\|\cdot\|_1$-neighborhood $\mathcal{V}(A_{\overline{\psi}}^{-1})$ of $A_{\overline{\psi}}^{-1}$, these extensions are also increasing functions on $[-\delta, |\Omega| + \delta]$. Using smooth cut-off functions at $-\delta$ and $|\Omega| + \delta$, these extensions can be adjusted so that the values are fixed to c^* and δ' at $-\delta$ and $|\Omega| + \delta$ respectively, and yet the extension remains increasing $[-\delta, |\Omega| + \delta]$ and is unchanged on $[0, |\Omega|]$. This extension, denoted $\mathcal{E}(H)$, is a *left*-inverse for H: for $\mu \in [0, |\Omega|]$,

$$\mathcal{E}(H)^{-1} \circ H(\mu) = \mathcal{E}(H)^{-1} \circ \mathcal{E}(H)(\mu) = \mu. \tag{5.212}$$

This can be done in such a way that $\mathcal{E}(H)$ is smooth tame with degree 0 and base 0:

$$\|\mathcal{E}(H)\|_{n,\alpha} \le C \cdot (\|H\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.213}$$

for all H in $\mathcal{V}(A_{\overline{\psi}}^{-1})$. $\square$

$\boxed{3}$ **The right-inverse** $VB(F) \cdot g$ Finally, fix $\underline{c} \in (c^*, \overline{c})$, then make $\mathcal{V}(A_{\overline{\psi}}^{-1})$ into a smaller $\|\cdot\|_1$-neighborhood if necessary so that $[\underline{c}, 0]$ is contained in range(H) for all $H \in \mathcal{V}(A_{\overline{\psi}}^{-1})$. One *then* constructs the solution operator $\psi = S(F)$ for

$$F \in \mathcal{V}_2(\overline{F}) \tag{5.214}$$

as in Proposition 15, Section 2.5. One should choose

$$\mathcal{V}_3(\overline{F}) \subset \mathcal{V}_2(\overline{F}) \tag{5.215}$$

a sufficiently small $\|\cdot\|_{2,\alpha}$-neighborhood so that the corresponding A_{ψ}^{-1} is in $\mathcal{V}(A_{\overline{\psi}}^{-1})$. (Shortly in this proof we will need to restrict F to a $\|\cdot\|_{3,\alpha}$-neighborhood so that tame estimates hold, hence we will reduce $\mathcal{V}_3(\overline{F})$ to a smaller $\|\cdot\|_{3,\alpha}$-neighborhood.)

Define now

$$VB(F) \cdot g := \left(\mathcal{E}g \circ \left(\mathcal{E}(A_{\psi}^{-1}) \right)^{-1} \right) \Big|_{[\underline{c},0]} \tag{5.216}$$

for $g \in C^{\infty}_{[0,|\Omega|]}$ and $F \in \mathcal{V}_3(\overline{F})$ and derive tame estimates for it. First recall from the *proof* of Proposition 32, Section 5.3, that

$$\|A_{\psi}^{-1}\|_{n,\alpha} \le C(\|F\|_{n-2,\alpha} + 1), \qquad n \ge 2 \tag{5.217}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood. Reduce $\mathcal{V}_3(\overline{F})$ further so that it is contained in such $\|\cdot\|_{2,\alpha}$-neighborhood. Since the extension operator is tame with degree 0 and base 0, we have

$$\|\mathcal{E}(A_\psi^{-1})\|_{n,\alpha} \leq C(\|F\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.218}$$

for $F \in \mathcal{V}_3(\overline{F})$, and observe that $\mathcal{E}(A_\psi^{-1})$ remains in a $\|\cdot\|_{1,\alpha}$-neighborhood by reducing $\mathcal{V}_3(\overline{F})$ to a smaller $\|\cdot\|_{3,\alpha}$-neighborhood. The operation of taking inverse being tame with degree 0 and base 1 (Lemma 51, Section A.2.3), this gives

$$\|\mathcal{E}(A_\psi^{-1})^{-1}\|_{n,\alpha} \leq C(\|F\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.219}$$

for F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$, and again $\mathcal{E}(A_\psi^{-1})^{-1}$ remains in a $\|\cdot\|_{1,\alpha}$-neighborhood. On the other hand,

$$\|\mathcal{E}g\|_{n,\alpha} \leq C\|g\|_{n,\alpha}, \qquad n \geq 0 \tag{5.220}$$

for all g (without restriction), and note that $\mathcal{E}g$ remains in a $\|\cdot\|_{1,\alpha}$-neighborhood if g does. Now composition of functions is tame with degree 0 in both arguments and base 1, so that

$$\|\mathcal{E}g \circ \mathcal{E}(A_\psi^{-1})^{-1}\|_{n,\alpha} \leq C(\|g\|_{n,\alpha} + \|F\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.221}$$

for all g in a $\|\cdot\|_{1,\alpha}$-neighborhood and F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$. Finally, the operation of restricting functions defined on $[-\delta, |\Omega| + \delta]$ to $[\underline{c}, 0]$ is obviously tame linear with degree 0 and base 0, so that

$$\|VB(F)\cdot f\|_{n,\alpha} \leq C(\|g\|_{n,\alpha} + \|F\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.222}$$

for all g in a $\|\cdot\|_{1,\alpha}$-neighborhood and F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$. ■

5.5 $DT(F)$ is surjective with smooth tame right-inverse

As a result of the previous Sections, we are ready to show that $DT(F)$ is surjective for each F in a neighborhood of $\overline{F}$, and that furthermore it has a smooth tame family of right-inverses. We have introduced various neighborhoods of $\overline{F}$. We finally define the neighborhood for which the statement of Theorem 1 (Section 1.3) holds:

$$\mathcal{V}(\overline{F}) := \mathcal{V}_1(\overline{F}) \cap \mathcal{V}_2(\overline{F}) \cap \mathcal{V}_3(\overline{F}) \tag{5.223}$$

where $\mathcal{V}_1(\overline{F})$ is the $\|\cdot\|_0$-neighborhood defined in Lemma 14, Section 2.5, in which all F have at least on e solution with range in a common interval; $\mathcal{V}_2(\overline{F})$ is the $\|\cdot\|_{2,\alpha}$-neighborhood defined in Proposition 15, Section 2.5, in which a solution operator $\psi = S(F)$ may be defined and ψ has no critical points; and $\mathcal{V}_3(\overline{F})$ is the $\|\cdot\|_{3,\alpha}$-neighborhood defined in Lemma 33, Section 5.4, in which a smooth tame right-inverse for $B(F)\cdot f$ is defined.

We construct a smooth tame right-inverse for $DT(F)\cdot f$ as follows. Set

$$L(F) := DT(F)\cdot VB(F) = \mathrm{Id}_{C^\infty_{[0,|\Omega|]}} + K(F) \tag{5.224}$$

where

$$K(F)\cdot g := \tilde{K}(F)\cdot VB(F)\cdot g = \left(\frac{dA_\omega^{-1}}{d\mu}\right)\left(J_\psi \frac{\phi}{|\nabla\psi|}\circ A_\psi^{-1}\right), \tag{5.225}$$

and using the short-hand notation

$$\phi = DS(F)\cdot f, \qquad f = VB(F)\cdot g. \tag{5.226}$$

By composition of smooth tame maps, $L(F)\cdot g$ is smooth tame. We show that $h = L(F)\cdot g$ is invertible for $F \in \mathcal{V}(\overline{F})$ and all $g \in C^\infty_{[0,|\Omega|]}$, and denote the family of inverses by $VL(F)\cdot h$. Then, we show that $VL(F)\cdot h$ is tame, and in fact $VL(F)\cdot h$ is smooth tame by Proposition 46 of Section A.2.2. Finally, we have a right-inverse $VT(F)\cdot h$ for $DT(F)\cdot g$ by setting

$$VT(F)\cdot h := VB(F)\cdot VL(F)\cdot h \tag{5.227}$$

for then $DT(F)\cdot VT(F)\cdot h = L(F)\cdot VL(F)\cdot h = h$.

Lemma 34 ($L(F)\cdot g$ is invertible) *$L(F)\colon C^\infty_{[0,|\Omega|]} \to C^\infty_{[0,|\Omega|]}$ is invertible for $F \in \mathcal{V}(\overline{F})$.*

Proof We have seen at the end of Section 5.3 that $\tilde{K}(F)$ extends to compact operators $C^{n,\alpha}_{[\underline{c},0]} \to C^{n,\alpha}_{[\underline{c},0]}$ for $n \geq 1$ and in fact satisfies

$$\|\tilde{K}(F)\cdot f\|_{n+2,\alpha} \leq C(\|F\|_{n+3,\alpha} + \|f\|_{n,\alpha} + 1), \qquad n \geq 0 \tag{5.228}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f in a $\|\cdot\|_{1,\alpha}$-neighborhood. As for $f = VB(F)\cdot g$, we have from Lemma 33, Section 5.4,

$$\|f\|_{n+2,\alpha} \le C(\|g\|_{n+2,\alpha} + \|F\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.229}$$

for F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$ and g in a $\|\cdot\|_{1,\alpha}$-neighborhood. Thus

$$\|K(F)\cdot g\|_{n+2,\alpha} \le C(\|F\|_{n+3,\alpha} + \|g\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.230}$$

for F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$ and g in a $\|\cdot\|_{1,\alpha}$-neighborhood. Since $K(F)\cdot g$ is linear in g, this can be converted (see Lemma 45, Section A.2.2) into

$$\|K(F)\cdot g\|_{n+2,\alpha} \le C(\|F\|_{n+3,\alpha}\|g\|_{1,\alpha} + \|g\|_{n,\alpha}) \le C(\|F\|_{n+3,\alpha} + 1)\|g\|_{n,\alpha}, \qquad n \ge 1 \tag{5.231}$$

for F in the $\|\cdot\|_{3,\alpha}$-neighborhood $\mathcal{V}_3(\overline{F})$ and all g without restriction. In turn, $K(F)$ extends to bounded linear operators operators

$$C^{n,\alpha}_{[\underline{c},0]} \to C^{n+2,\alpha}_{[0,|\Omega|]}, \qquad n \ge 1 \tag{5.232}$$

and in fact to compact operators

$$C^{n,\alpha}_{[\underline{c},0]} \to C^{n,\alpha}_{[0,|\Omega|]}, \qquad n \ge 1. \tag{5.233}$$

In other words, $L(F)$ is a compact perturbation of the identity on $C^{n,\alpha}_{[\underline{c},0]}$ for each $n \ge 1$ and the Fredholm alternative can be used. $\square$

Let $F \in \mathcal{V}(\overline{F})$ and suppose $g \in C^{\infty}_{[0,|\Omega|]}$ is in the kernel of $L(F)\colon C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]}$. Set $f = VB(F)\cdot g$ so that $DT(F)\cdot f = 0$ and thus $\nu = \{\alpha, \omega\}$ for some $\alpha \in \mathcal{U}$ (Lemma 26, Section 4.4). Then $f = 0$, precisely by the non-degeneracy condition (ND) and in turn $g = B(F)\cdot VB(F)\cdot g = B(F)\cdot f = 0$, i.e. $\ker\{L(F)\colon C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]}\} = 0$.

Next, we show that this implies that $L(F)\colon C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]}$ is an isomorphism of Fréchet spaces. (In particular, this implies that $L(F)$ is surjective and in turn $DT(F)$ must be surjective as well.) For each $n \ge 1$, we have that $\ker\{L(F)\colon C^{n,\alpha}_{[0,|\Omega|]} \to C^{n,\alpha}_{[0,|\Omega|]}\} = 0$, for if $g \in \ker\{L(F)\colon C^{n,\alpha}_{[0,|\Omega|]} \to C^{n,\alpha}_{[0,|\Omega|]}\}$ then $g = -K(F)\cdot g \in C^{n+1,\alpha}_{[0,|\Omega|]}$ (since $K(F)$ actually maps into these better spaces) and by induction on n, $g \in C^{\infty}_{[0,|\Omega|]}$, so $g = 0$. Let now $h \in C^{\infty}_{[0,|\Omega|]} \subset C^{n,\alpha}_{[0,|\Omega|]}$, $n \ge 1$. Then the Fredholm alternative (in Banach spaces) guarantees existence of a unique $g \in C^{n,\alpha}_{[0,|\Omega|]}$ such that $L(F) \cdot g = h$. But this g is the same for all n, hence $g \in C^{\infty}_{[0,|\Omega|]}$. ■

Observe that, by the open mapping theorem (valid in the Fréchet category, see p. 69, [14]) we have that for each $F \in \mathcal{V}(\overline{F})$ the map

$$VL(F)\colon C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]} \tag{5.234}$$

is continuous as a map of Fréchet spaces. But we need more, namely that $VL(F) \cdot h$ is continuous in *both* F and h, and furthermore that it is tame. As already explained, once this is established, we actually have that the family of right-inverses $VL(F) \cdot h$ for $L(F) \cdot g$ is smooth tame.

Proposition 35 (Smooth tame family of right-inverses) *The family of inverses $VL\colon \mathcal{V}_{(\overline{F})} \times C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]}$ forms a smooth tame map and $g = VL(F) \cdot h$ has degree 1 in F and 0 in h:*

$$\|VL(F) \cdot h\|_{n,\alpha} \le C \cdot (\|F\|_{n+1,\alpha} + \|h\|_{n,\alpha} + 1), \qquad n \ge 0 \tag{5.235}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and h in a $\|\cdot\|_{1,\alpha}$-neighborhood.

Proof That the inverses $VL(F)$ exist for $F \in \mathcal{V}_{(\overline{F})}$ is a consequence of Lemma 34. $L\colon \mathcal{V}_{(\overline{F})} \times C^{\infty}_{[0,|\Omega|]} \to C^{\infty}_{[0,|\Omega|]}$ is a smooth tame family of linear maps since K defined in (5.225) is. Thus, to prove that VL is a smooth tame map it is sufficient (see Proposition 46, Section A.2.2) to show that VL is a tame map, namely it is continuous and satisfies tame estimates; smoothness of VL and all its derivatives automatically follows from smoothness of L.

$\boxed{1}$ $VL(F) \cdot h$ **is (jointly) continuous in** F **and** h Fix $(F_1, h_1) \in \mathcal{V}_{(\overline{F})} \times C^{\infty}_{[0,|\Omega|]}$ and $n \in \mathbb{N}$. For $(F, h) \in \mathcal{V}_{(\overline{F})} \times C^{\infty}_{[0,|\Omega|]}$,

$$\|VL(F) \cdot h - VL(F_1) \cdot h_1\|_{n,\alpha} \le \|(VL(F) - VL(F_1)) \cdot h\|_{n,\alpha} + \|VL(F_1) \cdot (h_1 - h)\|_{n,\alpha} \tag{5.236}$$

Since F_1 is fixed and $L(F_1)\colon C^{n}_{[0,|\Omega|]} \to C^{n}_{[0,|\Omega|]}$ is continuous, $VL(F_1)\colon C^{n}_{[0,|\Omega|]} \to C^{n}_{[0,|\Omega|]}$ is also continuous by the open mapping theorem (for Banach spaces). In turn the second term is small for $\|h_1 - h\|_{n,\alpha}$ small. As for the first term, note that $L(F)\cdot g - L(F_1)\cdot g = K(F)\cdot g - K(F_1)\cdot g$. Thus, letting $\mathcal{V}^{m,\alpha}(\overline{F})$ be the completion of $\mathcal{V}(\overline{F})$ in the $\|\cdot\|_{m,\alpha}$-norm, (so that it satisfies the same desirable conditions as those in $\mathcal{V}(\overline{F})$), it is enough to show that

$$F \in (\mathcal{V}^{n+3,\alpha}(\overline{F}) \subset C^{n+3,\alpha}) \mapsto K(F) \in \mathcal{L}(C^{n,\alpha}, C^{n,\alpha}) \tag{5.237}$$

is continuous as a map of Banach spaces so that the operator norm $\|K(F) - K(F_1)\|_{\mathcal{L}(C^{n,\alpha},C^{n,\alpha})}$ is small for $\|F - F_1\|_{n+3,\alpha}$ small, and in turn the operator norm $\|VL(F) - VL(F_1)\|_{\mathcal{L}(C^{n,\alpha},C^{n,\alpha})}$ is small.

Pose

$$u := \frac{VE(-F' \circ \psi)\cdot(f \circ \psi)}{|\nabla\psi|}, \qquad u_1 := \frac{VE(-F_1' \circ \psi_1)\cdot(f \circ \psi_1)}{|\nabla\psi_1|} \tag{5.238}$$

using notation from Corollary 12, Section 2.3 (we also recall that $f = VB(F)\cdot g$ is linear in g). u is a tame map of F and f, and linear in f, and u_1 is a tame linear map of f (F_1 is fixed). Then (see (5.225))

$$
\begin{aligned}
& K(F)\cdot g - K(F_1)\cdot g && (5.239)\\
= \;& \left(\frac{dA_\omega^{-1}}{d\mu}\right)\left(J_\psi u \circ A_\psi^{-1}\right) - \left(\frac{dA_{\omega_1}^{-1}}{d\mu}\right)\left(J_{\psi_1} u_1 \circ A_{\psi_1}^{-1}\right) && (5.240)\\
= \;& \left(\frac{dA_\omega^{-1}}{d\mu} - \frac{dA_{\omega_1}^{-1}}{d\mu}\right)\left(J_\psi u \circ A_\psi^{-1}\right) && (5.241)\\
& + \frac{dA_{\omega_1}^{-1}}{d\mu}\left\{J_\psi u \circ A_\psi^{-1} - J_{\psi_1} u \circ A_{\psi_1}^{-1}\right\} && (5.242)\\
& + \frac{dA_{\omega_1}^{-1}}{d\mu}\left\{J_{\psi_1}(u - u_1) \circ A_{\psi_1}^{-1}\right\} && (5.243)\\
= \;& I + II + III. && (5.244)
\end{aligned}
$$

Estimates on u In this paragraph we show the estimate

$$\|u\|_{n,\alpha} \le C\cdot(\|F\|_{n-1,\alpha} + \|g\|_{n-2,\alpha} + 1), \qquad n \ge 2 \tag{5.245}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and g in a $\|\cdot\|_{1,\alpha}$-neighborhood.

First use tame estimates for product of functions (Lemma 48, Section A.2.3) and of the solution operator $\psi = S(F)$ (Proposition 15, Section 2.5) to write

$$\begin{aligned} \|F' \circ \psi\|_{m,\alpha} &\leq C \cdot (\|F'\|_{m,\alpha} + \|\psi\|_{m,\alpha} + 1), \qquad m \geq 0, &(5.246)\\ &\leq C \cdot (\|F'\|_{m,\alpha} + \|\psi\|_{m+2,\alpha} + 1), \qquad m \geq 0, &(5.247)\\ &\leq C \cdot (\|F\|_{m+1,\alpha} + \|F\|_{m,\alpha} + 1), \qquad m \geq 0, &(5.248)\\ &\leq C \cdot (\|F\|_{m+1,\alpha} + 1), \qquad m \geq 0 &(5.249)\end{aligned}$$

which holds for F in a $\|\cdot\|_{2,\alpha}$-neighborhood, and

$$\begin{aligned} \|f \circ \psi\|_{m,\alpha} &\leq C \cdot (\|f\|_{m,\alpha} + \|\psi\|_{m,\alpha} + 1), \qquad m \geq 0, &(5.250)\\ &\leq C \cdot (\|f\|_{m,\alpha} + \|\psi\|_{m+2,\alpha} + 1), \qquad m \geq 0, &(5.251)\\ &\leq C \cdot (\|f\|_{m,\alpha} + \|F\|_{m,\alpha} + 1), \qquad m \geq 0 &(5.252)\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f in a $\|\cdot\|_{1,\alpha}$-neighborhood. Also, Lemma 49, Section A.2.3 shows that

$$\left\|\frac{1}{|\nabla\psi|}\right\|_{n,\alpha} \leq C(\|\psi\|_{n+1,\alpha} + 1) \leq C(\|F\|_{n-1,\alpha} + 1), \qquad n \geq 1 \tag{5.253}$$

valid for F in a $\|\cdot\|_{2,\alpha}$-neighborhood.

With these, and tame estimates for product of functions (Lemma 48, Section A.2.3) and on the solution operator VE to elliptic equations (Corollary 12, Section 2.3) we obtain

$$\begin{aligned} &\|u\|_{n,\alpha} &(5.254)\\ \leq\ & C\left(\left\|\frac{1}{|\nabla\psi|}\right\|_{n,\alpha} + \|-F' \circ \psi\|_{n-2,\alpha} + \|f \circ \psi\|_{n-2,\alpha} + 1\right), \quad n \geq 2 &(5.255)\\ \leq\ & C \cdot (\|F\|_{n-1,\alpha} + \|f\|_{n-2,\alpha} + 1), \qquad n \geq 2 &(5.256)\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and f in a $\|\cdot\|_{1,\alpha}$-neighborhood. Finally,

$$\|u\|_{n,\alpha} \leq C \cdot (\|F\|_{n-1,\alpha} + \|g\|_{n-2,\alpha} + 1), \qquad n \geq 2 \tag{5.257}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and g in a $\|\cdot\|_{1,\alpha}$-neighborhood. Alternatively, since u is linear in g,

$$\begin{aligned} \|u\|_{n,\alpha} &\leq C \cdot (\|F\|_{n-1,\alpha}\|g\|_{1,\alpha} + \|g\|_{n-2,\alpha}), \qquad n \geq 2 &(5.258)\\ &\leq C \cdot (\|F\|_{n-1,\alpha} + 1)\|g\|_{n-2,\alpha}, \qquad n \geq 2 &(5.259)\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and g without restriction.

Estimate on I Using estimates (5.64) on $J_\psi u \circ A_\psi^{-1}$, and the estimates on u just established,

$$\begin{aligned}\left\|J_\psi u \circ A_\psi^{-1}\right\|_{n,\alpha} &\leq C\cdot(\|\psi\|_{n+1,\alpha}+1)\|u\|_{n,\alpha}, \qquad n\geq 1, &(5.260)\\ &\leq C\cdot(\|\psi\|_{n+1,\alpha}+1)(\|F\|_{n-1,\alpha}+1)\|g\|_{n-2,\alpha},\ n\geq 2 &(5.261)\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and all g without restriction. In particular, for F in the intersection of a $\|\cdot\|_{2,\alpha}$-neighborhood and a $\|\cdot\|_{n-1,\alpha}$-neighborhood,

$$\left\|J_\psi u \circ A_\psi^{-1}\right\|_{n,\alpha} \leq C\cdot\|g\|_{n-2,\alpha}, \qquad n\geq 2 \tag{5.262}$$

With this we conclude

$$\|I\|_{n,\alpha} \leq C\left\|A_\omega^{-1}-A_{\omega_1}^{-1}\right\|_{n+1,\alpha}\|g\|_{n,\alpha}, \qquad n\geq 2 \tag{5.263}$$

for F in the intersection of $\|\cdot\|_{2,\alpha}$-neighborhood and a $\|\cdot\|_{n-1,\alpha}$-neighborhood and all g (without restriction).

Estimate on II Pose $\psi_t := (1-t)\psi + t\psi_1$ and $N_t := \frac{\nabla\psi_t}{|\nabla\psi_t|}$ for $t\in[0,1]$. The fundamental theorem of calculus tells us that

$$\left(J_{\psi_1}u\circ A_{\psi_1}^{-1} - J_\psi u\circ A_\psi^{-1}\right) = \int_0^1 D_\psi(J_{\psi_t}u\circ A_{\psi_t}^{-1})\cdot(\psi_1-\psi)dt \tag{5.264}$$

and the integrand is given explicitly by (5.158):

$$\begin{aligned}&D_\psi(J_{\psi_t}u\circ A_{\psi_t}^{-1})\cdot(\psi_1-\psi) &(5.265)\\ = \quad & -J_{\psi_t}\left(\frac{(\psi_1-\psi)\mathrm{div}(uN_t)}{|\nabla\psi_t|}\right)\circ A_{\psi_t}^{-1} &(5.266)\\ &+\left(J_{\psi_t}\left(\frac{\mathrm{div}(uN_t)}{|\nabla\psi_t|}\right)\circ A_{\psi_t}^{-1}\right)\left(\frac{J_{\psi_t}\frac{\psi_1-\psi}{|\nabla\psi_t|}\circ A_{\psi_t}^{-1}}{J_{\psi_t}\frac{1}{|\nabla\psi_t|}\circ A_{\psi_t}^{-1}}\right). &(5.267)\\ & &(5.268)\end{aligned}$$

Now recall tame estimates on $J_{\psi_t}v\circ A_{\psi_t}^{-1}$ in the form (5.64):

$$\|J_{\psi_t}v\circ A_{\psi_t}^{-1}\|_{n,\alpha} \leq C\cdot(\|\psi_t\|_{n+1,\alpha}+1)\|v\|_{n,\alpha}, \qquad n\geq 0 \tag{5.269}$$

where the constant is independent of v and ψ_t as long as the latter remains in a $\|\cdot\|_{4,\alpha}$-neighborhood. Thus,

$$
\begin{aligned}
& \|D_\psi(J_{\psi_t} u \circ A_{\psi_t}^{-1}) \cdot (\psi_1 - \psi)\|_{n,\alpha} && (5.270)\\
\leq\ & C \cdot (\|\psi_t\|_{n+1,\alpha} + 1) \left\| \frac{(\psi_1 - \psi)\mathrm{div}(uN_t)}{|\nabla \psi_t|} \right\|_{n,\alpha} && (5.271)\\
& + C \cdot (\|\psi_t\|_{n+1,\alpha} + 1)^2 \left\| \frac{1}{J_{\psi_t} \frac{1}{|\nabla \psi_t|} \circ A_{\psi_t}^{-1}} \right\|_{n,\alpha} && (5.272)\\
& \times \left\| \frac{\mathrm{div}(uN_t)}{|\nabla \psi_t|} \right\|_{n,\alpha} \left\| \frac{\psi_1 - \psi}{|\nabla \psi_t|} \right\|_{n,\alpha} && (5.273)\\
\leq\ & C \cdot \|\mathrm{div}(uN_t)\|_{n,\alpha} \|\psi_1 - \psi\|_{n,\alpha} && (5.274)\\
\leq\ & C \cdot \|u\|_{n+1,\alpha} \|N_t\|_{n+1,\alpha} \|\psi_1 - \psi\|_{n,\alpha} && (5.275)\\
\leq\ & C \cdot \|u\|_{n+1,\alpha} \|\psi_1 - \psi\|_{n,\alpha} && (5.276)
\end{aligned}
$$

with constants independent of u and ψ_t as long as the latter remains in the intersection of a $\|\cdot\|_{n+2,\alpha}$-neighborhood and a $\|\cdot\|_{4,\alpha}$-neighborhood. Integrating,

$$
\left\| J_{\psi_1} u \circ A_{\psi_1}^{-1} - J_\psi u \circ A_\psi^{-1} \right\|_{n,\alpha} \leq C \|\psi_1 - \psi\|_{n,\alpha} \|u\|_{n+1,\alpha}, \qquad n \geq 0 \qquad (5.277)
$$

for ψ_1 in the said neighborhood.

Finally, with the estimates on u, we find

$$
\begin{aligned}
& \left\| J_{\psi_1} u \circ A_{\psi_1}^{-1} - J_\psi u \circ A_\psi^{-1} \right\|_{n,\alpha} && (5.278)\\
\leq\ & C \|\psi_1 - \psi\|_{n,\alpha} (\|F\|_{n,\alpha} + 1) \|g\|_{n-1,\alpha}, \qquad n \geq 1 && (5.279)\\
\leq\ & C \|\psi_1 - \psi\|_{n,\alpha} \|g\|_{n-1,\alpha}, \qquad n \geq 1 && (5.280)\\
& && (5.281)
\end{aligned}
$$

for F in the intersection of a $\|\cdot\|_{n,\alpha}$-neighborhood and a $\|\cdot\|_{3,\alpha}$-neighborhood (so we will restrict to $n \geq 3$), and all g without restriction.

Estimate on III Write

$$u - u_1 = \left(\frac{1}{|\nabla\psi|} - \frac{1}{|\nabla\psi_1|}\right) VE(-F' \circ \psi) \cdot (f \circ \psi) \quad (5.282)$$

$$+\frac{1}{|\nabla\psi_1|} VE(-F' \circ \psi) \cdot (f \circ \psi - f \circ \psi_1) \quad (5.283)$$

$$+\frac{1}{|\nabla\psi_1|} \big(VE(-F' \circ \psi) \cdot (f \circ \psi_1) - VE(-F_1' \circ \psi_1) \cdot (f \circ \psi_1)\big) \quad (5.284)$$

$$= I' + II' + III'. \quad (5.285)$$

The solution operator VE of elliptic equations has degree -2 in both variables (see (2.111)) and composition of functions has degree 0 in both arguments (see (A.52)) so that the first two terms are estimated by

$$\|I' + II'\|_{n,\alpha} \leq C \cdot \left(\left\|\frac{1}{|\nabla\psi|} - \frac{1}{|\nabla\psi_1|}\right\|_{n,\alpha} \|f\|_{n-2,\alpha} + \|f \circ \psi - f \circ \psi_1\|_{n-2,\alpha}\right) \quad (5.286)$$

for F in a $\|\cdot\|_{n-1,\alpha}$-neighborhood, and all f (without restriction). With $\psi_t := (1-t)\psi + t\psi_1$ for $t \in [0,1]$, Taylor's formula gives

$$f \circ \psi - f \circ \psi_1 = \int_0^1 \nabla f(\psi_t) \cdot (\psi - \psi_1)\, dt \quad (5.287)$$

and

$$\|\nabla f(\psi_t) \cdot (\psi - \psi_1)\|_{n-2,\alpha} \leq C \cdot \|\nabla f \circ \psi_t\|_{n-2,\alpha} \|\psi - \psi_1\|_{n-2,\alpha} \quad (5.288)$$

$$\leq C \cdot (\|\psi_t\|_{n-2,\alpha} + 1) \|f\|_{n-1,\alpha} \|\psi - \psi_1\|_{n-2,\alpha}. \quad (5.289)$$

Integrating, and using estimate (5.207) on $f = VB(F) \cdot g$,

$$\|I' + II'\|_{n,\alpha} \leq C \cdot \left(\left\|\frac{1}{|\nabla\psi|} - \frac{1}{|\nabla\psi_1|}\right\|_{n,\alpha} + \|\psi - \psi_1\|_{n-2,\alpha}\right) \|g\|_{n-1,\alpha} \quad (5.290)$$

for F in a $\|\cdot\|_{n-1,\alpha}$-neighborhood, and all g (without restriction).

As for III', set $c_t := -(1-t)F' \circ \psi - tF_1' \circ \psi_1$. Taylor's formula gives

$$VE(-F' \circ \psi) \cdot (f \circ \psi_1) - VE(-F_1' \circ \psi_1) \cdot (f \circ \psi_1) \quad (5.291)$$

$$= \int_0^1 DVE(c_t) \cdot (f \circ \psi_1, -F' \circ \psi + F_1' \circ \psi_1) dt \quad (5.292)$$

where the derivative of VE is given in (2.164). But then from (2.111), (A.52), and (5.207),

$$\begin{aligned}
& \|DVE(c_t)\cdot(f\circ\psi_1, -F'\circ\psi + F_1'\circ\psi_1)\|_{n,\alpha} && (5.293)\\
\leq\ & \|VE(c_t)\cdot\big((F'\circ\psi - F_1'\circ\psi_1)(VE(c_t)\cdot(f\circ\psi_1)\big)\|_{n,\alpha} && (5.294)\\
\leq\ & C\cdot(\|c_t\|_{n-2,\alpha}+1)\|(F'\circ\psi - F_1'\circ\psi_1)(VE(c_t)\cdot(f\circ\psi_1)\|_{n-2,\alpha} && (5.295)\\
\leq\ & C\cdot(\|c_t\|_{n-2,\alpha}+1)\|F'\circ\psi - F_1'\circ\psi_1\|_{n-2,\alpha}\times && (5.296)\\
& \times(\|c_t\|_{n-4,\alpha}+1)(\|\psi_1\|_{n-4,\alpha}+1)\|f\|_{n-4,\alpha} && (5.297)\\
\leq\ & C\cdot\|F'\circ\psi - F_1'\circ\psi_1\|_{n-2,\alpha}\|g\|_{n-4,\alpha} && (5.298)
\end{aligned}$$

for F in a $\|\cdot\|_{n+1,\alpha}$-neighborhood, and all g (without restriction).

With this we conclude on the estimate for III:

$$\begin{aligned}
& \|III\|_{n,\alpha} && (5.299)\\
\leq\ & \left(\left\|\frac{1}{|\nabla\psi|} - \frac{1}{|\nabla\psi_1|}\right\|_{n,\alpha} + \|\psi-\psi_1\|_{n-2,\alpha} + \|F'\circ\psi - F_1'\circ\psi_1\|_{n-2,\alpha}\right) && (5.300)\\
& \times\|g\|_{n-1,\alpha} && (5.301)
\end{aligned}$$

for F in a $\|\cdot\|_{n+1,\alpha}$-neighborhood, and all g (without restriction).

From (5.263), (5.281), and (5.301) we have

$$\begin{aligned}
& \|K(F)\cdot g - K(F_1)\cdot g\|_{n,\alpha} && (5.302)\\
\leq\ & \|I\|_{n,\alpha} + \|II\|_{n,\alpha} + \|III\|_{n,\alpha} && (5.303)\\
\leq\ & C\cdot\Big(\|A_\omega^{-1} - A_{\omega_1}^{-1}\|_{n+1,\alpha} + \left\|\frac{1}{|\nabla\psi|} - \frac{1}{|\nabla\psi_1|}\right\|_{n,\alpha} && (5.304)\\
& +\|\psi-\psi_1\|_{n-2,\alpha} + \|F'\circ\psi - F_1'\circ\psi_1\|_{n-2,\alpha}\Big) && (5.305)\\
& \times\|g\|_{n,\alpha} && (5.306)
\end{aligned}$$

for F in a $\|\cdot\|_{n+1,\alpha}$-neighborhood (with $n\geq 3$) and all g without restriction. Therefore, the operator norm $\|K(F)-K(F_1)\|_{\mathcal{L}(C^{n,\alpha},C^{n,\alpha})}$ $(n\geq 3)$ is small provided $\|F-F_1\|_{n+1,\alpha}$ is small.

This finishes the proof that $VL(F)\cdot g$ is continuous. $\square$

$\boxed{2}$ **Tame estimates on** $K(F)\cdot g$ Recall that

$$K(F)\cdot g = \left(\frac{dA_\omega^{-1}}{d\mu}\right)\left(J_\psi \frac{\phi}{|\nabla\psi|}\circ A_\psi^{-1}\right). \tag{5.307}$$

We prove the tame estimates

$$\|K(F)\cdot g\|_{n,\alpha} \leq C\cdot(\|F\|_{n+1,\alpha} + \|g\|_{n,\alpha} + 1), \qquad n \geq 2 \tag{5.308}$$

for F in a $\|\cdot\|_{2,\alpha}$-neighborhood and g in a $\|\cdot\|_{1,\alpha}$-neighborhood. For F in a $\|\cdot\|_{2,\alpha}$-neighborhood, A_ω^{-1} remains in a $\|\cdot\|_{1,\alpha}$ neighborhood, and for u in a $\|\cdot\|_{1,\alpha}$-neighborhood, $J_\psi u \circ A_\psi^{-1}$ remains in a $\|\cdot\|_{0,\alpha}$-neighborhood, so that from estimates (A.42) on product of functions, the estimates (5.63) on $J_\psi u\circ A_\psi^{-1}$, and the estimates (5.245) on u,

$$\begin{aligned}
&\|K(F)\cdot g\|_{n,\alpha} && (5.309)\\
\leq\ & C\cdot\left(\left\|\frac{dA_\omega^{-1}}{d\mu}\right\|_{n,\alpha} + \|J_\psi u\circ A_\psi^{-1}\|_{n,\alpha} + 1\right), \qquad n\geq 0 && (5.310)\\
\leq\ & C\cdot\left(\|A_\omega^{-1}\|_{n+1,\alpha} + \|\psi\|_{n+1,\alpha} + \|u\|_{n,\alpha} + 1\right), \qquad n\geq 0 && (5.311)\\
\leq\ & C\cdot(\|A_\omega^{-1}\|_{n+1,\alpha} + \|\psi\|_{n+1,\alpha} + \|F\|_{n-1,\alpha} + \|g\|_{n-2,\alpha} + 1), \quad n\geq 2 && (5.312)\\
\leq\ & C\cdot(\|F\|_{n+1,\alpha} + \|g\|_{n-2,\alpha} + 1), \qquad n\geq 2 && (5.313)
\end{aligned}$$

for F in a $\|\cdot\|_{2,\alpha}$ neighborhood and g in a $\|\cdot\|_{1,\alpha}$-neighborhood. This implies the desired tame estimates.

$\boxed{3}$ **Tame estimates on** $g = VB(F)\cdot h$ Now $g = h - K(F)\cdot g$ so that by interpolation inequalities

$$\begin{aligned}
\|g\|_{n,\alpha} &\leq \|h\|_{n,\alpha} + \|K(F)\cdot g\|_{n,\alpha} && (5.314)\\
&\leq C\cdot(\|h\|_{n,\alpha} + \|F\|_{n+1,\alpha} + \|g\|_{n-2,\alpha} + 1) && (5.315)\\
&\leq C\cdot(\|h\|_{n,\alpha} + \|F\|_{n+1,\alpha} + \epsilon\|g\|_{n,\alpha} + C(\epsilon)\|g\|_{1,\alpha} + 1) && (5.316)
\end{aligned}$$

so that choosing $\epsilon > 0$ small (depending on n),

$$\|g\|_{n,\alpha} \leq C\cdot(\|h\|_{n,\alpha} + \|F\|_{n+1,\alpha} + \|g\|_{1,\alpha} + 1) \tag{5.317}$$

for F in a $\|\cdot\|_{3,\alpha}$-neighborhood and as long as g remains in a $\|\cdot\|_{1,\alpha}$-neighborhood. But by continuity of $g = VB(F)\cdot h$, this is the case for h in a $\|\cdot\|_{1,\alpha}$-neighborhood and F

in a $\|\cdot\|_{3,\alpha}$-neighborhood and we conclude with

$$\|g\|_{n,\alpha} \leq C \cdot (\|h\|_{n,\alpha} + \|F\|_{n+1,\alpha} + 1). \tag{5.318}$$

■

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Appendix A

Smooth tame maps of tame Fréchet spaces

The main reference for this Appendix is [14].

A.1 The Fréchet category

$\boxed{1}$ C^∞_K **as Fréchet space** The function spaces we are going to consider are all going to be constructed from spaces of the form

$$C^\infty_K := \{F : K \to \mathbb{R}^d \text{ of class } C^\infty\} \tag{A.1}$$

where K is a compact subset of a Euclidean space with smooth boundary and $d \geq 1$ is an integer. Namely, we will consider affine subspaces of such spaces, and products of such affine subspaces. We will give C^∞_K the topology of a Fréchet space. We will only recall the necessary material. First observe that, as sets,

$$C^\infty_K = \bigcap_{n\geq 0} C^{n,\alpha}_K = \bigcap_{n\geq 0} C^n_K \tag{A.2}$$

where C^n_K denotes the space of n-times continuously differentiable functions $F\colon K \to \mathbb{R}$, and $C^{n,\alpha}_K$ denotes the space of functions in C^n_K all of whose n-th derivatives are Hölder continuous with exponent α. C^n_K is a Banach space with the norm

$$\|F\|_n := \sup_{0\leq j\leq n} \sup_K |\nabla^j F| \tag{A.3}$$

and $C_K^{n,\alpha}$ is a Banach space with the norm

$$\|F\|_{n,\alpha} := \|F\|_n + [\nabla^n F]_\alpha, \qquad [G]_\alpha := \sup_{x \neq y \in K} \frac{|G(x) - G(y)|}{|x-y|^\alpha}. \tag{A.4}$$

The countable family of norms $\|\cdot\|_n$ will be referred to as the C^n-grading, and the countable family of norms $\|\cdot\|_{n,\alpha}$ will be referred to as the $C^{n,\alpha}$-grading. Either grading determines the same topology on C_K^∞ (which we will refer to as the C^∞-topology) since for each $n \geq 0$ there exist constants C_n', C_n'' (independent of F) such that

$$C_n' \cdot \|F\|_n \leq \|F\|_{n,\alpha} \leq C_n'' \cdot \|F\|_{n+1} \tag{A.5}$$

(K has smooth boundary; see [8]).[1] With this topology, C_K^∞ is a **Fréchet space**, namely a complete Hausdorff topological vector space. A basis for this topology consists of the open sets from the C^n-grading

$$\mathcal{O}_{F_0,\epsilon,n} := \{F \in C_K^\infty \mid \|F - F_0\|_n < \epsilon\}. \tag{A.6}$$

Another basis consists of the open sets from the $C^{n,\alpha}$-grading

$$\mathcal{O}_{F_0,\epsilon,n,\alpha} := \{F \in C_K^\infty \mid \|F - F_0\|_{n,\alpha} < \epsilon\}. \tag{A.7}$$

□

$\boxed{2}$ **New Fréchet spaces from old ones** With Γ_i, $i = 1, 2, \ldots$ the components of the boundary $\partial\Omega$, and for constants $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots$,

$$\{F \in C_K^\infty \mid \int_{\Gamma_i} \frac{\partial F}{\partial N} = \gamma_i, i = 1, 2, \ldots \quad F_{|\partial\Omega} = \text{locally constant}\} \tag{A.8}$$

is an affine subspace of C_K^∞, closed in the C^∞-topology and the induced topology is the usual one. (N denotes the unit outer normal to the boundary and $\frac{\partial}{\partial N}$ is the outer normal derivative.)

[1] This means that continuity of maps between Fréchet spaces such as those considered in this thesis can be tested using either grading. On the other hand, the Nash-Moser theorem is valid in a subcategory (the tame Fréchet category) with a stronger notion of equivalence, which the above norms actually satisfy. In addition, maps in the tame Fréchet category must satisfy so-called tame estimates and it will be more natural to work in the $C^{n,\alpha}$-grading.

The topology on a product of m such spaces is defined by the collection of norms (with some abuse of notation)

$$\|(F_1, \dots, F_m)\|_n := \|F_1\|_n + \cdots + \|F_m\|_n, \qquad n = 0, 1, \dots \tag{A.9}$$

and likewise if the $C^{n,\alpha}$-grading is used. Note that, in general, the product of Fréchet spaces is an affine space (over the linear space obtained as the product of the linear spaces subtending the factors in the product), and a linear space if all factors are linear. The induced topology on a subset of such a product is the usual one.

In these notes, a **Fréchet space** (denoted $\mathcal{F}, \mathcal{F}', \mathcal{G}, \dots$) will always refer to a product of (vector) spaces C^∞_K and/or affine spaces of the form (A.8). We will say that it is equipped with the **C^∞-topology** where we use the product topology, and the C^∞-topology is used for each factor.

Now that we have a topology, we have a notion of **continuity**, which is of course characterized by the condition that the preimage of any open set is again an open set. For a map $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{F}'$ defined on an open subset $\mathcal{V}$ of a Fréchet space $\mathcal{F}$ with values in another Fréchet space $\mathcal{F}'$, we will simply say that P is a **map of Fréchet spaces**. In practise, continuity of a map of Fréchet spaces is established using the most convenient gradings. For example, continuity for solution operators of elliptic equations from Chapter 2 is established in the $C^{n,\alpha}$-grading as Schauder-type estimates are readily available. On the other hand, most other maps introduced in Chapter 5 (dealing with the proof of Theorem 1, Section 1.3 *per se*) are proved to be continuous using the C^n-grading. In the special case where the map $L : \mathcal{F} \to \mathcal{G}$ is a linear map of Fréchet spaces, we have the following characterization of continuity. L is continuous if and only for each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ and a constant $C = C_n$ such that

$$\|L(F)\|_n \leq C_n \cdot \|F\|_{k(n)}. \tag{A.10}$$

Note that there is no restriction on the behavior of $k(n)$ as a function of n. $\square$

$\boxed{3}$ **Directional derivatives** The notion of differentiability in Fréchet spaces is a subtle issue. We follow [14] in these notes.

Let $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{F}'$ be a continuous map of Fréchet spaces. The **derivative of** P **at** $F \in \mathcal{V}$ **in the direction**[2] $f \in \mathcal{F}$ is the (unique) element, if it exists, defined by

$$DP(F) \cdot f := \lim_{t \to 0} \frac{P(F + tf) - P(F)}{t} \tag{A.11}$$

where the limit is taken in the C^∞-topology. We say that P is **continuously differentiable** if $DP(F) \cdot f$ exists for all $F \in \mathcal{V}$ and $f \in \mathcal{F}$ (or the linear subspace subtending $\mathcal{F}$ it if is affine but not linear) and if

$$DP : \mathcal{V} \times \mathcal{F} \to \mathcal{F}' \tag{A.12}$$

is continuous as a map of *both* variables F and f. □

Remark 36 One of the main difference between the Banach category and the Fréchet category is in the definition of continuous differentiability. The definitions for continuity and directional derivatives of a continuous map of Banach spaces $P : (U \subset X) \to Y$ are the obvious analogues to those for a map of Fréchet spaces. In contrast, a map P of Banach spaces is said to be **continuously differentiable** if $DP : F \in U \mapsto DP(F) \in \mathcal{L}(X, Y)$ is continuous, where $\mathcal{L}(X, Y)$ is the space of bounded linear operators $X \to Y$. With the operator norm, $\mathcal{L}(X, Y)$ is itself a Banach space.

If however X, Y are Fréchet spaces, it is not true in general that the space $L(X, Y)$ of continuous linear maps is a Fréchet space. This explains the different definitions.

Note also that Banach spaces X, Y are also Fréchet spaces, and that the notion of continuous differentiability of a map between X and Y, when X, Y are viewed as Fréchet spaces, is weaker than that when they are viewed as Banach spaces. ■

$\boxed{4}$ **Partial derivatives** For a map

$$P : \big((\mathcal{V}_1 \subset \mathcal{F}_1) \times (\mathcal{V}_2 \subset \mathcal{F}_2)\big) \to \mathcal{G} \tag{A.13}$$

where we distinguish two "variables" F_1 and F_2, the **partial derivative with respect to** $F_1$is defined by

$$D_{F_1} P(F_1, F_2) \cdot f_1 := \lim_{t \to 0} \frac{P(F_1 + tf_1, F_2) - P(F_1, F_2)}{t}. \tag{A.14}$$

[2] In case $\mathcal{F}$ is affine, f should be taken in the linear space subtending $\mathcal{F}$. For example, if the affine space $\mathcal{F}$ is of the form (A.8) with γ_i not necessarily 0, then it is subtended by the linear space of the same form with $\gamma_i = 0$ for all $i = 1, 2, \ldots$.

The partial derivative in F_2 is defined in a completely analogous way. This notion is especially useful to the extent that the following holds.

Theorem 37 (Theorem I.3.4.3, p. 79, [14]) *The partial derivatives $D_{F_1}P$ and $D_{F_2}P$ exist and are continuous if and only if the (total) derivative DP exists and is continuous. In this case,*

$$DP(F_1, F_2) \cdot (f_1, f_2) = D_{F_1}P(F_1, F_2) \cdot f_1 + D_{F_2}P(F_1, F_2) \cdot f_2. \tag{A.15}$$

□

$\boxed{5}$ **Properties of continuously differentiable maps** Continuously differentiable maps of Fréchet spaces enjoy many of the properties of maps of Banach spaces.

Let $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{G}$ and $Q : (\mathcal{W} \subset \mathcal{G}) \to \mathcal{H}$ be continuously differentiable maps of Fréchet spaces (we assume $P(\mathcal{V}) \subset \mathcal{W}$).

- **Linearity** $DP(F) \cdot f$ is linear in f.
- **The chain rule** $Q \circ P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{H}$ is continuously differentiable, with derivative
$$D(Q \circ P)(F) \cdot f = DQ(P(F)) \cdot DP(F) \cdot f. \tag{A.16}$$
- **The Fundamental Theorem of Calculus** If the segment from F to $F + f$ lies within $\mathcal{V}$, then
$$P(F + f) = P(F) + \int_0^1 DP(F + tf) \cdot f dt. \tag{A.17}$$
- **Taylor's formula** If P is twice continuously differentiable,
$$P(F + f) = P(F) + DP(F) \cdot f + \int_0^1 (1 - t)D^2P(F + tf) \cdot (f, f)dt. \tag{A.18}$$
- **The Open Mapping Theorem** If $\mathcal{F}$, $\mathcal{G}$ are Fréchet spaces and $L \colon \mathcal{F} \to \mathcal{G}$ is continuous, linear, and invertible, then its inverse $V \colon \mathcal{G} \to \mathcal{F}$ is also continuous (and thus L is an isomorphism of Fréchet spaces).

Linearity is not immediate and requires the use of integrals with values in Fréchet spaces, as obviously do the Fundamental Theorem of Calculus and Taylor's formula. We refer to § I.2 in [14] for a more rigorous treatment of Riemann integrals of Fréchet space-valued functions. For our purpose, all properties valid for real-valued integrals will hold in the class of Fréchet spaces (and that of tame Fréchet spaces, which we will see later). □

$\boxed{6}$ **Families of linear maps** We will encounter maps of Fréchet spaces $L : (\mathcal{V} \subset F) \times \mathcal{G} \to \mathcal{H}$, where $\mathcal{G}$ and H are linear, which are linear in the second argument, as reflected in the notation from [14] and adopted in these notes,

$$L(F) \cdot g := L(F, g). \tag{A.19}$$

In this case we will say that $L(F) \cdot g$ is a **family of linear maps**. We already have a first example, namely the derivative $DP(F) \cdot f$ of a continuously differentiable map P.

Proposition 38 (Corollary I.3.4.4, p. 80, [14]) *If $L(F) \cdot g$ is a family of linear maps, continuous jointly in F and g, then it is continuously differentiable jointly in F and g if and only if it is continuously differentiable in F. In this case,*

$$DL(F) \cdot (f, \tilde{g}) := DL(F, g) \cdot (f, \tilde{g}) = D_F L(F, g) \cdot f + L(F, \tilde{g}) \tag{A.20}$$

(and one checks that the notation makes sense).

□

If a scalar function h of one variable is continuously differentiable and invertible, then its inverse has derivative $\frac{1}{h' \circ h^{-1}}$ and thus by induction is as smooth as h is. This can be generalized to families of linear maps of Fréchet spaces.

Proposition 39 (Theorem I.5.3.1, p. 102) *Let $L \colon (\mathcal{V} \subset \mathcal{F}) \times \mathcal{H} \to \mathcal{K}$ be a family of invertible linear maps of Fréchet spaces and let $V \colon (\mathcal{U} \subset \mathcal{F}) \times \mathcal{K} \to \mathcal{H}$ be the family of inverses. If L is smooth and V is continuous, then V is smooth and*

$$DV(F) \cdot (k, g) = -V(F) \cdot DL(F) \cdot (V(F) \cdot k, g). \tag{A.21}$$

$\boxed{7}$ **Second derivatives** The **second derivative** of a (continuously differentiable) map $P : (\mathcal{V} \subset F) \to \mathcal{G}$ is the derivative of the (first) derivative $DP(F) \cdot f$ (which we recall is a map of the two variables F and f). Owing to Theorem 37, it is determined by the partial derivatives with respect to F and f, and by Proposition 38, it is enough to compute the derivative with respect to the first variable only. We shall pose

$$D^2P(F) \cdot (f_1, f_2) := \lim_{t \to 0} \frac{DP(F + tf_2) \cdot f_1 - DP(F) \cdot f_1}{t} \tag{A.22}$$

the limit being taken in the C^∞-topology. □

$\boxed{8}$ **Smooth maps of Fréchet spaces** Derivatives of higher order are defined inductively:

$$D^{m+1}P(F) \cdot (f_1, \ldots, f_{m+1}) := \lim_{t \to 0} \frac{D^m P(F + tf_{m+1}) \cdot (f_1, \ldots, f_m) - D^n P(F) \cdot (f_1, \ldots, f_m)}{t} \tag{A.23}$$

the limit being taken in the C^∞-topology. A map $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{G}$ is said to be **smooth**, or **(of class)** C^∞, if derivatives of all orders exist and are continuous. □

A.2 The tame Fréchet category

The inverse function theorem does not hold for maps of Fréchet spaces. See I.5.5, p. 121, [14] for counterexamples. However, it holds in the tame category of smooth tame maps on tame Fréchet spaces, and the aim of this section is to define and discuss this category. We have explained in Section 1.4 of the Introduction that the Nash-Moser theorem is based on a modification of Newton's method. A major ingredient is that the (abstract) Fréchet spaces on which it is used must possess smoothing operators with suitable behavior, namely

$$\|S_t f\|_n \leq Ce^{(n-m)t}\|f\|_m, \qquad \|(\mathrm{Id} - S_t)f\|_m \leq Ce^{(m-n)t}\|f\|_n. \tag{A.24}$$

It is a *consequence* of the existence of such smoothing operators that interpolation inequalities

$$\|\cdot\|_m^{n-l} \leq C\|\cdot\|_n^{m-l}\|\cdot\|_l^{n-m}, \qquad l \leq m \leq n \tag{A.25}$$

hold. (See Corollary III.1.4.2, p. 176, [14].) A Fréchet space possessing such smoothing operators is called a **tame Fréchet space**.[3]

We will work in spaces of smooth functions, for which such inequalities are standard (see Section A.2.1), and one can construct smoothing operators with the desired properties. This can be done, for example, with the use of truncation of Fourier transforms. Thus, all Fréchet spaces that are involved in this thesis are actually also tame Fréchet spaces (the underlying domain is always a compact subset of some Euclidean space) with either the C^n- or $C^{n,\alpha}$-grading. We have seen that the C^n- and $C^{n,\alpha}$-gradings are equivalent in the Fréchet category by (A.5) but this also shows that they are equivalent in the tame Fréchet category. Therefore we may use either the C^n- or the $C^{n,\alpha}$-grading to show continuity and smoothness depending on which one is more convenient. On the other hand tame estimates (defined and discussed next) will all be derived in the $C^{n,\alpha}$-grading.

See also Corollary II.1.3.7, p. 138 of [14] for a proof that $C^\infty(X)$, where X is a compact manifold, is a tame Fréchet space in the precise sense developed by Hamilton in Sections III.1.1-III.1.3, [14]. Essentially, a **tame Fréchet space** in that sense is modelled on some space $\Sigma(B)$ defined p. 133, [14]. In Section B, we prove the Nash-Moser inverse function theorem for maps $C^\infty_{\mathbb{S}^1} \to C^\infty_{\mathbb{S}^1}$, which essentially covers the case of the map T (the choice of periodicity is for simplicity). This proof emphasizes the fact that the interpolation inequalities (A.28) are really the only essential feature of the function spaces for the algorithm to work.

The smoothing operators introduce some errors in the modified Newton scheme and thus limit the class of maps for which the algorithm will converge. The behavior of the map to be inverted cannot be too wild (hence the terminology of **tame maps**). Specifically, a tame map satisfies tame estimates. This is discussed in Section A.2.2 of the Appendix. There we state results for spaces of smooth functions which are true more

[3] We have seen that various gradings may give the same topology as a Fréchet space. There is a notion that two gradings are tame equivalent, and two such gradings will satisfy interpolation inequalities (A.25). We will not need to go into such detail, but it is enough to point out that the inequalities (A.5) imply that the C^n- and $C^{n,\alpha}$-gradings are tame equivalent. In particular, it is readily visible that then since one grading satisfies interpolation inequalities (A.25), then so must the other grading. In addition, we will shortly introduce the notion of tame maps of (tame) Fréchet spaces. A map is tame if it satisfies so-called tame estimates in a certain grading. Then, this map is also tame in a different grading if this grading is tame equivalent to the first one.

generally for tame Fréchet spaces, and thus we refer to those as such for consistency.

A.2.1 Interpolation inequalities

We record in this section elementary results which are used throughout when deriving tame estimates. The inequalities of this section are in fact also true with $C^{n,\alpha}$-norms replaced by C^n-norms, but we will not need these.

We begin with the elementary inequality

$$x^\delta y^{1-\delta} \leq x + y, \qquad x, y \geq 0, \quad \delta \in [0,1]. \tag{A.26}$$

Lemma 40 (Interpolation inequalities) *For all $n \geq 0$ and $\epsilon > 0$, there exists a constant $C = C(\epsilon, n)$ such that*

$$\|u\|_{n,\alpha} \leq C\|u\|_{0,\alpha} + \epsilon\|u\|_{n+1,\alpha}. \tag{A.27}$$

Proof Standard. See [8], [13], or [18]. ■

Lemma 41 (Multiplicative interpolation inequalities)

$$l \leq m \leq n, \qquad \|u\|_{m,\alpha}^{n-l} \leq C\|u\|_{l,\alpha}^{n-m}\|u\|_{n,\alpha}^{m-l}. \tag{A.28}$$

Proof This inequality is proved for the $\|\cdot\|_n$-norms in Theorem II.2.2.1, p. 143, [14]. For the $[\cdot]_\alpha$-contribution to the $\|\cdot\|_{n,\alpha}$-norms, see Exercise 3.3.7, p. 40, [18]. See also See [8]. ■

Corollary 42

$$\|u\|_{n+r,\alpha}\|v\|_{s+m,\alpha} \leq C \cdot (\|u\|_{n+r+m,\alpha}\|v\|_{s,\alpha} + \|u\|_{s,\alpha}\|v\|_{n+r+m,\alpha}). \tag{A.29}$$

In particular, when $u = v$,

$$\|u\|_{n+r,\alpha}\|u\|_{s+m,\alpha} \leq C \cdot \|u\|_{n+r+m,\alpha}\|u\|_{s,\alpha}. \tag{A.30}$$

Proof See Corollary II.2.2.3, p. 144, [14]. ■

A.2.2 Tame maps

$\boxed{1}$ **Tame linear maps and tame isomorphisms** Let $\mathcal{F}$ and $\mathcal{G}$ be linear Fréchet spaces. A linear map of Fréchet spaces $L\colon \mathcal{F} \to \mathcal{G}$ is a **tame (linear) map** if it satisfies so-called **tame estimates**

$$\|L \cdot f\|_{n,\alpha} \le C_n \cdot \|f\|_{n+r,\alpha}, \qquad n \ge b \tag{A.31}$$

where the **degree** r and the **base** b are independent of n and F.[4] (The constant may depend on n.) Note that by the characterization (A.10) of continuity for linear maps of Fréchet spaces, a tame linear map is necessarily continuous. Also, the degree of the tame estimates *depends on the grading* (for example, solutions to elliptic equations gain two derivatives from the right-hand side in the $C^{n,\alpha}$-grading (Schauder estimates), but no such estimates hold in the C^n-grading).

A **tame isomorphism** L is a linear isomorphism of tame Fréchet spaces such that both L and $V := L^{-1}$ are tame.

Composition of linear maps preserves tameness: if L is tame linear with degree r and base b,

$$\|L \cdot g\|_{n,\alpha} \le C \cdot \|g\|_{n+r,\alpha}, \qquad n \ge b \tag{A.32}$$

and if M is a tame linear map of degree s and base[5] $c = b + r$

$$\|M \cdot f\|_{n,\alpha} \le C \cdot \|f\|_{n+s,\alpha}, \qquad n \ge b + r \tag{A.33}$$

then $L \cdot M$ is a tame linear map of degree $r + s$ and base b:

$$\|LM \cdot f\|_{n,\alpha} \le C \cdot \|f\|_{n+r+s,\alpha}, \qquad n \ge b. \tag{A.34}$$

$\square$

$\boxed{2}$ **Tame maps** A (possibly nonlinear) map of Fréchet spaces $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{G}$ is **tame** if it is continuous and satisfies a **tame estimate** in a neighborhood of each point, namely

$$\|P(F)\|_{n,\alpha} \le C_n \cdot (\|F\|_{n+r,\alpha} + 1), \qquad n \ge b \tag{A.35}$$

[4] Clearly, L also satisfies tame estimates of degree $\tilde{r} \ge r$ and base $\tilde{b} \ge b$.
[5] Increase b or c if necessary so that this relation holds.

where the **degree** r and **base** b may depend on the neighborhood of F, but otherwise not on n or F. (The constant may depend on n and the neighborhood of F.)

Remark 43 In [14], Hamilton follows the habit of choosing neighborhoods in which tame estimates are valid to be $\|\cdot\|_{b+r,\alpha}$-neighborhoods. This is always possible since the open sets (A.7) form a basis for the C^∞-topology, and one may increase r and b if necessary. However, I do not follow this rule in the proof of Theorem 1, Section 1.3, since the tame estimates for the map T are calculated by composing tame maps, and this practise inflates quite artificially the base of the estimates and can make the derivations untractable. ■

Composition of tame maps preserves tameness: let $P : (\mathcal{V} \subset \mathcal{F}) \to \mathcal{G}$ and $Q : (\mathcal{W} \subset G) \to \mathcal{H}$ be tame maps such that $P(\mathcal{V}) \subset \mathcal{W}$, let $F_0 \in \mathcal{V}$ and set $G_0 = P(F_0)$. If Q satisfies tame estimates of degree r and base b in a neighborhood of G_0 and P satisifies tame estimates of degree s and base $b + r$ in a neighborhood of F_0, then $Q \circ P$ satisfies tame estimates of degree $r + s$ and base b in a neighborhood of F_0.

In case the map is linear, this definition of a tame map is consistent with that of a tame linear map:

Theorem 44 (Theorem I.2.1.5, p. 141, [14]) *A map is a tame linear map if and only if it is linear and tame.*

$\boxed{3}$ **Maps of two variables** For a map $P : (\mathcal{V} \subset \mathcal{F}) \times (\mathcal{W} \subset G) \to \mathcal{H}$ we may assign different degrees for the two variables: if P satisfies an estimate of the form

$$\|P(F,G)\|_{n,\alpha} \leq C \cdot (\|F\|_{n+r,\alpha} + \|G\|_{n+s,\alpha} + 1), \qquad n \geq b, \tag{A.36}$$

for F in a $\|\cdot\|_{b+r,\alpha}$-neighborhood and G in a $\|\cdot\|_{b+s,\alpha}$-neighborhood, then we say that P has degree r in F and s in G. □

$\boxed{4}$ **Tame estimates for families of linear maps**

Lemma 45 (Lemma II.2.1.7, p. 143, [14]) *If $L(F) \cdot g$ is a family of linear maps which is tame with degree r in F and s in f, and base b,*

$$\|L(F) \cdot g\|_{n,\alpha} \leq C \cdot (\|F\|_{n+r,\alpha} + \|g\|_{n+s,\alpha} + 1), \qquad n \geq b, \tag{A.37}$$

for all F in a $\|\cdot\|_{b+r,\alpha}$-neighborhood of some F_0 and all g in a $\|\cdot\|_{b+s,\alpha}$-neighborhood of 0, then it satisfies estimates

$$\|L(F) \cdot g\|_{n,\alpha} \leq C_n \cdot (\|F\|_{n+r,\alpha}\|g\|_{b+s,\alpha} + \|g\|_{n+s,\alpha}), \qquad n \geq b \tag{A.38}$$

for all F in a $\|\cdot\|_{b+r,\alpha}$-neighborhood of F_0 and all $g \in \mathcal{G}$ without restriction.[6] *For each F, $L(F)$ is a bounded linear map of Banach spaces:*

$$\|L(F) \cdot g\|_{n,\alpha} \leq C \cdot (\|F\|_{n+r,\alpha} + 1)\|g\|_{n+s,\alpha}, \qquad n \geq b. \tag{A.39}$$

$\boxed{5}$ **Smooth tame maps** A map of tame Fréchet spaces is a **smooth tame map** if it is smooth (as a map of Fréchet spaces) and all its derivatives are tame. □

$\boxed{6}$ **Inverses of families of linear maps** Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be tame Fréchet spaces, where $\mathcal{G}$ and $\mathcal{H}$ are linear.

Proposition 46 (Theorem I.5.3.1, p. 102 and Theorem II.3.1.1, p. 150, in [14]) *Let $L : (\mathcal{V} \subset \mathcal{F}) \times \mathcal{G} \to \mathcal{H}$ be a smooth tame family of linear maps. Suppose that $L(F)\colon \mathcal{G} \to \mathcal{H}$ is invertible for each F, and that the family of inverses $VL\colon (\mathcal{V} \subset \mathcal{F}) \times \mathcal{H} \to \mathcal{G}$ is a tame map of Fréchet spaces . Then, VL is also a smooth tame map.*

A.2.3 Examples of smooth tame maps

K will always denote a compact subset of Euclidean space with smooth boundary. We state continuity of maps of Fréchet spaces both in the C^n- and $C^{n,\alpha}$-grading, but we only give tame estimates in the $C^{n,\alpha}$-grading.

$\boxed{7}$ **Linear differential operators with constant coefficients**

[6] *The converse is obvious.*

Lemma 47 *A linear differential operator of order r with constant coefficients* $L : C_K^\infty \to C_K^\infty$ *is a smooth tame map of Fréchet spaces:* $L\colon C_K^{n+r} \to C_K^n$ *is continuous for each* $n \geq 0$*.* Lu *has degree* r *and base* 0*:*

$$\|Lu\|_{n,\alpha} \leq C \cdot \|u\|_{n+r,\alpha}, \qquad n \geq 0 \tag{A.40}$$

for all $u \in C_K^\infty$*.*

$\boxed{8}$ Product of functions

Lemma 48 *The bilinear map*

$$\begin{array}{rccccc} B: & C_K^\infty & \times & C_K^\infty & \to & C_K^\infty, \\ & (F & , & G) & \mapsto & FG \end{array} \tag{A.41}$$

is a smooth tame map of Fréchet spaces: for each $n \geq 0$*,* $B\colon C_K^n \times C_K^n \to C_K^n$ *is continuous as well as* $B\colon C_K^{n,\alpha} \times C_K^{n,\alpha} \to C_K^{n,\alpha}$*.* $B(F,G)$ *has degree* 0 *in* F *and* G*, and base* 0*:*

$$\|B(F,G)\|_{n,\alpha} \leq C \cdot (\|G\|_{0,\alpha}\|F\|_{n,\alpha} + \|F\|_{0,\alpha}\|G\|_{n,\alpha}), \qquad n \geq 0 \tag{A.42}$$

for all F *and* G*. The first derivative is given by*

$$B(F,G) \cdot (f,g) = fG + Fg. \tag{A.43}$$

There are obvious generalizations of the above for the product of an arbitrary number of functions $(F_1, \ldots, F_l) \mapsto F_1 \cdots F_l$.

$\boxed{9}$ **The Nemitskii operator** The Nemitskii operator performs the operation of "functional substitution". Namely, if $p(x,z) = p : K \times \mathbb{R} \to \mathbb{R}$ is a smooth function, define

$$P(F)(x) := p(x, F(x)), \qquad x \in K, \quad F \in \mathcal{V} \subset C_K^\infty. \tag{A.44}$$

Lemma 49 $P : \mathcal{V} \to C_K^\infty$ *is a smooth tame map of Fréchet spaces: for each* $n \geq 0$*,* $P\colon C_K^n \to C_K^n$ *is continuous as well as* $P\colon C_K^{n,\alpha} \to C_K^{n,\alpha}$*.* $P(F)$ *has degree* 0 *in* F *and base* 1*:*

$$\|P(F)\|_{n,\alpha} \leq C \cdot (\|F\|_{n,\alpha} + 1), \qquad n \geq 1 \tag{A.45}$$

for F in a neighborhood where $\|F\|_{1,\alpha}$ is bounded. The first derivative $DP(F)\cdot f \in C^\infty_K$ is given by

$$(DP(F)\cdot f)(x) = D_z p(x, F(x)) f(x), \qquad x \in K. \tag{A.46}$$

$\boxed{10}$ **Composition of functions** Let $K \subset \mathbb{R}^d, K' \subset \mathbb{R}^{d'}, K'' \subset \mathbb{R}^{d''}$ be compact subsets of Euclidean spaces, and let $G_0 \in C^\infty_{(K,\mathbb{R}^{d'})}$ such that $G_0(K) \subset V' \subset K'$ for some open set V'. If ϵ is sufficiently small, any $G \in \mathcal{O}_{G_0,0,\epsilon}$ has $G(K) \subset K'$. Thus, we may define the **composition operator**

$$\begin{array}{ccccccc} C & : C^\infty_{(K',\mathbb{R}^{d''})} & \times & \mathcal{O}_{G_0,0,\epsilon} & \to & C^\infty_{(K,\mathbb{R}^{d''})} \\ & (F & , & G) & \mapsto & F\circ G. \end{array} \tag{A.47}$$

Lemma 50 *C is a smooth tame map of Fréchet spaces. In the C^n-grading we have that*

$$C\colon C^n_{(K',\mathbb{R}^{d''})} \times C^n_{K,\mathbb{R}^{d'}} \to C^n_{(K,\mathbb{R}^{d''})}, \qquad n \geq 0 \tag{A.48}$$

is continuous, while in the $C^{n,\alpha}$-grading we only have that

$$C\colon C^{n+1,\alpha}_{(K',\mathbb{R}^{d''})} \times C^{n,\alpha}_{K,\mathbb{R}^{d'}} \to C^{n,\alpha}_{(K,\mathbb{R}^{d''})}, \qquad n \geq 0 \tag{A.49}$$

is continuous. Yet, $C(F,G) = F\circ G$ has degree 0 in F and G, and base 1:

$$\|F\circ G\|_{n,\alpha} \leq C\cdot(\|F\|_{n,\alpha} + \|G\|_{n,\alpha} + 1), \qquad n \geq 1 \tag{A.50}$$

for F, G in neighborhoods where $\|F\|_{1,\alpha}$ and $\|G\|_{1,\alpha}$ are bounded. Owing to Lemma 45, Section A.2.2, this becomes

$$\|F\circ G\|_{n,\alpha} \leq C\cdot(\|F\|_{n,\alpha} + \|G\|_{n,\alpha}\|F\|_{1,\alpha}), \qquad n \geq 1 \tag{A.51}$$

and

$$\|F\circ G\|_{n,\alpha} \leq C\cdot(\|G\|_{n,\alpha} + 1)\|F\|_{n,\alpha}, \qquad n \geq 1 \tag{A.52}$$

for G in a $\|\cdot\|_{1,\alpha}$-neighborhood, and all F (without restriction). The derivative is given by

$$DC(F,G)\cdot(f,g) = F'(G)g + f(G). \tag{A.53}$$

Proof See [8]. ■

$\boxed{11}$ **The inversion operator** Let $I, I' \subset \mathbb{R}$ be compact intervals and introduce the spaces of functions

$$\mathcal{D}^{\infty}_{I,I'} := \{M \in C^{\infty}_{(I,\mathbb{R})} \mid M' > 0 \text{ on } I,\ M(I) = I'\}, \tag{A.54}$$

$$\mathcal{D}^{\infty}_{I',I} := \{N \in C^{\infty}_{(I',\mathbb{R})} \mid N' > 0 \text{ on } I',\ N(I') = I\}. \tag{A.55}$$

Lemma 51 *The inversion operator*

$$\begin{array}{rccc} V: & \mathcal{D}^{\infty}_{I,I'} & \rightarrow & \mathcal{D}^{\infty}_{I',I} \\ & M & \mapsto & M^{-1} \end{array} \tag{A.56}$$

is a smooth tame map (and in fact $V: \mathcal{D}^{n+1}_{I,I'} \rightarrow \mathcal{D}^{n}_{I',I}$ is continuous for each $n \geq 1$). VM has degree 0 *in M and base* 1*:*

$$\|V(M)\|_{n,\alpha} \leq C \cdot (\|M\|_{n,\alpha} + 1), \qquad n \geq 1 \tag{A.57}$$

for M in a neighborhood where $\|M\|_{1,\alpha}$ is bounded. The first derivative is given by

$$DV(M) \cdot m = -\frac{m(M^{-1})}{M'(M^{-1})}. \tag{A.58}$$

Appendix B

The Nash-Moser inverse function theorem

B.1 Preliminary: Newton's algorithm in Banach spaces

Consider a smooth[1] map

$$P\colon X \to X \qquad \text{such that} \quad P(0) = 0 \tag{B.1}$$

on a Banach space X with norm $|\cdot|$. The (surjective part of the) classical inverse function theorem states that

$$P(f) = g \tag{B.2}$$

has a solution for any sufficiently small g, provided the derivative $DP(0)$ at $f = 0$ has a (bounded) right-inverse $VP(0)$.

One method of proof is via the Newton algorithm, which is very popular in its "discrete" (iterative) version. However, I will follow Hamilton's [14] and present the "continuous" version of the algorithm. A solution to (B.2) will be found by solving an ODE. This ODE, with solution f_t, is designed so that $P(f_t)$ goes along a straight line from 0 to g. First observe that since $DP(0) \cdot VP(0) = \mathrm{Id}$, then $DP(f) \cdot VP(0)$ is invertible for f in a neighborhood of 0, say $|f| < \delta$ (X is a Banach space). In turn,

[1] "Smooth" means that P is of class C^1 as a map of Banach spaces. For the Nash-Moser theorem for Fréchet spaces, we will assume P to be of class C^2 as a map of Fréchet spaces.

$DP(f)$ has a (bounded) right-inverse which satisfies

$$\|VP(f)\| < M, \qquad |f| < \delta \tag{B.3}$$

for some M. With this, the following ODE makes sense and achieves our desired goal:

$$\dot{f}_t = cVP(f_t) \cdot (g - P(f_t)), \qquad f_0 = 0 \tag{B.4}$$

where $c > 0$ is a constant which can in fact be arbitrarily chosen (but see discussion at the end of this section). Specifically, we show that

- the ODE (B.4) has a solution for all t,
- the solution f_t converges to some $f_\infty \in X$, and
- this limit solves (B.2): $P(f_\infty) = g$.

It is standard that (B.4) has a solution for some time. To show that the solution exists for all time, we will use the following, also standard fact: there exists $\delta > 0$ and $T > 0$ such that, if $|\tilde{f}_0| < \delta$, then the ODE

$$\dot{f}_t = cVP(f_t) \cdot (g - P(f_t)), \qquad f_0 = \tilde{f}_0 \tag{B.5}$$

has a solution for $t \in [0, T]$. (Making it smaller, δ works for both (B.3) and (B.5).) The error $k_t := g - P(f_t)$ satisfies

$$\dot{k}_t + ck_t = 0, \qquad k_t = ge^{-ct} \tag{B.6}$$

and Newton's algorithm, if it converges, does so exponentially fast. (The Nash-Moser algorithm takes advantage of this.) As long as the solution f_t exists and $|f_t| < \delta$,

$$|\dot{f}_t| \leq cM|k_t| \tag{B.7}$$

and so

$$|f_t| \leq \int_0^t |\dot{f}_\theta| d\theta < cM|g| \int_0^t e^{-c\theta} d\theta < M|g|. \tag{B.8}$$

(Observe how the integral converges owing to the exponential decay $|k_t| \leq |g|e^{-ct}$.) Therefore, provided

$$|g| < \frac{\delta}{M} \tag{B.9}$$

the solution f_t satisfies $|f_t| < \delta$ as long as it exists. If the maximal existence time T^* of f_t were finite, owing to the solvability of (B.5), the solution f_t could be extended to $[0, T^* + T)$, a contradiction. The solution exists for all time. In particular, $k_t \to 0$. It remains to show that f_t itself converges, and that the limit is a solution to $P(f) = g$.

Letting $t \to \infty$ (as we may),

$$\int_0^\infty |\dot{f}_t| dt \le M|g|. \tag{B.10}$$

In turn, $f_t - f_\tau = \int_\tau^t \dot{f}_\theta d\theta$ shows that f_t is Cauchy hence converges to some $f_\infty \in X$. Also, $k_t \to 0$ and taking limits in (B.4), we find $VP(f_\infty) \cdot (g - P(f_\infty)) = 0$. Taking $DP(f_\infty)$ gives $P(f_\infty) = g$. $\square$

A characteristic feature of the Newton algorithm is its exponential convergence. This convergence rate is much faster than is really needed or, alternatively, the bound (B.3) on the operator norms is more than is really needed. Suppose that one somehow has a bound of the form $\|VP(f)\| < Me^{\gamma t}$ for some $\gamma > 0$.[2] Then, choosing $c > \gamma$ (note that up to now we have let c be any positive real number) we conclude that $|f_t| \le M|g|\frac{c}{c-\gamma}$ and we still have that, as long as it exists, f_t remains small provided g is sufficiently small. In essence, this is how the Nash-Moser inverse function theorem exploits the rate of convergence of the Newton algorithm.

B.2 The modified Newton algorithm

When the map P (on the Banach space X) has unbounded derivatives, the ODE (B.4) cannot be used as such and a modified Newton algorithm taking advantage of the exponentional rate of convergence can be used. The algorithm is implemented via smoothing operators S_t by considering the ODE

$$\dot{f}_t = cVP(S_t f_t) \cdot S_t(g - P(f_t)), \qquad f_0 = 0. \tag{B.11}$$

[2] This assumption is entirely artificial in the present setting of a map of Banach spaces, but it will be much more natural when dealing with Fréchet spaces.

Following [14], we recast this problem for a map P of Fréchet spaces. The smoothing operators have the property that

$$S_t \to \mathrm{Id}, \qquad t \to \infty \tag{B.12}$$

so that, for large t, (B.11) very much ressembles (B.4), and the main ideas in the proof for Banach spaces developed in Section B.1 remain the same: a solution f_t to (B.11) is shown to exist for some time, and it is shown that f_t remains in a neighborhood where the solution can always be extended for some time T independent of f_t. Thus, f_t exists for all time, and furthermore it is shown that f_t converges to some f_∞ and that $P(f_\infty) = g$. Similar to the case for Banach spaces, the crucial point is to show that, as long as the solution exists,

$$\int_0^t \|\dot{f}_\theta\|_n d\theta \leq C\|g\|_n, \qquad n = 1, 2, \ldots \tag{B.13}$$

where the constants are independent of t. For then $\|f_t - f_\tau\|_n$ is Cauchy for each n, hence f_t converges in the Fréchet space.

Focusing for purpose of illustration on the lowest norm $n = 2r$,[3] the ODE (B.11)gives the estimate

$$\|\dot{f}_t\|_{2r} \leq Ce^{2rt}\|k_t\|_0 \tag{B.14}$$

where the factor e^{2rt} is introduced by the smoothing operator S_t. (Recall that in a similar way the estimate (B.7) was obtained from the ODE (B.4)).

In the case of Banach spaces, the exponential decay (B.6) of the error $k_t = g - P(f_t)$ gives a lot of slack for the integral in (B.8) to converge. In the present case, k_t satisfies

$$\dot{k}_t + cS_t k_t = l_t, \qquad l_t := [DP(S_t f_t) - DP(f_t)] \cdot \dot{f}_t. \tag{B.15}$$

For large t, the left-hand side $\dot{k}_t + cS_t k_t$ behaves much like $\dot{k}_t + ck_t$. As for the right-hand side l_t, the operator $DP(S_t f_t) - DP(f_t)$ "approaches" 0 (in some loose sense). This is enough for $\|k_t\|_0$ to converge. Choosing $c > 2r$, this convergence of $\|k_t\|_0$ can be made to beat the factor e^{2rt} (recall the discussion at the end of Section B.1) and one finds that, as long as the solution f_t to (B.11) exists,

$$\left(\int_0^t e^{2rt}\|k_\theta\|_0 d\theta\right) \leq C\|g\|_{2r} + \left(\int_0^t e^{2rt}\|k_\theta\|_0 d\theta\right)^2 \tag{B.16}$$

[3] We may assume, see Section B.3.2, that $P(f)$, $DP(f) \cdot h$, $VP(f) \cdot k$, and $D^2P(f) \cdot (h_1, h_2)$ have degree $2r$ in f, r in h, h_1, h_2, 0 in k, and base 0.

(the constant is independent of t). Therefore, $\int_0^t e^{2rt}\|k_\theta\|_0 d\theta$ remains bounded (independently of t) provided $\|g\|_{2r}$ is small, as desired.

B.3 The surjective part of the Nash-Moser theorem

We begin with a more detailed outline of the proof. Then we give the details of a normalization, which simplify the proof and which are only briefly sketched in [14]. Next, we derive an equivalent formulation for the special case of a map

$$P\colon (\mathcal{V} \subset C^\infty_{\mathbb{S}^1}) \to C^\infty_{\mathbb{S}^1}. \tag{B.17}$$

(This covers essentially the case of this thesis, the choice of periodicity being for simplicity.) We construct smoothing operators on $C^\infty_{\mathbb{S}^1}$ with the desired properties. These amount to truncation of higher Fourier modes. The rest of the section is devoted to the proof of the surjective part of the Nash-Moser inverse function theorem *per se* for the special case (B.17).

B.3.1 Outline of proof

The proof begins with some normalizations: one may assume that $P(f)$, $DP(f)\cdot h$, $D^2P(f)\cdot(h,\tilde{h})$, and the inverse $VP(f)\cdot k$ of DP are tame with degree $2r$ in f, r in $h,\tilde{h}$, and 0 in k. (See Section B.3.2.)After a second normalization, one may assume that the neighborhood in which the initial data for which $\dot{f}_t = cVP(S_tf_t)\cdot S_t(g - P(f_t))$ has a solution for some time is of the form

$$\|f_0\|_{2r} < 1. \tag{B.18}$$

Estimates (B.13) are obtained by studying the error term k_t which satisfies

$$k_t := g - P(f_t), \qquad \dot{k}_t + cS_tk_t = l_t, \qquad l_t := [DP(S_tf_t) - DP(f_t)]\cdot \dot{f}_t. \tag{B.19}$$

Note now that k_t satisfies a first order linear ODE and the usual method gives

$$k_t = a_{0,t}k_0 + \int_{\theta=0}^t a_{\theta,t}l_\theta d\theta \qquad (k_0 = g) \tag{B.20}$$

where $a_{\theta,t}$ is obtained from the integrating factor.

The ODE (B.11) can be written as

$$\dot{f}_t = cVP(S_t f_t) \cdot S_t k_t. \tag{B.21}$$

The smoothing operators satisfy estimates

$$\|S_t f\|_n \leq Ce^{(n-m)t}\|f\|_m, \qquad \|(\mathrm{Id} - S_t)f\|_m \leq Ce^{(m-n)t}\|f\|_n \tag{B.22}$$

and we can control the norms of $\dot{f}_t$ with those of f_t and k_t (of course, this statement has to be made very precise, as it could work against or for us). With this, we have a first estimate

$$\|\dot{f}_t\|_{2r} \leq Ce^{2rt}\|k_t\|_0 \tag{B.23}$$

as long as $\|f_t\|_{2r} < 1$, and it will be verified that this remains the case as long as the solution f_t exists. Now k_t satisfies the ODE (B.19) and this allows to show that

$$\int_0^t e^{2r\theta}\|k_\theta\|_0 \leq C\|g\|_{2r} + C\left(\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta\right)^2 \tag{B.24}$$

where the constant is independent of t. For $\|g\|_{2r}$ sufficiently small, this implies

$$\|f_t\|_{2r} \leq \int_0^t \|\dot{f}_\theta\|_{2r} d\theta \leq C\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta \leq C\|g\|_{2r} \tag{B.25}$$

where C is independent of t, and moreover this shows that $\|f_t\|_{2r} < 1$ for all t. This verifies the initial hypothesis of an induction process on $n \geq 2r$ which establishes the estimates $\int_0^T \|\dot{f}_t\|_n dt \leq C\|g\|_{2n}$. This induction is carried out by establishing the more general inequalities

$$\int_0^T \|\dot{f}_t\|_{n+q} dt \leq Ce^{qT}\|g\|_n \tag{B.26}$$

with constants independent of T, and which holds for all g such that $\|g\|_{2r}$ is small, independently of n and q. These estimates are obtained as follows. The ODE (B.21) shows that one can control the norms of $\dot{f}_t$ with those of f_t and k_t (note that the properties of the smoothing operators play a rôle here). Now k_t solves the linear first-order ODE (B.19), and thus one controls the norms of k_t with those of l_t, i.e. those of f_t and $\dot{f}_t$. How all of this is done must be stated precisely, as otherwise the argument may seem to go in circles.

In the end one gets the estimates

$$\int_0^T \|\dot{f}_t\|_{n+1+q} dt \le Ce^{qT}(\|g\|_{n+1} + \|g\|_n\|g\|_{2r+1}) \tag{B.27}$$

and by interpolation inequalities these become

$$\int_0^T \|\dot{f}_t\|_{n+1+q} dt \le Ce^{qT}\|g\|_{n+1} \tag{B.28}$$

as desired.

Remark 52 For emphasis, the smoothing operators and their properties have been used when controlling the norms of $\dot{f}_t$ in terms of those of f_t and k_t, and the interpolation inequalities also play a rôle in the last step described. On the other hand, tameness of VP also comes in when estimating $\dot{f}_t$ in terms of f_t and k_t, tameness of D^2P comes in when estimating l_t (see (B.19)) in terms of f_t and k_t. As for tameness of P and DP, they are not used directly in the proof that P is surjective (although tame estimates for P and DP can be obtained by integration from those for D^2P) but are used explicitly in establishing tame estimates for the inverse of P. ■

B.3.2 Normalizations

Consider a map P of Fréchet spaces

$$P\colon (\mathcal{U} \subset \mathcal{F}) \to \mathcal{G}. \tag{B.29}$$

The proof of the Nash-Moser inverse function theorem relies on tame estimates on the maps

$$P(f), \quad DP(f)\cdot h, \quad D^2P(f)\cdot(h_1,h_2), \quad VP(f)\cdot k. \tag{B.30}$$

Each map has a degree in f, h, and k, and a base, defined on a certain neighborhood. We will show that we may assume, without loss of generality, that all these maps have tame estimates with base 0, degree $2r$ in f and r in h (for some $r > 0$ sufficiently large r), 0 in k, and valid in a neighborhood $\|f\|_0 < 1$.

That this is possible can be seen from the following. The proof of the Nash-Moser theorem hinges on interpolation inequalities

$$\|f\|_m^{n-l} \le C\|f\|_l^{n-m}\|f\|_n^{m-l}, \qquad l \le m \le n \qquad (C = C(l,m,n)). \tag{B.31}$$

However, shifting the indices

$$|\cdot|_n := \|\cdot\|_{n+p} \qquad (p \in \mathbb{Z}) \tag{B.32}$$

preserves these interpolation inequalities:

$$|f|_m^{n-l} \leq C|f|_l^{n-m}|f|_n^{m-l}, \qquad l \leq m \leq n \qquad (C = C(l,m,n)). \tag{B.33}$$

(To avoid running into trouble with the indices, we first extend the $\|\cdot\|_n$-grading (for $n \geq 0$) into the negative indices with the 0 seminorms:

$$\ldots 0 = 0 \leq \|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots) \tag{B.34}$$

Therefore, we may replace the grading $\|\cdot\|_n$ by the tame equivalent grading $|\cdot|_n$. In this way it is easy to see that a given tame map can be made to have degree 0. On the other hand, we know that a base can be increased: if a map has base b than it also has base b' with $b' \geq b$. Likewise for the degree. This leaves some flexibility.

For clarity, we write $\|\cdot\|_n^{\mathcal{F}}$ and $\|\cdot\|_n^{\mathcal{G}}$, since we will be using different shift on the indices for the gradings on $\mathcal{F}$ and $\mathcal{G}$.

Suppose then that we have the following tame estimates:[4]

$$\begin{aligned}
\|P(f)\|_n^{\mathcal{G}} &\leq C(1 + \|f\|_{n+r_0}^{\mathcal{F}}), \qquad n \geq b_0, \qquad \|f\|_{a_0}^{\mathcal{F}} < \delta && \text{(B.35)}\\
\|DP(f)\cdot h\|_n^{\mathcal{G}} &\leq C(\|h\|_{n+s_1}^{\mathcal{F}} + \|f\|_{n+r_1}^{\mathcal{F}}\|h\|_{b_1+s_1}^{\mathcal{F}}), \quad n \geq b_1, \quad \|f\|_{a_1}^{\mathcal{F}} < \delta && \text{(B.36)}\\
\|VP(f)\cdot k\|_n^{\mathcal{F}} &\leq C(\|k\|_{n+s_{-1}}^{\mathcal{G}} + \|f\|_{n+r_{-1}}^{\mathcal{F}}\|k\|_{b_{-1}+s_{-1}}^{\mathcal{G}}), \; n \geq b_{-1}, \; \|f\|_{a_{-1}}^{\mathcal{F}} < \delta && \text{(B.37)}
\end{aligned}$$

and

$$\begin{aligned}
\|D^2P(f)\cdot(h_1,h_2)\|_n^{\mathcal{G}} \leq\; & C\Big(\|h_1\|_{b_2+s_2}^{\mathcal{F}}\|h_2\|_{n+s_2}^{\mathcal{F}} && \text{(B.38)}\\
& +\|h_1\|_{n+s_2}^{\mathcal{F}}\|h_2\|_{b_2+s_2}^{\mathcal{F}} && \text{(B.39)}\\
& +\|h_1\|_{s_2}^{\mathcal{F}}\|h_2\|_{s_2}^{\mathcal{F}}\|f\|_{n+r_2}^{\mathcal{F}}\Big), && \text{(B.40)}\\
& \quad n \geq b_2, \qquad \|f\|_{a_2}^{\mathcal{F}} < \delta && \text{(B.41)}
\end{aligned}$$

[4] As remarked earlier, the proof that P is surjective only uses the tame estimates for D^2P and VP, whilethose for P and DP are used to establish tame estimates for the (right-)inverse of P, once it is proved that one such exists. However, for sake of clarity, and in order to provide more explicit details than are given in Lemma III.1.2.3, p. 173, of [14], we include them here.

for f in the indicated neighborhoods, and any h, k (without restrictions). Set

$$|\cdot|_m^{\mathcal{F}} := \|\cdot\|_{m+p}^{\mathcal{F}}, \qquad |\cdot|_m^{\mathcal{G}} := \|\cdot\|_{m+q}^{\mathcal{G}} \tag{B.42}$$

where p and q will be specified later. The tame estimates become

$$\begin{aligned}
|P(f)|_{n-q}^{\mathcal{G}} &\leq C(1+|f|_{n+r_0-p}^{\mathcal{F}}), && \text{(B.43)}\\
& n \geq b_0, \qquad |f|_{a_0-p}^{\mathcal{F}} < \delta && \text{(B.44)}\\
|DP(f)\cdot h|_{n-q}^{\mathcal{G}} &\leq C(|h|_{n+s_1-p}^{\mathcal{F}} + |f|_{n+r_1-p}^{\mathcal{F}}|h|_{b_1+s_1-p}^{\mathcal{F}}), && \text{(B.45)}\\
& n \geq b_1, \qquad |f|_{a_1-p}^{\mathcal{F}} < \delta && \text{(B.46)}\\
|VP(f)\cdot k|_{n-p}^{\mathcal{F}} &\leq C(|k|_{n+s_{-1}-q}^{\mathcal{G}} + |f|_{n+r_{-1}-p}^{\mathcal{F}}|k|_{b_{-1}+s_{-1}-q}^{\mathcal{G}}), && \text{(B.47)}\\
& n \geq b_{-1}, \qquad |f|_{a_{-1}-p}^{\mathcal{F}} < \delta && \text{(B.48)}
\end{aligned}$$

and

$$\begin{aligned}
|D^2P(f)\cdot(h_1,h_2)|_{n-q}^{\mathcal{G}} &\leq C\Big(|h_1|_{b_2+s_2-p}^{\mathcal{F}}|h_2|_{n+s_2-p}^{\mathcal{F}} && \text{(B.49)}\\
&\quad +|h_1|_{n+s_2-p}^{\mathcal{F}}|h_2|_{b_2+s_2-p}^{\mathcal{F}} && \text{(B.50)}\\
&\quad +|h_1|_{s_2-p}^{\mathcal{F}}|h_2|_{s_2-p}^{\mathcal{F}}|f|_{n+r_2-p}^{\mathcal{F}}\Big), && \text{(B.51)}\\
& n \geq b_2, \qquad |f|_{a_2-p}^{\mathcal{F}} < \delta && \text{(B.52)}
\end{aligned}$$

Relabeling indices,

$$\begin{aligned}
|P(f)|_{m}^{\mathcal{G}} &\leq C(1+|f|_{m+r_0+q-p}^{\mathcal{F}}), && \text{(B.53)}\\
& m \geq b_0-q, \qquad |f|_{a_0-p}^{\mathcal{F}} < \delta && \text{(B.54)}\\
|DP(f)\cdot h|_{m}^{\mathcal{G}} &\leq C(|h|_{m+s_1+q-p}^{\mathcal{F}} + |f|_{m+r_1+q-p}^{\mathcal{F}}|h|_{b_1+s_1-p}^{\mathcal{F}}), && \text{(B.55)}\\
& m \geq b_1-q, \qquad |f|_{a_1-p}^{\mathcal{F}} < \delta && \text{(B.56)}\\
|VP(f)\cdot k|_{l}^{\mathcal{F}} &\leq C(|k|_{l+s_{-1}-q+p}^{\mathcal{G}} + |f|_{l+r_{-1}}^{\mathcal{F}}|k|_{b_{-1}+s_{-1}-q}^{\mathcal{G}}), && \text{(B.57)}\\
& l \geq b_{-1}-p, \qquad |f|_{a_{-1}-p}^{\mathcal{F}} < \delta && \text{(B.58)}
\end{aligned}$$

and

$$\begin{aligned}
|D^2P(f)\cdot(h_1,h_2)|_{m}^{\mathcal{G}} &\leq C\Big(|h_1|_{b_2+s_2-p}^{\mathcal{F}}|h_2|_{m+s_2+q-p}^{\mathcal{F}} && \text{(B.59)}\\
&\quad +|h_1|_{m+s_2+q-p}^{\mathcal{F}}|h_2|_{b_2+s_2-p}^{\mathcal{F}} && \text{(B.60)}\\
&\quad +|h_1|_{s_2-p}^{\mathcal{F}}|h_2|_{s_2-p}^{\mathcal{F}}|f|_{m+r_2+q-p}^{\mathcal{F}}\Big), && \text{(B.61)}\\
& m \geq b_2-q, \qquad |f|_{a_2-p}^{\mathcal{F}} < \delta && \text{(B.62)}
\end{aligned}$$

Equivalently,

$$
\begin{aligned}
|P(f)|_m^{\mathcal{G}} &\leq C(1+|f|^{\mathcal{F}}_{m+r_0+q-p}), && \text{(B.63)}\\
& \qquad m \geq b_0 - q, \qquad |f|^{\mathcal{F}}_{a_0-p} < \delta && \text{(B.64)}\\
|DP(f)\cdot h|_m^{\mathcal{G}} &\leq C(|h|^{\mathcal{F}}_{m+s_1+q-p} + |f|^{\mathcal{F}}_{m+r_1+q-p}|h|^{\mathcal{F}}_{b_1-q+s_1+q-p}), && \text{(B.65)}\\
& \qquad m \geq b_1 - q, \qquad |f|^{\mathcal{F}}_{a_1-p} < \delta && \text{(B.66)}\\
|VP(f)\cdot k|_l^{\mathcal{F}} &\leq C(|k|^{\mathcal{G}}_{l+s_{-1}+p-q} + |f|^{\mathcal{F}}_{l+r_{-1}}|k|^{\mathcal{G}}_{b_{-1}-p+s_{-1}+p-q}), && \text{(B.67)}\\
& \qquad l \geq b_{-1} - p, \qquad |f|^{\mathcal{F}}_{a_{-1}-p} < \delta && \text{(B.68)}
\end{aligned}
$$

and

$$
\begin{aligned}
|D^2P(f)\cdot(h_1,h_2)|_m^{\mathcal{G}} &\leq C\Big(|h_1|^{\mathcal{F}}_{b_2+s_2-q+q-p}|h_2|^{\mathcal{F}}_{m+s_2+q-p} && \text{(B.69)}\\
&\quad +|h_1|^{\mathcal{F}}_{m+s_2+q-p}|h_2|^{\mathcal{F}}_{b_2+s_2-q+q-p} && \text{(B.70)}\\
&\quad +|h_1|^{\mathcal{F}}_{s_2-q+q-p}|h_2|^{\mathcal{F}}_{s_2-q+q-p}|f|^{\mathcal{F}}_{m+r_2+q-p}\Big), && \text{(B.71)}\\
& \qquad m \geq b_2 - q, \qquad |f|^{\mathcal{F}}_{a_2-p} < \delta && \text{(B.72)}
\end{aligned}
$$

Now we want $VP(f)\cdot k$ to have degree 0 in k, and P, DP, and VP to have base 0 and be valid in a neighborhood $|f|^{\mathcal{F}}_0 < \delta$. Thus, we impose

$$
\begin{aligned}
& s_{-1}+p-q \leq 0, && && \text{(B.73)}\\
& b_0 - q \leq 0, && a_0 - p \leq 0, && \text{(B.74)}\\
& b_1 - q \leq 0, && a_1 - p \leq 0, && \text{(B.75)}\\
& b_{-1} - p \leq 0, && a_{-1} - p \leq 0, && \text{(B.76)}\\
& b_2 - q \leq 0, && a_2 - p \leq 0 && \text{(B.77)}
\end{aligned}
$$

since, as mentioned earlier, the base can be increased and neighborhoods shrunk. These conditions can be satisfied by choosing p sufficiently large first, and then q sufficiently large. Note that then $b_{-1}+s_{-1}-q\leq 0$. In turn,

$$|P(f)|^{\mathcal{G}}_m \leq C(1+|f|^{\mathcal{F}}_{m+r_0+q-p}), \tag{B.78}$$

$$m\geq 0, \qquad |f|^{\mathcal{F}}_0<\delta \tag{B.79}$$

$$|DP(f)\cdot h|^{\mathcal{G}}_m \leq C(|h|^{\mathcal{F}}_{m+s_1+q-p}+|f|^{\mathcal{F}}_{m+r_1+q-p}|h|^{\mathcal{F}}_{s_1+q-p}), \tag{B.80}$$

$$m\geq 0, \qquad |f|^{\mathcal{F}}_0<\delta \tag{B.81}$$

$$|VP(f)\cdot k|^{\mathcal{F}}_l \leq C(|k|^{\mathcal{G}}_l+|f|^{\mathcal{F}}_{l+r_{-1}}|k|^{\mathcal{G}}_0), \tag{B.82}$$

$$l\geq 0, \qquad |f|^{\mathcal{F}}_0<\delta \tag{B.83}$$

These inequalities show that:

P	has degree in	f	equal to	r_0+q-p
DP		f		r_1+q-p
		h		s_1+q-p
VP		f		r_1
		k		0
D^2P		f		r_2+q-p
		h		s_2+q-p

Increasing degrees if necessary, we may assume that these maps have degree $2r$ in f and r in h for some $r>0$. (The reason to choose a degree for f double that for h is that $D^2P(f)\cdot(h_1,h_2)$ is quadratic in h_1 and h_2. More specifically, the need for these choices can be seen when deriving estimates for l_t, which involves D^2P via Taylor's formula, interpolation inequalities and the properties of the smoothing operators (B.22). See Lemma 59, Section B.3.6.) □

B.3.3 Equivalent formulation

In this section we translate the problem from a map $P\colon C^\infty_{\mathbb{S}^1}\to C^\infty_{\mathbb{S}^1}$ into a map on the Fourier modes.

Any function in $C^\infty_{(\mathbb{S}^1)}$ can be decomposed into its Fourier modes:

$$f=\sum_{j\in\mathbb{Z}} f_j e^{ijx}. \tag{B.84}$$

Since f is $\mathbb{R}$-valued, $f_{-j} = f_j^*$ (complex conjugation) and so f is completely determined by $\{f_j\}_{j\in\mathbb{N}}$. It is standard that the norm in $H^{n,2}_{(\mathbb{S}^1)}$ is equivalent to

$$\|f\|^2_{H^{n,2}} := \sum_{j\in\mathbb{Z}}(1+|j|)^{2n}|f_j|^2. \tag{B.85}$$

These (countably many) norms give $C^\infty_{(\mathbb{S}^1)}$ the structure of a Fréchet space.

Next we exhibit a tame equivalent grading. The n-th derivative is given by

$$f^{(n)} = \sum_{j\in\mathbb{Z}}(ij)^n f_j e^{ijx} \tag{B.86}$$

and thus

$$\|f\|_{C^n} \leq C\sum_{j\in\mathbb{N}}(1+j)^n|f_j| \qquad (C = C(n)). \tag{B.87}$$

We set (overriding notation from the main body of this thesis)

$$\|f\|_n := \sum_{j\in\mathbb{N}}(1+j)^n|f_j|. \tag{B.88}$$

By Hölder's inequality,

$$\begin{aligned}
\|f\|_n &= \sum_{j\in\mathbb{N}}\frac{1}{1+j}(1+j)^{n+1}|f_j| && \text{(B.89)}\\
&\leq \sqrt{\sum_{j\in\mathbb{N}}\frac{1}{(1+j)^2}}\sqrt{\sum_j(1+j)^{2(n+1)}|f_j|^2} && \text{(B.90)}\\
&\leq C\|f\|_{H^{n+1,2}}. && \text{(B.91)}
\end{aligned}$$

Finally, since $\mathbb{S}^1$ is bounded,

$$\|f\|_{H^{n,2}} \leq C\|f\|_{C^{n,2}} \qquad (C = C(n)) \tag{B.92}$$

so that

$$C_n\|f\|_{C^n} \leq \|f\|_n \leq C_n'\|f\|_{C^{n+1}} \tag{B.93}$$

which shows that the $\|\cdot\|_n$-grading is tame equivalent to the C^n-grading.

Instead of working with functions in $C^\infty_{(\mathbb{S}^1)}$, we will work with sequences with sufficiently fast decay by identifying a function $f \in C^\infty_{\mathbb{S}^1}$ with (the sufficient part of) its Fourier coefficients $\{f_j\}_{j\in\mathbb{N}}$. Namely, $f \in C^\infty_{\mathbb{S}^1}$ if and only if $\{f_j\}_{j\in\mathbb{N}}$ satisfies (with some abuse of notation) $\|f\|_n < \infty$ for all n. Drawing on Hamilton's notation in [14], we denote the space of such sequences

$$l_1^\infty(\mathbb{N}, \delta_j, w_j), \qquad w_j := \ln(1+j) \tag{B.94}$$

and δ_j is the delta function. The (semi)norms $\|\cdot\|_n$ give this space the structure of Fréchet space which is tame equivalent to $C^\infty_{\mathbb{S}^1}$. (See [14] for the precise definition. Essentially this means that the proof of the Nash-Moser theorem can be equivalently carried out on $l_1^\infty(\mathbb{N}, \delta_j, w_j)$ instead of $C^\infty_{\mathbb{S}^1}$.) Interpreting $\mu = \delta_j$ as counting measure on $\mathbb{N}$, we may write

$$\|f\|_n = \int_{\mathbb{N}} e^{nw}|f|d\mu \tag{B.95}$$

which notation is as in Lemma II.1.3.5, p. 136 of [14]. In conclusion, we have the tame linear isomorphism

$$\iota\colon f \in C^\infty_{(\mathbb{S}^1)} \mapsto \{f_j\}_j \in l_1^\infty(\mathbb{N}, \delta_j, w_j). \tag{B.96}$$

In turn we work on the map $\iota \circ P \circ \iota^{-1}$ which is tame if P is. With some abuse of notation, we will write this map as P again:

$$P\colon (\mathcal{V} \subset l_1^\infty(\mathbb{N}, \delta_j, w_j)) \to l_1^\infty(\mathbb{N}, \delta_j, w_j) \tag{B.97}$$

defined on an open subset of $\mathcal{V}$.

B.3.4 Smoothing operators

We construct a first family of smoothing operators denoted T_σ. For this, let s be a smooth function on $\mathbb{R}$ such that $s(u) = 0$ for $u \le 0$, $s(u) = 1$ for $u \ge 1$, and $0 \le s(u) \le 1$ in between. Define

$$T_\sigma\colon l_1^\infty(\mathbb{N}, \delta_j, w_j) \to l_1^\infty(\mathbb{N}, \delta_j, w_j), \qquad (T_\sigma f)_j = s(\sigma - j)f_j, \qquad j \in \mathbb{N}. \tag{B.98}$$

Note that

$$0 \le |(T_\sigma f)_j| \le |f_j| \quad \text{for} \quad 0 \le j \le \sigma, \qquad 0 \le |f_j - (T_\sigma f)_j| \le |f_j| \quad \text{for} \quad \sigma \le j \tag{B.99}$$

Let $d \in \mathbb{N}$ and estimate

$$\begin{aligned}
\|(T_\sigma f)^{(d)}\|_n &= \sum_{j\in\mathbb{N}} j^d(1+j)^n |(T_\sigma f)_j| && \text{(B.100)}\\
&\le \sum_{j=0}^{\sigma} j^d(1+j)^n |f_j| && \text{(B.101)}\\
&\le \sigma^d \sum_{j\in\mathbb{N}} (1+j)^n |f_j| && \text{(B.102)}\\
&= \sigma^d \|f\|_n && \text{(B.103)}
\end{aligned}$$

Similarly, for $\sigma \ge 1$,

$$\|T_\sigma f\|_n \le \sum_{j=0}^{\sigma}(1+j)^n|f_j| \le \frac{1}{\sigma^d}\sum_{j\in\mathbb{N}} j^d(1+j)^n|f_j| = \frac{1}{\sigma^d}\|f^{(d)}\|_n \tag{B.104}$$

These inequalities imply

$$\|T_\sigma f\|_n \le C\sigma^{n-m}\|f\|_m \qquad (\sigma \ge 1) \tag{B.105}$$

for a constant depending only on $m, n \in \mathbb{N}$.

Likewise,

$$\|(f - T_\sigma f)^{(d)}\|_m \le \sum_{j=\lfloor\sigma\rfloor}^{\infty} j^d(1+j)^m|f_j| \le \sigma^d \sum_{j\in\mathbb{N}}(1+j)^m|f_j| = \sigma^d\|f\|_m \tag{B.106}$$

and

$$\|f - T_\sigma f\|_m \le \sum_{j=\lfloor\sigma\rfloor}^{\infty}(1+j)^m|f_j| \le \sigma^d \sum_{j=\sigma*}\frac{1}{j^d}j^d(1+j)^m|f_j| = \frac{1}{\sigma^d}\|f^{(d)}\|_m \tag{B.107}$$

and we have more generally

$$\|T_\sigma f\|_m \le C\sigma^{m-n}\|f\|_n \qquad (\sigma \ge 1) \tag{B.108}$$

for a constant depending only on $m, n \in \mathbb{N}$.

The smoothing operators that will be used in (B.11) are defined by

$$S_t = T_{e^t}. \tag{B.109}$$

B.3.5 Solvability of $\dot{f}_t = cVP(S_t f_t) \cdot S_t(g - P(f_t))$

One must naturally verify that the ODE (B.11) has a solution at all, even for short time.

Lemma 53 *There exists $\epsilon > 0$ such that a solution to*

$$\dot{f}_t = cVP(S_t f_t) \cdot (g - P(f_t)) \tag{B.110}$$

exists for $0 \leq t \leq \epsilon$ and for any initial data f_0 in the neighborhood $\|f_0\|_{2r} < 1$.

Proof For clarity, we distinguish the domain and target spaces by denoting them $\mathcal{F}$ and $\mathcal{G}$ respectively. Write the ODE as

$$\dot{f}_t = V(t, f_t, g), \qquad V(t, f, g) := cVP(S_t f) \cdot S_t(g - P(f)) \tag{B.111}$$

and the smooth map V takes an open set in $\mathbb{R} \times \mathcal{F} \times \mathcal{G}$ into $\mathcal{F}$. Over an interval $[0, T]$ for some fixed (finite) $T > 0$, V coincides with a map $[0, T] \times \mathcal{F}_T \times \mathcal{G}_T \to \mathcal{F}_T$ on finite-dimensional spaces corresponding to the Fourier modes that have not been suppressed by S_t for $t \in [0, T]$. Namely,

$$\mathcal{F}_T := \{f \in \mathcal{F} \mid f_k = 0, |k| \geq e^T\}, \qquad \mathcal{G}_T := \{g \in \mathcal{G} \mid g_k = 0, |k| \geq e^T\}. \tag{B.112}$$

Thus, a unique solution exists on a subinterval of $[0, T]$. Furthermore, the solution to this finite-dimensional system depends smoothly on t and the initial data. Returning to the original system (B.11), there exists $\epsilon > 0$ such that a solution exists for $0 \leq t \leq \epsilon$ and for any initial data f_0 in the neighborhood $\|f_0\|_{2r} < 1$. (See also p. 180, and Theorem I.5.6.2, p. 129, [14].) ∎

B.3.6 Proof

As a preliminary to the proof of the Nash-Moser theorem, we establish the expression (B.20) for k_t using the integrating factor for

$$\dot{k}_t + cS_t k_t = l_t \tag{B.113}$$

and *a priori* estimates for k_t. For simplicity, we assume that the maps $P(f)$, $DP(f)\cdot h$, $VP(f)\cdot k$, and $D^2P(f)\cdot(h_1,h_2)$ have degrees $2r$ in f, r in h, 0 in k, and base 0, directly in the $\|\cdot\|_n$-grading (see (B.88)), i.e. without normalizations.

We follow [14] very closely, although we work in the space $l_1^\infty(\mathbb{N},\delta_j,w_j)$ while Hamilton embeds this space into a model space of the form $\Sigma(l_1(\mathbb{N},\delta_j))$.

The integrating factor is

$$a_{t,j} = \exp(c\int_0^t s(e^\tau - j)d\tau). \tag{B.114}$$

It is immediate that

$$\dot{a}_{t,j} = cs(e^t - j)a_{t,j}, \qquad \dot{a}_t = cS_t a_t \tag{B.115}$$

and that

$$\frac{d}{dt}(a_t k_t) = a_t l_t, \qquad a_t k_t = a_0 k_0 + \int_{\theta=0}^t a_\theta l_\theta d\theta. \tag{B.116}$$

Setting

$$a_{\theta,t,j} := \frac{a_{\theta,j}}{a_{t,j}} = \exp(-c\int_\theta^t s(e^\tau - j)d\tau) \tag{B.117}$$

this proves

Lemma 54 *If $\dot{k}_t + cS_t k_t = l_t$, then*

$$k_t = a_{0,t}k_0 + \int_{\theta=0}^t a_{\theta,t} l_\theta d\theta \tag{B.118}$$

which is (B.20).

Lemma 55 *For all θ, t, and j, we have*

$$0 \le a_{\theta,t,j} \le 1, \qquad e^{ct}a_{\theta,t,j} \le e^{c\theta} + (1+j)^c. \tag{B.119}$$

Proof The first inequality is obvious by definition of $a_{\theta,t,j}$. If $\theta \ge \ln(1+j)$, then $s(e^\tau - j) = 1$ for all $\tau \ge \theta$ and $a_{\theta,t,j} = e^{-c(t-\theta)}$. If $\theta \le \ln(1+j)$, then

$$a_{\theta,t,j} = \exp(-c\int_\theta^t s(e^\tau - j)d\tau) \le \exp(-c\int_{\ln(1+j)}^t s(e^\tau - j)d\tau) = e^{-ct}(1+j)^c. \tag{B.120}$$

■

Lemma 56 *If $0 < q < c$, then*

$$\int_{t=\theta}^{\infty} e^{qt} a_{\theta,t,j} dt \leq C(e^{q\theta} + (1+j)^q). \tag{B.121}$$

Proof If $e^\theta \geq 1+j$, then $a_{\theta,t,j} \leq 2e^{c(\theta-t)}$ hence

$$\begin{aligned}
\int_{t=\theta}^{\infty} e^{qt} a_{\theta,t,j} dt &\leq 2\int_{t=\theta}^{\infty} e^{qt} e^{c(\theta-t)} dt & \text{(B.122)}\\
&= 2e^{c\theta} \int_{t=\theta}^{\infty} e^{(q-c)t} dt & \text{(B.123)}\\
&= \frac{2}{c-q} e^{q\theta}. & \text{(B.124)}
\end{aligned}$$

If $e^\theta \leq 1+j$, break the integral in two:

$$\int_{t=\theta}^{\ln(1+j)} e^{qt} a_{\theta,t,j} dt \leq \int_{-\infty}^{\ln(1+j)} e^{qt} dt = \frac{1}{q}(1+j)^c \tag{B.125}$$

and

$$\int_{t=\ln(1+j)}^{\infty} e^{qt} a_{\theta,t,j} dt \leq 2\int_{t=\ln(1+j)}^{\infty} e^{qt} e^{-ct} (1+j)^c dt = \frac{2}{q(c-q)} (1+j)^q. \tag{B.126}$$

■

Lemma 57 (Lemma III.1.4.6, p. 178, [14]) *If $\dot{k}_t + cS_t k_t = l_t$ for $0 \leq t \leq T$, then for all $p \geq 0$ and $0 < q < c$*

$$\int_0^T e^{qt} \|k_t\|_p dt \leq C\|k_0\|_{p+q} + C\int_0^T e^{qt} \|l_t\|_p + \|l_t\|_{p+q} dt \tag{B.127}$$

where the constants depend on q and c, but not on T.

Proof From the expression (B.20) and the definition of the (semi)norms $\|\cdot\|_n$ (which are 0 if $n < 0$), we have

$$\int_0^T e^{qt} \|k_t\|_p dt = \int_0^T \sum_{j\in\mathbb{N}} e^{qt} (1+j)^p |k_{k,j}| dt \tag{B.128}$$

$$\leq \int_{t=0}^{T} \sum_{j\in\mathbb{N}} e^{qt} (1+j)^p a_{0,t,j} |k_{0,j}| dt + \int_{t=0}^{T} \sum_{j\in\mathbb{N}} e^{qt} (1+j)^p \int_{\theta=0}^{t} a_{\theta,t,j} |l_{\theta,j}| d\theta dt \tag{B.129}$$

Using Lemma 56 and that $0 \leq a_{\theta,t,j} \leq 1$, the first term is bounded by

$$\sum_{j\in\mathbb{N}}(1+j)^p \int_{t=0}^{T} e^{qt} a_{0,t,j}dt|k_{0,j}| \leq C\sum_{j\in\mathbb{N}}(1+j)^{p+q}|k_{0,j}| = C\|k_0\|_{p+q}. \tag{B.130}$$

Interchanging the integrals and with Lemma 56, the second term gives

$$\begin{aligned} & \int_{\theta=0}^{T}\sum_{j\in\mathbb{N}}(1+j)^p \int_{t=\theta}^{T} e^{qt} a_{\theta,t,j}dt|l_{\theta,j}d\theta & \text{(B.131)}\\ \leq\ & C\int_{\theta=0}^{T}\sum_{j\in\mathbb{N}}(1+j)^p(e^{q\theta}+(1+j)^q)|l_{\theta,j}|d\theta & \text{(B.132)}\\ \leq\ & C\int_{\theta=0}^{T} e^{q\theta}\|l_\theta\|_p + \|l_\theta\|_{p+q}d\theta. & \text{(B.133)}\end{aligned}$$

■

The rest of the proof that P is surjective is literally the same as in [14], albeit with the difference that the ODE in [14] is in the model space $\Sigma(l_1(\mathbb{N},\delta_j))$, different from our space $l_1^\infty(\mathbb{N},\delta_j,w_j)$. We reproduce it (with only editorial differences, and at times additional explicit justifications) for convenience of the reader, with the exception of the one simplifying assumption (already enforced) that the maps $P(f)$, $DP(f)\cdot h$, $VP(f)\cdot k$, and $D^2P(f)\cdot(h_1,h_2)$ have degrees $2r$ in f, r in h, 0 in k, and base 0, directly in the $\|\cdot\|_n$-grading.

Assume now that $P(0)=0$, and let f_t be the solution to (B.11) (with $f_0=0$) which exists for $t\in[0,\omega)$ where ω is the maximal time of existence. We summarize the tame estimates:

$$\begin{aligned} \|P(f)\|_n &\leq C\|f\|_{n+2r} & \text{(B.134)}\\ \|DP(f)\cdot h\|_n &\leq C(\|h\|_{n+r}+\|f\|_{n+2r}\|h\|_r) & \text{(B.135)}\\ \|D^2P(f)\cdot(h_1,h_2)\|_n &\leq C(\|h_1\|_{n+r}\|h_2\|_r+\|h_1\|_r\|h_2\|_{n+r}+\|f\|_{n+2r}\|h_1\|_r\|h_2\|_r) & \text{(B.136)}\\ \|VP(f)\cdot k\|_n &\leq C(\|k\|_n+\|f\|_{n+2r}\|k\|_0) & \text{(B.137)}\end{aligned}$$

for $\|f\|_{2r}<1$, any h and k (without restriction) and all $n\geq 0$. The first estimate is obtained by integrating the second:

$$P(f) = P(0) + \int_0^1 DP(tf)\cdot f dt \tag{B.138}$$

which gives

$$\|P(f)\|_n \leq C(\|f\|_{n+r} + \|f\|_{n+2r}\|f\|_r) \leq C\|f\|_{n+2r} \tag{B.139}$$

since $\|f\|_{2r} < 1$.

Lemma 58 (Lemma III.1.6.1, p. 180, [14]) *For all $n \geq 0$ and $q \geq 0$,*

$$\|\dot{f}_t\|_{n+q} \leq Ce^{qt}(\|k_t\|_n + \|f_t\|_{n+2r}\|k_t\|_0) \tag{B.140}$$

while $\|f_t\|_{2r} < 1$.

Proof Use tame estimates (B.137) on VP and properties (B.22) of the smoothing operators to find

$$\|\dot{f}_t\|_{n+q} \leq C\|S_t k_t\|_{n+q} + C\|S_t f_t\|_{n+q+2r}\|S_t k_t\|_0 \tag{B.141}$$

which proves the lemma. ■

Lemma 59 (Lemma III.1.6.2, p. 180, [14]) *For all $n \geq 0$,*

$$\|l_t\|_n \leq C\|f_t\|_{n+2r}\|k_t\|_0 \tag{B.142}$$

while $\|f_t\|_{2r} < 1$.

Proof Write $l_t = (DP(S_t f_t) - DP(f_t)) \cdot \dot{f}_t$ as the integral

$$l_t = -\int_{\theta=0}^{1} D^2P((1-\theta)f_t + \theta S_t f_t) \cdot ((f_t - S_t f_t), \dot{f}_t) d\theta. \tag{B.143}$$

Tame estimates (B.136) give

$$\begin{aligned} \|l_t\|_n \quad \leq \quad & C\Big(\|(\mathrm{Id} - S_t)f_t\|_{n+r}\|\dot{f}_t\|_r + \|(\mathrm{Id} - S_t)f_t\|_r\|\dot{f}_t\|_{n+r} && \text{(B.144)} \\ & +(\|f_t\|_{n+2r} + \|S_t f_t\|_{n+2r})\|(\mathrm{Id} - S_t)f_t\|_r\|\dot{f}_t\|_r\Big). && \text{(B.145)} \end{aligned}$$

Use (B.22) to get

$$\begin{aligned} \|(\mathrm{Id} - S_t)f_t\|_{n+r} \quad &\leq \quad Ce^{-rt}\|f_t\|_{n+2r} && \text{(B.146)} \\ \|(\mathrm{Id} - S_t)f_t\|_r \quad &\leq \quad Ce^{-(n+r)t}\|f_t\|_{n+2r} && \text{(B.147)} \\ \|S_t f_t\|_{n+2r} \quad &\leq \quad C\|f_t\|_{n+2r} && \text{(B.148)} \\ \|(\mathrm{Id} - S_t)f_t\|_r \quad &\leq \quad Ce^{-rt}\|f_t\|_{2r} && \text{(B.149)} \end{aligned}$$

and, since $\|f_t\|_{2r} < 1$, Lemma 58 gives

$$\begin{aligned} \|\dot{f}_t\|_r &\leq Ce^{rt}\|k_t\|_0 && \text{(B.150)} \\ \|\dot{f}_t\|_{n+r} &\leq Ce^{(n+r)t}\|k_t\|_0. && \text{(B.151)} \end{aligned}$$

Putting these together proves the Lemma.

Note that these derivations explain the choice of degree r for h and $2r$ for f. ■

The following lemmas show that f_t remains in the neighborhood where solutions to (B.5) can be extended: $\|f_t\|_{2r} < 1$ for $t \in [0, \omega)$.

Lemma 60 (Lemma III.1.6.3, p. 181, [14]) *For any $T \in [0, \omega)$ and such that $\|f_t\|_{2r} < 1$ for $t \in [0, T]$,*

$$\int_0^T e^{2rt}\|k_t\|_0 dt \leq C\|g\|_{2r} + \left(\int_0^T e^{2rt}\|k_t\|_0 dt\right)^2 \tag{B.152}$$

where the constant C is independent of T.

Proof Choosing

$$c > 2r \tag{B.153}$$

in estimates of Lemma 57 (Section B.3.6) for $\dot{k}_t + cS_t k_t = l_t$, we find

$$\int_0^T e^{2rt}\|k_t\|_0 dt \leq C\|g\|_{2r} + C\int_0^T e^{2rt}\|l_t\|_0 + \|l_t\|_{2r} dt. \tag{B.154}$$

Estimates on $\dot{f}_t$ from Lemma 58 (with $n = 0$ and $q = 2r$, then $n = 0$ and $q = 4r$) give

$$\|\dot{f}_t\|_{2r} \leq Ce^{2rt}\|k_t\|_0, \qquad \|\dot{f}_t\|_{4r} \leq Ce^{4rt}\|k_t\|_0 \tag{B.155}$$

and integrating

$$\begin{aligned} \|f_t\|_{2r} &\leq C\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta && \text{(B.156)} \\ \|f_t\|_{4r} &\leq C\int_0^t e^{4r\theta}\|k_\theta\|_0 d\theta \leq Ce^{2rt}\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta. && \text{(B.157)} \end{aligned}$$

Estimate for l_t from Lemma 59 give

$$\|l_t\|_0 \leq C\|f_t\|_{2r}\|k_t\|_0, \qquad \|l_t\|_{2r} \leq C\|f_t\|_{4r}\|k_t\|_0. \tag{B.158}$$

Now

$$e^{2rt}\|l_t\|_0 + \|l_t\|_{2r} \leq Ce^{2rt}\|f_t\|_{2r}\|k_t\|_0 + C\|f_t\|_{4r}\|k_t\|_0 \tag{B.159}$$

$$\leq Ce^{2rt}\int_{\theta=0}^{t} e^{2r\theta}\|k_\theta\|_0 d\theta\|k_t\|_0 + \text{same}. \tag{B.160}$$

Taking $\int_0^T$ instead of $\int_0^t$ on the right-hand side, then integrating over $t \in [0, T]$, we find the desired quadratic inequality in $\int_0^T e^{2rt}\|k_t\|_0 dt$. ■

Lemma 61 *If $\|g\|_{2r} \leq \delta$ for δ sufficiently small, then $\|f_t\|_{2r} < 1$ for all $t \in [0, \omega)$, and there exists a constant independent of t such that*

$$\int_0^t e^{2rt}\|k_t\|_0 dt \leq C\|g\|_{2r}. \tag{B.161}$$

Proof For all $T \in [0, \omega)$ such that $\|f_t\|_{2r} < 1$ for $t \in [0, T]$, set

$$K_T := \int_0^T e^{2rt}\|k_t\|_0 dt. \tag{B.162}$$

K_T satisfies

$$K_t \leq C_1\|g\|_{2r} + C_1 K_T^2 \tag{B.163}$$

where we mark the constant explicitly. A moment's concentration easily convinces one that

$$\text{either} \qquad K_T \leq 2C_1\|g\|_{2r} \qquad \text{or} \qquad K_T \geq \frac{1}{2C_1}. \tag{B.164}$$

Since K_T is continuous in T and $K_0 = 0$, choosing

$$\delta < \frac{1}{4C_1^2} \tag{B.165}$$

we must have

$$K_T \leq 2C_1\|g\|_{2r} \leq 2C_1\delta \tag{B.166}$$

for all T such that $\|f_t\|_{2r} < 1$ for $t \in [0, T]$.

For any $t \in [0, T]$ with T as above,

$$\|f_t\|_{2r} \leq \int_0^t \|\dot{f}_\theta\|_{2r} d\theta \leq C_2 \int_0^T e^{2r\theta}\|k_\theta\|_0 d\theta = C_2 K_T \leq 2C_1C_2\delta \tag{B.167}$$

(where C_2 is the constant from Lemma 58) so that making δ smaller if necessary, this implies that in fact $\|f_t\|_{2r} < 1$ as long as the solution exists. ■

In the sequel δ will always be sufficiently small so that Lemma 61 is valid and we will assume that

$$\|g\|_{2r} < \delta. \tag{B.168}$$

Lemma 62 (Lemma III.1.6.7, p. 182, [14]) *Making δ sufficiently smaller, and with $\|g\|_{2r} < \delta$, we have: for any $T \in [0, \omega)$,*

$$\int_0^T e^{(2r+1)t}\|k_t\|_0 dt \leq C\|g\|_{2r+1}. \tag{B.169}$$

Proof Choose

$$c > 2r + 1 \tag{B.170}$$

in Lemma 57 of Section B.3.6 and let $p = 0$ and $q = 2r + 1$ to obtain

$$\int_0^T e^{(2r+1)t}\|k_t\|_0 dt \leq C\|g\|_{2r+1} + C\int_0^T e^{(2r+1)t}\|l_t\|_0 + \|l_t\|_{2r+1} dt. \tag{B.171}$$

Letting $n = 0$ and $q = 2r$, then $n = 0$ and $q = 4r + 1$ in Lemma 58,

$$\|\dot{f}_t\|_{2r} \leq Ce^{2rt}\|k_t\|_0, \qquad \|\dot{f}_t\|_{4r+1} \leq Ce^{(4r+1)t}\|k_t\|_0 \tag{B.172}$$

and from the estimate of Lemma 61,

$$\begin{aligned}
\|f_t\|_{2r} &\leq C\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta \leq C\|g\|_{2r} && \text{(B.173)}\\
\|f_t\|_{4r+1} &\leq C\int_0^t e^{(4r+1)\theta}\|k_\theta\|_0 d\theta \leq Ce^{(2r+1)t}\int_0^t e^{2r\theta}\|k_\theta\|_0 d\theta && \text{(B.174)}\\
&\leq Ce^{(2r+1)t}\|g\|_{2r}. && \text{(B.175)}
\end{aligned}$$

Estimates for l_t from Lemma 59 give

$$\|l_t\|_0 \leq C\|f_t\|_{2r}\|k_t\|_0, \qquad \|l_t\|_{2r+1} \leq C\|f_t\|_{4r+1}\|k_t\|_0 \tag{B.176}$$

and in turn

$$e^{(2r+1)t}\|l_t\|_0 + \|l_t\|_{2r} \leq Ce^{(2r+1)t}\|g\|_{2r}\|k_t\|_0. \tag{B.177}$$

Integrating,

$$\int_0^T e^{(2r+1)t}\|k_t\|_0 \leq C_3\|g\|_{2r+1} + C_3\|g\|_{2r}\int_0^T e^{(2r+1)t}\|k_t\|_0 dt. \tag{B.178}$$

Make δ smaller if necessary so that

$$C_3\delta < 1. \tag{B.179}$$

Then, the last term on the right-hand side can be subtracted from the left-hand side and the Lemma is proved. ■

As explained earlier, the proof of the surjective part of the Nash-Moser theorem hinges on the estimates (B.13). These are established by induction, for which we will need the more general estimates of the following Lemma.

Lemma 63 (Theorem III.1.6.8, p. 183, [14]) *Let δ be as in Lemma 61 and satisfying (B.179). Then there exist constants $C = C(n, q)$ such that for all $n \geq 2r$, all $q \geq 0$, and all $T \in [0, \omega)$, we have*

$$\int_0^T \|\dot{f}_t\|_{n+q} dt \leq Ce^{qT}\|g\|_n. \tag{B.180}$$

Proof The proof is an induction on $n \geq 2r$ or, writing $n = p + 2r$, on p. We let $q \geq 0$ be arbitrary. The case $p = 1$ is verified by Lemma 58 with $n = 0$ and replacing q by $1 + 2r + q$

$$\|\dot{f}_t\|_{1+2r+q} \leq Ce^{(1+2r+q)t}\|k_t\|_0 \tag{B.181}$$

and from Lemma 62

$$\int_0^T \|\dot{f}_t\|_{1+2r+q} dt \leq Ce^{qT}\int_0^T e^{(2r+1)t}\|k_t\|_0 dt \leq Ce^{qT}\|g\|_{2r+1}. \tag{B.182}$$

Suppose then that the inequality (B.180) is satisfied up to some $n \geq 2r + 1$ (or $p \geq 1$) and for all $q \geq 0$. In particular, with $q = 0$ and $q = 2r + 1$,

$$\|f_t\|_{p+2r} \leq C\|g\|_{p+2r}, \qquad \|f_t\|_{p+4r+1} \leq Ce^{(2r+1)t}\|g\|_{p+2r}. \tag{B.183}$$

In turn, estimates on l_t from Lemma 59 give

$$e^{(2r+1)t}\|l_t\|_p + \|l_t\|_{p+1+2r} \leq Ce^{(2r+1)t}\|g\|_{p+2r}\|k_t\|_0 \tag{B.184}$$

and choosing

$$c > 2r + 1 \tag{B.185}$$

and $q = 2r + 1$ in the estimates form $\dot{k}_t + cS_t k_t = l_t$ from Lemma 57 of Section B.3.6,

$$\int_0^T e^{(2r+1)t} \|k_t\|_p dt \leq C\|g\|_{p+1+2r} + C \int_0^T e^{(2r+1)t} \|g\|_{p+2r} \|k_t\|_0 dt. \tag{B.186}$$

Now Lemma 62 gives

$$\int_0^T e^{(2r+1)t} \|k_t\|_p dt \leq C\|g\|_{p+1+2r} + C\|g\|_{p+2r} \|g\|_{2r+1}. \tag{B.187}$$

Take $n = p$ and replace q by $2r + 1 + q$ in Lemma 58,

$$\|\dot{f}_t\| p + 2r + 1 + q \leq C e^{(q+2r+1)t} (\|k_t\|_p + \|f_t\|_{p+2r} \|k_t\|_0). \tag{B.188}$$

Now as just seen in this proof, $\|f_t\|_{p+2r} \leq C\|g\|_{p+2r}$, and using again Lemma 62, we have, after integration,

$$\begin{aligned}
\int_0^T \|\dot{f}_t\|_{p+2r+1+q} dt &\leq C e^{qT} (\|g\|_{p+2r+1} + \|g\|_{p+2r} \|g\|_{2r+1}) && \text{(B.189)} \\
&\leq C e^{qT} (\|g\|_{p+2r+1} + \|g\|_{p+2r+1} \|g\|_{2r}) && \text{(B.190)} \\
&\leq C e^{qT} \|g\|_{p+2r+1} && \text{(B.191)}
\end{aligned}$$

by interpolation inequalities and since $\|g\|_{2r} \leq \delta$. ■

Corollary 64 *If $\|g\|_{2r}$ is sufficiently small, then there exists f such that and $P(f) = g$.*

Proof Take $\|g\|_{2r} < \delta$ where δ is as in Lemma 61 and satisfies (B.179). Then the ODE (B.11) has a solution f_t, $t \in [0, \omega)$, such that $\|f_t\|_{2r} < 1$ and the estimates $\int_0^T \|\dot{f}_t\|_n dt \leq C\|g\|_{2n}$ discussed in Section B.3.1 are precisely those of (63) with $q = 0$. ■