# Generalized Sines, Multiway Curvatures, and the Multiscale Geometry of $d$-Regular Measures

A Dissertation

Submitted to The Faculty of the Graduate School

of the

University of Minnesota

by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

Advised by Professor Gilad Lerman

July 2009

## Acknowledgements

Many thanks to my advisor Gilad Lerman, especially for his clear vision of how this work was to proceed and his confidence in my ability to achieve it. I am fortunate to have had him as an advisor and a friend. Thanks also to the various faculty I've met at the U of M, such as Professors Paul Garrett and Dick Mcgehee, without whom I would definitely not have made it through graduate school. Special thanks also to Professors Sergey Bobkov, Peter Olver, and Guillermo Sapiro for being on my defense committee.

Thanks to my family for helping me to be in a situation to accomplish such a thing, my mother for her support, my uncle for his example, my aunts for the encouragement, and especially my wife for her patience. Great thanks are also due to Charles Royals, a benefactor without whom my trajectory in life would have been fundamentally different.

# Dedication

The work is dedicated to two people. The first is Leon Faure at the College of San Mateo who saw me as a mathematician long before I myself did. I have tried to emulate his clear thinking and common sense in both my research and teaching. The second is my son Martín, who I hope will one day at least read this dedication.

## Abstract

We define discrete Menger-type curvatures of $d+2$ points in a real separable Hilbert space $H$ by an appropriate scaling of the squared volume of the corresponding $(d+1)-$simplex. We then form a continuous curvature of an Ahlfors regular measure $\mu$ on $H$ by integrating the discrete curvature according to products of $\mu$ (or its restriction to balls). The essence of this work is estimating multiscale least squares approsimations of $\mu$ by the Menger-type curvature. We show that the continuous $d$-dimensional Menger-type curavture of $\mu$ is comparable to the "Jones-type flatness" of $\mu$. The latter quantity sums the scaled errors of approximations of $\mu$ by $d$-planes at different scales and locations, and is typically used to characterize the uniform rectifiability of $\mu$.

This work is divided into three basic parts, with the first part dealing with various geometric inequalities for the $d$-dimensional polar sine and hyper sine functions, which are higher-dimensional generalizations of the ordinary trigonometric sine function of an angle. The polar sine function is then used to formulate the Menger-type curvature in terms of a scaled volume. The second two parts use these geometric inequalities and their interaction with the geometry of $d$-regular measures to establish both an upper bound and a lower bound for the Menger-type curvature of $\mu$ restricted to a ball in terms of the Jones-type flatness of $\mu$ restricted to a ball. In addition to the Menger-type curvatures, we give a brief exploration of various other curvatures in the context of comparisons to the the Jones-type flatness and their use in the context of uniform rectifiability.

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# Introduction

### Notation

In this work, we fix a real, separable Hilbert, denoted by $H$, of inner product denoted by $\langle \cdot, \cdot \rangle$ and induced norm by $\| \cdot \|$. We denote its dimension (possibly infinite) by $\dim(H)$. We also denote intrinsic dimension of interest by $d \in \mathbb{N}$, and we reserve $d$ strictly for this purpose. Furthermore, we enforce the inequality $\dim(H) \geq d + 1$. We denote elements of $H$ by $u$, $v$ and $x$ among others, where these elements are often indexed by natural numbers (including 0), such as $v_0, \ldots, v_{d+1}$. We typically denote scalars with values at least 1 by upper-case plain letters, e.g., $C$; arbitrary integers by lower case letter, e.g., $i, j$ and large integers by $M$ and $N$; and arbitrary real numbers by lower-case Greek or lower-case letters, e.g., $\rho$, $r$.

We use the typical conventions of set notation, such as denoting the complement of a set $A \subseteq H$ by $A^c$, and the closed ball of center $x \in H$ and finite radius $t > 0$ by $B(x, t)$. At times we denote such a ball by $B$ for convenience, always assuming that the radius is *finite*. Given a closed ball $B$ of finite radius $t > 0$ and a constant $\gamma > 0$, then $\gamma \cdot B$ denotes the "blow up of $B$ by $\gamma$", that is a ball with the same center as $B$ and a radius $\gamma \cdot t > 0$. For a subset $A \subseteq H$, we denote its set diameter by $\operatorname{diam}(A)$. Also, for such a subset $A$ and $k \in \mathbb{N}$ we denote its $k$-fold cartesian product by $A^k \subseteq H^k$.

For a Borel measure $\mu$ on $H$, we denote its support (the smallest closed set of full measure) by $\operatorname{supp}(\mu)$. Given a fixed measure $\mu$, we say that the points $x \in \operatorname{supp}(\mu)$ and *finite* length scales $0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu))$ are the set of "reasonable locations and scales" for $\mu$. For a subset $A \subseteq H$, we denote the restriction of $\mu$ to $A$ by $\mu|_A$, and for $k \in \mathbb{N}$ we denote the $k$-fold product measure of $\mu$ on $H^k$ by $\mu^k$.

We say that a real-valued function $f$ is controlled by a real-valued function $g$, which we denote by $f \lessapprox g$, if there exists a positive constant $C$ such that $f \leq C \cdot g$. Similarly, we say that $f$ is comparable to $g$, denoted by $f \approx g$, if $f \lessapprox g$ and $g \lessapprox f$. The constants of control or comparability are *assumed to be independent* of the arguments of $f$ and $g$, which we make sure to indicate if unclear from the context.

If $f$ is defined on $H^k$ for $k \in \mathbb{N}$ and $k \geq 2$, then we denote the evaluation of $f$ on the ordered set of vectors $v_1, \ldots, v_{k+1} \in H$ with $v_j$ removed by $f(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k+1})$. We remark that we maintain this notation for all $1 \leq j \leq k$, in particular, $j = 1$ and $j = k+1$. Similarly, for $1 \leq j \leq k$, then $f(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_k)$ is $f$ evaluated on the ordered set of $k$ vectors $v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_k \in H$, where $v_j$ is replaced by $u$. We may remove two vectors, $v_i$ and $v_j$, from the ordered set $\{v_1, \ldots, v_{k+2}\}$ and denote the function $f$ evaluated on the resulting set by $f(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k+2})$, regardless of the order of $i$ and $j$ and whether or not either is 1 or $k+2$. In this case the convention is always that $i \neq j$.

## 1.1 Background

This work focuses on the geometry of a $d$-regular measure on $H$, which is simply a measure $\mu$ satisfying the following estimate for some $C \geq 1$

$$C^{-1} \cdot t^d \leq \mu(B(x,t)) \leq C \cdot t^d, \tag{1.1}$$

for all closed balls at all reasonable locations and scales. The smallest constant satisfying equation (1.1) is denoted by $C_\mu$, and is called the *regularity constant* of $\mu$. Such $d$-regular measures are intimately related to the $d$-dimensional Hausdorff measure on $H$, denoted by $\mathcal{H}_d$, the relationship being that $\mu \approx \mathcal{H}_d|_{\mathrm{supp}(\mu)}$ [9].

This work investigates quantitative relationships between some *continuous descriptions* of the "multiscale flatness" of $\mu$ and the Euclidean geometry of certain *discrete structures* sampled from $\mathrm{supp}(\mu)$.

### Multiscale Quantifications of Flatness

We first quantify how well $\mathrm{supp}(\mu)$ can be locally approximated by an affine $d$-dimensional subspace, and we use a least squares description called the Jones' $\beta_2$-number [10, 11, 23, 27]. This is simply a *normalized* minimal local least squares error of $\mu$ with respect to $d$-dimensional affine subspaces. Fixing $x \in H$ and $t > 0$, the formal definition is the following:

$$\beta_2(x,t) = \begin{cases} \inf\limits_{\text{Affine } L:\ \dim(L)=d} \sqrt{\displaystyle\int_{B(x,t)} \left(\frac{\mathrm{dist}(y,L)}{2 \cdot t}\right)^2 \frac{\mathrm{d}\mu(y)}{\mu(B(x,t))}}, & \text{if } \mu(B(x,t)) > 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The function $\beta_2(x,t)$ gives rise to a *multiscale* quantification of flatness called the *Jones'-type flatness*, which is basically just the cumulative error over a given set of locations and

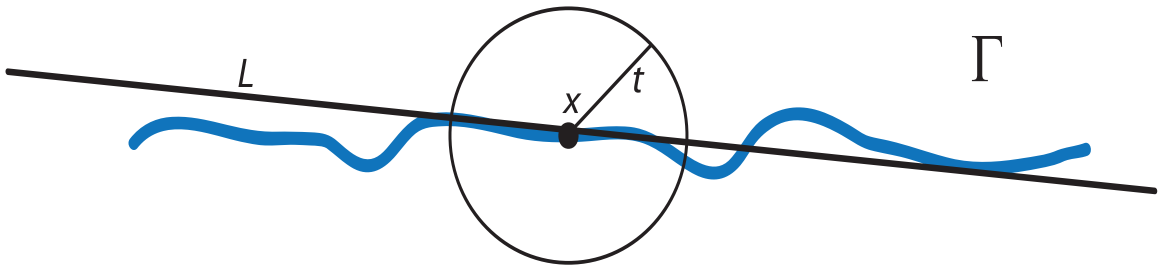


Figure 1.1: Taking $\mu$ to $\mathcal{H}_1$ restricted to a rectifiable curve $\Gamma$: Fix a location, $x$, a length scale, $t$, a line $L$, and then integrate the squared distance from $L$. The quantity $\beta_2(x,t)$ is obtained by normalizing appropriately, taking the square root, and finally taking the infimum over all such lines.

reasonable length scales. That is, if we fix a closed ball $B$, the formal definition of the Jones'-type flatness of $\mu$ restricted to $B$ is the following:

$$J_2(\mu|_B) = \int_0^{\mathrm{diam}(B)} \int_B \beta_2^2(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}. \tag{1.3}$$

The corresponding global version is given by

$$J_2(\mu) = \int_0^{\infty} \int_H \beta_2^2(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}. \tag{1.4}$$

The Jones'-type flatness is closely related to questions about the geometry of $\mathrm{supp}(\mu)$, among other things, deciding whether or not there exists a "sufficiently smooth" $d$-dimensional surface of full measure (see Subsection 1.4.2 below). In the 1-dimensional case, an example of such a relationship between the Jones'-type flatness and the existence of a "nice" parameterization is reflected in the following statement regarding the Analysts Traveling Salesman's Problem for 1-regular measures.

**Proposition 1.1.1.** *If $\mu$ is a 1-regular measure of bounded support, then there exists a rectifiable curve $\Gamma$ of full measure if and only if $J_2(\mu) < \infty$, and the length of a shortest* *such parameterizing curve* $\Gamma$ *is such that*

$$length(\Gamma) \approx \operatorname{diam}(\operatorname{supp}(\mu)) + J_2(\mu),$$

*where the comparison is independent of the ambient dimension.*

Proposition 1.1.1 is a corollary of a similar statement about sets rather than 1-regular measures. Peter Jones originally proved the more general statement for in the complex plane $\mathbb{C}$ using complex analysis [23], and this work was later extended to $\mathbb{R}^n$ and eventually arbitrary separable Hilbert spaces in [38, 41] via more geometric methods. While Proposition 1.1.1 shows that the global Jones'-type flatness of $\mu$ indicates the *rectifiability* of a global 1-dimensional parameterization of $\operatorname{supp}(\mu)$, the local quantification given by $J_2(\mu|_B)$ indicates the type of smoothness one can expect from such parameterizations, and this smoothness is formulated in terms of something called *uniform rectifiability* [11]. When $d = 1$, this condition can be characterized as the existence of a parameterizing *Ahlfors regular curve*, that is, a Lipshitz curve $\Gamma$ containing $\operatorname{supp}(\mu)$ such that for some $C \geq 1$, all $x \in \Gamma$, and all finite $r > 0$,

$$length(\Gamma \cap B(x, r)) \leq C \cdot r.$$

While there are many equivalent definitions, and at least a few uses for uniform rectifiability (see [10, 11, 34, 44]), one relatively non-technical characterization is in terms of a quantitative restriction on the notion of *d-rectifiability* in geometric measure theory [32]. Denoting the $d$-dimensional Hausdorff measure on $H$ by $\mathcal{H}_d$, a set $E \subseteq H$ is said to be $d$-rectifiable if $0 < \mathcal{H}_d(E) < \infty$ and there exists a countable collection of Lipshitz maps $f_i : \mathbb{R}^d \to H$ such that

$$\mathcal{H}_d\left(E \setminus \bigcup_i f_i(\mathbb{R}^d)\right) = 0.$$

In this framework, the uniform rectifiability of a $d$-regular $\mu$ is characterized in terms of uniform lower control at all reasonable scales and locations on *how much* of a closed ball is comprised of a given $d$-dimensional Lipshitz image $f(\mathbb{R}^d)$, where the Lipshitz norm of the underlying maps are assumed to be *uniformly bounded.* The set of uniformly rectifiable measures is a restricted class of $d$-regular measures, and it is a stronger condition which implies ordinary rectifiability [11]. A fundamental result connecting the Jones'-type flatness to uniform rectifiability is the following theorem.

**Theorem 1.1.1** (David and Semmes)**.** *A $d$-regular measure $\mu$ on $\mathbb{R}^n$ is uniformly rectifiable if and only if $J_2(\mu|_B) \lessapprox \mu(B)$ for all closed balls at all reasonable scales and locations, $B \subseteq H$.*

In Section 1.4.2 we give a characterization of uniform rectifiability clarifying the type of smoothness referred to above based on the existence of a special $d$-dimensional surface containing $\text{supp}(\mu)$. That is, uniform rectifiability can be characterized by the existence of a "sufficiently smooth" solution to the"$d$-dimensional Analysts Traveling Salesman's Problem".

### A Discrete Curvature for $d = 1$

Given the above framework we can begin the discussion of relating these *continuous* descriptions of flatness to the geometry of *discrete structures* sampled from $\text{supp}(\mu)$. In order to avoid needless abstractions at this point, we will first focus on how the geometry of *triangles* has already been extensively used to characterize/quantify various geometric and analytic properties of "1-dimensional" sets and measures in $H$. This 1-dimensional background will serve as a good springboard into the higher-dimensional versions that are the actual material of this work.

We begin by defining the sine of the angle between two vectors $v_1, v_2 \in H$, whereby we simply mean to make the dependence on $v_1$ and $v_2$ explicit by defining

$$\sin(v_1, v_2) := \sin(\theta),$$

for $\theta$ denoting the angle between $v_1$ and $v_2$ induced by the inner product on $H$. Indeed, viewing the sine function in such a way one might as well make it a function of *three vector arguments* by introducing $v_0 \in H$ (see Figure 1.2) and making the definition

$$\sin_{v_0}(v_1, v_2) := \sin(v_1 - v_0, v_2 - v_0).$$

Such a function quantifies the geometry of the triangle through the three vertices, with the subscript $v_0$ reflecting the quantification of the angular information at the vertex $v_0$. Clearly, given the the three points there are two other sine values at the vertices $v_1$ and $v_2$ respectively.

While the sine function at a vertex gives angular information about the triangle, by applying a simple normalization one obtains a *scale dependent* invariant of the triangle known as the Menger-curvature. That is, taking a triangle with three *non-collinear* vertices and normalizing the sine at a vertex by the length of the opposite side, we have the following invariant (see Figure 1.2)

$$c_M(v_0, v_1 v_2) := \frac{1}{R(v_0, v_1, v_2)} = \frac{2 \cdot \sin_{v_0}(v_1, v_2)}{\|v_1 - v_2\|}, \tag{1.5}$$

where $R(v_0, v_1, v_2)$ is the *radius* of the circum-circle through the vertices $v_0, v_1, v_2$. This is extended to collinear vertices, defining $c_M(v_0, v_1, v_2) = 0$ in this case.

The one-dimensional Menger curvature, $c_M$, quantifies a great deal of information about the geometric and analytic properties of sets and measures in Euclidean spaces. Specifically,

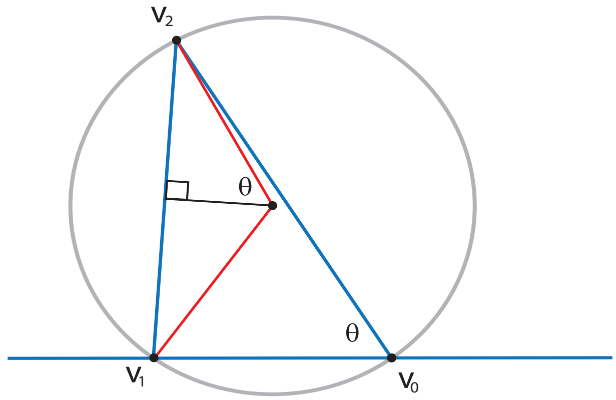


Figure 1.2: The ratio of equation (1.5) is independent of the vertex used to calculate it due to the law of sines. The red segment indicates the radius of the circle, and the blue segments the edges connecting the vertices.

for a locally finite Borel measure $\mu$ on $H$, we define the quantity

$$c_M^2(\mu) := \int_H \int_H \int_H c_M(v_0, v_1, v_2)^2 \, \mathrm{d}\mu(v_0) \, \mathrm{d}\mu(v_1) \, \mathrm{d}\mu(v_2),$$

as well as the corresponding local version for a closed ball $B \subseteq H$,

$$c_M^2(\mu|_B) := \int_B \int_B \int_B c_M(v_0, v_1, v_2)^2 \, \mathrm{d}\mu(v_0) \, \mathrm{d}\mu(v_1) \, \mathrm{d}\mu(v_2).$$

These two quantities are useful for answering some inter-related questions in both geometric measure theory and harmonic analysis, such as:

1. Is a given set $E \subseteq H$ 1-rectifiable with respect to $\mathcal{H}_1$? In [25], Léger showed that if $0 < \mathcal{H}_1(E) < \infty$ and $c_M^2(\mathcal{H}_1|_E) < \infty$, then $E$ is 1-rectifiable.

2. When is a 1-regular measure $\mu$ uniformly rectifiable? In [33], Mattila, Melnikov, and Verdera showed that that $\mu$ is uniformly rectifiable if and only if $c_M^2(\mu|_B) \lessapprox \mu(B)$. This inequality is often referred to as the *local curvature condition*. Furthermore, showed

that local curvature condition holds if and only if specific singular integral operator on $L_2(\mu)$, that is the Cauchy operator, is bounded. These results depended on some quantitative relationships between the Menger curvature, uniform rectifiability, and the Jones'-type flatness, all encompassed by unpublished results of Jones recorded in [39].

3. What is a metric/geometric characterization of removable sets for bounded analytic functions when $H = \mathbb{C}$ [2]? In [43], Tolsa solved the so called Painlevé problem by showing that a compact set $E \subseteq \mathbb{C}$ is *non-removable* for bounded analytic functions if and only if it supports a positive Radon measure $\mu$ with linear growth (that is, $\mu(B) \lessapprox \operatorname{diam}(B)$ for all balls, $B$) and $c_M^2(\mu) < \infty$.

The utility of this "1-dimensional" curvature for answering such geometric questions about sets and measures suggests that a higher-dimensional generalization might be useful. In fact, the 1-dimensional case also indicates the types of discrete objects we would like to sample from $\operatorname{supp}(\mu)$, that is, we "ought" to focus on quantifying the geometry of $(d+1)$-dimensional simplices using curvatures in order to quantify the "$d$-dimensional" geometry of $\mu$. By "curvature" we simply mean a function defined on the collection of $(d+2)$ vertices quantifying some aspect of the Euclidean/metric geometry of the simplex. For example, when $d = 1$ the Menger-curvature quantifies the geometry of a triangle by computing a quantity based on the Euclidean structure of the three vertices. When $d = 2$ then we would expect a curvature for 3-simplices, i.e., tetrahedra, based on the geometric structure of the 4 vertices. While a great deal of work has been done with the one-dimensional Menger-curvature, up until this thesis there was no suitable analogue for a higher-order curvature serving a similar function for the cases of $d \geq 2$. The reasons for this are three-fold.

One reason was work [18] showing that the so called "magic formula" of Melnikov [35] relating the Menger curvature to the $L_2$ Cauchy operator (leading to the work in [33]) could not possibly be replicated for the desired higher-dimensional singular integral operators and a non-negative algebraic curvature. As such, this created a great deal of pessimism about the utility of such curvatures for characterizing analytic properties of $d$-regular measures.

Another reason is that it was not entirely clear how to "correctly" generalize the Menger-curvature to higher-order simplices in such a way as to relate their geometry to the $d$-dimensional rectifiability of measures, both uniform and otherwise. Previously, a curvature of $(d+1)$-simplices was proposed by Léger for studying rectifiability of measures for $d \geq 1$ [25]. However, his analysis was done only for $d = 1$, and there was no proof given that his curvature actually had the stated relationship to rectifiability properties when $d > 1$. Furthermore, when one tries to work with such a curvature for $d > 1$, one finds that it is related to a geometric property weaker than uniform rectifiability (see Subsections 3.5.2 and 3.5.3), and that it is dubiously related to rectifiability at all.

Yet another reason is that the investigations into uniform rectifiability were related to questions in pure mathematics, and somewhat predated what has since then become an extremely active field in applied mathematics and computer science, that of *mathematical data analysis* (see [6, 13, 40] among others for such examples). Data analysis, (previously the domain of statisticians, social scientists, biologists, etc.,) received the attention of computer scientists and applied mathematicians beginning in the mid to late 1990's with the idea of re-introducing rigor and mathematical analysis into such pursuits. As such, the possibility of *computational applications* for such curvatures in practical problems was un-explored. However, the author's advisor (who belonged to a small group who had worked on problems of uniform rectifiability, computational harmonic analysis, and data analysis) saw the potential for curvatures to yield significant practical information about data sets. The basic idea is that such curvatures encode geometric information about the possible continuous structures underlying data distributions, and that *sampling procedures* could then be used to gain or estimate such information.

## 1.2 Overview of Results

### 1.2.1 The Basic Objective

The broad aim was the definition of "Menger-type" curvatures, denoted by $c_{MT}$, for $(d+1)$-dimensional simplices whose multiple square integrals would yield "equivalent" geometric information about a $d$-regular measure $\mu$ as the local Jones'-type flatness, $J_2(\mu|_B)$. For convenience we suppress the dependence of $c_{MT}$ on the dimension $d$, since this is assumed to be fixed before hand.

Specifically, what was wanted was a curvature $c_{MT}(v_0, \ldots, v_{d+1})$ evaluated on the $(d+2)$ vertices, $v_0, \ldots, v_{d+1}$ of a $(d+1)$-simplex satisfying the following estimate on a large family of closed balls $B$:

$$J_2(\mu|_{\gamma_1 \cdot B}) \lessapprox \int_B \cdots \int_B c_{MT}(v_0, \ldots, v_{d+1})^2 \, \mathrm{d}\mu(v_0) \cdots \mathrm{d}\mu(v_{d+1}) \lessapprox J_2(\mu|_{\gamma_2 \cdot B}), \tag{1.6}$$

for fixed $\gamma_2 \geq 1 \geq \gamma_1 > 0$. In this case, using Theorem 1.1.1 and the $d$-regularity of $\mu$ such curvatures would provide a metric/geometric characterization of the uniform rectifiability of a measure, and would thus be an avenue for the investigation of new contexts for uniform rectifiability, as well as the possibility of investigation similar problems to those worked on in the 1-dimensional case in terms of geometry and rectifiability in general.

As such, there was sufficient motivation for this work for the reason that it would provide a novel characterization of rectifiability and uniform rectifiability for $d > 1$. An accompanying, and quite significant, motivation was the possibilities for using of such curvatures for the purposes of computational geometric data analysis. Initial work had indicated the usefulness of the 1-dimensional Menger curvature for solving a 1-dimensional version of the Hybrid Linear Modeling problem. The hope was that higher-dimensional curvatures could be used to solve higher-dimensional versions of Hybrid Linear Modeling [7, 47] (see [4, 5] for applications of multiway curvatures to this problem for $d \geq 1$), and the Multi-Manifold Modeling problem in general [31, 19, 8].

### Initial Progress

The first step of the research was demonstrating that the 1-dimensional Menger curvature satisfied an estimate of the form in (1.6). In this case, the lower bound had been demonstrated in [25], but the upper bound had never been published. By adapting some multiscale techniques previously used for $L_\infty$ variants of the $\beta_2$-numbers, denoted by $\beta_\infty$ (see Subsection 1.4.3), as well as introducing ideas and techniques which would be significantly generalized for the work on $d > 1$, the author was able to establish the following 1-dimensional results:

**Proposition 1.2.1** (W)**.** *If $\mu$ is a 1-regular measure on a real separable Hilbert space $H$, then for all closed balls $B \subseteq H$ at all reasonable locations and scales,*

$$J_2(\mu|_{\frac{1}{3}\cdot B}) \lessapprox c_M^2(\mu|_B) \lessapprox J_2(\mu|_{6\cdot B}).$$

The lower bound is an improvement on previous versions since it previously it had only been done on $\mathbb{R}^n$, and the blow-up coefficient on the ball now had a *fixed* value $\gamma_1 = \frac{1}{3}$.

Previously this coefficient had only been expressed in such a way that $\gamma_1$ depended on the regularity of $\mu$, i.e., $C_\mu$.

The importance of this was to show that basic comparability held between the Jones'-type flatness and the Menger curvature *independently of the ambient dimension.* This was interesting because it expanded the class of 1-regular measures that can be analyzed via their discrete structures to those on infinite dimensional spaces. Furthermore, it suggested that in computational problems one might be able use the Menger curvature to quantify the geometry of data sets embedded in very high-dimensional Euclidean spaces in such a way as to *downplay the size of the ambient dimension*, an issue which is typically a barrier in such computational problems.

The next portion of the work consisted of attempting to construct suitable $(d+1)$-dimensional curvatures for simplices. After various versions of *strictly geometric generalizations* of the 1-dimensional curvature (e.g., that of [25] and others based on the radius of circum-spheres etc.), it became apparent that the "best" approach would be to find the "correct" generalizations of the trigonometric sine function to higher-dimensional "generalized" sine functions, and then to find the correct normalization along the lines of equation (1.5), i.e.,

$$c_{MT}(v_0, \ldots, v_{d+1}) = \frac{\text{generalizedsine}(v_0, \ldots, v_{d+1})?}{\text{normalization?}},$$

where the actual functions and the normalizations were unknown.

### 1.2.2 Generalized Sine Functions and Simplex Inequalities

One approach for constructing higher-dimensional sine functions is to define real valued quantities called "vertex angles" and to compute their trigonometric sines, where the most basic vertex angle is the real valued angle at a vertex of a triangle. A generalization of this to higher-dimensions is the "solid" angle of the vertex of a tetrahedron, that is the area of the corresponding *spherical triangle* on the unit sphere [16].

For our purposes, it turns out to be more fruitful to ignore the notion of a vertex angle and to simply focus on functions defined on collections of vectors that have *properties qualifying them as "generalized sine functions"*, such as:

1. Having absolute values bounded by 1

2. Taking value $\pm 1$ for an orthogonal collection and value 0 for a linearly dependent one

3. Sub-linearity similar to the trigonometric sine function of a real angle

In the case of $d = 1$, the sub-linearity of property 3 takes the form of the *triangle inequality* for the trigonometric sine which holds regardless of the dimension of $H$:

$$|\sin(v_1, v_2)| \leq |\sin(v_1, u)| + |\sin(u, v_2)|, \text{ for all } v_1, v_2 \in H \text{ and } u \in H \setminus \{0\}. \tag{1.7}$$

This inequality is crucial in the proof of the upper bound of Proposition 1.2.1.

At least two different generalized sine functions, called the polar sine and the hyper sine [15], defined on collections of vectors are known to satisfy properties 1 and 2 above, and these have been known for more than a century. Euler [17] formulated the two-dimensional polar sine (for vertices of tetrahedra) and D'Ovidio [14] generalized it to the vertices of higher-dimensional simplices. Joachimsthal [22] suggested the two-dimensional hypersine (for tetrahedra) and Bartoš [3] extended it to simplices of any dimension. Different authors have explored their properties and applied them to a variety of problems on the discrete geometry of simplices (see e.g., [45], [26], [46], [24] and references therein). For our purposes, we have slightly modified the existing definitions (see [15] for the traditional versions), in particular allowing negative values if the $\dim(H) = d+1$, and taking the $d$-th root of the hyper sine.

This work, specifically Chapter 2 [30], is the first to show that the polar sine and the $d$-th root of the hyper sine are sub-linear in their arguments, that is they satisfy a generalization of the triangle inequality of equation (1.7), e.g., if $f : H^3 \to \mathbb{R}$ then $f$ satisfies a "tetrahedral inequality" if

$$|f(v_1, v_2, v_3)| \leq |f(u, v_2, v_3)| + |f(v_1, u, v_3)| + |f(v_1, v_2, u)| \text{ for all } u \in H \setminus \{0\}. \tag{1.8}$$

Furthermore, the work of Chapters 3 [29] and 4 [28] is also the first relating these functions (specifically the polar sine) to the continuous geometry of measures on $H$ for $d > 1$ via the use of Menger-type curvatures.

The polar sine and the hyper sine (for precise definitions see Subsection 2.1), denoted respectively by $\mathrm{p}_d\mathrm{sin}$ and $\mathrm{g}_d\mathrm{sin}$, are exemplified in Figure 1.3 and described as follows (see Figure 1.3). For any set of $(d+2)$ vectors $v_0, v_1, \ldots, v_{d+1} \in H$, and the parallelotope through the points $v_0, v_1, \ldots, v_{d+1}$, the quantity $|\,\mathrm{p}_d\mathrm{sin}_{v_0}(v_1, \ldots, v_{d+1})|$ is obtained by dividing the $(d+1)$-volume of that parallelotope by the $d+1$ edge lengths at the vertex $v_0$. Similarly, we define $|\,\mathrm{g}_d\mathrm{sin}_{v_0}(v_1, \ldots, v_{d+1})|$ to be the $(d+1)$-volume scaled by the $d$-th roots of the $d$-volumes of the faces containing the vertex $v_0$ (there are $d+1$ of these). We often assume that $v_0 = 0$ since the more general case can be obtained by a simple shift (see equations (2.2) and (2.3)). We note that when $d = 1$: $|\,\mathrm{p}_1\mathrm{sin}_0(v_1, v_2)| = |g_1 \sin_0(v_1, v_2)| = |\sin(v_1, v_2)|$.

We note that for the $(d+1)$-simplex through the vertices $\{v_0, \ldots, v_{d+1}\}$ there are exactly $(d+2)$ values of the generalized sine functions at each of the $(d+2)$ vertices.

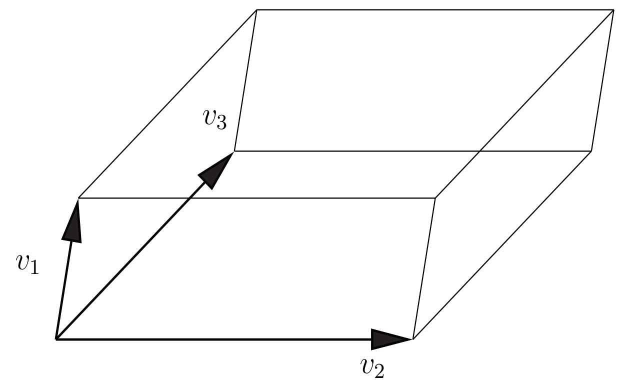


Figure 1.3: Exemplifying the computation of $\mathrm{p}_d\mathrm{sin}_0(v_1, v_2, v_3)$ and $\mathrm{g}_d\mathrm{sin}_0(v_1, v_2, v_3)$, when $d = 2$ and $H = \mathbb{R}^3$: The figure shows the parallelepiped generated by 0, $v_1$, $v_2$ and $v_3$. In this case, $\mathrm{p}_d\mathrm{sin}_0(v_1, v_2, v_3) = \frac{v_1 \bullet (v_2 \times v_3)}{\|v_1\| \cdot \|v_2\| \cdot \|v_3\|}$ and $\mathrm{g}_d\mathrm{sin}_0(v_1, v_2, v_3) = \dfrac{v_1 \bullet (v_2 \times v_3)}{\sqrt{\|v_1 \times v_2\| \cdot \|v_2 \times v_3\| \cdot \|v_1 \times v_3\|}}$.

### Simplex Inequalities

In addition to a search for curvatures, another motivation for investigating generalized sines is a search for high-dimensional versions of metrics and $d$-way kernel methods in machine learning [1, 42]. Deza and Rosenberg [12] defined a $d$-semimetric in the following way. If $E$ is a given set and $f : E^{d+1} \mapsto [0, \infty)$ is symmetric (invariant to permutations), then the pair $(E, f)$ is a $d$-semimetric if $f$ satisfies the following sub-linearity or "simplex-type inequality":

$$f(x_1, \ldots, x_{d+1}) \leq \sum_{i=1}^{d+1} f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{d+1}) \quad \text{for all} \quad x_1, \ldots, x_{d+1}, u \in E. \quad (1.9)$$

We typically refer to $f$ itself as a $d$-semimetric when the set E is clear.

Much of Chapter 2 works to establish the following examples of $d$-semimetrics.

**Theorem 1.2.1** (W). *If $H$ is a real pre-Hilbert space, $d \in \mathbb{N}$, and $\dim(H) \geq d + 1$, then the functions $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ are $d$-semimetrics with respect to the set $H \setminus \{0\}$.*

In general, the examples of $d$-semimetrics proposed by Deza and Rosenberg do not represent $d$-dimensional geometric properties, since they typically form them by averaging non-negative functions quantifying lower order geometric properties (see [12, Fact 2]). For example, to form a 2-semimetric on $H$ they average the pairwise distances between three points, i.e., the scaled perimeter of the corresponding triangle. In particular, they are $d$-dimensional semimetrics in the sense that they are 0 if and only if their $(d+2)$ arguments are affine linearly dependent, that is, they all lie in the same $d$-dimensional affine subspace. Furthermore, they are 1 if and only if their arguments are mutually orthogonal at the given vertex.

**Relaxed Two Term Inequalities**

With respect to the formation of curvatures from the polar sine, it turns out that something slightly different than Theorem 1.2.1 is needed. Below in Subsection 1.2.3 we use the polar sine to define the Menger-type curvature for $d > 1$, and in Chapters 3 and 4 we show that such curvatures satisfy the estimate of equation (1.6). The proof of the upper bound relies heavily on the fact that $|\,\mathrm{p}_d\mathrm{sin}\,|$ satisfies "a relaxed simplex inequality of two controlling terms with high local probability", at least when working with $d$-regular measures. We define this notion in a somewhat general setting as follows.

A symmetric function $f$ on $H^{d+1}$ satisfies a *relaxed simplex inequality* of 2 terms if for all $v_1, \ldots, v_{d+1} \in H$, all indices $1 \leq i < j \leq d+1$, and any $u \in H$ we have the relaxed inequality

$$f(v_1, \ldots, v_{d+1}) \lessapprox f(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}) + f(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{1.10}$$

For $S = \{v_0, v_1, \ldots, v_{d+1}\} \subseteq H$ and $C > 0$, we let $U_C(S, v_0)$ be the set of vectors $u$ giving rise to a relaxed simplex inequality of two terms and constant $C$ for $|\,\mathrm{p}_d\mathrm{sin}_{v_0}|$, that is,

$$\begin{aligned} U_C(S,v_0) = \Big\{u \in H :\ &|\,\mathrm{p}_d\mathrm{sin}_{v_0}(v_1,\ldots,v_{d+1})| \leq \\ C\cdot\big(|\,\mathrm{p}_d\mathrm{sin}_{v_0}(v_1,\ldots,v_{i-1},u,v_{i+1},\ldots,v_{d+1})| &+ |\,\mathrm{p}_d\mathrm{sin}_{v_0}(v_1,\ldots,v_{j-1},u,v_{j+1},\ldots,v_{d+1})|\big), \\ &\text{for all } 1 \leq i < j \leq d+1\Big\}. \end{aligned} \tag{1.11}$$

For any $v_0 \in \mathrm{supp}(\mu)$, the event $U_C(S, v_0)$ has *uniformly large probability* at any relevant ball in $H$, where the probability at a ball is obtained by scaling $\mu$ by the measure of the ball. We formulate this more precisely and even more generally as follows:

**Theorem 1.2.2.** *If $H$ is a pre-Hilbert space, $2 \leq d \in \mathbb{N}$, $0 < \epsilon < 1$, $\gamma \in \mathbb{R}$ is such that $d - 1 < \gamma \leq d$, $v_0 \in \mathrm{supp}(\mu)$, $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, and $\mu$ is a $\gamma$-regular measure on $H$ with regularity constant $C_\mu$, then there exists a constant $C_0 \geq 1$ depending only on $C_\mu$, $\epsilon$, $\gamma$, and $d$, such that for all $C \geq C_0$:*

$$\frac{\mu\left(U_C(S,v_0) \cap B(v_0,r)\right)}{\mu\left(B(v_0,r)\right)} \geq 1-\epsilon, \quad \text{for all } 0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu)). \tag{1.12}$$

Theorems 1.2.1 and 1.2.2 are proved in Chapter 2, and combined with a set of smaller results are also contained in [30].

### 1.2.3 Menger-Type Curvatures

The $d$-dimensional Menger-type curvature is defined on $(d+2)$ vectors $v_0, \ldots, v_{d+1} \in H$ as the symmetric average of the $d$-dimensional polar sine evaluated on each of the vertices, and normalized by an appropriate power of the diameter of the set $\{v_0, \ldots, v_{d+1}\}$, denoted by $\operatorname{diam}(v_1, \ldots, v_{d+1})$. Formally, we define

$$c_{MT}(v_0, \ldots, v_{d+1}) := \left( \frac{\displaystyle\sum_{i=0}^{d+1} \mathrm{p}_d \sin_{v_i}(v_0, \ldots, v_{d+1})}{(d+2) \cdot \operatorname{diam}(v_0, \ldots, v_{d+1})^{d(d+1)}} \right)^{1/2}, \tag{1.13}$$

if $\operatorname{diam}(v_0, \ldots, v_{d+1}) > 0$, and $c_{MT}(v_0, \ldots, v_{d+1}) = 0$ otherwise.

The curvature $c_{MT}$ generalizes the 1-dimensional Menger curvature in the following way. Recalling equation (1.5) and applying the law of sines we note that

$$\frac{1}{12} \cdot c_M^2(v_0, v_1, v_2) \;\leq\; c_{MT}^2(v_0, v_2, v_2) = \\ \frac{\sin^2_{v_0}(v_1, v_2) + \sin^2_{v_1}(v_0, v_2) + \sin^2_{v_2}(v_0, v_1)}{3 \cdot \operatorname{diam}^2(v_0, v_1, v_2)} \leq \frac{1}{4} \cdot c_M^2(v_0, v_1, v_2)\,.$$

Furthermore, similar to the 1-dimensional case, the curvature $c_{MT}(v_0, \ldots, v_{d+1})$ can be expressed in terms of the radius of the circum-sphere, $R(v_0, \ldots, v_{d+1})$, i.e.,

$$c_{MT}(v_0, \ldots, v_{d+1}) = \left( \frac{\Delta(v_0, \ldots, v_{d+1})}{R(v_0, \ldots, v_{d+1})} \right)^{\frac{d(d+1)}{2}},$$

where the function $\Delta(v_0, \ldots, v_{d+1})$ is independent of the embedding sphere for $v_0, \ldots, v_{d+1}$. However, for $d > 1$ the function $\Delta$ is *singular*, that is, one can find a sequence of simplices with vertices in the $d$-dimensional unit sphere such that the function $\Delta$ blows up on this sequence. When $d = 1$, equation (1.5) shows that $\Delta = 2$. While such an explicitly geometric characterization of the Menger-type curvature may be of interest, it plays no role in the rest of this work.

### 1.2.4 Menger-Type Curvatures and the Jones'-Type Flatness

The basic results of this work are estimates relating various integrals of the Menger-type curvature to different levels of information provided by the $\beta_2$-numbers and the Jones'-type

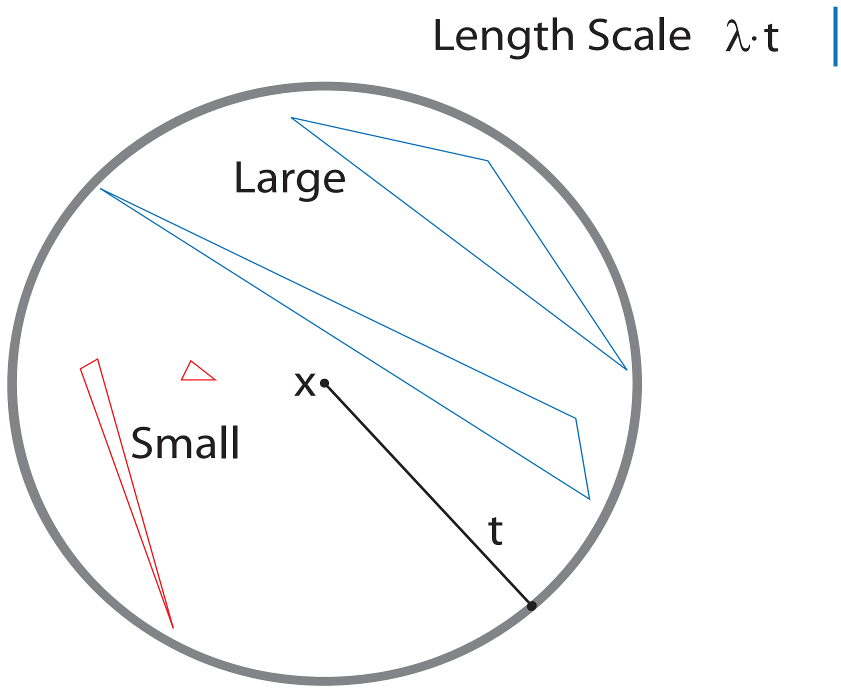


Figure 1.4: Exhibiting the well-separated elements in $U_\lambda(B)$ when $d = 1$. For the ball $B(x,t)$, the blue triangles satisfy the condition since all edge lengths are at least $\lambda \cdot t$, however, the red triangles do not either due to a small *diameter* or one small edge length.

flatness. In order to establish a full comparison as outlined in equation (1.6), a few intermediate comparisons must be done between the squared $\beta_2$-numbers (henceforth denoted by $\beta_2^2$) and some truncated integrals of the Menger-curvature. We clarify this as follows.

We first exemplify the use of the curvatures in the simplest setting of approximating $\mu$ by a $d$-dimensional plane at a given scale and location, $B = B(x,t)$. We show that by fixing $\lambda > 0$ and sampling well separated simplices in $B^{d+2}$, that is, simplices such that all edges are comparable to the overall length scale of the ball (see Figure 1.2.4)

$$U_\lambda(B) = \left\{ (v_0, \ldots, v_{d+1}) \in B^{d+2} : \min_{0 \le i < j \le d+1} \|v_i - v_j\| \ge \lambda \cdot t \right\}, \tag{1.14}$$

one can approximate $\beta_2^2(x,t)$ by averages of the curvature $c_{MT}^2$ in the following way. For

$\lambda > 0$ and a closed ball $B(x,t) \subseteq H$, we denote the restriction of the integral of the squared curvature to the set of well separated simplices by

$$c_{MT}^2(x,t,\lambda) = \int_{U_\lambda(B(x,t))} c_{MT}^2(v_0,\ldots,v_{d+1})\,\mathrm{d}\mu(v_1)\ldots\,\mathrm{d}\mu(v_{d+2}). \tag{1.15}$$

The function $c_{MT}^2(x,t,\lambda)$ satisfies the following estimate.

**Theorem 1.2.3.** *There exist constants $\lambda = \lambda(d, C_\mu)$ and $C_0 = C_0(d, C_\mu, \lambda)$ such that at all reasonable scales and locations*

$$C_0^{-1} \cdot \beta_2^2(x,t) \leq \frac{c_{MT}^2(x,t,\lambda_0)}{\mu(B(x,t))} \leq C_0 \cdot \beta_2^2(x,t)\,. \tag{1.16}$$

The lower bound is proved in Chapter 3, and the upper bound is established in Chapter 4.

We next extend the above estimates to multiscale least squares approximations, i.e., we estimate the Jones'-type flatness $J_2(\mu|_B)$ in the following way. Fixing $\lambda > 0$ and a ball $B \subseteq H$, we restrict the sampling to *well-scaled* simplices, that is the set of simplices with vertices in $B$ such that all edge lengths are comparable (see Figure 1.2.4):

$$W_\lambda(B) = \left\{(v_0,\ldots,v_{d+1}) \in B^{d+2} : \min_{0\leq i<j\leq d+1} \|v_i - v_j\| \geq \lambda \cdot \mathrm{diam}(v_0,\ldots,v_{d+1}) > 0\right\}, \tag{1.17}$$

In this case the following theorem holds.

**Theorem 1.2.4.** *There exist constants $\lambda = \lambda(d, C_\mu) > 0$ and $C_3 = C_3(d, \mu, \lambda)$ such that*

$$\frac{1}{C_3} \cdot J_d\left(\mu|_{\frac{1}{3}\cdot B}\right) \leq \int_{W_\lambda(B)} c_{MT}^2(v_0,\ldots,v_{d+1})\,\mathrm{d}\mu(v_1)\ldots\,\mathrm{d}\mu(v_{d+2}) \leq C_3 \cdot J_d\left(\mu|_{6\cdot B}\right), \tag{1.18}$$

*for any closed ball $B \subseteq H$ at all reasonable locations and scales.*

The lower bound of Theorem 1.2.4 is established in Chapter 3, and the upper bound is given in Chapter 4.

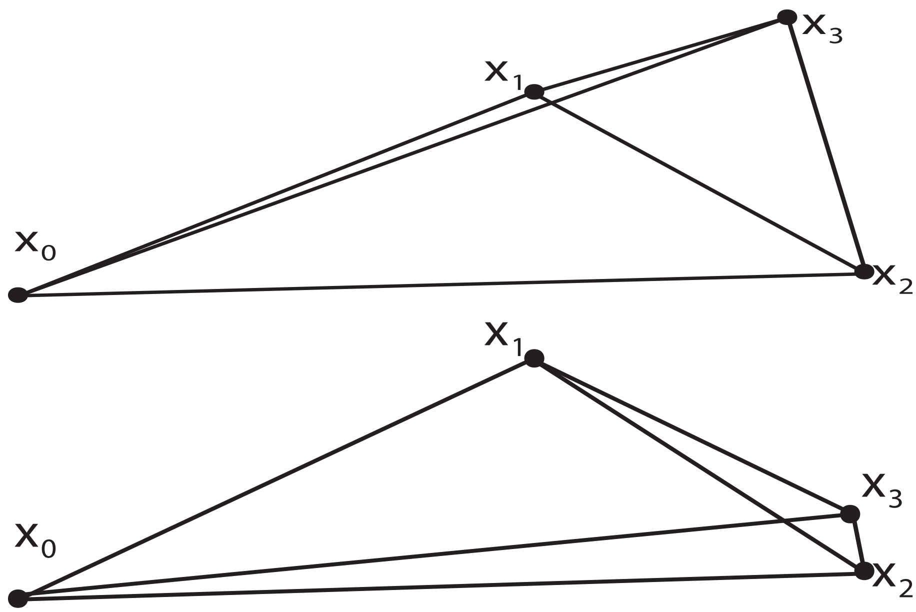


Figure 1.5: The tetrahedron on the top is well-scaled for some parameter $\lambda \approx 1$, whereas the bottom tetrahedron is poorly scaled for that same value of $\lambda$.

The following theorem is a much stronger version of Theorem 1.2.4, specifically in terms of the upper bound.

**Theorem 1.2.5.** *There exists a constant $C_1 = C_1(d, C_\mu)$ such that for all closed balls at all reasonable locations and scales, $B \subseteq H$, the following estimate holds:*

$$\frac{1}{C_1} \cdot J_d\left(\mu|_{\frac{1}{3}\cdot B}\right) \leq c^2_{MT}\left(\mu|_B\right) \leq C_1 \cdot J_d\left(\mu|_{6\cdot B}\right). \tag{1.19}$$

The lower bound holds trivially due to the fact that $W_\lambda(B) \subseteq B^{d+1}$ for all closed balls $B$, i.e.,

$$\int_{W_\lambda(B)} c^2_{MT}(v_0, \ldots, v_{d+1})\, \mathrm{d}\mu(v_1) \ldots \mathrm{d}\mu(v_{d+2}) \leq c^2_{MT}(\mu|_B).$$

However, the upper bound requires substantial development, which is given in Chapter 4.

Combining Theorem 1.2.5 with the $d$-regularity of $\mu$ and Theorem 1.1.1, we obtain the following characterization of uniform rectifiability.

**Theorem 1.2.6.** *If $\mu$ is $d$-regular $H$, then $\mu$ is uniformly rectifiable if and only if $c^2_{MT}(\mu|_B) \lessapprox \mu(B)$ for all closed balls at all reasonable scales and locations $B \subseteq H$.*

Theorem 1.2.6 is a fully geometric characterization of the uniform rectifiability of a $d$-regular measure in terms of its "discrete" geometry.

## 1.3 Organization

This work is divided into three major portions, excluding the introduction and the appendix. In Chapter 2 we develop some basic properties of the generalized sine functions. Useful decomposition formulas are given in Sections 2.2 and 2.3. The simplex inequalities of Theorem 1.2.1 and the relaxed two-term inequality of Theorem 1.2.2 are proved in Sections 2.4 and 2.5 respectively.

Chapter 3 then develops the techniques for establishing an upper bound for $J_2(\mu|_{\frac{1}{3}\cdot B})$ in terms of $c^2_{MT}(\mu|_B)$, that is the *lower bounds* of Theorems 1.2.3 and 1.2.4. These are proved in Sections 3.3 and 3.4 respectively. In addition, Section 3.2 contains the development of a crucial quantitative geometric property of $d$-regular measures necessary for the proof of Theorem 1.2.3. This property is called "$d$-separation", and among other things, it is further elucidated in a brief introduction contained in Chapter 3.

Chapter 4 gives the proofs of the various upper bounds of Theorems 1.2.3-1.2.5. The development in this chapter is substantial since it includes some elementary concepts (and resulting notation) that are not to be found elsewhere. Chapter 4 includes a brief introduction outlining these concepts and their utility in the proofs, as well as a more detailed description of the organization of the chapter.

Finally, computations that are too tedious, elementary, or distracting have been given at the end in Chapter 5.

## 1.4 Notation and Definitions

Here we develop some basic notation and definitions that will be used throughout this work. However, due to the specialized nature of the other three chapters, we do not introduce all notation initially since this would be unnecessarily tedious. We reserve the more particular notation, definitions, and propositions for the appropriate sections of the respective chapters.

We often denote subspaces of $H$ by $V$ or $W$, and the orthogonal complement of $V$ is denoted by $V^{\perp}$. For such a subspace $V$, we denote its dimension by $\dim(V)$. If $V$ is a complete subspace of $H$ (in particular finite dimensional), then we denote the orthogonal projection of $H$ onto $V$ by $P_V$. We denote the distance between $x, y \in H$ by $\operatorname{dist}(x, y)$ or equivalently $\|x - y\|$. Similarly, $\operatorname{dist}(x, V) = \|P_V(x) - x\|$ is the induced distance between $x \in H$ and a complete subspace $V \subseteq H$. If $n \leq \dim(H)$, we use the phrase $n$-plane to refer to an $n$-dimensional affine subspace of $H$.

For a closed ball $B(x, r)$ and $\gamma > 0$, let $\gamma \cdot B(x, r) = B(x, \gamma \cdot r)$. If $x \in H$ and $r_1, r_2 > 0$, then we define the annulus

$$A(x, r_1, r_2) = \{y \in H : r_1 < \|x - y\| \leq r_2\} = B(x, r_2) \setminus B(x, r_1).$$

We reserve $u$, $v$, $w$, $x$, $y$, and $z$ to denote elements of $H$; $X$ to denote elements of $H^m$ for $m \geq 3$; $L$ for a complete affine subspace of $H$ (possibly a linear subspace); $V$ for a complete linear subspace of $H$; $B$ to denote *closed* balls of finite radius in $H$; $t$ and $r$ for arbitrary *finite* length scales, in particular, radii of balls.

### 1.4.1 $d$-Regular Measures

**Definition 1.4.1.** *A locally finite Borel measure $\mu$ on $H$ is a $d$-regular measure if there exists a constant $C$ such that for all $x \in \operatorname{supp}(\mu)$ and $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$,*

$$C^{-1} \cdot r^d \leq \mu(B(x,r)) \leq C \cdot r^d.$$

We denote the smallest constant $C$ for which the inequality above holds by $C_\mu$. We refer to it as the *regularity constant* of $\mu$. We note that this definition extends easily to $\gamma$-regularity for $\gamma > 0$.

For a fixed $d$-regular $\mu$ on $H$, in addition to the regularity constant $C_\mu$, we fix the constant

$$C_{\mathrm{p}} = C_{\mathrm{p}}(d, C_\mu) = \begin{cases} \dfrac{\sqrt{5} \cdot \pi^2}{4 \cdot \arcsin\left(2^{-(5/2 \cdot d+1)} \cdot C_\mu^{-2}\right)}, & \text{if } d > 1; \\ \\ 1, & \text{if } d = 1. \end{cases} \tag{1.20}$$

We also fix the constant $\alpha_0$ and use its powers to provide appropriate scales for the work in Chapter 4:

$$\alpha_0 = \alpha_0(d, C_\mu) = \min\left\{\frac{1}{2 \cdot C_{\mathrm{p}}^2}, \left(\frac{1}{4 \cdot C_\mu^2}\right)^{1/d}\right\} = \begin{cases} \dfrac{1}{4 \cdot C_\mu^2}, & \text{if } d = 1; \\ \dfrac{1}{2 \cdot C_{\mathrm{p}}^2}, & \text{if } d > 1. \end{cases} \tag{1.21}$$

If $n \in \mathbb{Z}$, then we abbreviate our usual notation for an annulus in the following way:

$$A_n(x,r) = A(x, \alpha_0^{n+1} \cdot r, \alpha_0^n \cdot r). \tag{1.22}$$

Two very basic properties of $d$-regular measures that will be used throughout this work given in the following lemmas. Their proofs follow immediately from the definition of $d$-regularity.

**Lemma 1.4.1.** *If $x \in \operatorname{supp}(\mu)$, $y \in \operatorname{supp}(\mu)$, and $0 < r_1 \le r_2$, then*

$$\frac{\mu(B(x,r_2))}{\mu(B(y,r_1))} \le C_\mu^2 \cdot \left(\frac{r_2}{r_1}\right)^d .$$

**Lemma 1.4.2.** *If $x \in \operatorname{supp}(\mu)$, $0 < t \le \operatorname{diam}(\operatorname{supp}(\mu))$, and $0 < s < 1/C_\mu^{2/d}$, then*

$$\frac{\mu(A(x, s \cdot t, t))}{\mu(B(x,t))} \ge \left(1 - s^d \cdot C_\mu^2\right) > 0.$$

Using the constant $\alpha_0$ of equation (1.21) as a parameter, Lemma 1.4.2 implies the following for all $x \in \operatorname{supp}(\mu)$ and $0 < r \le \operatorname{diam}(\operatorname{supp}(\mu))$:

$$\frac{\mu(A_0(x,r))}{\mu(B(x,r))} \ge \left(1 - \alpha_0^d \cdot C_\mu^2\right) \ge \frac{3}{4}. \tag{1.23}$$

### 1.4.2 Uniform Rectifiability

We review here basic notions in the theory of uniform rectifiability [10, 11]. Even though the original theory is formulated in finite dimensional Euclidean spaces, the part presented here generalizes to any separable real Hilbert space.

#### $A_1$ Weights and $\omega$-Regular Surfaces

Let $\mathcal{L}_d$ denote the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$. Given a locally integrable function $\omega : \mathbb{R}^d \to [0, \infty)$, we can induce a measure on Borel subsets $A$ of $\mathbb{R}^d$ by defining

$$\omega(A) = \int_A \omega(x) \, \mathrm{d}\mathcal{L}_d(x).$$

We say that $\omega$ is an $A_1$ weight if for any ball $Q$ in $\mathbb{R}^d$,

$$\frac{\omega(Q)}{\mathcal{L}_d(Q)} \lessapprox \omega(x), \quad \text{for } \mathcal{L}_d \text{ a.e. } x \in Q.$$

We note that the measure induced by $\omega$ is doubling in the following sense: $\omega(Q) \approx \omega(2 \cdot Q)$ for any ball $Q$. Consequently, the following function is a quasidistance (i.e., a symmetric positive definite function satisfying a relaxed version of the triangle inequality with controlling constant $C \geq 1$ instead of one):

$$\operatorname{qdist}_\omega(x, y) = \sqrt[d]{\omega\left(B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right)\right)}, \qquad \text{for all } x, y \in \mathbb{R}^d.$$

Given an $A_1$ weight $\omega$, we define $\omega$-regular surfaces as follows.

**Definition 1.4.2.** *Let $\omega$ be an $A_1$ weight on $\mathbb{R}^d$. A subset $\Gamma$ of $H$ is called an* $\omega$-regular surface *if there exists a function $f\colon \mathbb{R}^d \to H$, as well as constants $L$ and $C$, such that $\Gamma = f(\mathbb{R}^d)$,*

$$\|f(x) - f(y)\| \leq L \cdot \operatorname{qdist}_\omega(x, y), \qquad \text{for all } x, y \in \mathbb{R}^d, \tag{1.24}$$

*and*

$$\omega\left(f^{-1}(B(x, r))\right) \leq C \cdot r^d, \qquad \text{for all } x \in H \text{ and } r > 0. \tag{1.25}$$

If $d = 1$, we consider the special case of the $A_1$ weight: $\omega = 1$. The corresponding $\omega$-regular surface is an Ahlfors regular curve [11].

### Uniformly Rectifiable Measures

Given a Borel measure $\mu$ on $H$, we let

$$\widehat{H} = \begin{cases} H \times \mathbb{R}, & \text{if } \dim(H) < 2 \cdot d; \\ H, & \text{otherwise,} \end{cases}$$

and we define the induced measure $\widehat{\mu}$ on $\widehat{H}$ to be $\widehat{\mu}(A) = \mu(A \cap H)$, for all Borel sets $A \subseteq \widehat{H}$.

We define $d$-dimensional uniformly rectifiable measures as follows:

**Definition 1.4.3.** *A Borel measure $\mu$ on $H$ is said to be* $d$-dimensional uniformly rectifiable *if it is $d$-regular and there exist an $A_1$ weight $\omega$ on $\mathbb{R}^d$ along with an $\omega$-regular surface $\Gamma \subseteq \widehat{H}$ such that $\widehat{\mu}\left(\widehat{H} \setminus \Gamma\right) = 0$.*

### 1.4.3 Jones'-Type Flatness for $1 \leq p \leq \infty$

For any fixed $1 \leq p < \infty$, $x \in H$ and $0 < t < \infty$, we define the $d$-dimensional Jones' $\beta_p$-numbers [10] as follows:

$$\beta_p(x,t) = \begin{cases} \inf\limits_{d-\text{planes } L} \left( \displaystyle\int_{B(x,t)} \left( \frac{\operatorname{dist}(y,L)}{2 \cdot t} \right)^p \frac{\mathrm{d}\mu(y)}{\mu(B(x,t))} \right)^{1/p}, & \text{if } \mu(B(x,t)) > 0; \\ 0, & \text{if } \mu(B(x,t)) = 0. \end{cases}$$

If $p = \infty$, then the $\beta_\infty$-numbers are defined as

$$\beta_\infty(x,t) = \inf_{d\text{-planes } L} \sup_{y \in B(x,t) \cap \operatorname{supp}(\mu)} \frac{\operatorname{dist}(y,L)}{2 \cdot t}.$$

For any fixed $1 \leq p < \infty$ and any ball $B$ in $H$, we define the continuous local Jones'-type $L_p$ flatness as follows:

$$J_p(\mu|_B) = \int_B \int_0^{\operatorname{diam}(B)} \beta_p^p(x,t) \frac{\mathrm{d}t}{t} \,\mathrm{d}\mu(x).$$

In this work we only work with the $L_2$ versions referred to in the introduction, denoted by $J_2(\mu|_B)$.

We also find it convenient to introduce the following definition, which simply denotes the calculation of a local, normalized $L_2$ error with respect to a *fixed* $d$-plane $L$:

$$\beta_2(x,t,L) = \begin{cases} \left( \displaystyle\int_{B(x,t)} \left( \frac{\operatorname{dist}(y,L)}{2 \cdot t} \right)^2 \frac{\mathrm{d}\mu(y)}{\mu(B(x,t))} \right)^{1/2}, & \text{if } \mu(B(x,t)) > 0; \\ 0, & \text{if } \mu(B(x,t)) = 0. \end{cases}$$

We note the trivial equality

$$\beta_2(x,t) = \inf_{d-\text{planes } L} \beta_2(x,t,L).$$

If $B$ is a ball in $H$ of indeterminate center $x$ and radius $t$, we denote the quantities $\beta_2(x,t,L)$ and $\beta_2(x,t)$ by $\beta_2(B,L)$ and $\beta_2(B)$ respectively.

# Chapter 2

# Generalized Sines and Simplex Inequalities

### Organization

This chapter is organized as follows. In Section 2.1 we give the formal definitions of the polar sine and the hyper sine, and in Sections 2.2 and 2.3 we develop some elementary properties and functional identities satisfies by these generalized sine functions. Section 2.4 contains the proof of Theorem 1.2.1, whereas Section 2.5 contains the proof of Theorem 1.2.2.

### Notation

For an affine subspace $L \subseteq H$, a point $x \in L$, and an angle $\theta$ such that $0 \leq \theta \leq \pi/2$, we define the *cone*, $\mathrm{C_{one}}(\theta, L, x)$, centered at $x$ on $L$ in the following way

$$\mathrm{C_{one}}(\theta, L, x) := \{u \in H : \mathrm{dist}(u, L) \leq \|u - x\| \cdot \sin(\theta)\}.$$

For an affine subspace, $L \subseteq H$ and $h > 0$, we define the *tube* of height $h$ on $L$, $\mathrm{T_{ube}}(L, h)$, as follows.

$$\mathrm{T_{ube}}(L, h) := \{u \in H : \mathrm{dist}(u, L) \leq h\}.$$

We also quite often use sets generated by vectors. If $v_1 \ldots, v_k \in H$, then the *parallelotope* spanned by these vectors is the set

$$\mathrm{P_{rll}}(v_1, \ldots, v_k) := \left\{ \sum_{i=1}^{k} t_i v_i \ : 0 \leq t_i \leq 1,\ i = 1, \ldots, k \right\}.$$

Similarly, the *polyhedral cone* spanned by $v_1, \ldots, v_k$ has the form

$$\mathrm{C_{poly}}(v_1, \ldots, v_k) := \left\{ \sum_{i=1}^{k} t_i v_i \ : t_i \geq 0,\ i = 1, \ldots, k \right\}.$$

The *affine plane* through the vectors $v_1, \ldots, v_k$ is defined by

$$\mathrm{A_{ffn}}(v_1, \ldots, v_k) := \left\{ \sum_{i=1}^{k} t_i v_i \ : \ \sum_{i=1}^{k} t_i = 1,\ \ t_i \in \mathbb{R},\ i = 1, \ldots, k \right\}.$$

The *convex hull* of $v_1, \ldots, v_k$ is the set

$$\mathrm{C_{hull}}(v_1, \ldots, v_k) := \ \mathrm{A_{ffn}}(v_1 \ldots, v_k) \cap \mathrm{C_{poly}}(v_1, \ldots, v_k).$$

If $S$ is a finite subset of $H$, we denote the span of $S$ by $\mathrm{L}_S$, and sometimes also by $\mathrm{Sp}(S)$.

## 2.1 Generalized Sines

### Determinants and Contents

If $H$ is finite-dimensional, $\dim(H) = k$, and $\Phi = \{\phi_1, \ldots, \phi_k\}$ is an arbitrary orthonormal basis for $H$, then we denote by $\det_\Phi$ the determinant function with respect to $\Phi$, that is, the unique alternating multilinear function such that $\det_\Phi(\phi_1, \ldots, \phi_k) = 1$. The following elementary property of the determinant will be fundamental in part of our analysis and hence we distinguish it.

**Proposition 2.1.1.** *If* $\dim(H) = k$, $v_1, \ldots, v_k \in H$ *and* $u \in \mathrm{A_{ffn}}(v_1, \ldots, v_k)$, *then for any orthonormal basis* $\Phi$

$$\det_\Phi(v_1, \ldots, v_k) = \sum_{i=1}^{k} \det_\Phi(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_k).$$

The arbitrary choice of $\Phi$ will not matter to us and thus will not be specified. Indeed, our major statements will involve only $|\det_\Phi|$, or will be related to Proposition 2.1.1, both of which are invariant under any choice of orthonormal basis $\Phi$. For this reason we will usually refer to "the determinant" and dispense with the subscript $\Phi$, i.e., $\det \equiv \det_\Phi$.

We define the $k$-content of the parallelotope $\mathrm{P_{rll}}(v_1, \ldots, v_k)$, denoted by $M_k(v_1, \ldots, v_k)$, as follows:

$$M_k(v_1, \ldots, v_k) := \begin{cases} \det_\Phi(v_1, \ldots, v_k), & \text{if } k = \dim(H) \text{ for fixed } \Phi, \\ \left[\det\left(\{\langle v_i, v_j\rangle\}_{i,j=1}^{k}\right)\right]^{\frac{1}{2}}, & \text{if } k < \dim(H). \end{cases} \tag{2.1}$$

We note that if $k = \dim(H)$, then the $k$-content may take negative values, and that the absolute value of the $k$-content can be expressed by the same formula for all $k \leq \dim(H)$, i.e.,

$$|M_k(v_1, \ldots, v_k)| = \left[\det\left(\{\langle v_i, v_j\rangle\}_{i,j=1}^{k}\right)\right]^{\frac{1}{2}}.$$

### 2.1.1 The Hyper Sine and Polar Sine Functions

Using the definition of $M_k$ in equation (2.1) and the Euclidean norm on $H$, we define the functions $\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})$ and $\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})$ respectively as (see Figure 1.3.)

$$\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) := \frac{M_{d+1}(v_1, \ldots v_{d+1})}{\left(\prod_{j=1}^{d+1} M_d(v_1, \ldots v_{j-1}, v_{j+1} \ldots v_{d+1})\right)^{1/d}}$$

and

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) := \frac{M_{d+1}(v_1, \ldots, v_{d+1})}{\prod_{j=1}^{d+1} \|v_j\|},$$

where if either of the denominators above is zero (and thus the numerator as well), then the corresponding function also obtains the value zero. We note that $|\,\mathrm{g}_d\mathrm{sin}\,|$ is the $d$-th root of the hyper sine as defined in [15].

For these functions and their vector arguments $v_1, \ldots, v_{d+1}$, we treat the point 0 as a distinguished vertex of the $(d{+}1)$-simplex through the vertices $\{0, v_1, \ldots, v_{d+1}\}$. More generally, we may add a vertex $w \in H$ other than 0, and we define the functions $\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ and $\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1})$ for vectors $v_1, \ldots, v_{d+1}$, $w \in H$ as follows:

$$\mathrm{g}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1}) = \mathrm{g}_d\mathrm{sin}_0(v_1 - w, \ldots, v_{d+1} - w), \tag{2.2}$$

and

$$\mathrm{p}_d\mathrm{sin}_w(v_1, \ldots, v_{d+1}) = \mathrm{p}_d\mathrm{sin}_0(v_1 - w, \ldots, v_{d+1} - w). \tag{2.3}$$

Whenever possible we refer to the functions $|\,\mathrm{p}_d\mathrm{sin}_w\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_w\,|$ so that we do not need to distinguish between the cases $\dim(H) = d + 1$ and $\dim(H) > d + 1$. We mainly use the notation $\mathrm{p}_d\mathrm{sin}_w$ or $\mathrm{g}_d\mathrm{sin}_w$ when $\dim(H) = d + 1$. In particular, we may use the absolute values even if it is clear that $\dim(H) > d + 1$ and thus the two sine functions are nonnegative.

## 2.2 Elementary Properties of the High-Dimensional Sine Functions

We frequently use the following elementary property of $\mathrm{p}_d\mathrm{sin}_w$ and $\mathrm{g}_d\mathrm{sin}_w$, whose proof is included in Appendix 5.1.1.

**Proposition 2.2.1.** *The functions* $|\operatorname{p}_d\!\sin_0|$ *and* $|\operatorname{g}_d\!\sin_0|$ *defined on* $H^{d+1}$ *are invariant under orthogonal transformations of* $H$ *and non-zero dilations of their arguments. Moreover, if* $\dim(H) = d+1$*, then* $\operatorname{p}_d\!\sin_0$ *and* $\operatorname{g}_d\!\sin_0$ *are invariant under dilations by positive coefficients.*

Finally, we describe a generalized law of sines for $\operatorname{g}_d\!\sin$ following Eriksson [15] (see also Bartoš [3]):

**Proposition 2.2.2.** *If* $\{0, v_1, \ldots, v_{d+1}\} \subseteq H$ *are vertices of a non-degenerate* $(d+1)$*-simplex, then for all* $1 \leq i \neq j \leq d+1$*:*

$$\frac{|\operatorname{g}_d\!\sin_0(v_1, \ldots, v_{d+1})|^d}{M_d(v_1 - v_{d+1}, \ldots, v_d - v_{d+1})} = \\ \frac{|\operatorname{g}_d\!\sin_{v_i}(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_{d+1})|^d}{M_d(v_1 - v_j, \ldots, v_{j-1} - v_j, v_{j+1} - v_j, \ldots, v_{i-1} - v_j, -v_j, v_{i+1} - v_j, \ldots, v_{d+1} - v_j)}\,.$$

The proof follows from the definition of $|\operatorname{g}_d\!\sin_0|$. A reformulation of this law is the invariance of the function $|\operatorname{g}_d\!\sin_u(v_1, \ldots, v_{d+1})|/M_d(v_1 - u, \ldots, v_{d+1} - u)^{1/d}$ with respect to permuting its arguments, $u$ included.

### 2.2.1 Elevation and Dihedral Angles

For a complete and non-trivial subspace $W \subseteq H$ and $u \in H \setminus \{0\}$, we define the *elevation angle* of $u$ with respect to $W$ to be the smallest angle that $u$ makes with any element $w \in W \setminus \{0\}$, and we denote this angle by $\theta(u, W)$. More formally, in this case

$$\theta(u, W) = \min_{w \in W \setminus \{0\}} \left\{ \arccos\left( \left\langle \frac{u}{\|u\|}, \frac{w}{\|w\|} \right\rangle \right) \right\}.$$

If $u = 0$, then we take $\theta(0, W) = 0$. We call the sines of these angles *elevation sines* and note the following formula for computing them:

$$\sin(\theta(u, W)) = \frac{\operatorname{dist}(u, W)}{\|u\|}\,. \tag{2.4}$$

If $V$ is a complete subspace of $H$ and $v_1, v_2 \in H$, we define the *maximal elevation angle* of $v_1$ and $v_2$ with respect to $V$, denoted by $\Theta(v_1, v_2, V)$, as follows:

$$\Theta(v_1, v_2, V) = \max\{\theta(v_1, V), \theta(v_2, V)\}. \tag{2.5}$$

Given finite dimensional subspaces $W$ and $V$ of $H$ such that $\dim(W) = \dim(V)$ and $\dim(W \cap V) = \dim(W) - 1$, we define the *dihedral angle* between $W$ and $V$ along $W \cap V$ to be the acute angle between the normals of $W \cap V$ in $W$ and $V$. We denote this angle by $\alpha(W, V)$. We call the sines of such angles *dihedral sines* and note the following formula for computing them:

$$\sin(\alpha(W, V)) = \frac{\operatorname{dist}(w, V)}{\operatorname{dist}(w, W \cap V)} = \frac{\operatorname{dist}(v, W)}{\operatorname{dist}(v, W \cap V)}, \quad \text{for all } w \in W \setminus V \text{ and } v \in V \setminus W. \tag{2.6}$$

### 2.2.2 Product Formulas for Generalized Sine Functions

Two of the most useful properties of the high-dimensional sine functions are their decompositions as products of lower-dimensional sines. For $v_1, \ldots, v_{d+1} \in H$ and $S = \{v_1, \ldots, v_{d+1}\}$, we formulate those decompositions as follows.

**Proposition 2.2.3.**

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})|^d = \left(\prod_{i=1}^{d} \sin\left(\alpha_0\left[\mathrm{L}_{S\setminus\{v_{d+1}\}}, \mathrm{L}_{S\setminus\{v_i\}}\right]\right)\right) \cdot |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_d)|^{d-1}.$$

**Proposition 2.2.4.** $|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| = \sin\left(\theta\left[v_{d+1}, \mathrm{L}_{S\setminus\{v_{d+1}\}}\right]\right) \cdot |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_d)|$.

Proposition 2.2.3 was established in [15, equation 7], and Proposition 2.2.4 can be established given the fact that

$$\begin{aligned} |M_{d+1}(v_1, \ldots, v_{d+1})| &= \operatorname{dist}(v_{d+1}, \mathrm{L}_{S\setminus\{v_{d+1}\}}) \cdot M_d(v_1, \ldots, v_d) \\ &= \|v_{d+1}\| \cdot \sin\left(\theta\left[v_{d+1}, \mathrm{L}_{S\setminus\{v_{d+1}\}}\right]\right) \cdot M_d(v_1, \ldots, v_d). \end{aligned} \tag{2.7}$$

## 2.3 Functional Identities for High-Dimensional Sine Functions

Throughout this section we assume that $\dim(H) = d+1$ and formulate identities for $\mathrm{p}_d\mathrm{sin}$ and $\mathrm{g}_d\mathrm{sin}$. We denote the vectors used for the arguments of the latter functions by $u$, $v_1, \ldots, v_{d+1} \in H$, and assume the following: $\{v_1, \ldots, v_{d+1}\}$ is a basis for $H$, $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$, and $u$ is not a scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$, in particular, $u \neq 0$.

The basics of our identities are exemplified in Figure 2.1 and described as follows. We introduce positive free parameters $\{\beta_i\}_{i=1}^{d+1}$, and we note that $\mathrm{C}_{\mathrm{poly}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$. We express the vector $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$ as a linear combination of $\{\beta_i v_i\}_{i=1}^{d+1}$ with coefficients $\{\lambda_i\}_{i=1}^{d+1}$, that is,

$$u = \sum_{i=1}^{d+1} \lambda_i \cdot \beta_i v_i \,. \tag{2.8}$$

We note that since $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$ and $u \neq 0$, we have that $\sum_{i=1}^{d+1} \lambda_i > 0$. We then define

$$\tilde{u} := \left( \sum_{i=1}^{d+1} \lambda_i \right)^{-1} u, \tag{2.9}$$

and observe that

$$\tilde{u} \in \mathrm{A}_{\mathrm{ffn}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}). \tag{2.10}$$

Finally, Proposition 2.1.1 gives the fundamental identity used to establish all of the following identities:

$$\det(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \sum_{i=1}^{d+1} \det(\beta_1 v_1, \ldots, \beta_{i-1} v_{i-1}, \tilde{u}, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1}). \tag{2.11}$$

In Subsection 2.3.1 we develop identities for $\mathrm{p}_d\mathrm{sin}_0$ by direct application of the above equations. Similarly, in Subsection 2.3.2 we develop identities for $\mathrm{g}_d\mathrm{sin}_0$ following the same equations. If $d = 1$, both identities for $\mathrm{p}_d\mathrm{sin}_0$ and $\mathrm{g}_d\mathrm{sin}_0$ reduce to a functional equation satisfied by the sine function.

### 2.3.1 Identities for $\mathrm{p}_d\mathrm{sin}_0$

Dividing both sides of equation (2.11) by $\prod_{i=1}^{d+1} \|\beta_i v_i\|$, we obtain

$$\mathrm{p}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \sum_{i=1}^{d+1} P_i \cdot \mathrm{p}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{i-1} v_{i-1}, \tilde{u}, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1}),$$

where

$$P_i \equiv P_i\left(\{\beta_i\}_{i=1}^{d+1},\, \{v_i\}_{i=1}^{d+1},\, u\right) = \frac{\|\tilde{u}\|}{\|\beta_i v_i\|}. \tag{2.12}$$

Applying either the law of sines or the formal definition of $\mathrm{p}_1\mathrm{sin}$, we express the coefficients $P_i$ as follows:

$$P_i = \frac{\mathrm{p}_1\mathrm{sin}_0(-\beta_i v_i, \tilde{u} - \beta_i v_i)}{\mathrm{p}_1\mathrm{sin}_0(\tilde{u}, -\tilde{u} + \beta_i v_i)}\,. \tag{2.13}$$

By the positive scale-invariance of $\mathrm{p}_d\mathrm{sin}_0$ we obtain that

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \sum_{i=1}^{d+1} P_i \cdot \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{2.14}$$

By choosing different coefficients $\{\beta_i\}_{i=1}^{d+1}$ we can obtain different identities for $\mathrm{p}_d\mathrm{sin}_0$. There are only $d$ degrees of freedom in forming such identities due to the restriction of equation (2.11). In Subsection 2.4.2 we will use the following choice of $\{\beta_i\}_{i=1}^{d+1}$:

$$\beta_i = \frac{1}{\|v_i\|},\ \ i = 1, \ldots, d+1. \tag{2.15}$$

The coefficients $\{P_i\}_{i=1}^{d+1}$, as described in equation (2.12), thus obtain the form,

$$P_1 = \ldots = P_{d+1} = \|\tilde{u}\| \tag{2.16}$$

and consequently equation (2.14) becomes

$$\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \|\tilde{u}\| \cdot \sum_{i=1}^{d+1} \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}). \tag{2.17}$$

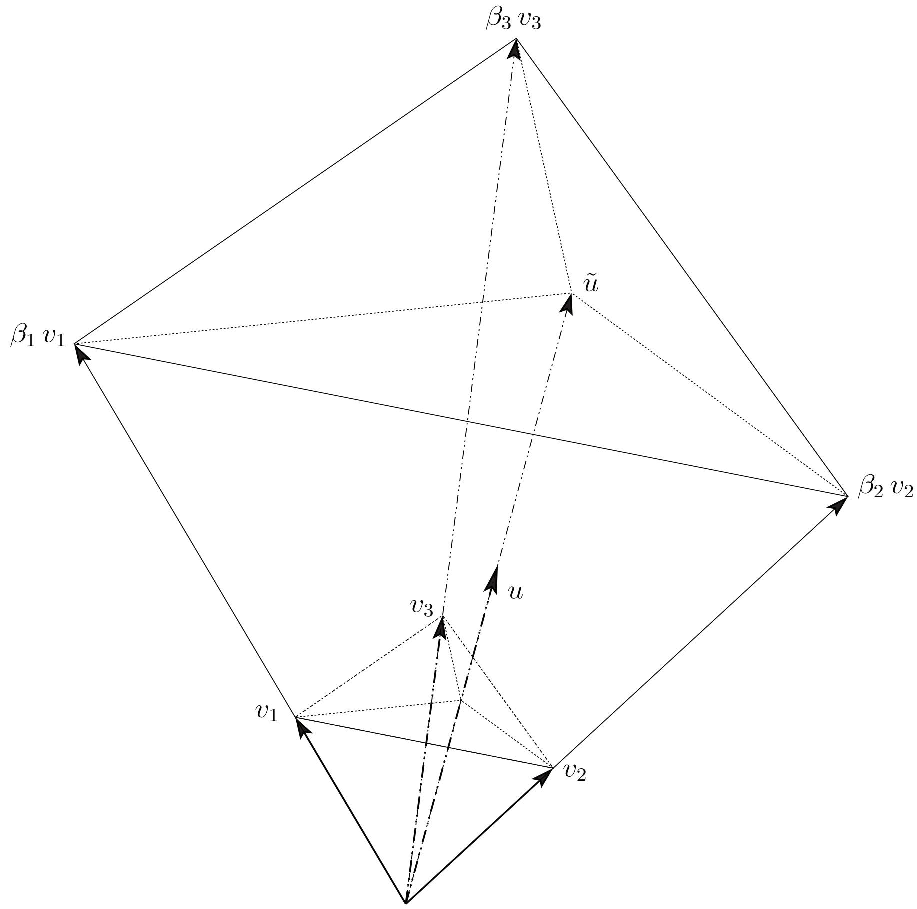


Figure 2.1: Exemplifying the basic construction of this section, when $d = 2$ and $H = \mathbb{R}^3$: We plot four particular vectors $v_1$, $v_2$, $v_3$ and $u$ and note that in this special case $u$ is not contained in the affine plane spanned by $v_1$, $v_2$ and $v_3$. We scale the latter three vectors arbitrarily by the positive parameters $\beta_1$, $\beta_2$ and $\beta_3$ and plot the resulting vectors. We form $\tilde{u}$ by scaling $u$ so that it is in the affine plane spanned by $\beta_1 v_1$, $\beta_2 v_2$ and $\beta_3 v_3$.

At last we exemplify the above identities when $d = 1$. We denote the angle between $v_1$ and $u$ by $\alpha > 0$, and the angle between $u$ and $v_2$ by $\beta > 0$, so that $\alpha + \beta$ is the angle between $v_1$ and $v_2$. We note that by the two assumptions of linear independence and $u \in \mathrm{C}_{\mathrm{poly}}(v_1, v_2)$ we have that $\alpha + \beta < \pi$. We denote the angle between $-u$ and $v_1 - u$ by $\delta$, where $\beta < \delta < \pi - \alpha$. The parameter $\delta$ represents the unique degree of freedom.

In this case, equations (2.13) and (2.14) reduce to the following trigonometric identity:

$$\sin(\alpha + \beta) = \frac{\sin(\alpha + \delta)}{\sin(\delta)} \cdot \sin(\beta) \; + \; \frac{\sin(\delta - \beta)}{\sin(\delta)} \cdot \sin(\alpha). \tag{2.18}$$

This identity generalizes to all $\alpha, \beta \in \mathbb{R}$ and $\delta \in \mathbb{R} \setminus \pi\mathbb{Z}$. It was used in [37] and is also very natural when establishing Ptolemy's theorem by trigonometry.

Furthermore, equation (2.17) reduces to the trigonometric identity

$$\sin(\alpha + \beta) = \frac{\sin(\frac{\alpha+\beta}{2})}{\sin(\frac{\alpha-\beta}{2})} \cdot \big(\sin(\alpha) - \sin(\beta)\big),$$

which can also be derived from equation (2.18) by setting $\delta = (\beta - \alpha)/2$.

### 2.3.2 Identities for $\mathrm{g}_d\mathrm{sin}_0$

We now establish similar identities for $\mathrm{g}_d\mathrm{sin}_0$. Dividing both sides of equation (2.11) by $\prod_{j=1}^{d+1} \left(M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})\right)^{1/d}$ we obtain that

$$\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1}) = \sum_{i=1}^{d+1} Q_i \cdot \mathrm{g}_d\mathrm{sin}_0(\beta_1 v_1, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1}), \tag{2.19}$$

where

$$Q_i = \left(\prod_{\substack{j=1\\ j\neq i}}^{d+1} \frac{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{i-1}v_{i-1}, \tilde{u}, \beta_{i+1}v_{i+1}, \ldots, \beta_{d+1}v_{d+1})}{M_d(\beta_1 v_1, \ldots, \beta_{j-1}v_{j-1}, \beta_{j+1}v_{j+1}, \ldots, \beta_{d+1}v_{d+1})}\right)^{1/d}. \tag{2.20}$$

By the positive scale-invariance of $\mathrm{g}_d\mathrm{sin}_0$, we rewrite equation (2.19) as

$$\mathrm{g}_d\mathrm{sin}_0(v_1,\dots,v_{d+1}) = \sum_{i=1}^{d+1} Q_i \cdot \mathrm{g}_d\mathrm{sin}_0(v_1,\dots,v_{i-1},u,v_{i+1},\dots,v_{d+1}). \tag{2.21}$$

We can express the coefficients $Q_i$ in different ways. First, we note that

$$Q_i = \prod_{\substack{j=1 \\ j\neq i}}^{d+1} \frac{\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1,\dots,\beta_{j-1}v_{j-1},0,\beta_{j+1}v_{j+1},\dots,\beta_{i-1}v_{i-1},\tilde{u},\beta_{i+1}v_{i+1},\dots,\beta_{d+1}v_{d+1})}{\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1,\dots,\beta_{j-1}v_{j-1},0,\beta_{j+1}v_{j+1},\dots,\beta_{d+1}v_{d+1})}. \tag{2.22}$$

The fact that the absolute values of both equations (2.20) and (2.22) are the same follows from the generalized law of sines (see Proposition 2.2.2). Moreover, the terms $\{Q_i\}_{i=1}^{d+1}$ in equation (2.22) are positive (see Appendix 5.1.2), as are the corresponding terms of equation (2.20).

A different expression for $\{Q_i\}_{i=1}^{d+1}$ can be obtained as follows. We set $S = \{v_1,\dots,v_{d+1}\}$ and notice that equation (2.7) implies that for all $1 \le i < j \le d+1$ :

$$\frac{M_d(\beta_1 v_1,\dots,\beta_{j-1}v_{j-1},\beta_{j+1}v_{j+1},\dots,\beta_{i-1}v_{i-1},\tilde{u},\beta_{i+1}v_{i+1},\dots,\beta_{d+1}v_{d+1})}{M_d(\beta_1 v_1,\dots,\beta_{j-1}v_{j-1},\beta_{j+1}v_{j+1},\dots,\beta_{d+1}v_{d+1})} = \frac{\mathrm{dist}\left(\tilde{u},\mathrm{L}_{S\setminus\{v_i,v_j\}}\right)}{\mathrm{dist}\left(\beta_i v_i,\mathrm{L}_{S\setminus\{v_i,v_j\}}\right)}. \tag{2.23}$$

Therefore, the coefficients $Q_i$, $i = 1,\dots,d+1$, have the form

$$Q_i = \prod_{\substack{j=1 \\ j\neq i}}^{d+1} \left(\frac{\mathrm{dist}\left(\tilde{u},\mathrm{L}_{S\setminus\{v_i,v_j\}}\right)}{\mathrm{dist}\left(\beta_i v_i,\mathrm{L}_{S\setminus\{v_i,v_j\}}\right)}\right)^{1/d}. \tag{2.24}$$

By further application of equation (2.4), we obtain that

$$Q_i = \frac{\|\tilde{u}\|}{\|\beta_i v_i\|} \cdot \prod_{\substack{j=1 \\ j\neq i}}^{d+1} \left(\frac{\sin\left(\theta\left[\tilde{u},\mathrm{L}_{S\setminus\{v_i,v_j\}}\right]\right)}{\sin\left(\theta\left[\beta_i v_i,\mathrm{L}_{S\setminus\{v_i,v_j\}}\right]\right)}\right)^{1/d}. \tag{2.25}$$

It thus follows from equations (2.12) and (2.25) that

$$Q_i = P_i \cdot \prod_{\substack{j=1 \\ j\neq i}}^{d+1} \left(\frac{\sin\left(\theta\left[\tilde{u},\mathrm{L}_{S\setminus\{v_i,v_j\}}\right]\right)}{\sin\left(\theta\left[\beta_i v_i,\mathrm{L}_{S\setminus\{v_i,v_j\}}\right]\right)}\right)^{1/d}. \tag{2.26}$$

There are different possible choices for the parameters $\{\beta_i\}_{i=1}^{d+1}$, and we present a specific choice and its consequence in Subsection 2.4.1.

## 2.4 Simplex Inequalities for High-Dimensional Sine Functions

In this section we prove Theorem 1.2.1, that is, we show that the functions $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ are $d$-semimetrics. We establish it separately for each of the functions in the following theorems.

**Theorem 2.4.1.** *If* $v_1, \ldots, v_{d+1} \in H$ *and* $u \in H \setminus \{0\}$*, then*

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|.$$

**Theorem 2.4.2.** *If* $v_1, \ldots, v_{d+1} \in H$ *and* $u \in H \setminus \{0\}$*, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|.$$

The proofs of both theorems are parallel. We first prove them when $\dim(H) = d + 1$ by applying the identities developed in Section 2.3. We then notice two phenomena of dimensionality reduction. The first is that projection reduces the values of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$. The second is that if $u \in (\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\}))^\perp$, then the corresponding simplex inequality for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ reduces to a relaxed simplex inequality of one term and constant 1. We remark that the second phenomenon of dimensionality reduction is not fully necessary for concluding the theorems, i.e., using the regular simplex inequality is fine, but we find it worth mentioning.

We prove Theorem 2.4.1 in Subsection 2.4.1 and Theorem 2.4.2 in Subsection 2.4.2.

### 2.4.1 The Proof of Theorem 2.4.1

#### The Case of $\mathbf{dim(H) = d + 1}$

We establish the following proposition.

**Lemma 2.4.1.** *If* $\dim(H) = d + 1$, $\{v_1, \ldots, v_{d+1}\} \subseteq H$ *and* $u \in H \setminus \{0\}$, *then*

$$|\operatorname{g}_d\!\sin_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\operatorname{g}_d\!\sin_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|. \tag{2.27}$$

*Proof of Lemma 2.4.1.* Let $S = \{v_1, \ldots, v_{d+1}\}$. If $S$ is linearly dependent, then $|\operatorname{g}_d\!\sin_0(v_1, \ldots, v_{d+1})| = 0$, and the inequality holds. Similarly, if $u$ is scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$, then the inequality holds as an equality. Thus, we may assume that $\operatorname{Sp}(S) = H$ and that $u$ is not a scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$.

Furthermore, we may assume that $u \in \operatorname{C}_{\text{poly}}(v_1, \ldots, v_{d+1})$. Indeed, if this is not the case, then we may apply the following procedure. We express $u$ as a linear combination of the vectors $\{v_i\}_{i=1}^{d+1}$ using the coefficients $\{\lambda_i\}_{i=1}^{d+1}$:

$$u = \sum_{i=1}^{d+1} \lambda_i \, v_i = \sum_{i=1}^{d+1} |\lambda_i| \, \operatorname{sign}(\lambda_i) \, v_i, \text{ where } \sum_{i=1}^{d+1} |\lambda_i| \neq 0.$$

For all $1 \leq i \leq d + 1$, we let

$$\hat{v}_i = \begin{cases} \operatorname{sign}(\lambda_i) \, v_i, & \text{if } \lambda_i \neq 0, \\ v_i, & \text{otherwise.} \end{cases}$$

We note that $u = \sum_{i=1}^{d+1} |\lambda_i| \cdot \hat{v}_i$, and therefore $u \in \operatorname{C}_{\text{poly}}(\hat{v}_1, \ldots, \hat{v}_{d+1})$. Moreover, by the scale-invariance of the function $|\operatorname{g}_d\!\sin_0|$ we obtain that the required inequality (equation (2.27)) holds if and only if

$$|\,\mathrm{g}_d\mathrm{sin}_0(\hat{v}_1,\ldots,\hat{v}_{d+1})| \le \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(\hat{v}_1,\ldots,\hat{v}_{i-1},u,\hat{v}_{i+1},\ldots,\hat{v}_{d+1})|.$$

Thus it is sufficient to consider the case where $u \in \mathrm{C}_{\mathrm{poly}}(v_1,\ldots,v_{d+1})$. We observe that this assumption and equation (2.10) imply that

$$\tilde{u} \in \mathrm{C}_{\mathrm{hull}}(v_1,\ldots,v_{d+1})\,. \tag{2.28}$$

We next obtain the desired inequality by using equation (2.21) together with the form of $\{Q_i\}_{i=1}^{d+1}$ set in equation (2.24). The question is how to choose the positive coefficients $\{\beta_i\}_{i=1}^{d+1}$ such that $Q_i \le 1$, $i = 1,\ldots,d+1$. Avoiding a messy optimization argument, we will show that there is a natural geometric choice for the parameters $\{\beta_i\}_{i=1}^{d+1}$. Indeed, letting

$$\beta_i = M_d(v_1,\ldots,v_{i-1},v_{i+1},\ldots v_{d+1}),\ \ i = 1,\ldots,d+1,$$

we have that for all $1 \le i \le d+1$

$$\begin{aligned} &M_d(\beta_1 v_1,\ldots \beta_{i-1}v_{i-1},\beta_{i+1}v_{i+1},\ldots,\beta_{d+1}v_{d+1}) = \\ &\quad M_d(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{d+1})\cdot\prod_{\substack{j=1\\ j\neq i}}^{d+1}\beta_j = \prod_{j=1}^{d+1}\beta_j = \prod_{j=1}^{d+1} M_d(v_1,\ldots,v_{j-1},v_{j+1},\ldots,v_{d+1}). \end{aligned} \tag{2.29}$$

In particular, for the simplex with vertices $\{0,\beta_1 v_1,\ldots,\beta_{d+1}v_{d+1}\}$, we obtain equal contents for all $d$-faces containing the vertex 0.

Another geometric property of the resulting simplex is that if $1 \le k \ne i \le d+1$, then $\beta_k v_k$ and $\beta_i v_i$ are of equal distance from the $(d-1)$-plane $\mathrm{L}_{S\setminus\{v_i,v_k\}}$. That is,

$$\mathrm{dist}(\beta_k v_k, \mathrm{L}_{S\setminus\{v_i,v_k\}}) = \mathrm{dist}(\beta_i v_i, \mathrm{L}_{S\setminus\{v_i,v_k\}}),\ \text{ where } 1 \le k \ne i \le d+1.$$

This is a direct result of equation (2.7) and the fact that the $d$-dimensional contents of the relevant faces are equal (recall equation (2.29)). Then, denoting the common distance for both $\beta_k v_k$ and $\beta_i v_i$ from $\mathrm{L}_{S\setminus\{v_i,v_k\}}$ by $d_{ik}$, we note that

$$\{\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}\} \subseteq \mathrm{T}_{\mathrm{ube}}\left(\mathrm{L}_{S\setminus\{v_i,v_k\}}, d_{ik}\right) \quad \text{for all} \;\; 1 \leq k \neq i \leq d+1\,.$$

Since $\mathrm{T}_{\mathrm{ube}}\left(\mathrm{L}_{S\setminus\{v_i,v_k\}}, d_{ik}\right)$ is convex,

$$\mathrm{C}_{\mathrm{hull}}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) \subseteq \mathrm{T}_{\mathrm{ube}}\left(\mathrm{L}_{S\setminus\{v_i,v_k\}}, d_{ik}\right) \quad \text{for all} \;\; 1 \leq k \neq i \leq d+1\,.$$

This observation together with equation (2.28) imply that

$$\tilde{u} \in \mathrm{T}_{\mathrm{ube}}\left(\mathrm{L}_{S\setminus\{v_i,v_k\}}, d_{ik}\right) \quad \text{for all} \;\; 1 \leq k \neq i \leq d+1,$$

that is,

$$\frac{\mathrm{dist}\left(\tilde{u}, \mathrm{L}_{S\setminus\{v_i,v_k\}}\right)}{\mathrm{dist}\left(\beta_i v_i, \mathrm{L}_{S\setminus\{v_i,v_k\}}\right)} \leq 1 \quad \text{for all } 1 \leq k \neq i \leq d+1. \tag{2.30}$$

It follows from equations (2.24) and (2.30) that $0 \leq Q_i \leq 1$ for all $1 \leq i \leq d+1$ and the desired inequality is concluded. $\square$

### Dimensionality Reduction I

We show that projections reduce the value of $|\,\mathrm{g}_d\mathrm{sin}_0\,|$.

**Lemma 2.4.2.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $\{v_1, \ldots, v_d\} \subseteq V$, $u \in H$, and $P_V : H \to V$ is the orthogonal projection onto $V$, then*

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_d, P_V(u))| \leq |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_d, u)|. \tag{2.31}$$

*Proof of Lemma 2.4.2.* We form the sets $B = \{v_1, \ldots, v_d\}$, $S = \{v_1, \ldots, v_d, u\}$ and $\tilde{S} = \{v_1, \ldots, v_d, P_V(u)\}$. In order to conclude the lemma it is sufficient to prove the following inequality for dihedral angles:

$$\sin\left(\alpha_0\left[\mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}},\mathrm{L}_{\tilde{S}\setminus\{v_i\}}\right]\right)\leq\sin\left(\alpha_0\left[\mathrm{L}_{S\setminus\{u\}},\mathrm{L}_{S\setminus\{v_i\}}\right]\right),\quad\text{for all } 1\leq i\leq d. \tag{2.32}$$

Indeed, equation (2.31) is a direct consequence of both equation (2.32) and the product formula for $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ of Proposition 2.2.3.

In order to prove the bound of equation (2.32) it will be convenient to use the following orthogonal projections, while recalling that $B=\{v_1,\ldots,v_d\}$:

$$P_B : H \to \mathrm{L}_B,$$

$$N_B : H \to (\mathrm{L}_B)^{\perp}\cap V,$$

$$P_i : H \to \mathrm{L}_{B\setminus\{v_i\}},\ 1\leq i\leq d,$$

$$N_i : H \to (\mathrm{L}_{B\setminus\{v_i\}})^{\perp}\cap \mathrm{L}_B,\ 1\leq i\leq d.$$

We also define

$$N_V := I - P_V.$$

We note that $u=P_V(u)+N_V(u)=P_i(u)+N_i(u)+N_B(u)+N_V(u)$, for all $1\leq i\leq d$.

If $N_B(u)=0$, then $P_V(u)=P_B(u)$ and the set $\{v_1,\ldots,v_d,P_V(u)\}$ is linearly dependent. Hence, $|\,\mathrm{g}_d\mathrm{sin}_0(v_1,\ldots,v_d,P_V(u))|=0$ and the inequality holds in this case.

If $N_B(u)\neq 0$, we apply equation (2.6) and obtain that for all $1\leq i\leq d$

$$\sin\left(\alpha_0\left[\mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}},\mathrm{L}_{\tilde{S}\setminus\{v_i\}}\right]\right)=\frac{\mathrm{dist}\left(P_V(u),\mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right)}{\mathrm{dist}\left(P_V(u),\mathrm{L}_{\tilde{S}\setminus\{P_V(u),v_i\}}\right)}=\frac{\|N_B(u)\|}{\|N_B(u)+N_i(u)\|}, \tag{2.33}$$

and

$$\sin\left(\alpha_0\left[\mathrm{L}_{S\setminus\{u\}},\mathrm{L}_{S\setminus\{v_i\}}\right]\right)=\frac{\mathrm{dist}\left(u,\mathrm{L}_{S\setminus\{u\}}\right)}{\mathrm{dist}\left(u,\mathrm{L}_{S\setminus\{u,v_i\}}\right)}=\frac{||N_B(u)+N_V(u)||}{||N_i(u)+N_B(u)+N_V(u)||}. \tag{2.34}$$

For any fixed $1 \leq i \leq d$, the vectors $N_B(u)$, $N_V(u)$, and $N_i(u)$ are mutually orthogonal, and therefore

$$\sin\left(\alpha_0\left[\mathrm{L}_{S\setminus\{u\}}, \mathrm{L}_{S\setminus\{v_i\}}\right]\right) = \frac{\|N_B(u)\|}{\|N_i(u) + N_B(u)\|}\sqrt{\frac{\frac{||N_V(u)||^2}{||N_B(u)||^2} + 1}{\frac{||N_V(u)||^2}{||N_B(u)||^2+||N_i(u)||^2} + 1}} \geq \frac{||N_B(u)||}{||N_i(u) + N_B(u)||}\,. \tag{2.35}$$

Equation (2.32) follows from equations (2.33) and (2.35), and thus the lemma is concluded.

□

**Dimensionality Reduction II**

We show how to relax the simplex inequality stated in Theorem 2.4.1 in the following special case.

**Lemma 2.4.3.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $\{v_1, \ldots, v_{d+1}\} \subseteq V$, and $u \in V^\perp \setminus \{0\}$, then*

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots v_{d+1})| \leq |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|, \quad \text{for all } 1 \leq i \leq d+1. \tag{2.36}$$

*Proof of Lemma 2.4.3.* We assume without loss of generality that $\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\}) = V$ (otherwise equation (2.36) follows trivially). We define $S_i = \{v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}\}$ for all $1 \leq i \leq d+1$. Since $u \in V^\perp \setminus \{0\}$ we obtain from equation (2.6) that

$$\sin\left(\alpha_0\left[\mathrm{L}_{S_i\setminus\{u\}}, \mathrm{L}_{S_i\setminus\{v_j\}}\right]\right) = 1, \quad \text{for all } 1 \leq j \neq i \leq d+1. \tag{2.37}$$

Combining equation (2.37) with the product formula for $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ (Proposition 2.2.3) we get the following equality for all $1 \leq i \leq d+1$,

$$
\begin{aligned}
&|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|^d \\
&\qquad = |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})|^{d-1} \prod_{\substack{j=1 \\ j \neq i}}^{d+1} \sin\left(\alpha_0\left[\mathrm{L}_{S_i \setminus \{u\}}, \mathrm{L}_{S_i \setminus \{v_j\}}\right]\right) \\
&\qquad = |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})|^{d-1}.
\end{aligned} \tag{2.38}
$$

By further application of the product formula for $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ we obtain that for all $1 \leq i \leq d{+}1$,

$$|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots v_{d+1})|^d \leq |\,\mathrm{g}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_{i-1}, v_{i+1} \ldots, v_{d+1})|^{d-1}. \tag{2.39}$$

Equation (2.36) thus follows from equations (2.38) and (2.39). □

**Conclusion of Theorem 2.4.1**

Let $P$ denote the orthogonal projection from $H$ onto $\mathrm{Sp}\{v_1, \ldots, v_{d+1}\}$. If $P(u) = 0$, then we conclude the Theorem from Lemma 2.4.3.

If $P(u) \neq 0$, then we conclude the theorem by applying Lemmata 2.4.1 and 2.4.2 successively as follows:

$$
\begin{aligned}
|\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| &\leq \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, P(u), v_{i+1}, \ldots, v_{d+1})| \\
&\leq \sum_{i=1}^{d+1} |\,\mathrm{g}_d\mathrm{sin}_0(v_1, \ldots v_{i-1}, u, v_{i+1} \ldots, v_{d+1})|.
\end{aligned}
$$ □

### 2.4.2 The Proof of Theorem 2.4.2

Here we prove essentially the same three lemmata of Subsection 2.4.1 for the function $|\,\mathrm{p}_d\mathrm{sin}_0\,|$.

### The Case of $\mathbf{dim(H) = d + 1}$

We establish here the following proposition.

**Lemma 2.4.4.** *If* $\dim(H) = d+1$*,* $v_1, \ldots, v_{d+1} \in H$ *and* $u \in H \setminus \{0\}$*, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|. \tag{2.40}$$

*Proof of Lemma 2.4.4.* Similarly as in the proof of Lemma 2.4.1, we can assume that $v_1, \ldots, v_{d+1}$ are linearly independent, $u$ is not a scalar multiple of any of the individual basis vectors $v_1, \ldots, v_{d+1}$ and $u \in \mathrm{C}_{\mathrm{poly}}(v_1, \ldots, v_{d+1})$. Using the choice of $\{\beta_i\}_{i=1}^{d+1}$ specified in equation (2.15), we have that $\|\beta_i v_i\| \leq 1$ for all $1 \leq i \leq d+1$. In view of equation (2.10) we can extend this bound to $\tilde{u}$, i.e., we have that $\|\tilde{u}\| \leq 1$. The lemma then follows by combining equation (2.17) with the latter bound. $\square$

### Dimensionality Reduction I

We show that projections reduce the value of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$.

**Lemma 2.4.5.** *If* $V$ *is a* $(d+1)$*-dimensional subspace of* $H$*,* $v_1, \ldots, v_d \in V$*,* $u \in H$*, and* $P_V : H \to V$ *is the orthogonal projection onto* $V$*, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_d, P_V(u))| \leq |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_d, u)|. \tag{2.41}$$

*Proof of Lemma 2.4.5.* We form the sets $B = \{v_1, \ldots v_d\} \subseteq V$, $S = \{v_1, \ldots, v_d, u\}$ and $\tilde{S} = \{v_1, \ldots, v_d, P_V(u)\}$. In order to conclude the lemma, it is sufficient to prove that

$$\sin\left(\theta\left[P_V(u), \mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right]\right) \leq \sin\left(\theta\left[u, \mathrm{L}_{S\setminus\{u\}}\right]\right). \tag{2.42}$$

Indeed, equation (2.41) is a direct consequence of equation (2.42) and the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ (Proposition 2.2.4).

In order to prove equation (2.42), it will be convenient to use the following orthogonal projections:

$$P_B : H \to \mathrm{L}_B,$$

$$N_B : H \to (\mathrm{L}_B)^{\perp} \cap V.$$

We also define

$$N_V := I - P_V\,.$$

We note that $u = P_V(u) + N_V(u) = P_B(u) + N_B(u) + N_V(u)$.

If $N_B(u) = 0$, then $P_V(u) = P_B(u) \in \mathrm{L}_B$, and the inequality (equation (2.41)) holds trivially since the set $\tilde{S} = \{v_1, \ldots, v_d, P_V(u)\}$ is linearly dependent.

If $N_B(u) \neq 0$, we apply equation (2.4) to obtain that

$$\sin\left(\theta\left[u, \mathrm{L}_{S\setminus\{u\}}\right]\right) = \frac{\operatorname{dist}\left(u, \mathrm{L}_{S\setminus\{u\}}\right)}{\|u\|} = \frac{\|N_B(u) + N_V(u)\|}{\|P_B(u) + N_B(u) + N_V(u)\|}\,,$$

and

$$\sin\left(\theta\left[P_V(u), \mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right]\right) = \frac{\operatorname{dist}\left(P_V(u), \mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right)}{\|P_V(u)\|} = \frac{\|N_B(u)\|}{\|P_B(u) + N_B(u)\|}\,.$$

Thus,

$$\sin\left(\theta\left[u, \mathrm{L}_{S\setminus\{u\}}\right]\right) = \sin\left(\theta\left[P_V(u), \mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right]\right)\sqrt{\frac{1 + \frac{\|N_V(u)\|^2}{\|N_B(u)\|^2}}{1 + \frac{\|N_V(u)\|^2}{\|P_B(u)+N_B(u)\|^2}}} \geq \sin\left(\theta\left[P_V(u), \mathrm{L}_{\tilde{S}\setminus\{P_V(u)\}}\right]\right).$$

That is, equation (2.42) is verified and the lemma is concluded. $\square$

**Dimensionality Reduction II**

We show how to relax the simplex inequality stated in Theorem 2.4.2 in the following special case.

**Lemma 2.4.6.** *If $V$ is a $(d+1)$-dimensional subspace of $H$, $v_1, \ldots, v_{d+1} \in V$, and $u \in V^{\perp} \setminus \{0\}$, then*

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq |\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})|, \quad \text{for } i = 1, \ldots, d+1\,. \tag{2.43}$$

*Proof of Lemma 2.4.6.* We assume without loss of generality that $\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\}) = V$. We define the sets $S_i = \{v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1}\}$ for all $1 \leq i \leq d+1$. Since $u \in V^{\perp} \setminus \{0\}$ we obtain from equation (2.4) that

$$\sin\left(\theta\left[u, \mathrm{L}_{S_i \setminus \{u\}}\right]\right) = 1, \quad \text{for all } 1 \leq i \leq d+1\,. \tag{2.44}$$

Combining equation (2.44) with the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ (Proposition 2.2.4), we get the following equality for all $i = 1, \ldots, d+1$

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| = |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0\,(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})\,|\,. \tag{2.45}$$

By further application of the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$, we obtain that for all $i = 1, \ldots, d+1$:

$$|\,\mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq |\,\mathrm{p}_{d\text{-}1}\mathrm{sin}_0(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1})|. \tag{2.46}$$

Combining equations (2.45) and (2.46) we conclude equation (2.43). □

**Conclusion of Theorem 2.4.2**

Let $P$ denote the orthogonal projection of $H$ onto $\mathrm{Sp}(\{v_1, \ldots, v_{d+1}\})$. If $P(u) = 0$, then we conclude the theorem from Lemma 2.4.6.

If $P(u) \neq 0$, then applying Lemmata 2.4.4 and 2.4.5 successively we obtain that

$$
\begin{aligned}
|\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{d+1})| &\leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{i-1},P(u),v_{i+1},\ldots,v_{d+1})| \\
&\leq \sum_{i=1}^{d+1} |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots v_{i-1},u,v_{i+1}\ldots,v_{d+1})|\,. \quad \square
\end{aligned}
$$

## 2.5 Concentration Inequalities for the Polar Sine

In this section we prove Theorem 1.2.2. As explained in the introduction, we interpret this theorem as indicating that the polar sine $|\,\mathrm{p}_d\mathrm{sin}_w\,|$ satisfies a relaxed simplex inequality of two terms with "high probability at all reasonable scales and locations" with respect to a $\gamma$-regular measure, $\mu$, for $d-1<\gamma\leq d$. The "local probabilities" are given by scaling $\mu$ by the measure of the ball.

### Notation, Definitions and Elementary Propositions

For convenience of our notation, we assume that $w=0$ and $0\in\mathrm{supp}(\mu)$, and thus establish most of the propositions for $\mathrm{p}_d\mathrm{sin}_0$. They can be generalized for $\mathrm{p}_d\mathrm{sin}_w$ via equation (2.3).

Throughout this section we extensively use the definitions and notation for elevation, maximal elevation, and dihedral angles formulated in Subsection 2.2. We often fix $S=\{v_1,\ldots,v_{d+1}\}\subseteq H$. If $0\leq\epsilon\leq 1$ and $1\leq i\leq d+1$, then we denote by $\mathrm{C_{one}}^i(\epsilon)$, the cone

$$\mathrm{C_{one}}^i(\epsilon)=\mathrm{C_{one}}\left(\epsilon\cdot\theta\left[v_i,\mathrm{L}_{S\setminus\{v_i\}}\right],\mathrm{L}_{S\setminus\{v_i\}},0\right).$$

If $0\leq\epsilon\leq 1$ and $1\leq i<j\leq d+1$, then we denote by $\mathrm{C_{one}}^{i,j}(\epsilon)$ the set

$$\mathrm{C_{one}}^{i,j}(\epsilon)=\mathrm{C_{one}}^i(\epsilon)\cap\mathrm{C_{one}}^j(\epsilon)\,. \tag{2.47}$$

If $1\leq i<j\leq d+1$, then we denote by $\Theta_{i,j}$ the following maximal elevation angle

$$\Theta_{i,j}=\Theta(v_i,v_j,\mathrm{L}_{S\setminus\{v_i,v_j\}}). \tag{2.48}$$

Throughout the rest of the paper we fix a real parameter $\gamma \in \mathbb{R}$, $d-1 < \gamma \leq d$ (the most natural choice is $\gamma = d$) and assume that $H$ is equipped with a $\gamma$-regular measure.

The following proposition and its immediate corollary, will be useful for us. We prove them in Appendix 5.1.3.

**Proposition 2.5.1.** *If $\gamma > 1$, $m \in \mathbb{N}$ such that $1 \leq m < \gamma$, $\mu$ a $\gamma$-regular measure on $H$ with regularity constant $C_\mu$, $0 \leq \epsilon \leq 1$, and $L$ an $m$-dimensional affine subspace of $H$, then for all $x \in \operatorname{supp}(\mu) \cap L$ and $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$*

$$\mu(\mathrm{T_{ube}}(L, \epsilon \cdot r) \cap B(x,r)) \;\leq\; 2^{m+\frac{3\gamma}{2}} \cdot C_\mu \cdot \epsilon^{\gamma-m} \cdot r^\gamma. \tag{2.49}$$

**Corollary 2.5.1.** *If $\gamma > 1$, $m \in \mathbb{N}$ such that $1 \leq m < \gamma$, $\mu$ a $\gamma$-regular measure on $H$ with regularity constant $C_\mu$, $0 \leq \theta \leq \pi/2$, and $L$ an $m$-dimensional affine subspace of $H$, then for all $x \in \operatorname{supp}(\mu) \cap L$ and $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$*

$$\mu(\mathrm{C_{one}}(\theta, L, x) \cap B(x,r)) \;\leq\; 2^{m+\frac{3\gamma}{2}} \cdot C_\mu \cdot \sin(\theta)^{\gamma-m} \cdot r^\gamma\,.$$

We will frequently use the following elementary inequalities for the one-dimensional sine:

**Lemma 2.5.1.** *If $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq c \leq 1$, then*

$$c \cdot \sin(\theta) \leq \sin(c \cdot \theta) \tag{2.50}$$

*and*

$$\sin(c \cdot \theta) \leq \frac{\pi}{2} \cdot c \cdot \sin(\theta). \tag{2.51}$$

Both inequalities can be derived by noting that the function $\sin(c\,\theta)/(c\,\sin(\theta))$ is increasing in $\theta$ and thus obtains its lower bound, 1, as $\theta$ approaches 0 and its maximum value, bounded by $\pi/2$, at $\theta = \frac{\pi}{2}$.

### 2.5.1 Conic Regions and Relaxed Two-Term Inequalities for the Polar Sine

We establish here the following relation between the set $U_C(S,0)$ defined in equation (1.11) and the intersection of various cones.

**Proposition 2.5.2.** *If $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, $C \geq 1$, $U_C(S,0)$ is the set defined in equation (1.11) with $w = 0$, and $\mathrm{C_{one}}^{i,j}(C^{-1})$ for $1 \leq i < j \leq d+1$ are the intersections of cones defined in equation (2.47) with $\epsilon = C^{-1}$, then*

$$H \setminus \left( \bigcup_{1 \leq i < j \leq d+1} \mathrm{C_{one}}^{i,j}(C^{-1}) \right) \subseteq U_C(S,0).$$

*Proof.* We note that

$$H \setminus \bigcup_{1 \leq i < j \leq d+1} \mathrm{C_{one}}^{i,j}(C^{-1}) = \bigcap_{1 \leq i < j \leq d+1} \left( H \setminus \mathrm{C_{one}}^{i,j}(C^{-1}) \right),$$

and

$$U_C(S,0) = \bigcap_{1 \leq i < j \leq d+1} U_C^{i,j}(S,0),$$

where

$$U_C^{i,j}(S,0) = \Big\{ u \in H : |\, \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{d+1})| \leq C \cdot \Big( |\, \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_{d+1})| + |\, \mathrm{p}_d\mathrm{sin}_0(v_1, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_{d+1})| \Big) \Big\}. \quad (2.52)$$

Therefore, in order to conclude the proposition we only need to prove that

$$H \setminus \mathrm{C_{one}}^{i,j}(C^{-1}) \subseteq U_C^{i,j}(S,0), \quad \text{for all } 1 \leq i < j \leq d+1. \quad (2.53)$$

If $u \in H \setminus \mathrm{C_{one}}^{i,j}(C^{-1})$ for some $i$ and $j$, where $1 \leq i < j \leq d+1$, then either $u \in H \setminus \mathrm{C_{one}}^{i}(C^{-1})$ or $u \in H \setminus \mathrm{C_{one}}^{j}(C^{-1})$. Assume without loss of generality that $u \in$

$H \setminus \mathrm{C_{one}}^{i}(C^{-1})$, then

$$\sin\left(\theta\left[u, \mathrm{L}_{S\setminus\{v_i\}}\right]\right) \geq \sin\left(C^{-1}\cdot\theta\left[v_i, \mathrm{L}_{S\setminus\{v_i\}}\right]\right). \tag{2.54}$$

Combining the product formula for $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ (Proposition 2.2.4) with equations (2.54) and (2.50), we obtain that

$$C\cdot|\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{i-1},u,v_{i+1},\ldots,v_{d+1})| \geq |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{d+1})|.$$

In particular, $u \in U_C^{i,j}(S,0)$, and equation (2.53), and consequently the proposition, is concluded. $\square$

### 2.5.2 Controlling the Intersection of Two Cones

The main part of the proof of Theorem 1.2.2 is to show that for any $0 \leq \epsilon \leq 1$ we can control the measure of the sets $\mathrm{C_{one}}^{i,j}(\epsilon)$ which are defined in equation (2.47). We accomplish this by showing that such sets are contained in specific cones on $(d-1)$-dimensional subspaces of $H$, and then applying Corollary 2.5.1 to control the measure of the latter cones. The crucial proposition is the following.

**Proposition 2.5.3.** *If $0 \leq s \leq 1$, $k \geq 3$, $V$ is a $k$-dimensional subspace of $H$, $V_1$ and $V_2$ are two different $(k-1)$-dimensional subspaces of $V$, $v_1 \in V_1 \setminus V_2$, $v_2 \in V_2 \setminus V_1$, and $\theta(v_1,V_2)$, $\theta(v_2,V_1)$, as well as $\Theta(v_1,v_2,V_1\cap V_2)$ are the corresponding elevation and maximal elevation angles, then*

$$\mathrm{C_{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi}\cdot\theta\left(v_1,V_2\right),V_2,0\right)\bigcap \mathrm{C_{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi}\cdot\theta\left(v_2,V_1\right),V_1,0\right)\subseteq$$
$$\mathrm{C_{one}}\left(s\cdot\Theta(v_1,v_2,V_1\cap V_2),V_1\cap V_2,0\right).$$

*Proof.* Note that $\dim(V_1 \cap V_2) = k - 2$, and that

$$V = \mathrm{Sp}\{v_1\} + V_2 = \mathrm{Sp}\{v_2\} + V_1 = \mathrm{Sp}\{v_1, v_2\} + V_1 \cap V_2.$$

However, the above sum is not direct (the subspaces in the sum are not mutually orthogonal). We thus create a few orthogonal subspaces which are expressed via the following orthogonal projections:

$$\begin{aligned} P_i &: H \to V_i, \ i = 1, 2, \\ N_i &: H \to V_i \cap (V_1 \cap V_2)^{\perp}, \ i = 1, 2, \\ P_{1,2} &: H \to V_1 \cap V_2. \end{aligned}$$

Note that

$$P_i = P_{1,2} + N_i, \quad i = 1, 2, \tag{2.55}$$

and consequently

$$(I - P_2) \cdot P_1 = (I - P_2) \cdot N_1. \tag{2.56}$$

We denote

$$\begin{aligned} \tilde{\mathrm{C}}^1_{\mathrm{one}} &= \mathrm{C}_{\mathrm{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi} \cdot \theta\,(v_2, V_1)\,, V_1, 0\right), \\ \tilde{\mathrm{C}}^2_{\mathrm{one}} &= \mathrm{C}_{\mathrm{one}}\left(\frac{2\,s}{\sqrt{5}\,\pi} \cdot \theta\,(v_1, V_2)\,, V_2, 0\right), \\ \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}} &= \tilde{\mathrm{C}}^1_{\mathrm{one}} \cap \tilde{\mathrm{C}}^2_{\mathrm{one}}, \\ \tilde{\Theta}_{1,2} &= \Theta(v_1, v_2, V_1 \cap V_2). \end{aligned}$$

Following our notation and the definition of a cone, we need to prove that

$$\|u - P_{1,2}(u)\| \leq \sin\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|, \quad \text{for all } u \in \tilde{\mathrm{C}}^{1,2}_{\mathrm{one}},$$

or equivalently (via equation (2.55)),

$$\|N_1(u) + u - P_1(u)\| \leq \sin\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|, \quad \text{for all } u \in \tilde{\mathrm{C}}^{1,2}_{\text{one}}. \tag{2.57}$$

For $u \in \tilde{\mathrm{C}}^{1,2}_{\text{one}}$, we will bound $\|N_1(u)\|$ and $\|u - P_1(u)\|$ separately and then combine the two estimates to conclude the above inequality and the current proposition.

Our bound for $\|u - P_1(u)\|$ is straightforward. Indeed, if $u \in \tilde{\mathrm{C}}^{1,2}_{\text{one}}$ respectively, then $u \in \tilde{\mathrm{C}}^{1}_{\text{one}}$, and by the definition of $\tilde{\mathrm{C}}^{1}_{\text{one}}$ as well as the application of equation (2.51) we obtain that

$$\|u - P_1(u)\| \leq \sin\left(\frac{2\,s}{\sqrt{5}\,\pi} \cdot \theta(v_2, V_1)\right) \cdot \|u\| \leq \frac{s}{\sqrt{5}} \cdot \sin\left(\theta(v_2, V_1)\right) \cdot \|u\|. \tag{2.58}$$

Our bound for $\|N_1(u)\|$ has the following form:

$$\|N_1(u)\| \leq \frac{s}{\sqrt{5}} \left[\sin(\theta(v_1, V_1 \cap V_2)) + \sin(\theta(v_2, V_1 \cap V_2))\right] \cdot \|u\|. \tag{2.59}$$

In order to verify it, we assume without loss of generality that $N_1(u) \neq 0$ and note that equation (2.6) implies the following relation

$$\sin(\alpha(V_1, V_2)) = \frac{\operatorname{dist}(N_1(u), V_2)}{\operatorname{dist}(N_1(u), V_1 \cap V_2)} = \frac{\|N_1(u) - P_2 \cdot N_1(u)\|}{\|N_1(u)\|}. \tag{2.60}$$

Combining equations (2.56) and (2.60) we obtain that

$$\|N_1(u)\| = \frac{\|P_1(u) - P_2 \cdot P_1(u)\|}{\sin\left(\alpha(V_1, V_2)\right)}. \tag{2.61}$$

We bound $\|P_1(u) - P_2 \cdot P_1(u)\|$ as follows.

$$\begin{aligned} \|P_1(u) - P_2 \cdot P_1(u)\| &= \|(I - P_2) \cdot P_1(u)\| \\ &= \|(I - P_2)(u) - (I - P_2) \cdot (I - P_1)(u)\| \leq \|u - P_2(u)\| + \|u - P_1(u)\|. \end{aligned} \tag{2.62}$$

Equation (2.58) gives a bound for $\|u - P_1(u)\|$, and similarly we obtain that

$$\|u - P_2(u)\| \leq \frac{s}{\sqrt{5}} \cdot \sin(\theta(v_1, V_2) \cdot \|u\|. \tag{2.63}$$

Combining equations (2.58) and (2.61)-(2.63) we get that

$$\|N_1(u)\| \leq \frac{s}{\sqrt{5}} \left( \frac{\sin\left(\theta\left(v_1, V_2\right)\right)}{\sin(\alpha(V_1, V_2))} + \frac{\sin\left(\theta\left(v_2, V_1\right)\right)}{\sin(\alpha(V_1, V_2))} \right) \cdot \|u\|. \tag{2.64}$$

At last we note that equations (2.4) and (2.6) imply that

$$\sin(\alpha(V_1, V_2)) = \frac{\sin(\theta(v_1, V_2))}{\sin(\theta(v_1, V_1 \cap V_2))} = \frac{\sin(\theta(v_2, V_1))}{\sin(\theta(v_2, V_1 \cap V_2))}.$$

Applying this identity in (2.64), we achieve the bound for $\|N_1(u)\|$ stated in equation (2.59).

Finally, noting that $N_1(u) \perp (u - P_1(u))$ and applying the bounds of equations (2.58) and (2.59) we obtain that

$$\begin{aligned} &\|N_1(u) + u - P_1(u)\|^2 = \|N_1(u)\|^2 + \|u - P_1(u)\|^2 \\ &\quad \leq \left(\frac{s}{\sqrt{5}}\right)^2 \cdot 5 \cdot \sin^2\left(\tilde{\Theta}_{1,2}\right) \cdot \|u\|^2 = s^2 \cdot \sin^2\left(\tilde{\Theta}_{1,2}\right) \cdot \|u\|^2 \leq \sin^2\left(s \cdot \tilde{\Theta}_{1,2}\right) \cdot \|u\|^2. \end{aligned}$$

Equation (2.57), and consequently the proposition, is thus concluded. □

**Remark 2.5.1.** *The proposition extends trivially to $k = 2$, where the intersection of two cones, centered around two vectors $w_1$ and $w_2$ respectively with opening angles less than half the angle between, is the origin, which is a degenerate cone.*

Proposition 2.5.3 implies the following corollary:

**Corollary 2.5.2.** *If $0 \leq s \leq 1$, $2 \leq k \leq \dim(H)$, $1 \leq i < j \leq d+1$, $S = \{v_1, \ldots, v_k\} \subseteq H$ is a linearly independent set and $\Theta_{i,j}$ as well as ${\mathrm{C_{one}}}^{i,j}\left(\frac{2\cdot s}{\sqrt{5}\,\pi}\right)$ are defined by equations (2.48) and (2.47) respectively, then*

$${\mathrm{C_{one}}}^{i,j}\left(\frac{2 \cdot s}{\sqrt{5}\,\pi}\right) \subseteq \mathrm{C_{one}}\left(s \cdot \Theta_{i,j}, \mathrm{L}_{S \setminus \{v_i, v_j\}}, 0\right). \tag{2.65}$$

Indeed, Corollary 2.5.2 is obtained as a special case of Proposition 2.5.3 by setting $V = \mathrm{L}_S$, $V_1 = \mathrm{L}_{S\setminus\{v_i\}}$, and $V_2 = \mathrm{L}_{S\setminus\{v_j\}}$, and noting that $V_1 \cap V_2 = \mathrm{L}_{S\setminus\{v_i,v_j\}}$.

### 2.5.3 Conclusion of Theorem 1.2.2

Theorem 1.2.2 follows directly from Proposition 2.5.2 and Corollaries 2.5.1 and 2.5.2.

In view of equation (2.3), we note that it is sufficient to prove the theorem when $w = 0$ and $0 \in \mathrm{supp}(\mu)$. We assume an arbitrary parameter $0 < s \leq 1$ and set

$$C = \frac{\sqrt{5}\,\pi}{2 \cdot s}\,. \tag{2.66}$$

At the end of the proof we further restrict the values of $s$ from above and consequently restrict those of $C$ from below.

Let $S = \{v_1, \ldots, v_{d+1}\} \subseteq H$, $\ 0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$, and $\mathrm{C_{one}}^{i,j}(C^{-1})$ be defined according to equation (2.47). We assume without loss of generality that the set $S$ is linearly independent. Proposition 2.5.2 implies that

$$B(0,r) \setminus \left( \bigcup_{1 \leq i \neq j \leq d+1} \mathrm{C_{one}}^{i,j}(C^{-1}) \right) \subseteq B(0,r) \bigcap U_C(S,0)\,.$$

Using the additivity and monotonicity of $\mu$, we get

$$\mu\left(B(0,r) \cap U_C(S,0)\right) \geq \mu\left(B(0,r)\right) \; - \sum_{1 \leq i < j \leq d+1} \mu\left(B(0,r) \cap \mathrm{C_{one}}^{i,j}(C^{-1})\right). \tag{2.67}$$

Next, we combine Corollary 2.5.2 together with equation (2.66) to obtain that

$$\begin{aligned} B(0,r) \cap \mathrm{C_{one}}^{i,j}(C^{-1}) \subseteq B(0,r) \cap \mathrm{C_{one}}\left(s \cdot \Theta_{i,j}, \mathrm{L}_{S\setminus\{v_i,v_j\}}, 0\right) \subseteq \\ B(0,r) \cap \mathrm{C_{one}}\left(s \cdot \frac{\pi}{2}, \mathrm{L}_{S\setminus\{v_i,v_j\}}, 0\right). \end{aligned} \tag{2.68}$$

Now, Corollary 2.5.1, Definition 1.4.1, and equation (2.68) imply that for all $1 \leq i \neq j \leq$

$d+1$,

$$\mu\left(B(0,r)\cap \mathrm{C_{one}}^{i,j}(C^{-1})\right) \leq \mu\left(B(0,r)\cap \mathrm{C_{one}}\left(s\cdot\frac{\pi}{2}, \mathrm{L}_{S\setminus\{v_i,v_j\}}, 0\right)\right)$$
$$\leq 2^{\frac{3\gamma}{2}+d-1}\cdot C_\mu^2\cdot\left(\sin\left(s\cdot\frac{\pi}{2}\right)\right)^{\gamma+1-d}\cdot\mu(B(0,r))\,. \quad (2.69)$$

Combining equations (2.67) and (2.69), we get that

$$\frac{\mu\big(B(0,r)\cap U_C(S,0)\big)}{\mu\left(B(0,r)\right)} \geq 1\ -\ \binom{d+1}{2}\cdot 2^{\frac{3\gamma}{2}+d-1}\cdot C_\mu^2\cdot\left(\sin\left(s\cdot\frac{\pi}{2}\right)\right)^{\gamma+1-d}. \qquad (2.70)$$

By setting the parameter $s$ so that

$$\binom{d+1}{2}\cdot 2^{\frac{3\gamma}{2}+d-1}\cdot C_\mu^2\cdot\left(\sin\left(s\cdot\frac{\pi}{2}\right)\right)^{\gamma+1-d}\leq\epsilon\,,$$

that is,

$$s\leq s_0' = \frac{2}{\pi}\cdot\arcsin\left[\left(\frac{\epsilon}{2^{\frac{3\gamma}{2}+d-1}\cdot C_\mu^2\cdot\binom{d+1}{2}}\right)^{\frac{1}{\gamma+1-d}}\right],$$

we obtain that equation (1.12) is satisfied for all $C\geq C_0'$, where

$$C_0' = \frac{\sqrt{5}\,\pi}{2\,s_0'} = \sqrt{5}\,\left(\frac{\pi}{2}\right)^2\left(\arcsin\left[\left(\frac{\epsilon}{2^{\frac{3\gamma}{2}+d-1}\cdot C_\mu^2\cdot\binom{d+1}{2}}\right)^{\frac{1}{\gamma+1-d}}\right]\right)^{-1}$$
$$= O\left(\left(2^{\frac{3\gamma}{2}+d}\cdot C_\mu^2\cdot\binom{d+1}{2}\cdot\epsilon^{-1}\right)^{\frac{1}{\gamma+1-d}}\right)\quad \text{as } \epsilon\to 0 \ \text{ or }\ d\to\infty\,.$$

The theorem is thus concluded, where $C_0'$ provides an upper bound for the best possible choice for the constant $C_0$. □

**Remark 2.5.2.** *Note that Theorem 1.2.2 extends trivially to the case where $\gamma > d$. In fact, in this case, it is possible to replace the set $U_C(S,v_0)$ by*

$$U_C'(S,w) = \big\{u\in H:\ |\,\mathrm{p}_d\mathrm{sin}_w(v_1,\ldots,v_{d+1})|\leq$$
$$C\cdot|\,\mathrm{p}_d\mathrm{sin}_w(v_1,\ldots,v_{i-1},u,v_{i+1},\ldots,v_{d+1})|,\ \text{for all } 1\leq i\leq d+1\big\}.$$

*That is, if $\gamma > d$, then the polar sine satisfies a relaxed simplex inequality of one term "with high probability at all scales and locations". This fact is a direct consequence of Corollary 2.5.1 and analogues of Proposition 2.5.2 and equation (2.67) obtained by replacing $U_C(S,0)$ with $U'_C(S,0)$ and $\{\mathrm{C_{one}}^{i,j}(C^{-1})\}_{1\leq i<j\leq d+1}$ with $\{\mathrm{C_{one}}^{i}(C^{-1})\}_{1\leq i<d+1}$.*

*Nevertheless, if $d-1 < \gamma \leq d$, then one cannot replace the set $U_C(S, v_0)$ in Theorem 1.2.2 by $U'_C(S,w)$.*

**Remark 2.5.3.** *Let us slightly reformulate the above results so that they could be directly applied in Chapter 4. For $S = \{v_1, \ldots, v_{d+1}\}$ as above, $C > 0$, and an arbitrarily fixed pair of indices $i$ and $j$, where $1 \leq i < j \leq d+1$, we form the set $U_C(S,i,j,0)$ as follows:*

$$U_C(S,i,j,0) = \Big\{u \in H : \ |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{d+1})| \leq$$

$$C \cdot \big(|\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{i-1},u,v_{i+1},\ldots,v_{d+1})| \ + |\,\mathrm{p}_d\mathrm{sin}_0(v_1,\ldots,v_{j-1},u,v_{j+1},\ldots,v_{d+1})|\big)\Big\}.$$

*If $\gamma = d$ and $0 < \epsilon < 1$, then for all $C \geq C''_0$, where*

$$C''_0 = \sqrt{5}\cdot\left(\frac{\pi}{2}\right)^2\cdot\left(\arcsin\left(\frac{\epsilon}{2^{\frac{5\,d}{2}-1}\cdot C_\mu^2}\right)\right)^{-1},$$

*we have that*

$$\frac{\mu\big(B(0,r)\cap U_C(S,i,j,0)\big)}{\mu\left(B(0,r)\right)} \geq 1-\epsilon.$$

# Chapter 3

# Controlling the Jones-Type Flatness by the Menger-Type Curvature

### 3.0.4 Notation

#### Elements of $H^{n+1}$ and Corresponding Subsets of $H$

Fixing $n \geq 1$, we denote an element of $H^{n+1}$ by $X = (x_0, \ldots, x_n)$ and we typically refer to it as an $n$-simplex, or at times just a simplex when the dimension is clear. For $X = (x_0, \ldots, x_n) \in H^{n+1}$, we say that $X$ is *non-degenerate* if the set $\{x_1 - x_0, \ldots, x_n - x_0\}$ is linearly independent, and we say that $X$ is *degenerate* otherwise. Also, for fixed $0 \leq i \leq n$, we let $(X)_i = x_i$ denote the projection of $X$ onto its $i^{\text{th}}$ $H$-valued *coordinate*. The $0^{\text{th}}$ coordinate $(X)_0 = x_0$ is special in many of our calculations. For $X \in H^{n+1}$ as above, let $L[X]$ denote the affine subspace of $H$ of minimal dimension containing the coordinates of

$X$, and let $V[X]$ be the linear subspace parallel to $L[X]$. We note that $\dim(V[X]) \leq n$.

For $0 \leq i \leq n$ and $X = (x_0, \dots, x_n) \in H^{n+1}$, let $X(i)$ be the following element of $H^n$:

$$X(i) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \tag{3.1}$$

that is, $X(i)$ is the projection of $X$ onto $H^n$ that eliminates its $i^{\text{th}}$ coordinate. Furthermore, for $n \geq 2$ and $0 \leq i < j \leq n$, let $X(i;j)$ be the following element of $H^{n-1}$:

$$X(i;j) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n). \tag{3.2}$$

If $1 \leq i \leq n$, $X(i) \in H^n$, and $y \in H$, we form $X(y,i) \in H^{n+1}$ as follows:

$$X(y,i) = (x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n).$$

Finally, if $y \in H$ and $z \in H$, $n \geq 2$, and $1 \leq i < j \leq n$, then we define the elements $X(y,i;j)$, $X(i;z,j) \in H^n$ and $X(y,i;z,j) \in H^{n+1}$ by the following formulas:

$$X(y,i;j) = (x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \tag{3.3}$$

$$X(i;z,j) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n), \tag{3.4}$$

and

$$X(y,i;z,j) = (x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n). \tag{3.5}$$

**Remark 3.0.4.** *We usually take $n = d+1$, and without referring directly to $X \in H^{d+2}$ we often denote elements of $H^{d+1}$ by $X(i)$ for some $1 \leq i \leq d+1$ and elements of $H^d$ by $X(i;j)$ where $0 \leq i < j \leq d$. It is important to note that in this case, we reserve the indexing conventions of equations* (3.1) *and* (3.2) *for the aforemention elements in $H^{d+1}$ and $H^d$.*

**Functions of Simplices**

Two very basic functions of a simplex $X = (x_0, \ldots, x_n) \in H^{n+1}$ that we use in this work are

$$\min(X) = \min_{0 \leq i < j \leq n} \|x_i - x_j\| \quad \text{and} \quad \operatorname{diam}(X) = \max_{0 \leq i < j \leq n} \|x_i - x_j\|. \tag{3.6}$$

These two functions quantify the minimal and maximal length scales of the simplex $X$, and we typically restrict our attention to simplices such that $\min(X) > 0$.

We also make use of the following notations for content functions, polar sines, and the Menger-type curvatures:

$$M_k(X) := M_k(x_1 - x_0, \ldots, x_k - x_0),$$

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) := \mathrm{p}_d\mathrm{sin}_{x_0}(x_1, \ldots, x_{d+1}) \text{ for } X \in H^{d+2},$$

and finally

$$c_{MT}(X) := \sqrt{\frac{\sum_{i=0}^{d+1} \mathrm{p}_d\mathrm{sin}^2_{x_i}(X)}{(d+2) \cdot \operatorname{diam}(X)^{d(d+1)}}}. \tag{3.7}$$

### 3.0.5 Introduction

Here we establish the lower bounds of Theorems 1.2.3 and 1.2.4. Despite the fact that Léger's curvature is unsuitable for our purposes, his basic analysis is instrumental in this chapter. Similar to his work in [25], the main ingredients of our analysis include repeated applications of both Fubini's Theorem and Chebychev's inequality as well as various metric inequalities and identities. However, we needed to develop additional analytic and combinatoric propositions for the case where $d > 1$. In particular, we have generalized the separation of points by pairwise distances employed in [25] to a $d$-dimensional separation of simplices (see Section 3.2). This $d$-separation plays a fundamental role in the proof of

Theorem 3.0.1, which is the main result of this chapter. A weaker notion of $d$-separation has been applied earlier by David and Semmes [10, Lemma 5.8].

It should be noted that our work with the Menger-type curvatures lead us to find curvatures which are related to various normalized versions of the polar sine. One such curvature is

$$\tilde{c}_{MT}(X) := \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X)}{\displaystyle\prod_{1\le i<j\le d+1} \|x_i - x_j\|}\,. \tag{3.8}$$

The curvature $\tilde{c}_{MT}$ is a well defined invariant of the simplex $X$, since $\tilde{c}_{MT}$ is simply the $(d+1)$-dimensional volume of the simplex normalized by *all* of the side lengths. This curvature is *much* more singular than $c_{MT}$, and we still do not know if it satisfies an estimate of the form in equation (1.6). However, due to a simple algebraic comparison (given in Section 3.5) between $c_{MT}$ and $\tilde{c}_{MT}$ restricted to $X \in U_\lambda(B)$ (see equation (1.14) and Figure 1.2.4), we see that $\tilde{c}_{MT}$ must also satisfy Theorem 1.2.3. We discuss this curvature and others in Section 3.5

For $\lambda > 0$ and a closed ball $B(x,t) \subseteq H$, let

$$c^2_{MT}(x,t,\lambda) = \int_{U_\lambda(B(x,t))} c^2_{MT}(X)\,\mathrm{d}\mu^{d+2}(X). \tag{3.9}$$

Here we establish the following inequality.

**Theorem 3.0.1.** *There exist constants* $0 < \lambda_0 = \lambda_0(d, C_\mu) < 2$ *and* $C_1 = C_1(d, C_\mu) \geq 1$ *such that*

$$\beta_2^2(x,t)\cdot\mu(B(x,t)) \leq C_1 \cdot c^2_{\mathrm{MT}}(x,t,\lambda_0)\,,$$

*for all* $x \in \mathrm{supp}(\mu)$ *and* $0 < t \leq \mathrm{diam}(\mathrm{supp}(\mu))$.

We note that this is simply a way of rewriting the lower bound of Theorem 1.2.3.

We arbitrarily fix $0 < \lambda \leq 1$ and a ball $B$ in $H$ and define the following set of well scaled simplices in $B$ (see Figure 1.2.4)

$$W_\lambda(B) = \{X \in B^{d+2} : \ \min(X) \geq \lambda \cdot \operatorname{diam}(X) > 0\}.$$

The local curvature of $\mu$ with respect to $B$ and $\lambda$ has the form

$$c^2_{\mathrm{MT}}\left(\mu|_B, \lambda\right) = \int_{W_\lambda(B)} c^2_{\mathrm{MT}}(X) \, \mathrm{d}\mu^{d+2}(X) \, .$$

Clearly, for any $\lambda > 0$ we have that

$$c^2_{\mathrm{MT}}\left(\mu|_B, \lambda\right) \leq c^2_{\mathrm{MT}}\left(\mu|_B\right) .$$

Using this notation and the constant $\lambda_0$ of Theorem 1.2.3 we formulate the following theorem.

**Theorem 3.0.2.** *There exists a constant $C_3 = C_3(d, C_\mu, \lambda_0) \geq 1$ such that*

$$J_2\left(\mu|_B\right) \leq C_3 \cdot c^2_{\mathrm{MT}}\left(\mu|_{3B}, \lambda_0/2\right)$$

*for all balls $B \subseteq H$.*

Again, we note that this is simply the lower bound of Theorem 1.2.4 expressed in different notation.

## Organization of the Chapter

This chapter is organized as follows. In Section 3.1 we reformulate some previous propositions to fit with the current notation, and we give some useful elementary properties of the $d$-dimensional polar sine. Furthermore, we give a simplified expression for the Menger-type curvature, $c_{MT}(\mu|_B)$, which will be used throughout the rest of the paper. Section 3.2 states and proves a geometric proposition regarding the $d$-dimensional separation of points in the support of an arbitrary $d$-regular measure $\mu$ on $H$. Sections 3.3 and 3.4 contain the proofs of Theorem 3.0.1 and Theorem 3.0.2 respectively. Section 3.5 discusses other possible curvatures and their relation to the curvature $c_{\mathrm{MT}}$, as well as some problems with a previously proposed curvature [25].

## 3.1 Properties of the Polar Sine and a Reduction of $c_{MT}^2(\mu|_B)$

If $n \geq 1$, $X \in H^{n+2}$, and $1 \leq i \leq n+1$, then let $\theta_i(X)$ denote the elevation angle of $x_i - x_0$ with respect to $V\left[X(i)\right]$. We note that if $\min(X) > 0$, then

$$\sin(\theta_i(X)) = \frac{\operatorname{dist}(x_i, L[X(i)])}{\|x_i - x_0\|}. \tag{3.10}$$

The $d$-dimensional polar sine has the following elementary properties. First, $\mathrm{p}_d\mathrm{sin}_{x_i}(X) = 0$ for some $0 \leq i \leq d+1$ if and only if $X$ is degenerate. Second, for $d > 1$ we reformulate the product formula of Proposition 2.2.4:

**Proposition 3.1.1.** *If $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ and $1 \leq i \leq d+1$, then*

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) = \sin\left(\theta_i(X)\right) \cdot \mathrm{p}_{d\text{-}1}\mathrm{sin}_{x_0}\left(X(i)\right).$$

Proposition 3.1.1 and equation (3.10) imply the following lower bound for the polar sine.

**Lemma 3.1.1.** *If $x \in H$, $0 < t < \infty$, and $X \in B(x,t)^{d+2}$ is such that*

$$\mathrm{M}_d\left(X(i)\right) \geq \omega \cdot t^d \text{ for some } 0 < \omega \leq 1 \text{ and } 1 \leq i \leq d+1,$$

*then*

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \geq \frac{\omega}{2^{d+1}} \cdot \frac{\operatorname{dist}(x_i, L[X(i)])}{t}.$$

The definition of the polar sine implies the following generalization of the one-dimensional law of sines: If $X = (x_0, \dots, x_{d+1}) \in H^{d+2}$ is such that $\min(X) > 0$, then

$$\frac{\mathrm{p}_d\mathrm{sin}_{x_i}(X)}{\displaystyle\prod_{\substack{0\le s<r\le d+1\\ s,r\neq i}} \|x_s - x_r\|} = \frac{\mathrm{p}_d\mathrm{sin}_{x_j}(X)}{\displaystyle\prod_{\substack{0\le \ell<q\le d+1\\ \ell,q\neq j}} \|x_\ell - x_q\|} \text{ for all } 0 \le i < j \le d+1. \tag{3.11}$$

Finally, there is an invariance of $\mathrm{p}_d\mathrm{sin}_{x_0}(X)$ to permutations fixing the zeroth coordinate: If $\sigma$ is a permutation of $\{0, 1, \dots, d+1\}$ fixing 0, and if $\sigma(X)$ is defined by

$$\sigma(X) = \left(x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(d+1)}\right) = \left(x_0, x_{\sigma(1)}, \dots, x_{\sigma(d+1)}\right), \tag{3.12}$$

then

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) = \mathrm{p}_d\mathrm{sin}_{x_0}(\sigma(X)). \tag{3.13}$$

**Reducing the Local Menger-Type Curvature for a Closed Ball $B$**

For a permutation $\sigma$ of $(d+2)$ indices and $X = (x_0, \dots, x_{d+1}) \in H^{d+2}$, we let $\sigma(X) = (x_{\sigma(0)}, \dots, x_{\sigma(d+1)})$, and we say that a set $A \subseteq H^{d+2}$ is *permutation invariant* if $\sigma(A) = A$ for all permutations. For such permutation invariant sets, we have the following trivial equality for all indices $1 \le i \le d+1$:

$$\int_A \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X) = \int_A \frac{\mathrm{p}_d\mathrm{sin}^2_{x_i}(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X).$$

This equality leads to the following simplified expression of $c^2_{\mathrm{MT}}(\mu|_B)$ given in equation (3.7)

$$c^2_{\mathrm{MT}}\left(\mu|_B\right) = \int_{B^{d+2}} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X). \tag{3.14}$$

## 3.2 On $d$-Separation of $d$-Regular Measures

We introduce here a notion of $d$-dimensional separation of $(d+1)$-simplices and show that there are many such separated simplices in $\mathrm{supp}(\mu)^{d+2}$. Specifically, we show that independently of $x \in \operatorname{supp}(\mu)$ and $0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu))$ there exists a "sufficiently large" amount of $(d+1)$-simplices, $X \in [B(x,t) \cap \operatorname{supp}(\mu)]^{d+2}$, whose $d$-dimensional faces, $\{X(i)\}_{i=0}^{d+1}$, are "sufficiently large". We refer to this property as $d$-separation of the measure $\mu$ and also refer to the corresponding simplices as $d$-separated. A similar notion was already applied by David and Semmes [10, Lemma 5.8]. However, they did not need the notion of $d$-separability of $\mu$ which is essential for our development.

### 3.2.1 $n$-Separated Simplices

Let $X \in H^{d+2}$ with $\operatorname{diam}(X) > 0$. We say that $X$ is 1-*separated for* $\omega > 0$ if

$$\frac{\min(X)}{\operatorname{diam}(X)} \geq \omega\,.$$

We say that $X$ is $d$-*separated for* $\omega > 0$ if

$$\frac{\min_{0 \leq i \leq d+1} \mathrm{M}_d(X(i))}{\operatorname{diam}^d(X)} \geq \omega\,.$$

More generally, we say that $X$ is $n$-*separated for* $\omega > 0$ and $1 < n < d$ if the minimal $n$-content through its vertices scaled by $\operatorname{diam}^n(X)$ is larger than $\omega$. We typically do not mention the constant $\omega$, and we just say $n$-separated if $\omega$ is clear from the context.

We note that the $n$-separation of an element $X$ implies the $j$-separation for all $1 \leq j < n$. For example, given a $d$-separated element $X$, using the product formula for contents we have that

$$\omega \cdot \operatorname{diam}^d(X) \leq \min_{0 \leq i \leq d+1} \mathrm{M}_d(X(i)) \leq \min(X) \cdot \operatorname{diam}^{d-1}(X).$$

Hence, $X$ is 1-separated for $\omega$.

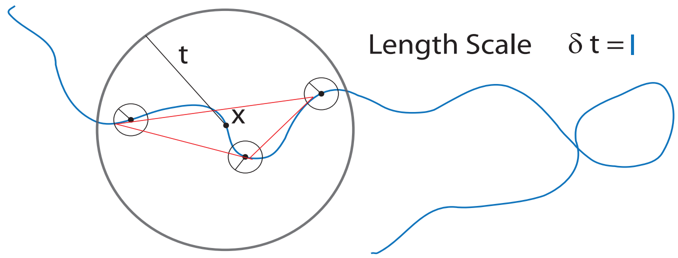


Figure 3.1: The three individual balls within the larger ball $B(x,t)$ are such that triangles with vertices in the balls have "sufficiently large" edge lengths. We note that the *area* of the triangle is irrelevant. Furthermore, each of individual the balls clearly has a "sufficiently large" measure, since each radius is "sufficiently large".

### 3.2.2 $n$-Separated Balls and Measures

Let $B(x,t) \subseteq H$, $m, n \in \mathbb{N}$, $m \geq n \geq 1$, and $\omega > 0$. We say that a collection of $m+1$ balls, $\{B_i\}_{i=0}^{m}$, is *$n$-separated in $B(x,t)$ for $\omega$* if

$$\bigcup_{0 \leq i \leq m} B_i \subseteq B(x,t),$$

and any $(n+1)$ points drawn without repetition from any sub-collection of $(n+1)$ distinct balls is $n$-separated for $\omega$. That is,

$$\min_{\widetilde{X} \in \prod_{i \in I} B_i} \mathrm{M}_n(\widetilde{X}) \geq \omega \cdot t^n, \text{ for each set of distinct indices, } I = \{i_1, \ldots, i_{n+1}\}.$$

We extend this definition to $d$-regular measures in the following way. For $x \in \mathrm{supp}(\mu)$ and $0 < t \leq \mathrm{diam}(\mathrm{supp}(\mu))$, we say that $\mu$ is *$n$-separated in $B(x,t)$* (for $0 < \delta < 1$ and $\omega > 0$) if there exist $(n+2)$ balls, $\{B_i\}_{i=0}^{n+1}$, which are $n$-separated (for $\omega$) in $B(x,t)$, and satisfying

$$\min_{0 \leq i \leq n+1} \frac{\mathrm{diam}(B_i)}{2 \cdot t} \geq \delta.$$

We show here that $\mu$ is $d$-separated at all scales and locations in the following sense (see Figures 3.2.2 and 3.2.2).

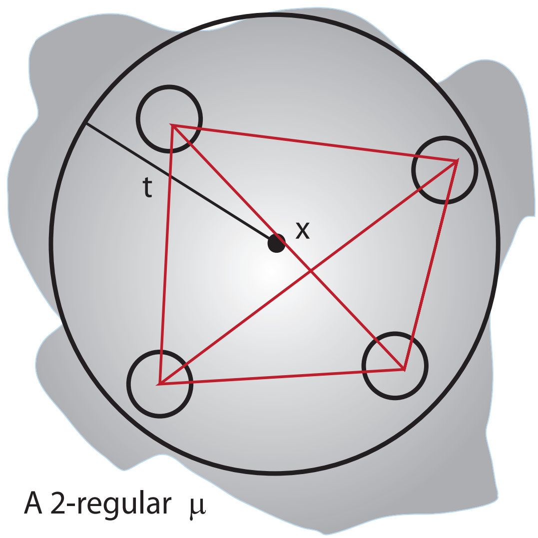


Figure 3.2: This time a 2-regular $\mu$ indicated by the grey area, and a location $x$ and a length scale $t$. The 4 balls are sufficiently large, and sufficiently separated in a $d$-dimensional sense so that any 3-simplex with vertices in each ball is such that the individual faces have a “sufficiently large” area with respect to $t^2$.

**Proposition 3.2.1.** *There exist* $0 < \delta_\mu = \delta_\mu(d, C_\mu) < 1$ *and* $\omega_\mu = \omega_\mu(d, C_\mu) > 0$ *such that for any ball* $B(x, t) \subseteq H$ *with* $x \in \mathrm{supp}(\mu)$ *and* $0 < t \leq \mathrm{diam}(\mathrm{supp}(\mu))$, *the following property is satisfied: There exists a d-separated collection of* $d + 2$ *balls,* $\{B(x_i, \delta_\mu \cdot t)\}_{i=0}^{d+1}$, *contained in* $B(x, t)$ *as well as centered on* $B(x, t) \cap \mathrm{supp}(\mu)$.

### 3.2.3 Proof of Proposition 3.2.1

For simplicity, we prove Proposition 3.2.1 by looking at the ball $B(x, 2 \cdot t)$ and reducing it to the following two parts.

**Part I:** We first prove the existence of constants $0 < \delta_d = \delta_d(d, C_\mu) \leq 1/2$ and $\omega_d = \omega_d(d, C_\mu) > 0$ such that for every $x \in \mathrm{supp}(\mu)$ and $0 < t \leq \mathrm{diam}(\mathrm{supp}(\mu))$, there exists a collection of $d+1$ balls, $\{B(x_i, \delta_d \cdot t)\}_{i=0}^{d}$, centered on $B(x, t) \cap \mathrm{supp}(\mu)$ and $d$-separated for $\omega_d$ in the ball $B(x, 2 \cdot t)$.

**Part II:** Given the $d$-separated balls, $\{B(x_i, \delta_d \cdot t)\}_{i=0}^{d}$ for $B(x, 2 \cdot t)$ constructed in Part I, we construct a point $x_{d+1} \in B(x, t) \cap \mathrm{supp}(\mu)$ and constants

$$0 < \widetilde{\delta}_\mu = \widetilde{\delta}_\mu(d, C_\mu) \leq \delta_d \text{ and } \widetilde{\omega}_\mu = \widetilde{\omega}_\mu(d, C_\mu) > 0$$

such that the collection of $(d + 2)$ balls, $\{B(x_i, \widetilde{\delta}_\mu \cdot t)\}_{i=0}^{d+1}$, is also $d$-separated in the ball $B(x, 2 \cdot t)$.

Parts I and II imply the desired proposition for the ball $B(x, t)$, with $\delta_\mu = \widetilde{\delta}_\mu/2$ and $\omega_\mu = \widetilde{\omega}_\mu/2^d > 0$.

We establish Part I and Part II in Subsections 3.2.3 and 3.2.3 respectively. An elementary lemma used in Subsection 3.2.3 is proved separately in Subsection 3.2.3.

**Part I of the Proof**

Our proof is inductive on $n$. If $n = 1$, then let $x_0 = x$ and $\delta_0 = \sqrt[d]{\frac{1}{2 \cdot C_\mu^2}}$. By Lemma 1.4.2 we have the inequality

$$\mu\left(A\left(x_0, \delta_0 \cdot t, t\right)\right) \geq \frac{1}{2} \cdot \mu(B(x_0, t)) > 0.$$

Then, we arbitrarily fix $x_1 \in A\left(x_0, \delta_0 \cdot t, t\right) \cap \operatorname{supp}(\mu)$ and set $\delta_1 = \delta_0/3$. For any $\widetilde{x}_0 \in B(x_0, \delta_1 \cdot t)$ and $\widetilde{x}_1 \in B(x_1, \delta_1 \cdot t)$, let $\widetilde{X}_1 = (\widetilde{x}_0, \widetilde{x}_1)$. Clearly we have

$$\mathrm{M}_1(\widetilde{X}_1) = \|\widetilde{x}_0 - \widetilde{x}_1\| \geq \delta_1 \cdot t,$$

and thus the statement holds for $n = 1$, where

$$\omega_1 = \delta_1 = \frac{1}{3} \cdot \left(\frac{1}{2 \cdot C_\mu^2}\right)^{1/d} \leq \frac{1}{2}.$$

Now, for some $1 \leq n < d$, we take the induction hypothesis to be the existence of $n+1$ points $\{x_0, \ldots, x_n\} \subseteq B(x, t) \cap \operatorname{supp}(\mu)$, and constants $0 < \delta_n \leq \frac{1}{2}$ and $\omega_n > 0$ such that the collection of $n+1$ balls $\{B(x_i, \delta_n \cdot t)\}_{i=0}^n$ is $n$-separated (for $\omega_n$) in $B(x, 2 \cdot t)$. We further assume that $x_0 = x$ (which was satisfied for $n = 1$). We will construct a point $x_{n+1} \in B(x, t) \cap \operatorname{supp}(\mu)$ and constants $0 < \delta_{n+1} \leq \delta_n$ and $\omega_{n+1} > 0$ such that the collection of balls $\{B(x_i, \delta_{n+1} \cdot t)\}_{i=0}^{n+1}$ is $(n+1)$-separated (for $\omega_{n+1}$) in $B(x, 2 \cdot t)$ for $\omega_{n+1}$.

For the set of balls of the induction hypothesis, $\{B(x_i, \delta_n \cdot t)\}_{i=0}^n$, let $X_n = (x_0, \ldots, x_n)$ denote the non-degenerate simplex generated by their centers, and furthermore let $P$ denote the orthogonal projection of $H$ onto the $n$-plane $L[X_n]$. Let $\delta$ be an arbitrary constant with $0 < \delta \leq \delta_n \leq 1/2$, where we will eventually specify a choice for $\delta$, i.e., the constant $\delta_{n+1}$ mentioned above.

We take an arbitrary element

$$\widetilde{X}_n = (\widetilde{x}_0, \ldots, \widetilde{x}_n) \in \prod_{i=0}^{n} B(x_i, \delta \cdot t), \tag{3.15}$$

and for such $\widetilde{X}_n$, we note that $\{\widetilde{x}_0, \ldots, \widetilde{x}_n\} \subseteq B\left(x_0, \frac{3}{2} \cdot t\right)$, and thus

$$\operatorname{diam}(\widetilde{X}_n) \leq 3 \cdot t. \tag{3.16}$$

Let $\widetilde{P}_\delta$ denote the orthogonal projection of $H$ onto the $n$-plane $L[\widetilde{X}_n]$. For convenience, we suppress the dependence of $P$ and $\widetilde{P}_\delta$ on the elements $X_n$ and $\widetilde{X}_n$ respectively.

The induction step consists of three parts. The first is the existence of a constant $\epsilon_n > 0$ (independent of $x$ and $t$) and an element $x_{n+1} \in B(x_0, t) \cap \operatorname{supp}(\mu)$ such that

$$\|x_{n+1} - P\left(x_{n+1}\right)\| \geq \epsilon_n \cdot t. \tag{3.17}$$

The second part is the existence of a constant $0 < \delta_{n+1} = \delta_{n+1}(n, \delta_n, \omega_n, \epsilon_n) \leq \delta_n$ such that

$$\|x_{n+1} - \widetilde{P}_{\delta_{n+1}}\left(x_{n+1}\right)\| \geq \frac{2 \cdot \epsilon_n}{3} \cdot t. \tag{3.18}$$

The last part of the induction proof is showing that for any $\widetilde{x}_{n+1} \in B(x_{n+1}, \delta_{n+1} \cdot t)$, we have the lower bound

$$\|\widetilde{x}_{n+1} - \widetilde{P}_{\delta_{n+1}}(\widetilde{x}_{n+1})\| \geq \frac{\epsilon_n}{3} \cdot t. \tag{3.19}$$

Then, we conclude the proof of part I by combining equation (3.19) with the induction hypothesis and the product formula for contents. That is, we obtain that for any $1 \leq n \leq d$ the family of balls $\{B(x_i, \delta_{n+1} \cdot t)\}_{i=0}^{n+1}$ is $(n+1)$-separated in $B(x, 2 \cdot t)$ for the constant

$$\omega_{n+1} = \frac{\epsilon_n \cdot \omega_n}{3}.$$

Now, to prove equation (3.17) for $1 \leq n < d$, let

$$\epsilon_n = \left(\frac{1}{2^{\frac{3 \cdot d}{2} + n + 1} \cdot C_\mu^2}\right)^{1/(d-n)}.$$

Noting that $\dim(L[X_n]) = n < d$, Proposition **??** implies that:

$$\mu\big(B(x,t) \setminus \mathrm{T_{ube}}\,(L[X_n], \epsilon_n \cdot t)\,\big) > \frac{1}{2} \cdot \mu(B(x,t)) > 0,$$

in particular,

$$[B(x,t) \cap \mathrm{supp}(\mu)] \setminus \mathrm{T_{ube}}\,(L[X_n], \epsilon_n \cdot t) \neq \emptyset.$$

We arbitrarily fix $x_{n+1} \in [B(x,t) \cap \mathrm{supp}(\mu)] \setminus \mathrm{T_{ube}}\,(L[x_n], \epsilon_n \cdot t)$, and we immediately obtain equation (3.17). We also note that

$$\|x_{n+1} - P(\widetilde{P}(x_{n+1}))\| \geq \epsilon_n \cdot t. \tag{3.20}$$

This follows from equation (3.17) and the fact that $P(x_{n+1})$ is the closest point to $x_{n+1}$ in the $n$-plane $L[X_n]$.

To establish equation (3.18), we will first show that there exists a constant $C_4 = C_4(n, \omega_n) > 0$ such that for any $0 < \delta \leq \delta_n$ we have the uniform upper bound

$$\|P(\widetilde{P}_\delta(y)) - \widetilde{P}_\delta(y)\| \leq C_4 \cdot \delta \cdot t, \text{ for all } y \in B(x_0, t). \tag{3.21}$$

Then, imposing the following restriction on $\delta$:

$$C_4 \cdot \delta \leq \frac{\epsilon_n}{3}, \tag{3.22}$$

and applying equations (3.17), (3.21), and (3.22), we derive equation (3.18) as follows

$$\begin{aligned}
&\|x_{n+1} - \widetilde{P}_\delta(x_{n+1})\| \geq \\
&\qquad \Big| \|x_{n+1} - P(\widetilde{P}_\delta(x_{n+1}))\| - \|P(\widetilde{P}_\delta(x_{n+1})) - \widetilde{P}_\delta(x_{n+1})\| \Big| \geq \\
&\qquad\qquad \|x_{n+1} - P(x_{n+1})\| - \|P(\widetilde{P}_\delta(x_{n+1})) - \widetilde{P}_\delta(x_{n+1})\| \geq \frac{2 \cdot \epsilon_n}{3} \cdot t.
\end{aligned}$$

To establish equation (3.21) and calculate the constant $C_4$, we first express the projection of any $y \in H$ onto $L[\widetilde{X}_n]$ as

$$\widetilde{P}_\delta(y) = \widetilde{x}_0 + \sum_{i=1}^{n} \widetilde{s}_i(y) \cdot (\widetilde{x}_i - \widetilde{x}_0)\,,$$

with $\widetilde{s}_i(y) \in \mathbb{R}$, $1 \leq i \leq n$. For $0 \leq i \leq n$, the points $\widetilde{x}_i$ have the decomposition

$$\widetilde{x}_i = x_i + \widetilde{z}_i + \widetilde{\varepsilon}_i, \tag{3.23}$$

where $\widetilde{z}_i \in \mathrm{Sp}\{x_1 - x_0, \ldots, x_n - x_0\}$ and $\widetilde{\varepsilon}_i$ is orthogonal to $\mathrm{Sp}\{x_1 - x_0, \ldots, x_n - x_0\}$. Therefore, we have the following equality for all $y \in H$:

$$P(\widetilde{P}_\delta(y)) - \widetilde{P}_\delta(y) = -\left(\widetilde{\varepsilon}_0 + \sum_{i=1}^{n} \widetilde{s}_i(y) \cdot (\widetilde{\varepsilon}_i - \widetilde{\varepsilon}_0)\right). \tag{3.24}$$

Furthermore, equations (3.15) and (3.23) imply the following inequality for all $0 \leq i \leq n$

$$\|\widetilde{\epsilon}_i\|^2 \leq \|\widetilde{z}_i\|^2 + \|\widetilde{\varepsilon}_i\|^2 \;\leq\; (\delta \cdot t)^2. \tag{3.25}$$

Thus, applying equation (3.25) and the triangle inequality to the RHS of equation (3.24), we have that

$$\|P(\widetilde{P}_\delta(y)) - \widetilde{P}_\delta(y)\| \leq \left(1 + \sum_{i=1}^{n} 2 \cdot |s_i(y)|\right) \cdot \delta \cdot t. \tag{3.26}$$

Now, to bound the RHS of equation (3.26) for $y \in B(x_0, t)$ (and thereby calculate an upper bound for the constant $C_4$ of equation (3.21)), we calculate a uniform bound for the quantities $\{|s_i(y)|\}_{i=1}^n$. In fact, we will establish the following inequality for all $y \in B(x_0, t)$:

$$\max_{1 \leq i \leq n} |\widetilde{s}_i(y)| \leq \frac{2 \cdot 3^{n-1}}{\omega_n}. \tag{3.27}$$

The combination of such a bound with equation (3.26) clearly implies equation (3.21), where

$$C_4 = \left(1 + n \cdot \frac{4 \cdot 3^{n-1}}{\omega_n}\right). \tag{3.28}$$

We first note that the coefficients $\widetilde{s}_i(y)$ satisfy the following equation for all $1 \leq i \leq n$:

$$\sin(\theta_i(\widetilde{X}_n)) \cdot |\widetilde{s}_i(y)| \cdot \|\widetilde{x}_i - \widetilde{x}_0\| = \operatorname{dist}(\widetilde{P}_\delta(y), L[\widetilde{X}_n(i)]). \tag{3.29}$$

Obtaining an upper bound on the RHS of equation (3.29) as well as a lower bound on the quantity $\sin(\theta_i(\widetilde{X}_n)) \cdot \|\widetilde{x}_i - \widetilde{x}_0\|$, will then establish equation (3.27).

We determine an upper bound by noting that $\widetilde{x}_0 \in B(x_0, t) \cap L[\widetilde{X}(i)]$, and thus for any $y \in B(x_0, t)$

$$\operatorname{dist}(\widetilde{P}_\delta(y), L[\widetilde{X}_n(i)]) \leq \left\| \widetilde{P}_\delta(y) - \widetilde{x}_0 \right\| \leq \|y - \widetilde{x}_0\| \leq 2 \cdot t. \tag{3.30}$$

In order to obtain the lower bound, we apply the product formula for contents as well as equation (3.16), and get that for any $0 < \delta \leq \delta_n$ and all $1 \leq i \leq n$

$$\mathrm{M}_n(\widetilde{X}_n) = \sin(\theta_i(\widetilde{X}_n)) \cdot \|\widetilde{x}_i - \widetilde{x}_0\| \cdot \mathrm{M}_{n-1}(\widetilde{X}_n(i)) \leq \sin(\theta_i(\widetilde{X}_n)) \cdot \|\widetilde{x}_i - \widetilde{x}_0\| \cdot 3^{n-1} \cdot t^{n-1}.$$

Combining this with the induction hypothesis, i.e., $\mathrm{M}_n(\widetilde{X}_n) \geq \omega_n \cdot t^n$, we obtain the inequality

$$\min_{1 \leq i \leq n} \sin(\theta_i(\widetilde{X}_n)) \cdot \|\widetilde{x}_i - \widetilde{x}_0\| \geq \frac{\omega_n}{3^{n-1}} \cdot t. \tag{3.31}$$

Applying the bounds of equations (3.31) and (3.30) to equation (3.29), we conclude equation (3.27), and consequently equations (3.21) and (3.28). We note that the constant $\delta = \delta_{n+1}$ needs to satisfy equation (3.22) and the requirement $0 < \delta_{n+1} \leq \delta_n$. We thus set its value in the following way:

$$\delta_{n+1} = \min \left\{ \frac{\epsilon_n}{3\left(1 + n \cdot \frac{4 \cdot 3^{n-1}}{\omega_n}\right)} \, , \, \delta_n \right\}. \tag{3.32}$$

To prove the final part of the induction argument, i.e., equation (3.19), we apply the triangle inequality and equations (3.15) (with $\delta = \delta_{n+1}$), (3.18) and (3.32), obtaining that for any $\widetilde{x}_{n+1} \in B\left(x_{n+1}, \delta_{n+1} \cdot t\right)$

$$\left\|\widetilde{x}_{n+1} - \widetilde{P}_{\delta_{n+1}}\left(\widetilde{x}_{n+1}\right)\right\| \geq \left| \left\|x_{n+1} - \widetilde{P}_{\delta_{n+1}}(\widetilde{x}_{n+1})\right\| - \|\widetilde{x}_{n+1} - x_{n+1}\| \right| \geq$$

$$\left\|x_{n+1} - \widetilde{P}_{\delta_{n+1}}(x_{n+1})\right\| - \|\widetilde{x}_{n+1} - x_{n+1}\| \geq \frac{\epsilon_n}{3} \cdot t.$$

**Part II of Proof**

Using the set of $d$-separated balls of Part I, $\{B(x_i, \delta_d \cdot t)\}_{i=0}^{d}$, we take the element $X_d = (x_0, \ldots, x_d)$, and for $0 < \rho < 1$ we define the constant

$$\epsilon_\rho = \frac{1-\rho}{2^{\frac{5 \cdot d}{2} - 1} \cdot C_\mu^2}.$$

We note that by Proposition **??**

$$\min_{0 \leq i \leq d} \mu\Big(B(x,t) \setminus \mathrm{T}_{\mathrm{ube}}\left(L[X_d(i)], \epsilon_\rho \cdot t\right)\Big) \geq \rho \cdot \mu\left(B(x,t)\right). \tag{3.33}$$

Hence, imposing the restriction $\rho > d/(d+1)$, and applying Lemma 3.2.1 with $\nu$ being the restricted measure $\mu|_{B(x,t)}$ scaled appropriately, $\xi = \rho$, $A_i = B(x,t) \setminus \mathrm{T}_{\mathrm{ube}}(L[X_d(i)], \epsilon_\rho \cdot t)$ for $0 \leq i \leq d$, and $k = d$, we get the following lower bound:

$$\mu\left(B(x,t) \setminus \bigcup_{i=0}^{d} \mathrm{T}_{\mathrm{ube}}\left(L[X_d(i)], \epsilon_\rho \cdot t\right)\right) > 0. \tag{3.34}$$

Therefore, for such $\rho$ there exists a point $x_{d+1} \in B(x,t) \cap \mathrm{supp}(\mu)$ so that

$$\min_{0 \leq i \leq d} \mathrm{dist}\left(x_{d+1}, L[X_d(i)]\right) > \epsilon_\rho \cdot t. \tag{3.35}$$

To choose the constants $\widetilde{\delta}_\mu = \widetilde{\delta}_\mu(d, C_\mu) > 0$ and $\widetilde{\omega}_\mu = \widetilde{\omega}_\mu(d, C_\mu) > 0$, as well as verify the claim of $d$-separation, we use practically the same arguments as those for proving equations (3.17)-(3.19). We arbitrarily fix $0 < \delta \leq \delta_d$, while later specifying its value, and an element

$$\widetilde{X}_d = (\widetilde{x}_0, \ldots, \widetilde{x}_d) \in \prod_{i=0}^{d} B(x_i, \delta \cdot t).$$

By the conclusion of Part I of the proof, we have that

$$\mathrm{M}_d(\widetilde{X}_d) \geq \omega_d \cdot t^d.$$

Furthermore, $\operatorname{diam}(\widetilde{X}_d) \leq 3 \cdot t$. Combining these with the product formula for contents, we obtain the inequality

$$\min_{0 \leq i \leq d} \mathrm{M}_{d-1}(\widetilde{X}_d(i)) \geq \frac{\omega_d}{3} \cdot t^{d-1}. \tag{3.36}$$

For $0 \leq i \leq d$, let $P_i$ and $\widetilde{P}_{\delta,i}$ denote the orthogonal projections of $H$ onto $L[X_d(i)]$ and $L[\widetilde{X}_d(i)]$, respectively. By virtually the same argument producing equation (3.21), while applying equation (3.36), we have that for all $y \in B(x,t)$,

$$\max_{0 \leq i \leq d} \left\| P_i\left(\widetilde{P}_{\delta,i}(y)\right) - \widetilde{P}_{\delta,i}(y) \right\| \leq \left(1 + (d-1) \cdot \frac{4 \cdot 3^{d-1}}{\omega_d}\right) \cdot \delta \cdot t\,.$$

Next, we impose the further restriction $\rho_0 = \frac{d+0.5}{d+1}$, and for this value of $\rho$ we set

$$\widetilde{\delta}_\mu = \min\left\{ \frac{\epsilon_{\rho_0}}{3 \cdot \left(1 + (d-1) \cdot \frac{4 \cdot 3^{d-1}}{\omega_d}\right)}, \delta_d \right\}.$$

By the same calculations producing equation (3.19), we get that

$$\min_{0 \leq i \leq d} \|\widetilde{x}_{d+1} - \widetilde{P}_{\widetilde{\delta}_\mu,i}(\widetilde{x}_{d+1})\| \geq \frac{\epsilon_{\rho_0}}{3} \cdot t \;\text{ for all } \widetilde{x}_{d+1} \in B(x_{d+1}, \widetilde{\delta}_\mu \cdot t). \tag{3.37}$$

Finally, combining Part I and equations (3.36) and (3.37) along with the product formula for contents, we have that for any

$$\widetilde{X}_{d+1} = (\widetilde{x}_0, \ldots, \widetilde{x}_{d+1}) \in \prod_{i=0}^{d+1} B(x_i, \delta \cdot t),$$

the following inequality is satisfied

$$\min_{0 \leq i \leq d+1} \mathrm{M}_d\left(\widetilde{X}_{d+1}(i)\right) \geq \frac{\omega_d \cdot \epsilon_{\rho_0}}{9} \cdot t^d.$$

Therefore, taking

$$\widetilde{\omega}_\mu = \frac{\epsilon_{\rho_0} \cdot \omega_d}{9},$$

the collection of balls $\{B(x_i, \widetilde{\delta}_\mu \cdot t)\}_{i=0}^{d+1}$ is $d$-separated in $B(x, 2 \cdot t)$ for $\widetilde{\omega}_\mu$. $\square$

**An Elementary Lemma**

We establish the following elementary proposition which was used in Subsection 3.2.3 and will also be used later in Subsection 3.3.3.

**Lemma 3.2.1.** *If $\nu$ is a Borel probability measure, $A_i \subseteq \operatorname{supp}(\nu)$ for $0 \leq i \leq d$, $0 < \xi < 1$, and*

$$\min_{0 \leq i \leq d} \nu(A_i) \geq \xi, \tag{3.38}$$

*then for any $0 \leq k \leq d$ the following inequality holds*

$$\nu\left(\bigcap_{i=0}^{k} A_i\right) \geq (k+1) \cdot \xi - k\,. \tag{3.39}$$

*Proof.* The proof is by induction. Equation (3.38) clearly implies the inequality of equation (3.39) when $k = 0$. Supposing that equation (3.39) holds for some $0 \leq k < d$, we note that

$$1 \geq \nu\left(\bigcap_{i=0}^{k} A_i \cup A_{k+1}\right) = \nu\left(\bigcap_{i=0}^{k} A_i\right) + \nu(A_{k+1}) - \nu\left(\bigcap_{i=0}^{k+1} A_i\right). \tag{3.40}$$

Thus, by the induction hypothesis and equation (3.40) we have that

$$\nu\left(\bigcap_{i=0}^{k+1} A_i\right) \geq \nu\left(\bigcap_{i=0}^{k} A_i\right) + \nu(A_{k+1}) - 1 \geq (k+1) \cdot \xi - k + \xi - 1 = (k+2) \cdot \xi - (k+1)\,.$$

□

## 3.3 The Proof of Theorem 3.0.1

In order to prove Theorem 3.0.1, we will establish the existence of constants $\lambda_0 = \lambda_0(d, C_\mu)$ and $C_1 = C_1(d, C_\mu)$ such that there exists a $d$-plane $L_{(x,t)}$ with

$$\int_{B(x,t)} \left(\frac{\operatorname{dist}\left(y, L_{(x,t)}\right)}{2 \cdot t}\right)^2 \mathrm{d}\mu(y) \leq C_1 \cdot c_{\mathrm{MT}}^2(x, t, \lambda_0), \tag{3.41}$$

for any $x \in \operatorname{supp}(\mu)$ and $0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu))$. Applying the definition of the $\beta_2$ numbers to equation (3.41) then proves Theorem 3.0.1.

Our approach for establishing equation (3.41) generalizes the proof of Léger [25, Lemma 2] for the case $d = 1$. In that case, constructing the line $L_{(x,t)}$ is relatively straightforward and short. However, for $d \geq 2$ there are combinatorial and geometric issues that do not manifest themselves when $d = 1$, e.g., the proofs of Proposition 3.3.1 and Lemma 3.3.1 below, and the notion of $d$-separation for $d \geq 2$ (Section 3.2 above). We present the overall argument in Subsection 3.3.2, and we leave the details to Subsections 3.3.3 and 3.3.4. Preliminary notation and observations are provided in Subsection 3.3.1

### 3.3.1 Notation and Preliminary Observations

For any $x \in \operatorname{supp}(\mu)$, $0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu))$, $0 < \lambda < 2$, $0 \leq i < j \leq d+1$, $X(i) \in H^{d+1}$, $X(i,j) \in H^d$, $y \in H$ and $X(y,i;j)$, we define the following slices of the set $U_\lambda(x,t) = U_\lambda(B(x,t))$ of equation (1.14):

$$U_\lambda\left(x,t \,\middle|\, X(i)\right) = \{y \in B(x,t) : X(y,i) \in U_\lambda(x,t)\},$$

$$U_\lambda\left(x,t \,\middle|\, X(i;j)\right) = \left\{(y,z) \in B(x,t)^2 : X(y,i;z,j) \in U_\lambda(x,t)\right\},$$

$$U_\lambda\left(x,t \,\middle|\, X(y,i;j)\right) = \left\{z \in B(x,t) : X(y,i;z,j) \in U_\lambda(x,t)\right\},$$

$$U_\lambda\left(x,t \,\middle|\, X(i;y,j)\right) = \left\{z \in B(x,t) : X(z,i;y,j) \in U_\lambda(x,t)\right\}.$$

We note that each of these sets can be empty depending on the elements $X(i)$, $X(i,j)$, and $X(y,i)$, which were defined in equations (3.1)-(3.3) (see also Remark 3.0.4). In addition, we fix the following constant of 1-separation

$$\lambda_0 = \frac{\delta_\mu}{2},$$

where $\delta_\mu$ is the constant suggested by Proposition 3.2.1.

For the remainder of the proof (i.e., the whole section) we arbitrarily fix $x \in \mathrm{supp}(\mu)$ and $0 < t \leq \mathrm{diam}(\mathrm{supp}(\mu))$, and some $d$-separated collection of balls $\{B(x_i, \delta_\mu \cdot t)\}_{i=0}^{d+1}$ in $B(x,t)$ for the constant $\omega_\mu$ (see Proposition 3.2.1). We denote $B_i = B(x_i, \delta_\mu \cdot t)$ for $0 \leq i \leq d+1$. Restricting our attention to only the first $(d+1)$ balls, we also form an arbitrary element

$$\widetilde{X}(d+1) = (\widetilde{x}_0, \ldots, \widetilde{x}_d) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i.$$

We note that

$$B(x,t) \setminus \bigcup_{i=0}^{d} B_i \;\subseteq\; U_{\lambda_0}(x,t|\widetilde{X}(d+1)) \tag{3.42}$$

and

$$B_i \nsubseteq U_{\lambda_0}(x,t|\widetilde{X}(d+1)),\ \text{for each } 0 \leq i \leq d. \tag{3.43}$$

### 3.3.2 The Essence of the Proof of Theorem 3.0.1

For $0 < \rho < \infty$ let

$$\mathcal{E}(\rho) = \Bigg\{ \widetilde{X}(d+1) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i : \int_{U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(d+1)\right)} \frac{\mathrm{p}_d \mathrm{sin}^2_{\widetilde{x}_0}\left(\widetilde{X}(y,d+1)\right)}{\mathrm{diam}\left(\widetilde{X}(y,d+1)\right)^{d(d+1)}} \,\mathrm{d}\mu(y) \leq \rho \cdot \frac{c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^{d(d+1)}} \Bigg\}. \tag{3.44}$$

We will show that $\mu^{d+1}(\mathcal{E}(\rho))$ is sufficiently large for some $0 < \rho < \infty$.

First, applying Chebychev's inequality to equation (3.44) we obtain that

$$\mu^{d+1}\left(\prod_{i=0}^{d} \frac{1}{2} \cdot B_i \setminus \mathcal{E}(\rho)\right) \leq \frac{t^{d(d+1)}}{\rho}. \tag{3.45}$$

Next, we note that the $d$-regularity of $\mu$ implies that

$$\frac{1}{C_\mu^{d+1}} \cdot (\lambda_0 \cdot t)^{d(d+1)} \leq \mu^{d+1}\left(\prod_{i=0}^{d} \frac{1}{2} \cdot B_i\right) \leq C_\mu^{d+1} \cdot (\lambda_0 \cdot t)^{d(d+1)}. \tag{3.46}$$

Thus, combining equations (3.45) and (3.46), and taking

$$\rho_1 = \rho_1(d, C_\mu) > \frac{2}{\lambda_0^{d(d+1)}} \cdot C_\mu^{d+1}, \tag{3.47}$$

we obtain the lower bound

$$\mu^{d+1}\left(\mathcal{E}(\rho_1)\right) > \frac{1}{2} \cdot \mu^{d+1}\left(\prod_{i=0}^{d} \frac{1}{2} \cdot B_i\right) > 0. \tag{3.48}$$

We will show that the desired $d$-plane, $L_{(x,t)}$, of equation (3.41) is obtained by $L[\widetilde{X}(d+1)]$ for some $\widetilde{X}(d+1) \in \mathcal{E}(\rho_1)$. In fact, for any such $\widetilde{X}(d+1)$ we immediately obtain control on a part of the integral on the LHS of equation (3.41) as follows. Since $\widetilde{X}(d+1) \in \mathcal{E}(\rho_1)$ is $d$-separated, by Lemma 3.1.1 the following lower bound holds for all $y \in H$

$$\frac{\mathrm{p}_d \sin^2_{\widetilde{x}_0}\left(\widetilde{X}(y, d+1)\right)}{\operatorname{diam}\left(\widetilde{X}(y, d+1)\right)^{d(d+1)}} \geq \frac{\omega_\mu^2}{2^{(d+1)(d+2)}} \cdot \left(\frac{\operatorname{dist}(y, L[\widetilde{X}(d+1)])}{t}\right)^2 \cdot \frac{1}{t^{d(d+1)}}\,. \tag{3.49}$$

Thus, by equations (3.44) and (3.49) we have that for any $\widetilde{X}(d+1) \in \mathcal{E}(\rho_1)$

$$\int_{U_{\lambda_0}\left(x,t \,\middle|\, \widetilde{X}(d+1)\right)} \left(\frac{\operatorname{dist}(y, L[\widetilde{X}(d+1)])}{t}\right)^2 \mathrm{d}\mu(y) \;\leq \frac{2^{(d+1)(d+2)}}{\omega_\mu^2} \cdot \rho_1 \cdot\; c^2_{\mathrm{MT}}(x, t, \lambda_0). \tag{3.50}$$

Combining this with the set inclusion of equation (3.42) implies that

$$\int_{B(x,t)\setminus\bigcup_{i=0}^{d} B_i} \left(\frac{\operatorname{dist}(y, L[\widetilde{X}(d+1)])}{t}\right)^2 \mathrm{d}\mu(y) \;\leq \frac{2^{(d+1)(d+2)}}{\omega_\mu^2} \cdot \rho_1 \cdot c^2_{\mathrm{MT}}(x, t, \lambda_0). \tag{3.51}$$

Despite the upper bound of equation (3.51), the condition of the set $\mathcal{E}(\rho_1)$ does not help us to obtain a bound for the integrals over the individual balls $B_i$, $0 \leq i \leq d$. This incompleteness follows from equation (3.43). In order to obtain such an upper bound (thus concluding equation (3.41)), we must impose further restrictions on the element $\widetilde{X}(d+1)$.

For $0 < \rho < \infty$, let

$$\mathcal{A}(\rho) = \Bigg\{ \widetilde{X}(d+1) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i :$$
$$\max_{0 \le i \le d} \int_{U_{\lambda_0}\left(x,t \,\middle|\, \widetilde{X}(i;d+1)\right)} \frac{\mathrm{p}_d \sin^2_{\left(\widetilde{X}(y,i;z,d+1)\right)_0}\left(\widetilde{X}(y,i;z,d+1)\right)}{\operatorname{diam}\left(\widetilde{X}(y,i;z,d+1)\right)^{d(d+1)}} \, \mathrm{d}\mu^2(y,z) \le \frac{\rho \cdot c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^{d^2}} \Bigg\}. \tag{3.52}$$

Below in Subsection 3.3.3 we prove the following lemma.

**Lemma 3.3.1.** *There exists a constant $\rho_2 = \rho_2(d, C_\mu) > 0$ such that*

$$\mu^{d+1}\left(\mathcal{A}(\rho_2)\right) > \frac{1}{2} \cdot \mu^{d+1}\left(\prod_{i=0}^{d} \frac{1}{2} \cdot B_i\right) > 0,$$

*for any $x \in \operatorname{supp}(\mu)$ and $0 < t \le \operatorname{diam}(\operatorname{supp}(\mu))$.*

The condition imposed by $\mathcal{A}(\rho_2)$ yields the following estimate which is proved in Subsection 3.3.4.

**Proposition 3.3.1.** *There exists a constant $C_5 = C_5(d, C_\mu)$ such that for any element $\widetilde{X}(d+1) \in \mathcal{A}(\rho_2)$:*

$$\max_{0 \le i \le d} \int_{B_i} \left(\frac{\operatorname{dist}(y, L[\widetilde{X}(d+1)])}{t}\right)^2 \mathrm{d}\mu(y) \le C_5 \cdot c^2_{\mathrm{MT}}(x,t,\lambda_0).$$

Finally, equation (3.48) and Lemma 3.3.1 imply that $\mathcal{E}(\rho_1) \cap \mathcal{A}(\rho_2) \neq \emptyset$. Thus, fixing an arbitrary $\widetilde{X}(d+1) \in \mathcal{E}(\rho_1) \cap \mathcal{A}(\rho_2)$, equation (3.41) is deduced from equation (3.51) and Proposition 3.3.1. □

### 3.3.3 The Proof of Lemma 3.3.1

For each $0 \le i \le d$ we define the following cartesian product

$$A_i = \prod_{0 \le j \ne i \le d} \frac{1}{2} \cdot B_j.$$

We note that the $d$-regularity of $\mu$ (and the fact that $\lambda_0 = \delta_\mu/2$) trivially implies the following estimate for each $0 \leq i \leq d$:

$$\frac{1}{C_\mu^d} \cdot (\lambda_0 \cdot t)^{d^2} \leq \mu^d(A_i) \leq C_\mu^d \cdot (\lambda_0 \cdot t)^{d^2}. \tag{3.53}$$

Then, for $0 < \rho < \infty$ and $0 \leq i \leq d$, we define the set

$$\begin{aligned}\mathcal{A}_i(\rho) = \Bigg\{ & \widetilde{X}(i; d+1) \in A_i : \\ & \int_{U_{\lambda_0}\left(x,t \,\middle|\, \widetilde{X}(i;d+1)\right)} \frac{\mathrm{p}_d \sin^2_{\left(\widetilde{X}(y,i;z,d+1)\right)_0}\left(\widetilde{X}(y,i;z,d+1)\right)}{\operatorname{diam}\left(\widetilde{X}(y,i;z,d+1)\right)^{d(d+1)}} \,\mathrm{d}\mu^2(y,z) \leq \frac{\rho \cdot c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^{d^2}} \Bigg\},\end{aligned} \tag{3.54}$$

and we embed it in the product $\prod_{j=0}^d \frac{1}{2} \cdot B_j$ by defining the set

$$\underline{\mathcal{A}_i\left(\rho\right)} = \left\{ \widetilde{X}(y,i;d+1) : \widetilde{X}(i,d+1) \in \mathcal{A}_i\left(\rho\right) \text{ and } y \in \frac{1}{2} \cdot B_i \right\}.$$

From this definition we see that

$$\mu^{d+1}\left(\underline{\mathcal{A}_i(\rho)}\right) = \mu^d(\mathcal{A}_i(\rho)) \cdot \mu\left(\frac{1}{2} \cdot B_i\right). \tag{3.55}$$

Furthermore, for the set $\mathcal{A}(\rho)$ of equation (3.52), we note the inclusion

$$\bigcap_{i=0}^{d} \underline{\mathcal{A}_i(\rho)} \subseteq \mathcal{A}(\rho) \subseteq \prod_{i=0}^{d} \frac{1}{2} \cdot B_i. \tag{3.56}$$

We next find $\rho$ such that $\mu^{d+1}(\mathcal{A}(\rho))$ is sufficiently large. We do this by first using equation (3.55) to find $\rho$ such that the individual $\mu^{d+1}(\underline{\mathcal{A}_i(\rho)})$, $0 \leq i \leq d$, are sufficiently large, and then applying equation (3.56) to get the desired conclusion about $\mathcal{A}(\rho)$.

Applying Chebychev's inequality to equation (3.54) implies that for all $0 \leq i \leq d$

$$\mu^d\left(\mathcal{A}_i(\rho)\right) \geq \mu^d\left(A_i\right) - \frac{t^{d^2}}{\rho}. \tag{3.57}$$

In order to choose $\rho$, for any $0 < \xi < 1$ we define

$$\rho(\xi) = \frac{C_\mu^d}{1-\xi} \cdot \left(\frac{1}{\lambda_0}\right)^{d^2},$$

and by applying the estimates of equations (3.53) and (3.57) we obtain that

$$\mu^d\left(\mathcal{A}_i(\rho(\xi))\right) \geq \xi \cdot \mu^d\left(A_i\right), \text{ for each } 0 \leq i \leq d. \tag{3.58}$$

Hence, by equations (3.55) and (3.58) we have the lower bound

$$\mu^{d+1}\left(\underline{\mathcal{A}_i(\rho(\xi))}\right) \geq \xi \cdot \mu^{d+1}\left(\prod_{j=0}^{d} \frac{1}{2} \cdot B_j\right), \text{ for all } 0 \leq i \leq d. \tag{3.59}$$

Therefore, letting $\rho_2 = \rho_2(\xi)$ where

$$\xi > \frac{d+1/2}{d+1}, \tag{3.60}$$

and applying Lemma 3.2.1 (with $\nu$ being the measure $\mu^{d+1}$ restricted to the set $\prod_{j=0}^{d} 1/2 \cdot B_j$ and scaled to 1 on that set, $A_i = \underline{\mathcal{A}_i(\rho(\xi))}$ for $0 \leq i \leq d$, and $k = d$) we get the following lower bound:

$$\mu^{d+1}\left(\bigcap_{i=0}^{d} \underline{\mathcal{A}_i(\rho_2(\xi))}\right) \;\geq\; \big((d+1)\cdot\xi - d\big) \cdot \mu^{d+1}\left(\prod_{j=0}^{d} \frac{1}{2} \cdot B_j\right) \;>\; \frac{1}{2} \cdot \mu^{d+1}\left(\prod_{j=0}^{d} \frac{1}{2} \cdot B_j\right). \tag{3.61}$$

Lemma 3.3.1 thus follows from equations (3.56) and (3.61). $\square$

### 3.3.4 The Proof of Proposition 3.3.1

Up until this point, we have not used the full statement of Proposition 3.2.1. We have only used the first $(d+1)$ balls, $B_0, \ldots, B_d$, in the definitions of the sets $\mathcal{E}(\rho_1)$ and $\mathcal{A}(\rho_2)$, and we have completely ignored the $(d+2)$-nd ball of the $d$-separated collection $\{B_j\}_{j=0}^{d+1}$. The proof of Proposition 3.3.1 requires the use of this final ball, which we have denoted by $B_{d+1}$. We use this ball to formulate the following lemma (whose proof is given in Subsection 3.3.4).

**Lemma 3.3.2.** *There exist constants $C_6 = C_6(d, C_\mu, \lambda_0)$ and $C_7 = C_7(d, C_\mu, \lambda_0)$ such that for any fixed $\widetilde{X}(d+1) \in \mathcal{E}(\rho_1) \cap \mathcal{A}(\rho_2)$ and fixed $0 \le i \le d$, the following property is satisfied: There exists a point*

$$\widetilde{x}_{d+1} \in \frac{1}{2} \cdot B_{d+1} \cap \operatorname{supp}(\mu)$$

*with*

$$\int_{U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(i;\widetilde{x}_{d+1},d+1)\right)} \frac{\mathrm{p}_d \sin^2_{\left(\widetilde{X}(y,i;\widetilde{x}_{d+1},d+1)\right)_0}\left(\widetilde{X}\,(y,i;\widetilde{x}_{d+1},d+1)\right)}{\operatorname{diam}\left(\widetilde{X}\,(y,i;\widetilde{x}_{d+1},d+1)\right)^{d(d+1)}}\,\mathrm{d}\mu(y) \le C_6 \cdot \frac{c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^{d^2+d}}, \tag{3.62}$$

*and*

$$\left(\frac{\operatorname{dist}(\widetilde{x}_{d+1}, L[\widetilde{X}(d+1)])}{t}\right)^2 \le C_7 \cdot \frac{c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^d}. \tag{3.63}$$

We will prove this lemma in Subsection 3.3.4, and will then use it in Subsection 3.3.4 to prove Proposition 3.3.1.

### Proof of Lemma 3.3.2

To construct the point $\widetilde{x}_{d+1}$, for any fixed $\widetilde{X}(d+1) \in \mathcal{A}(\rho_2)$ and any $0 \le i \le d$ we define the following two sets for $\rho_2$ of Lemma 3.3.1 and any $0 < \tau < \infty$:

$$\mathcal{Q}(\tau) = \left\{ z \in \frac{1}{2} \cdot B_{d+1} : \int_{U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(i;\,z,d+1)\right)} \frac{\mathrm{p}_d \sin^2_{\left(\widetilde{X}(y,i;z,d+1)\right)_0}\left(\widetilde{X}\,(y,i;z,d+1)\right)}{\operatorname{diam}\left(\widetilde{X}\,(y,i;z,d+1)\right)^{d(d+1)}}\,\mathrm{d}\mu(y) \le \frac{\tau}{t^d} \cdot \rho_2 \cdot \frac{c^2_{\mathrm{MT}}(x,t,\lambda_0)}{t^{d^2}} \right\},$$

and

$$\mathcal{G}(\tau) = \left\{ z \in \frac{1}{2} \cdot B_{d+1} : \left(\frac{\operatorname{dist}(z, L[\widetilde{X}(d+1)])}{t}\right)^2 \le \frac{\tau}{t^d} \cdot \rho_1 \cdot \frac{2^{(d+1)(d+2)}}{\omega_\mu^2} \cdot c^2_{\mathrm{MT}}(x,t,\lambda_0) \right\}.$$

The idea is to find a large enough $\tau$ so that the intersection of these two sets is non-empty.

We first focus on the set $\mathcal{Q}(\tau)$ and specify a value for $\tau$ such that $\mu(\mathcal{Q}(\tau))$ is sufficiently large. Trivially, we have the inclusion

$$\frac{1}{2}\cdot B_{d+1} \subseteq U_{\lambda_0}\left(x,t\middle|\widetilde{X}(d+1)\right),$$

and thus

$$\widetilde{X}(y,i;z,d+1) \in U_{\lambda_0}(x,t), \tag{3.64}$$

for all $z \in \frac{1}{2}\cdot B_{d+1}$ and $y \in U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(i;\,z,d+1)\right)$.

Since $\widetilde{X}(d+1) \in \mathcal{A}(\rho_2)$, by equations (3.52) and (3.64) we get that

$$\int_{\frac{1}{2}\cdot B_{d+1}}\left(\int_{U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(i;\,z,d+1)\right)} \frac{\mathrm{p}_d\sin^2_{\left(\widetilde{X}(y,i;z,d+1)\right)_0}\left(\widetilde{X}\,(y,i;z,d+1)\right)}{\operatorname{diam}\left(\widetilde{X}\,(y,i;z,d+1)\right)^{d(d+1)}}\,\mathrm{d}\mu(y)\right)\mathrm{d}\mu(z) \leq$$
$$\int_{U_{\lambda_0}\left(x,t\,\middle|\,\widetilde{X}(i;d+1)\right)} \frac{\mathrm{p}_d\sin^2_{\left(\widetilde{X}(y,i;z,d+1)\right)_0}\left(\widetilde{X}\,(y,i;z,d+1)\right)}{\operatorname{diam}\left(\widetilde{X}\,(y,i;z,d+1)\right)^{d(d+1)}}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \leq \frac{\rho_2}{t^{d^2}}\cdot c^2_{\mathrm{MT}}(x,t,\lambda_0).$$

Hence, by Chebychev's inequality we obtain

$$\mu\left(\mathcal{Q}(\tau)\right) \geq \mu\left(\frac{1}{2}\cdot B_{d+1}\right) - \frac{t^d}{\tau}.$$

We thus fix

$$\tau_0 = \tau_0(d,C_\mu) > \frac{2\cdot C_\mu}{\lambda_0^d},$$

and by the $d$-regularity of $\mu$ we have the lower bound

$$\mu(\mathcal{Q}(\tau_0)) > \frac{1}{2}\cdot\mu\left(\frac{1}{2}\cdot B_{d+1}\right). \tag{3.65}$$

Clearly, one can choose any $\widetilde{x}_{d+1}$ in $\mathcal{Q}(\tau_0) \neq \emptyset$ and it will satisfy equation (3.62) with $C_6 = \tau_0\cdot\rho_2$.

Next, to choose $\tau$ such that $\mu(\mathcal{G}(\tau))$ is also sufficiently large, i.e., to find a point $\widetilde{x}_{d+1}$ which satisfies equation (3.63) as well, we apply equation (3.50) and Chebychev's inequality to obtain

$$\mu\left(\mathcal{G}(\tau)\right) \geq \mu\left(\frac{1}{2}\cdot B_{d+1}\right) - \frac{t^d}{\tau}.$$

Hence, for $\tau = \tau_0$, by the $d$-regularity of $\mu$ we get that

$$\mu\left(\mathcal{G}(\tau_0)\right) > \frac{1}{2}\cdot\mu\left(\frac{1}{2}\cdot B_{d+1}\right). \tag{3.66}$$

Finally, the combination of equations (3.65) and (3.66) results in the inequality

$$\mu\left(\mathcal{Q}(\tau_0)\cap\mathcal{G}(\tau_0)\right) > 0,$$

and therefore the lemma is established with $C_6$ (as specified above) and

$$C_7 = \tau_0\cdot\rho_1\cdot\frac{2^{(d+1)(d+2)}}{\omega_\mu}.$$

□

**Deriving Proposition 3.3.1 from Lemma 3.3.2**

We arbitrarily fix an index $0 \leq i \leq d$ and prove Proposition 3.3.1 by specifying a constant $C_5 = C_5(d, C_\mu, \lambda_0)$ such that

$$\int_{B_i}\left(\frac{\operatorname{dist}(y, L[\widetilde{X}(d+1)])}{t}\right)^2 \mathrm{d}\mu(y) \leq C_5\cdot c^2_{\mathrm{MT}}(x,t,\lambda_0),\ \text{for all } 0\leq i\leq d\,. \tag{3.67}$$

Our strategy for proving equation (3.67) is to first show that for any point $\widetilde{x}_{d+1}$ satisfying Lemma 3.3.2, the following inequality holds (for the fixed index $i$ and $C_6$ as in Lemma 3.3.2)

$$\int_{B_i}\left(\frac{\operatorname{dist}(y, L[\widetilde{X}(i\,;\widetilde{x}_{d+1}, d+1)])}{t}\right)^2 \mathrm{d}\mu(y) \leq \frac{2^{(d+1)(d+2)}}{\omega_\mu^2\cdot\lambda_0^{d(d+1)}}\cdot C_6\cdot c^2_{\mathrm{MT}}(x,t,\lambda_0). \tag{3.68}$$

Then, a basic geometric argument shows that equation (3.68) implies equation (3.67).

Now, if $\widetilde{x}_{d+1}$ is a point satisfying Lemma 3.3.2, then since $\widetilde{x}_{d+1}\in B_{d+1}$ and the collection of balls $\{B_i\}_{i=0}^{d+1}$ is $d$-separated we have that

$$B_i \subseteq U_{\lambda_0}(x,t\,|\,\widetilde{X}(i;\widetilde{x}_{d+1}, d+1)) \tag{3.69}$$

and

$$\mathrm{M}_d(\widetilde{X}(i;\widetilde{x}_{d+1},d+1)) \geq \omega_\mu \cdot t^d. \tag{3.70}$$

If $1 \leq i \leq d$, then Lemma 3.1.1 implies that for any $y \in B_i$

$$\frac{\mathrm{p}_d\mathrm{sin}^2_{\widetilde{x}_0}\left(\widetilde{X}(y,i\,;\widetilde{x}_{d+1},d+1)\right)}{\mathrm{diam}\left(\widetilde{X}(y,i\,;\widetilde{x}_{d+1},d+1)\right)^{d(d+1)}} \geq \frac{\omega_\mu^2}{2^{(d+1)(d+2)}} \cdot \left(\frac{\mathrm{dist}(y,L[\widetilde{X}(i\,;\widetilde{x}_{d+1},d+1)])}{t}\right)^2 \cdot \frac{1}{t^{d(d+1)}}. \tag{3.71}$$

This inequality immediately implies equation (3.68) in this case.

However, if $i=0$, then we cannot directly apply Lemma 3.1.1. Instead, we first note that for all $y \in B_0$

$$\delta_\mu \cdot t = 2 \cdot \lambda_0 \cdot t \leq \min(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)) \leq \mathrm{diam}(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)) \leq 2 \cdot t. \tag{3.72}$$

Then, using these bounds we apply the law of sines for the polar sine (see equation (3.11)) to obtain the lower bound

$$\begin{aligned}
&\mathrm{p}_d\mathrm{sin}_y\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right) \geq \\
&\quad\left(\frac{\min\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right)}{\mathrm{diam}\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right)}\right)^{\frac{d(d+1)}{d}} \cdot \mathrm{p}_d\mathrm{sin}_{\widetilde{x}_1}\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right) \geq \\
&\qquad \lambda_0^{\frac{d(d+1)}{2}} \cdot \mathrm{p}_d\mathrm{sin}_{\widetilde{x}_1}\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right).
\end{aligned} \tag{3.73}$$

Applying Lemma 3.1.1 to the RHS of equation (3.73), and then applying equation (3.72) to the resulting equation gives the inequality

$$\begin{aligned}
&\frac{\mathrm{p}_d\mathrm{sin}^2_y\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right)}{\mathrm{diam}\left(\widetilde{X}(y,0\,;\widetilde{x}_{d+1},d+1)\right)^{d(d+1)}} \geq \\
&\qquad \frac{\omega_\mu^2 \cdot \lambda_0^{d(d+1)}}{2^{(d+1)(d+2)}} \cdot \left(\frac{\mathrm{dist}(y,L[\widetilde{X}(0\,;\widetilde{x}_{d+1},d+1)])}{t}\right)^2 \cdot \frac{1}{t^{d(d+1)}}.
\end{aligned} \tag{3.74}$$

We replace equation (3.71) (where $1 \leq i \leq d$) and equation (3.74) (where $i = 0$) by the following equation which holds for all $0 \leq i \leq d$

$$\frac{\mathrm{p}_d \sin^2_{\left(\widetilde{X}(y,i\,;\widetilde{x}_{d+1},d+1)\right)_0}\left(\widetilde{X}(y,i\,;\widetilde{x}_{d+1},d+1)\right)}{\operatorname{diam}\left(\widetilde{X}(y,i\,;\widetilde{x}_{d+1},d+1)\right)^{d(d+1)}} \geq \frac{\omega_\mu^2 \cdot \lambda_0^{d(d+1)}}{2^{(d+1)(d+2)}} \cdot \left(\frac{\operatorname{dist}(y, L[\widetilde{X}(i\,;\widetilde{x}_{d+1},d+1)])}{t}\right)^2 \cdot \frac{1}{t^{d(d+1)}}. \quad (3.75)$$

Combining equation (3.75) with equations (3.62) and (3.69) implies equation (3.68) for the fixed index $i$.

Next, equation (3.68) implies equation (3.67) via the following argument. We first note that the elements $\widetilde{X}(d+1)$, $\widetilde{X}(i;\widetilde{x}_{d+1},d+1)$, and $\widetilde{X}(i;d+1)$ are all non-degenerate, and we define the orthogonal projections

$$P_{d+1} \ : H \to L[\widetilde{X}(d+1)]\,,$$

$$P_i \ : H \to L[\widetilde{X}(i;\widetilde{x}_{d+1},d+1)]\,,$$

$$P_{i,d+1} \ : H \to L[\widetilde{X}(i;d+1)]\,.$$

Using these projections we can reduce the situation to two cases.

The first is when $P_{d+1} = P_i$, that is, $L[\widetilde{X}(d+1)] = L[\widetilde{X}(i\,;\widetilde{x}_{d+1},d+1)]$. In this case equation (3.67) holds trivially by equation (3.68) with

$$C_5 \geq \frac{2^{(d+1)(d+2)}}{\omega_\mu^2 \cdot \lambda_0^{d(d+1)}} \cdot C_6.$$

The second is when $P_{d+1} \neq P_i$. In this case, we rely on the following inequality.

$$\left(\frac{\operatorname{dist}\left(y, P_{d+1}(y)\right)}{t}\right)^2 \leq \left(\frac{\operatorname{dist}(y, P_{d+1}\left(P_i(y)\right)}{t}\right)^2 \leq 2 \cdot \left[\left(\frac{\operatorname{dist}\left(y, P_i(y)\right)}{t}\right)^2 + \left(\frac{\operatorname{dist}\left(P_i(y), P_{d+1}(P_i(y))\right)}{t}\right)^2\right]. \quad (3.76)$$

Integrating the inequality of equation (3.76) over the ball $B_i$ and applying the inequality of equation (3.68), we obtain the bound

$$\int_{B_i} \left( \frac{\operatorname{dist}(y, P_{d+1}(y))}{t} \right)^2 \mathrm{d}\mu(y) \leq$$
$$2 \cdot \frac{2^{(d+1)(d+2)}}{\omega_\mu^2 \cdot \lambda_0^{d(d+1)}} \cdot C_8 \cdot c_{\mathrm{MT}}^2(x, t, \lambda_0) + 2 \int_{B_i} \left( \frac{\operatorname{dist}(P_i(y), P_{d+1}(P_i(y)))}{t} \right)^2 \mathrm{d}\mu(y). \quad (3.77)$$

The only thing remaining is to bound the second term on the RHS of equation (3.77).

Since $P_{d+1} \neq P_i$, the $d$-planes $L[\widetilde{X}(d+1)]$ and $L[\widetilde{X}(i\,; \widetilde{x}_{d+1}, d+1)]$ are distinct. Let $\alpha$ denote the dihedral angle between these two $d$-planes along their intersection, the $(d-1)$-plane $L[\widetilde{X}(i; d+1)]$. We note that $\sin(\alpha) > 0$. Furthermore, for all $y \in B(x, t)$ we have that

$$\operatorname{dist}(P_i(y), P_{d+1}(P_i(y))) = \sin(\alpha) \cdot \operatorname{dist}(P_i(y), P_{i,d+1}(P_i(y)))\,. \quad (3.78)$$

We can bound the RHS of equation (3.78) by bounding each of the factors separately. For any $j \neq i, d+1$, we have the inclusion

$$(\widetilde{X}(d+1))_j \in B(x, t) \cap L[\widetilde{X}(i; d+1)] \subseteq L[\widetilde{X}(i; \widetilde{x}_{d+1}, d+1)].$$

Hence we obtain that for all $y \in B(x, t)$

$$\operatorname{dist}(P_i(y), P_{i,d+1}(P_i(y))) \leq \|P_i(y) - (\widetilde{X}(i; d+1))_j\| =$$
$$\|P_i(y - (\widetilde{X}(i; d+1))_j)\| \leq \|y - (\widetilde{X}(i; d+1))_j\| \leq 2 \cdot t\,. \quad (3.79)$$

To bound $\sin(\alpha)$, we observe that

$$\sin(\alpha) = \frac{\operatorname{dist}(\widetilde{x}_{d+1}, P_{d+1}(\widetilde{x}_{d+1}))}{\operatorname{dist}(\widetilde{x}_{d+1}, P_{i,d+1}(\widetilde{x}_{d+1}))}. \quad (3.80)$$

By equation (3.70) and the product formula for contents we have that

$$\operatorname{dist}(\widetilde{x}_{d+1}, P_{i,d+1}(\widetilde{x}_{d+1})) \geq \frac{\omega_\mu}{2^{d-1}} \cdot t.$$

Combining this lower bound with equation (3.80) we have the upper bound

$$\sin(\alpha) \leq \frac{2^{d-1}}{\omega_\mu} \cdot \frac{\operatorname{dist}\left(\widetilde{x}_{d+1}, P_{d+1}(\widetilde{x}_{d+1})\right)}{t} = \frac{2^{d-1}}{\omega_\mu} \cdot \frac{\operatorname{dist}\left(\widetilde{x}_{d+1}, L[\widetilde{X}(d+1)]\right)}{t}. \tag{3.81}$$

Applying equations (3.79) and (3.81) to the RHS of equation (3.78), we have the following uniform upper bound for all $y \in B(x,t)$

$$\operatorname{dist}\left(P_i(y), P(P_i(y))\right) \leq \frac{2^d}{\omega_\mu} \cdot \operatorname{dist}\left(\widetilde{x}_{d+1}, L[\widetilde{X}(d+1)]\right). \tag{3.82}$$

Equation (3.82), Lemma 3.3.2 and the $d$-regularity of $\mu$ imply that

$$\int_{B_i} \left(\frac{\operatorname{dist}\left(P_i(y), P_{d+1}(P_i(y))\right)}{t}\right)^2 \mathrm{d}\mu(y) \leq \frac{4^d}{\omega_\mu^2} \cdot \left(\frac{\operatorname{dist}\left(\widetilde{x}_{d+1}, L[\widetilde{X}(d+1)]\right)}{t}\right)^2 \mu(B_i) \leq \frac{4^d \cdot C_\mu \cdot C_7}{\omega_\mu^2} \cdot c_{\mathrm{MT}}^2(x,t,\lambda_0). \tag{3.83}$$

Finally, applying equation (3.83) to the right hand side of equation (3.77) finishes the proof of equation (3.67), and thus concludes Proposition 3.3.1. □

## 3.4 The Proof of Theorem 3.0.2

Theorem 3.0.2 is an easy consequence of Theorem 3.0.1 and the following proposition, which actually holds for any non-negative function on $H^{d+2}$ instead of $c_{\mathrm{MT}}$.

**Proposition 3.4.1.** *If $\lambda > 0$, then*

$$\int_B \int_0^{\operatorname{diam}(B)} c_{\mathrm{MT}}^2(x,t,\lambda) \frac{\mathrm{d}t}{t^{d+1}} \mathrm{d}\mu(x) \leq \left(\frac{2}{\lambda}\right)^d \cdot \frac{C_\mu}{d} \cdot c_{\mathrm{MT}}^2(\mu|_{3\cdot B}, \lambda/2), \tag{3.84}$$

*for any ball $B \subseteq H$.*

Indeed, Theorem 3.0.1 and the $d$-regularity of $\mu$ imply that

$$\beta_2^2(x,t) \lessapprox \frac{c_{\mathrm{MT}}^2(x,t,\lambda_0)}{t^d}, \text{ for all } x \in \operatorname{supp}(\mu),\ 0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu)),$$

where the comparison depends on $d$, $C_\mu$, and $\lambda_0$. Thus, by Proposition 3.4.1 we obtain the following estimate for all balls $B$ such that $\mathrm{diam}(B) \leq \mathrm{diam}(\mathrm{supp}(\mu))$:

$$J_2(\mu|_B) = \int_B \int_0^{\mathrm{diam}(B)} \beta_2^2(x,t)\frac{\mathrm{d}t}{t}\mathrm{d}\mu(x) \lessapprox c_{\mathrm{MT}}^2(\mu|_{3\cdot B}, \lambda_0/2) \leq c_{\mathrm{MT}}^2\left(\mu|_{3\cdot B}\right),$$

where again the comparison depends on $d$, $C_\mu$, and $\lambda_0$.

Furthermore, if $B$ is a ball such that $\mathrm{diam}(B) \geq \mathrm{diam}(\mathrm{supp}(\mu))$, then for any ball $B'$ such that $B' \subseteq B$ and $\mathrm{diam}(B') = \mathrm{diam}(\mathrm{supp}(\mu))$, we have that $J_2(\mu|_B) \approx J_2(\mu|_{B'}) \lessapprox c_{\mathrm{MT}}^2(\mu|_{3\cdot B'}) \leq c_{\mathrm{MT}}^2(\mu|_{3\cdot B})$. □

The rest of this section proves Proposition 3.4.1.

### 3.4.1 Proof of Proposition 3.4.1

#### Preliminary Notation and Observations

For any $x \in H$ and $0 < t < \infty$ we note the following trivial inclusion

$$U_\lambda(x,t) \subseteq W_{\lambda/2}(B(x,t)). \tag{3.85}$$

If $B$ is a ball of finite diameter in $H$, let

$$W(B) = \left\{(x, X, t) \in\ B \times H^{d+2} \times (0, \mathrm{diam}(B)] :\ X \in U_\lambda(x,t)\right\}.$$

For fixed $(x, X) \in B \times H^{d+2}$, we define the slice of $W(B)$ corresponding to $(x, X)$:

$$W\left(B\middle|\, x, X\right) = \{\, t > 0 : (x, X, t) \in W(B)\}\,.$$

We note that if $W\left(B\middle|\, x, X\right)$ is non-empty, then

$$\max_{0\leq i\leq d+1} \|x_i - x\| \leq t \leq \frac{\min(X)}{\lambda},\ \text{ for all } t \in W\left(B\middle|\, x, X\right)$$

and moreover

$$W\left(B\middle|\, x,X\right) = [u_1(x,X), u_2(x,X)] := \left[\max_{0\le i\le d+1} \|x_i - x\|\,,\ \frac{\min(X)}{\lambda}\right]. \tag{3.86}$$

We define the following two projections. Let $P_{1,2} : H \times H^{d+2} \times (0,\infty) \to H \times H^{d+2}$ be such that $P_{1,2}(x,X,t) = (x,X)$, and let $P_2 : H \times H^{d+2} \to H^{d+2}$ be the projection such that $P_2(x,X) = X$. We also adopt the harmless convention of taking $P_2(x,X,t) = P_2(x,X) = X$.

At last we note that the combination of equation (3.85) and the definition of $W(B)$ implies the inclusion

$$P_2\left(W(B)\right) \subseteq W_{\lambda/2}(3\cdot B). \tag{3.87}$$

**Details of the Proof**

We first apply Fubini's Theorem and the definition of $c^2_{\mathrm{MT}}(x,t,\lambda)$ to rewrite the integral on the LHS of equation (3.84) in the following form

$$\int_B \int_0^{\mathrm{diam}(B)} c^2_{\mathrm{MT}}(x,t,\lambda)\frac{\mathrm{d}t}{t^{d+1}}\,\mathrm{d}\mu(x) \;=\; \int_{P_{1,2}(W(B))} c^2_{\mathrm{MT}}(X)\left(\int_{W(B|x,X)} \frac{\mathrm{d}t}{t^{d+1}}\right)\mathrm{d}\mu^{d+3}(x,X). \tag{3.88}$$

Then, for $(x,X) \in P_{1,2}\left(W(B)\right)$ such that $W\left(B\middle|\, x,X\right) \neq \emptyset$, let

$$I(x,X) = \int_{W(B|x,X)} \frac{\mathrm{d}t}{t^{d+1}}. \tag{3.89}$$

In view of equation (3.86) we get that

$$I(x,X) = \int_{u_1(x,X)}^{u_2(x,X)} \frac{\mathrm{d}t}{t^{d+1}}.$$

We note that $u_1(x,X) > 0$ a.e. on $P_{1,2}\left(W(B)\right)$ (w.r.t. $\mu^{d+3}$), and thus $I(x,X) < \infty$ a.e. on $P_{1,2}\left(W(B)\right)$. This yields the inequality

$$I(x,X) = \frac{1}{d}\cdot\left(\frac{1}{u_1(x,X)^d} - \frac{1}{u_2(x,X)^d}\right) \le \frac{1}{d}\cdot\frac{1}{\max\limits_{0\le i\le d+1}\|x_i - x\|^d} \quad \text{a.e. on } P_{1,2}\left(W(B)\right). \tag{3.90}$$

Moreover, we can restrict our attention to $(x, X)$ such that $I(x, X) > 0$. Defining the set

$$I^{-1}(0,\infty) = \{(x, X) \in P_{1,2}\left(W(B)\right) : 0 < I(x, X) < \infty\}, \tag{3.91}$$

and combining equations (3.88)-(3.90) we obtain the inequality

$$\int_B \int_0^{\operatorname{diam}(B)} c_{\mathrm{MT}}^2(x,t,\lambda)\frac{\mathrm{d}t}{t^{d+1}}\,\mathrm{d}\mu(x) \leq \frac{1}{d}\int_{I^{-1}(0,\infty)} \frac{c_{\mathrm{MT}}^2(X)}{\max\limits_{0\leq i\leq d+1}\|x_i - x\|^d}\,\mathrm{d}\mu^{d+3}(x,X). \tag{3.92}$$

In order to estimate the right hand RHS of equation (3.92) we again apply Fubini's Theorem. More specifically, for any $X \in P_2\left(I^{-1}(0,\infty)\right)$ we define

$$I^{-1}(0,\infty)\big|X = \{x \in H : (x,X) \in I^{-1}(0,\infty)\},$$

and thus rewrite equation (3.92) as follows.

$$\int_B \int_0^{\operatorname{diam}(B)} c_{\mathrm{MT}}^2(x,t)\frac{\mathrm{d}t}{t^{d+1}}\,\mathrm{d}\mu(x) \leq$$
$$\frac{1}{d}\int_{P_2(I^{-1}(0,\infty))} c_{\mathrm{MT}}^2(X)\left[\int_{I^{-1}(0,\infty)\big|X} \frac{\mathrm{d}\mu(x)}{\max\limits_{0\leq i\leq d+1}\|x_i - x\|}\right]\mathrm{d}\mu^{d+2}(X)\,. \tag{3.93}$$

We next bound the inner integral of the above equation for a.e. $X \in P_2\left(I^{-1}(0,\infty)\right)$, that is, the integral

$$\int_{I^{-1}(0,\infty)\big|X} \frac{\mathrm{d}\mu(x)}{\max\limits_{0\leq i\leq d+1}\|x_i - x\|^d}\,. \tag{3.94}$$

If $X \in P_2\left(I^{-1}(0,\infty)\right)$ and $x \in I^{-1}(0,\infty)\big|X$, then

$$\frac{\min(X)}{\lambda} > \max_{0\leq i\leq d+1}\|x_i - x\| > 0.$$

Hence, for fixed $X \in P_2\left(I^{-1}(0,\infty)\right)$, with $x_0 = (X)_0$, we have the set inclusion

$$I^{-1}(0,\infty)\big|X \subseteq B\left(x_0, \lambda^{-1}\cdot\min(X)\right).$$

Thus the integral of equation (3.94) is bounded by

$$\int_{B(x_0,\lambda^{-1}\cdot\min(X))} \frac{\mathrm{d}\mu(x)}{\max\limits_{0\leq i\leq d+1}\|x_i - x\|^d}.$$

Furthermore, $\min(X) > 0$ a.e. on $H^{d+2}$, and by the triangle inequality we obtain

$$\min(X) \leq 2 \cdot \max_{0 \leq i \leq d+1} \|x_i - x\|.$$

Combining these observations with the upper bound of equation (**??**), we have the following inequality for a.e. $X \in P_2\left(I^{-1}(0,\infty)\right)$:

$$\int_{I^{-1}(0,\infty)\big|X} \frac{\mathrm{d}\mu(x)}{\max\limits_{0 \leq i \leq d+1} \|x_i - x\|^d} \leq 2^d \int_{B(x_0, \lambda^{-1} \cdot \min(X))} \frac{\mathrm{d}\mu(x)}{\min(X)^d} \leq \left(\frac{2}{\lambda}\right)^d \cdot C_\mu.$$

Applying this uniform bound to the RHS of equation (3.93) we get that

$$\int_B \int_0^{\operatorname{diam}(B)} c_{\mathrm{MT}}^2(x,t) \frac{\mathrm{d}t}{t^{d+1}} \, \mathrm{d}\mu(x) \leq \left(\frac{2}{\lambda}\right)^d \cdot \frac{C_\mu}{d} \int_{P_2(I^{-1}(0,\infty))} c_{\mathrm{MT}}^2(X) \, \mathrm{d}\mu^{d+2}(X). \quad (3.95)$$

Further application of equations (3.87) and (3.91) bounds the integral on the RHS of equation (3.95) by $c_{\mathrm{MT}}^2(\mu|_{3 \cdot B}, \lambda/2)$ and thus concludes the proof. $\square$

## 3.5 A Menagerie of Curvatures

Here we discuss a variety of curvatures for $d$-regular measures on $H$. In Subsection 3.5.1 we describe some curvatures that can be used to characterize uniform rectifiability, while indicating two levels of information needed for this purpose. In Subsection 3.5.2 we briefly give an example of continuous curvatures that can be used to quantify the $(p,p)$-geometric property $(1 \leq p < \infty)$ of David and Semmes [11]. Finally, in Subsection 3.5.3 we discuss our doubts about the utility of a previously suggested curvature for the purposes of implying the rectifiability of $\mu$ [25, Theorem 0.3].

### 3.5.1 Curvatures Characterizing Uniform Rectifiability

We start with a few continuous curvatures that are completely equivalent to the Jones'-type flatness (in the sense of Theorem 1.2.5). They thus characterize the uniform rectifiability of $\mu$ by the criterion that the ratios between the curvatures of $\mu|_B$ and the corresponding measures $\mu(B)$ are uniformly bounded for all balls $B$ in $H$. We remark that here the continuous curvature of $\mu|_B$ is obtained by integrating a corresponding discrete curvature over all $(d+1)$-simplices in $B^{d+2}$.

It is also possible to use a coarser level of information by introducing the parameter $\lambda$ and modifying the continuous curvature of $\mu|_B$ by integrating over the well-scaled set of simplices $W_\lambda(B)$ (see equation (3.0.5)). In the case of the Menger-type curvatures, both types of continuous curvatures are comparable (up to possible blow ups of the ball $B$). We thus say that the Menger-type curvature is stable (when $\lambda$ approaches zero). In Subsection 3.5.1 we present a discrete curvature for which we can easily compare the Jones'-type flatness with the latter type of continuous curvature (with parameter $\lambda$). Currently, we cannot decide if this curvature is stable and thus cannot use the former version of continuous curvature to characterize uniform rectifiability.

#### Additional Stable Curvatures

We define the following discrete curvatures

$$c_{\min}(X) = \sqrt{\frac{\min\limits_{0\le i\le d+1} \mathrm{p}_d \sin^2_{x_i}(X)}{\operatorname{diam}(X)^{d(d+1)}}}\,,$$

$$c_{\mathrm{vol}}(X) = \sqrt{\frac{\mathrm{M}_{d+1}(X)}{\operatorname{diam}(X)^{(d+1)(d+2)}}}\,,$$

and

$$c_{\max}(X) = \sqrt{\frac{\max_{0\le i\le d+1} \mathrm{p}_d \sin^2_{x_i}(X)}{\operatorname{diam}(X)^{d(d+1)}}}\,.$$

We note that for all $X \in H^{d+2}$ with $\min(X) > 0$:

$$c^2_{\mathrm{MT}}(X) \approx c^2_{\max}(X) \ge c^2_{\min}(X) \ge c^2_{\mathrm{vol}}(X)\,.$$

Furthermore, we note that for $\lambda > 0$ and any 1-separated element $X$,

$$c_{\mathrm{vol}}^2(X) \geq \lambda^{2(d+1)} \cdot c_{\mathrm{MT}}^2(X).$$

As such, analogous estimates to those of Theorems 4.0.2 and 3.0.2 hold for the curvatures $c_{\min}$, $c_{\mathrm{vol}}$, and $c_{\max}$. In particular, the continuous curvatures (integrated over all simplices in corresponding products of balls) $c_{\min}$, $c_{\mathrm{MT}}$, $c_{\mathrm{vol}}$ and $c_{\max}$ are comparable (up to possible blow ups of the underlying balls).

We can further extend this collection of stable curvatures. Indeed, we may include any order statistics of the polar sines of vertices of a given simplex (replacing the maximum or the minimum, which are used in $c_{\max}$ and $c_{\min}$ respectively).

### An Algebraic Curvature with Questionable Stability

For $X \in H^{d+2}$, let

$$\tilde{c}_{MT}(X) = \begin{cases} \dfrac{\mathrm{p}_d\mathrm{sin}_{x_0}(X)}{\displaystyle\prod_{1 \leq i < j \leq d+1} \|x_i - x_j\|}, & \text{if } \min(X) > 0, \\ \\ 0, & \text{otherwise.} \end{cases}$$

We see that unlike the curvatures of Subsection 3.5.1, this one is algebraic. In fact, it is the invariant ratio of the law of polar sines expressed in equation (3.11).

We trivially have the inequality

$$\tilde{c}_{MT}^2(X) \geq c_{\mathrm{MT}}^2(X), \text{ for all } X \in H^{d+2}.$$

Furthermore, if $X$ is 1-separated for $\lambda > 0$, then we also have the opposite inequality:

$$c_{\mathrm{MT}}^2(X) \geq \lambda^{d(d+1)} \cdot \tilde{c}_{MT}^2(X).$$

Hence, for any $0 < \lambda \leq \lambda_0$ and all balls $B \subseteq H$ we have the estimate (with constants depending on $d$, $C_\mu$, and $\lambda$):

$$J_2(\mu|_{\frac{1}{3}\cdot B}) \lessapprox \tilde{c}^2_{MT}(\mu|_B, \lambda/2) \lessapprox J_2(\mu|_{6\cdot B})\,. \tag{3.96}$$

Thus, fixing such $\lambda$ one can use the curvatures $\{\tilde{c}^2_{MT}(\mu|_B, \lambda/2)\}_{B\subseteq H}$ to characterize uniform rectifiability.

At this point we are not sure whether one can replace the curvature $\tilde{c}^2_{MT}(\mu|_B, \lambda/2)$ in equation (3.96) by the curvature $\tilde{c}^2_{MT}(\mu|_B)$. That is, we cannot decide at this point if the algebraic curvature is stable or not. It is clear though that our methods for controlling $c^2_{\mathrm{MT}}(\mu|_B)$ by $J_2(\mu|_{6\cdot B})$ (as expressed in Theorem 4.0.2 and established in [28]) are insufficient for controlling $\tilde{c}^2_{MT}(\mu|_B)$ by $J_2(\mu|_{C\cdot B})$ for some $C > 1$ (independent of $B$).

### 3.5.2 Curvatures Characterizing the $(p, p)$-Geometric Property $(1 \leq p < \infty)$

For $1 \leq p < \infty$, recall the $L_p$ Jones'-type flatness

$$J_p(\mu|_B) = \int_B \int_0^{\mathrm{diam}(B)} \beta_p^p(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}.$$

A $d$-regular measure $\mu$ on $H$ satisfies the $(p, p)$-geometric property [11, Part IV] if

$$J_p(\mu|_B) \lessapprox \mu(B), \text{ for all balls } B \subseteq H.$$

The methods of Chapters 3 and 4 extend to comparing $J_p(\mu|_B)$ with a modified curvature as follows.

**Theorem 3.5.1.** *If $\mu$ is a $d$-regular measure on $H$ and $1 \leq p < \infty$, then there exists a constant $C_9 = C_9(d, C_\mu, p)$ such that for all balls $B \subseteq H$ at reasonable locations and scales*

$$\frac{1}{C_9}\cdot J_p\left(\mu|_{\frac{1}{3}\cdot B}\right) \leq \int_{B^{d+2}} \frac{\mathrm{p}_d \sin^p_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) \leq C_9\cdot J_p\left(\mu|_{6\cdot B}\right).$$

We thus obtain the following characterization of the $(p,p)$-geometric property.

**Corollary 3.5.1.** *If $1 \leq p < \infty$ and $\mu$ is a $d$-regular measure on $H$, then $\mu$ satisfies the $(p,p)$-geometric property if and only if for all balls $B \subseteq H$ at reasonable locations and scales*

$$\int_{B^{d+2}} \frac{\mathrm{p}_d \sin_{x_0}^p(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) \lessapprox \mu(B).$$

If $p = 2$ then Theorem 3.5.1 coincides with the combination of Theorems 4.0.2 and 3.0.2. Similarly, in that case Corollary 3.5.1 coincides with [28, Theorem 1.3]

### 3.5.3 A Previously Suggested Curvature

Léger [25] proposed the following discrete curvature for $(d{+}1)$-simplices $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ where $d \geq 1$:

$$c_L^{d+1}(X) = \frac{\operatorname{dist}(x_0, L[X(0)])^{d+1}}{\prod_{i=1}^{d+1} \|x_i - x_0\|^{d+1}},$$

and the corresponding continuous curvature for $\mu$ restricted to any ball $B \subseteq H$

$$c_L^{d+1}(\mu|_B) = \int_{B^{d+2}} c_L^{d+1}(X) \, \mathrm{d}\mu^{d+2}(X).$$

If $d = 1$, then his curvature coincides with the Menger curvature [33, 36] (up to multiplication by a constant). He showed how to use his curvature in that case to infer rectifiability properties of $\mu$. In particular, he established Theorem 3.0.2 when $d = 1$.

Léger's approach for proving the same type of results for $d \geq 2$ (while using the curvature $c_L^{d+1}(X)$) ostensibly requires a bound of the form

$$J_2(\mu|_B) \lessapprox c_L^{d+1}(\mu|_{3 \cdot B}),$$

thus generalizing [25, Lemma 2.5]. However, any analysis or adaptation of the proof of [25, Lemma 2.5] to the curvature $c_L^{d+1}(X)$, where $d > 1$, seems to give at best the following lower bound.

**Proposition 3.5.1.** *If $\mu$ is $d$-regular, then*

$$J_{d+1}\left(\mu|_{\frac{1}{3}\cdot B}\right) \lessapprox c_L^{d+1}(\mu|_B)$$

*for any ball $B \subseteq H$ at all reasonable locations and scales.*

If $d > 1$, then the function $J_{d+1}$ (discussed in the previous subsection) can be significantly smaller than the required quantity $J_2$ (especially for very large $d$). This function also characterizes the $(d+1, d+1)$-geometric property (see Corollary 3.5.1), which includes measures that are not uniformly rectifiable whenever $d > 1$.

# Chapter 4

# Controlling the Menger-Type Curvature by the Jones-Type Flatness

This chapter develops two very different kinds of methods. We refer to the first one as geometric multipoles, which is a way of decomposing the multivariate integral over the set of well-scaled simplices $W_\lambda(B)$ according to multiscale regions in $H$, while emphasizing in each such region approximations by $d$-planes. This method applies very nicely to the integral over the well-scaled simplices since their nice configuration results in good control on the polar sine in terms of distances from $d$-planes (see Proposition 4.2.1 of Subsection 4.2.4). We view it as a $d$-way analog of the fast multipoles method [20], which is based on decomposing an integral according to dyadic grids of $\mathbf{R}^n$ and emphasizing near-field interactions corresponding to $d = 0$, i.e., distances from points instead of $d$-planes.

Our second method is a technique developed to bound the integrals over the poorly scaled simplices. This situation is more complicated than the well-scaled situation because for such simplices it is more difficult to obtain control on their polar sine in terms of distances from $d$-planes. Although its ultimate goal is an application of the method of geometric multipoles, it really depends on a technique which is probabilistic in nature. The basic idea behind the technique is referred to as *multiscale sequences*, and relies on a relatively technical procedure for exchanging a poorly scaled simplex for a predictable sequence of well-scaled ones. Constructing such a multiscale sequence relies on the concentration inequality of Theorem 1.2.2, and we use this to introduce simplices satisfying a point-wise weighted inequality, which we then turn into an *integral* inequality using a procedure referred to as *augmentation.*

In this chapter we establish the upper bounds of Theorems 1.2.3-1.2.5. We formulate these respectively as follows.

**Proposition 4.0.2.** *There exists a constant* $C_5 = C_5(d, C_\mu) \geq 1$ *such that*

$$\int_{U_\lambda(B(x,t))} c_{MT}^2(X)\,\mathrm{d}\mu^{d+2}(X) \leq \frac{C_5}{\lambda^{d(d+1)+4}} \cdot \beta_2^2(x,t) \cdot \mu(B(x,t)), \tag{4.1}$$

*for all* $\lambda > 0$, $x \in \operatorname{supp}(\mu)$, *and* $t \in \mathbb{R}$ *with* $0 < t \leq \operatorname{diam}(\operatorname{supp}(\mu))$.

**Proposition 4.0.3.** *There exists a constant* $C_6 = C_6(d, C_\mu, \alpha_0)$ *such that*

$$\int_{W_\lambda(B)} c_{MT}^2(X)\,\mathrm{d}\mu^{d+2}(X) \leq \frac{C_6}{\lambda^2} \cdot J_d^{\mathcal{D}}\left(\mu|_B\right) \tag{4.2}$$

*for all* $\lambda > 0$ *and all balls* $B \subseteq H$.

**Theorem 4.0.2.** *There exists a constant* $C_7 = C_7(d, C_\mu, \alpha_0)$ *such that*

$$c_{MT}^2\left(\mu|_B\right) \leq C_7 \cdot J_d^{\mathcal{D}}\left(\mu|_B\right)$$

*for all balls* $B \subseteq H$.

We prove Proposition 4.0.2 in Section 4.4 and Proposition 4.0.3 in Section 4.5. Section 4.6 reduces Theorem 4.0.2 into more elementary propositions, whereas Sections 4.7 and 4.8 contain their proofs.

## 4.1 Basic Notation and Definitions

If $\mathcal{B}$ is a family of balls, then we denote the union of its elements by $\bigcup \mathcal{B}$, and we distinguish the latter notation from $\bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$, which is a family of balls formed by the countable union of other families. For such a family of balls, $\mathcal{B}$, then we denote the corresponding blow up of the family by a constant $\gamma > 0$ by $\gamma \cdot \mathcal{B} = \{\gamma \cdot B : B \in \mathcal{B}\}$.

Finally, if $x \in \mathbb{R}$, then we denote the corresponding ceiling and floor functions by $\lceil x \rceil$ and $\lfloor x \rfloor$. More specific notation and definitions used throughout the chapter, as well as related propositions, are described in the following subsections according to topic.

## 4.2 Geometry of Simplices and the Polar Sine Function

Here we review some geometric properties of simplices that will be used frequently throughout the paper. We also define the $d$-dimensional polar sine function and the Menger-type curvature, and establish some of their properties.

We are only interested in simplices without any coinciding vertices, that is, simplices represented by elements in the set

$$S = \{X \in H^{d+2} : \min(X) > 0\}. \tag{4.3}$$

We note that $\mu^{d+2}\left(H^{d+2} \setminus S\right) = 0$, and thus we restrict our analysis to the set $S$.

### 4.2.1 The Scale of the Simplex $X$ at $x_0$

Given $X \in S$, we quantify the disparity between the largest and smallest edges of $X$ at $x_0$ using the functions $\max_{x_0}(X) = \max_{1\leq i\leq d+1} \|x_i - x_0\|$ and $\min_{x_0}(X) = \min_{1\leq i\leq d+1} \|x_i - x_0\|$, as well as the function

$$\text{scale}_{x_0}(X) = \frac{\min_{x_0}(X)}{\max_{x_0}(X)}.$$

For a ball $B \subseteq H$ and $\lambda > 0$, we define the set of *one-separated simplices at $x_0$ in $B$* to be the set

$$S_\lambda(B) = \{X \in B^{d+2} : \text{scale}_{x_0}(X) \geq \lambda\}, \tag{4.4}$$

and we denote $S_\lambda = S_\lambda(H)$. Furthermore, we trivially have the inclusion

$$W_\lambda(Q) \subseteq S_\lambda(Q), \tag{4.5}$$

where $W_\lambda(Q)$ was defined in equation (3.0.5).

### 4.2.2 Decomposing the Set of Simplices $S$

Multiscale analysis on $H$ typically involves a countable number of "scales" and corresponding "dyadic" covers or partitions of $H$ (or one of its subsets), see e.g., Section 4.3. In order to apply such a multiscale analysis of $\text{supp}(\mu)$ to the multivariate integrals $c^2_{MT}(\mu|_Q)$, we find it useful to first decompose the set of simplices $S$.

The organizing principle is the classification of simplices according to degeneracies of scale. Specifically, for a fixed $k \in \mathbb{N}_0$ and $p \in \mathbb{N}$, we define the following subset of $S$

$$S_{k,p} = \left\{X \in S : \ \alpha_0^{k+p} < \text{scale}_{x_0}(X) \leq \alpha_0^k\right\}. \tag{4.6}$$

We also denote the distinguished set of simplices

$$\widehat{S} = S_{0,3} = S_{\alpha_0^3}.$$

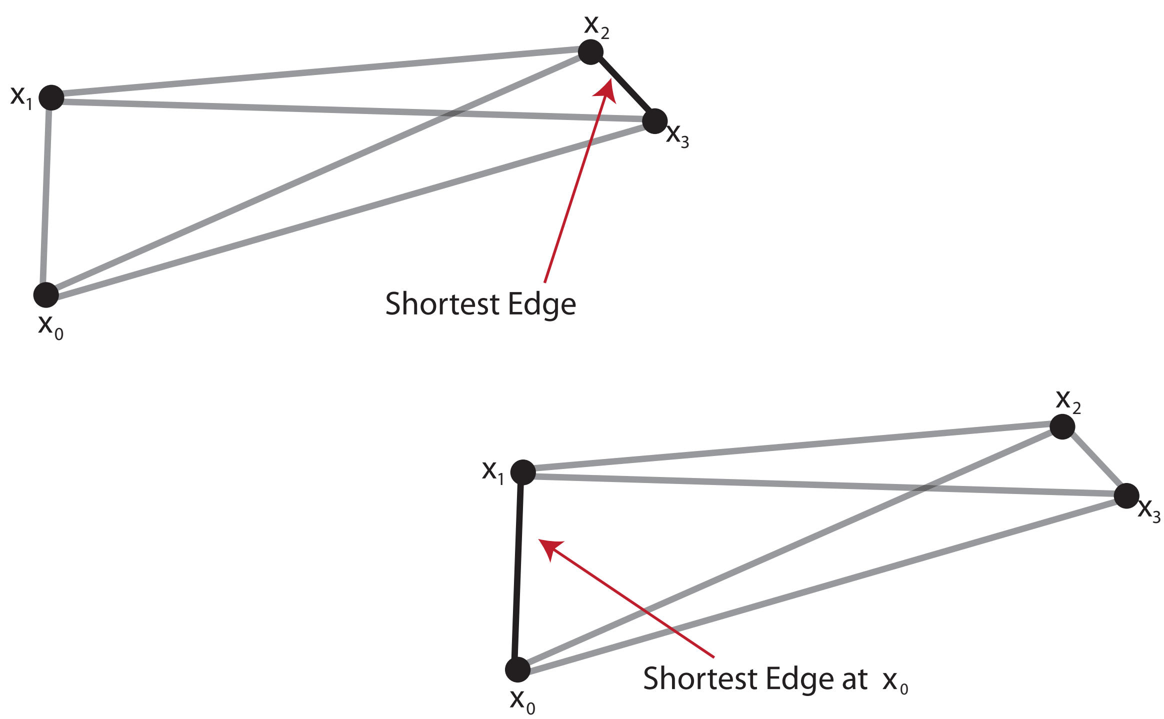


Figure 4.1: Distinguishing between $\min(X)$ and $\min_{x_0}(X)$ for a tetrahedron $X = (x_0, x_1, x_2, x_3)$.

We refer to the elements of $\widehat{S}$ as *well-scaled simplices at* $x_0$, and the elements of $S \setminus \widehat{S}$ as *poorly-scaled simplices at* $x_0$. Quantifiably, $X \in S$ is well-scaled at $x_0$ if and only if

$$\frac{\min_{x_0}(X)}{\max_{x_0}(X)} > \alpha_0^3 \,. \tag{4.7}$$

For a ball $Q$ in $H$ and $p \in \mathbb{N}$, we often use localized versions of the sets $S$, $\widehat{S}$, and $S_{k,p}$, $k \geq 3$, defined as

$$S(Q) = S \cap Q^{d+2}\,, \quad \widehat{S}(Q) = \widehat{S} \cap Q^{d+2} \quad \text{and} \quad S_{k,p}(Q) = S_{k,p} \cap Q^{d+2}\,. \tag{4.8}$$

In most of the paper it will be sufficient to assume $p = 1$, however in parts of Sections 4.6 through 4.8 we will need to consider the case $p = 2$. We note that if $p = 1$, then the sets $S_{k,1}$, $k \geq 3$, partition $S \setminus \widehat{S}$, whereas if $p \geq 1$, then they cover it.

Arbitrarily fixing $k \geq 3$ and $p \in \mathbb{N}$, there is a further division of the sets $S_{k,p}$ according to configuration. While we wait until Subsection 4.6.1 for the technical details, we at least give a bit of terminology here. If $X = (x_0, \ldots, x_{d+1}) \in S_{k,p}$, then we say that an edge connecting $x_0$ and $x_i$, $1 \leq i \leq d+1$, is a *handle* if

$$\frac{\|x_0 - x_i\|}{\max_{x_0}(X)} > \alpha_0^k \tag{4.9}$$

and a *tine* otherwise, i.e.,

$$\alpha_0^{k+p} < \frac{\|x_i - x_0\|}{\max_{x_0}(X)} \leq \alpha_0^k.$$

Clearly $X \in S_{k,p}$ has at least one handle (obtaining the maximal edge length at $x_0$) and at most $d$ of them (excluding the one of minimal edge length at $x_0$). We say that $X$ is a *rake*, or a single-handed rake, if it only has one handle. Similarly, $X$ is a *multi-handed rake* if it has more than one handle (see Figure 4.2.2). We remark that these notions depend on our fixed choice of $p$ which will be clear from the context.

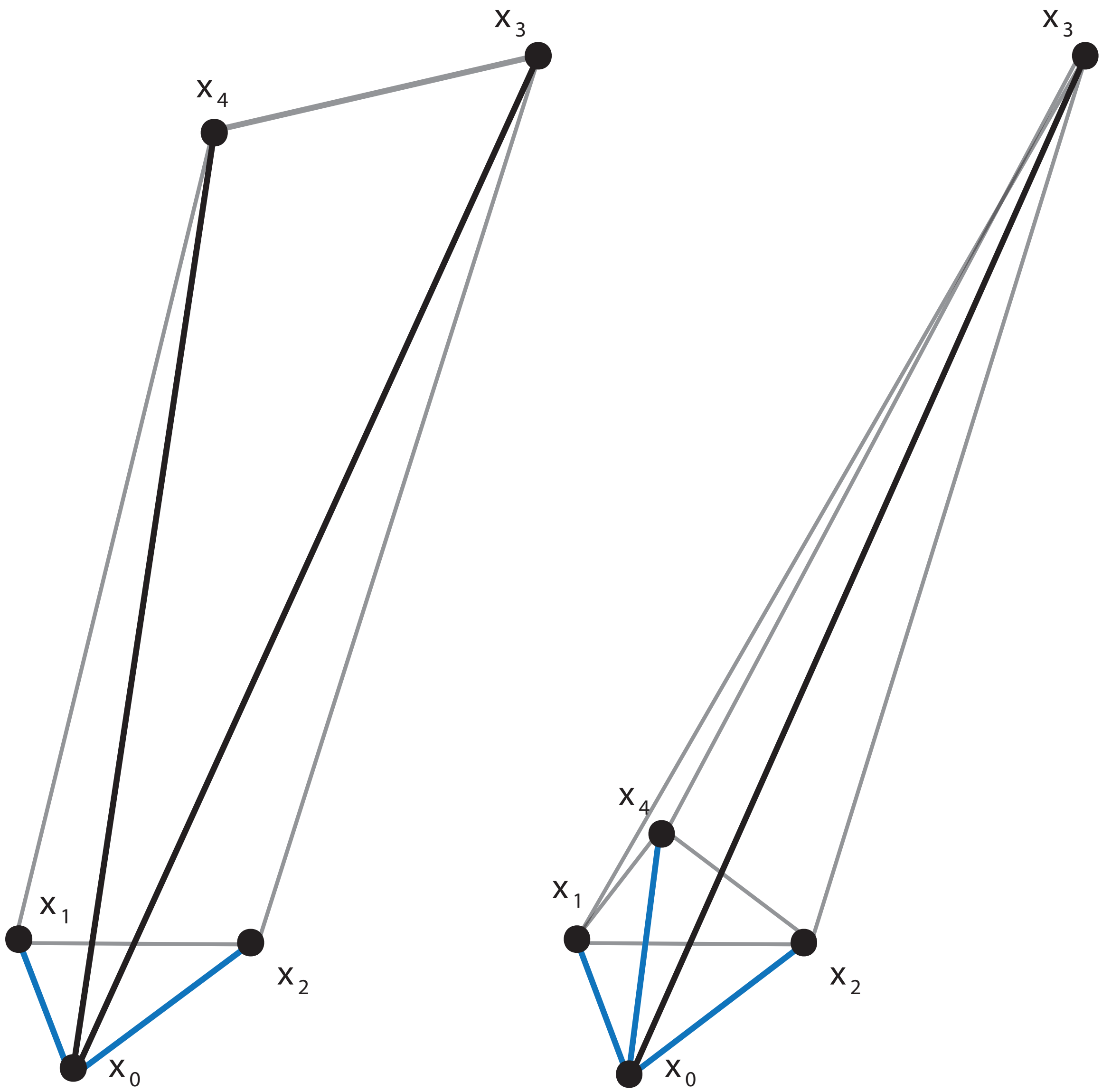


Figure 4.2: Two different 4-simplices, contrasting a multi handled rake with a single handled rake, handles at $x_0$ being indicated by the black edges, and tines by the blue edges. All edges *not connected* to $x_0$ are indicated in grey. The simplex on the left has two handles and three tines, and is furthermore an element of the set $S^2_{k,p}(0,0,1,1)$ defined in Subsection 4.6.1. The simplex on the right is a single-handled rake, and is in $S^1_{k,p}(0,0,1,0)$.

### 4.2.3 The Minimal Height and Width of a Simplex

Fixing $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ and $0 \leq i \leq d+1$, we define the height of the simplex $S[X]$ through the vertex $x_i$ as

$$h_{x_i}(X) = \operatorname{dist}\left(x_i, L[X(i)]\right). \tag{4.10}$$

We denote the minimal height by

$$h(X) = \min_{x_i} \; h_{x_i}(X). \tag{4.11}$$

The width, $w(X)$, of the simplex $S[X]$ is given by the following infimum over all $d$-planes $L$:

$$w(X) = 2 \min_{L} \; \max_{x \in S[X]} \operatorname{dist}(x, L). \tag{4.12}$$

Equivalently, the width $w(X)$ is the shortest distance between any two parallel $d$-planes supporting $S[X]$, i.e., $S[X]$ is trapped between them and its boundary touches them.

When $d = 1$, the minimal height and width of a *triangle* are the same, but for higher order simplices, i.e., $d \geq 2$, this does not hold. However, the width and the minimal height of the simplex $S[X]$ are comparable as follows:

$$w(X) \leq h(X) \leq \left\lceil \frac{d+1}{2} \right\rceil \cdot w(X). \tag{4.13}$$

The inequality on the LHS of equation (4.13) follows directly from the definitions of $w(X)$ and $h(X)$, and the inequality on the RHS of equation (4.13) is established in [21, Lemma 3].

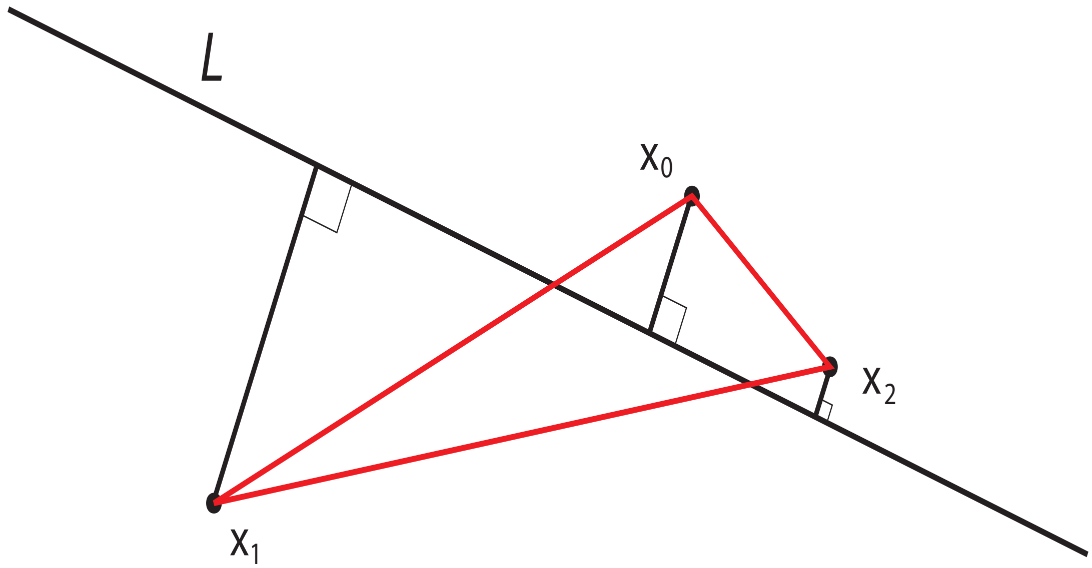


Figure 4.3: Exemplifying $D_2(X, L)$ when $d = 1$ for the triangle $X = (x_0, x_1, x_2)$ and the line $L$.

### 4.2.4 Linear Deviations and Their Use in Bounding the Polar Sine

Fixing $X \in H^{d+2}$ and $L$ an affine subspace of $H$, we define the $\ell_2$ deviation of $X$ from $L$, denoted by $D_2(X, L)$, as follows:

$$D_2(X, L) = \left( \sum_{i=0}^{d+1} \text{dist}^2\left(x_i, L\right) \right)^{1/2} . \tag{4.14}$$

Using this quantity, we get the following upper bound on the polar sine, which we establish in Appendix 5.2.1.

**Proposition 4.2.1.** *If $X \in S$ and $L$ is an arbitrary $d$-plane of $H$, then*

$$\text{p}_d\text{sin}_{x_0}(X) \leq \frac{\sqrt{2} \cdot (d+1) \cdot (d+2)}{\text{scale}_{x_0}(X)} \, \frac{D_2(X, L)}{\text{diam}(X)} .$$

## 4.3 Multiscale Resolutions and Partitions

Our multiscale analysis requires covers and partitions of $\text{supp}(\mu)$ that will be used to construct covers and partitions of $S \cap [\text{supp}(\mu)]^{d+2}$. Using the scales $\{\alpha_0^n\}_{n \in \mathbb{Z}}$, we construct a countable sequence of covers of $\text{supp}(\mu)$ by balls, denoted by $\{\mathcal{B}'_n\}_{n\in\mathbb{Z}}$, and we use them to construct a sequence of partitions of $\text{supp}(\mu)$, denoted by $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$.

We say that $E_n$ is an *$n$-net* for $\text{supp}(\mu)$ if

1. $E_n \subseteq \text{supp}(\mu)$.

2. $\|x-y\| > \alpha_0^n$, for all $x$ and $y$ in $E_n$.

3. $\text{supp}(\mu) \subseteq \bigcup_{x\in E_n} B(x,\alpha_0^n)$.

For each $n \in \mathbb{Z}$, we arbitrarily form an $n$-net, $E_n$, and then take the family of balls (see Figure 4.3)

$$\mathcal{B}'_n = \{B(x, 4\cdot\alpha_0^n)\}_{x\in E_n}.$$

Since $H$ is separable, $\text{supp}(\mu)$ is also separable and hence $E_n$ and $\mathcal{B}'_n$ are countable. Furthermore, we note that the family $\frac{1}{4}\cdot\mathcal{B}'_n$ is a cover of $\text{supp}(\mu)$. Since the collection $\mathcal{B}'_n$ can be possibly much larger than needed, we select a sub-collection $\mathcal{B}_n \subseteq \mathcal{B}'_n$ such that $\frac{1}{4}\cdot\mathcal{B}_n$ is a maximal mutually disjoint sub-collection of $\frac{1}{4}\cdot\mathcal{B}'_n$. We index the elements of $\mathcal{B}_n$ by $\Lambda_n = \{1,2,\ldots\}$, which is either finite or $\mathbb{N}$, so that

$$\mathcal{B}_n = \{B_{n,j}\}_{j\in\Lambda_n}. \tag{4.15}$$

We call the family $\mathcal{B}_n$ an *$n$-resolution* for $\text{supp}(\mu)$, and we define the corresponding *multiresolution family* for $\text{supp}(\mu)$, $\mathcal{D}$, to be the collection of all $n$-resolutions, i.e.,

$$\mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n. \tag{4.16}$$

These resolutions can be replaced by multiscale partitions of $\text{supp}(\mu)$ in the following manner.

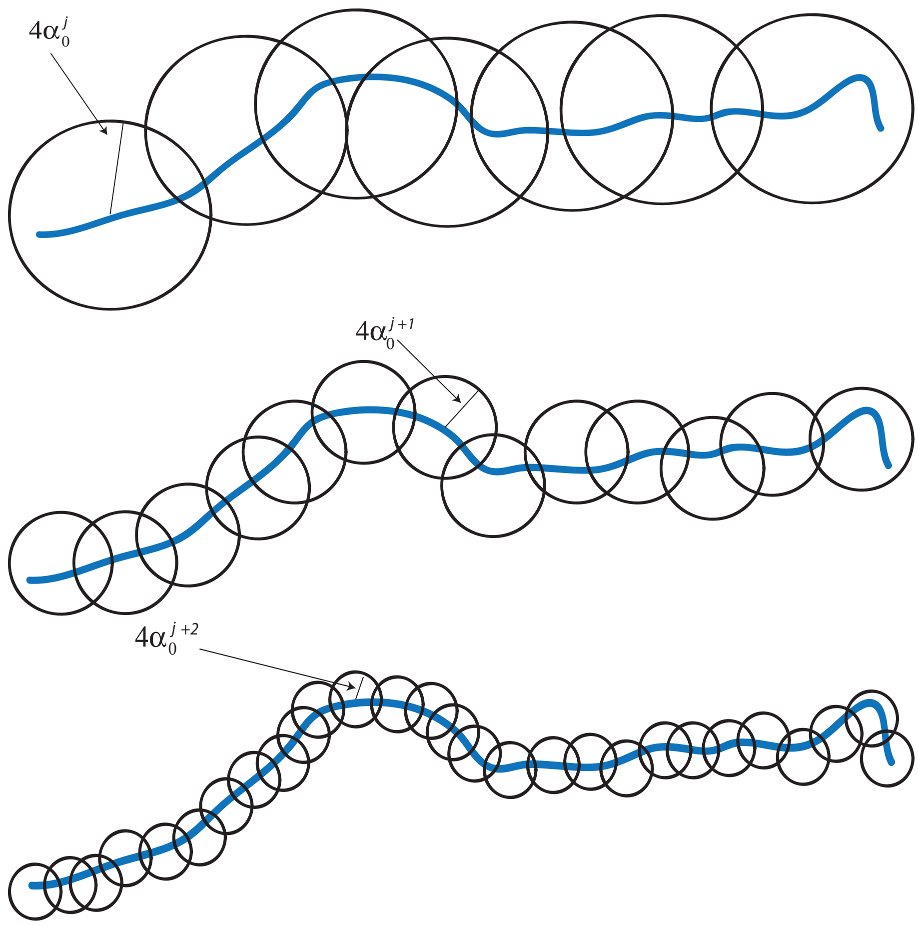


Figure 4.4: Illustrating three levels of a multiresolution family, i.e., the families $\mathcal{B}'_j, \mathcal{B}'_{j+1}$ and $\mathcal{B}'_{j+2}$ when $\mu$ is $\mathcal{H}_1$ supported on a curve.

**Lemma 4.3.1.** *For any $n \in \mathbb{Z}$ there exists a partition $\mathcal{P}_n = \{P_{n,j}\}_{j \in \Lambda_n}$ of* $\operatorname{supp}(\mu)$*, such that for any $j \in \Lambda_n$ there exists a unique $B_{n,j} \in \mathcal{B}_n$, so that*

$$\operatorname{supp}(\mu) \bigcap \frac{1}{4} \cdot B_{n,j} \subseteq P_{n,j} \subseteq \operatorname{supp}(\mu) \bigcap \frac{3}{4} \cdot B_{n,j} \,.$$

The constructive proof appears in Appendix 5.2.2. We typically work with localized resolutions which we define as follows. If $Q$ is a ball in $H$, then we define the integer

$$m(Q) = \left\lceil \frac{\ln(\operatorname{diam}(Q))}{\ln(\alpha_0)} \right\rceil , \tag{4.17}$$

that is, $m(Q)$ is the smallest integer $m$ such that $\alpha_0^m \leq \operatorname{diam}(Q)$. For $n \geq m(Q)$ we define the *local $n$-resolution* to be

$$\mathcal{B}_n(Q) = \{B_{n,j} \in \mathcal{B}_n : B_{n,j} \cap Q \neq \emptyset\} \,. \tag{4.18}$$

We define the *local multiresolution family* to be the collection

$$\mathcal{D}(Q) = \bigcup_{n \geq m(Q)} \mathcal{B}_n(Q) \,. \tag{4.19}$$

We also define the set of indices (possibly empty)

$$\Lambda_n(Q) = \{j \in \Lambda_n : P_{n,j} \cap Q \neq \emptyset\}.$$

Using the multiscale resolution, we can form a discretized version of the continuous local Jones'-type flatness,

$$J_d^{\mathcal{D}}(\mu|_Q) = \sum_{B \in \mathcal{D}(Q)} \beta_2^2(B) \cdot \mu(B).$$

Peter Jones [23] has suggested a quantity similar to $J_d^{\mathcal{D}}$ when $d = 1$, while David and Semmes [10] formed $J_d(\mu|_Q)$ and an analog of $J_d^{\mathcal{D}}(\mu|_Q)$ for $d \geq 1$. We will use the following inequality whose proof is provided in Appendix 5.2.3 (an inverse inequality can also be formulated).

**Proposition 4.3.1.** *There exists a constant $C_4 = C_4(d, C_\mu)$ such that for any multiresolution family $\mathcal{D}$ on* $\mathrm{supp}(\mu)$ *:*

$$J_d^{\mathcal{D}}(\mu|_Q) \leq C_4 \cdot J_d(\mu|_{6 \cdot Q}) \quad \text{for any ball } Q \text{ in } H\,. \tag{4.20}$$

## 4.4 Proof of Proposition 4.0.2

We first note that the set $U_\lambda(B(x,t))$ is invariant under any permutation of the coordinates. Thus, by the same argument producing equation (3.14) we have the equality

$$\int_{U_\lambda(B(x,t))} c_{MT}^2(X)\,\mathrm{d}\mu^{d+2}(X) = \int_{U_\lambda(B(x,t))} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X). \tag{4.21}$$

Furthermore, $\mathrm{diam}(X) \geq \lambda \cdot t$ and $\mathrm{scale}_{x_0}(X) \geq \lambda$ for all $X \in U_\lambda(B(x,t))$. Applying these estimates together with Proposition 4.2.1 to the RHS of equation (4.21) we get that for any $d$-plane $L$

$$\int_{U_\lambda(B(x,t))} c_{MT}^2(X)\,\mathrm{d}\mu^{d+2}(X) \leq \frac{2 \cdot (d+1)^2 \cdot (d+2)^2}{\lambda^{d(d+1)+4}} \int_{U_\lambda(B(x,t))} \frac{D_2^2(X,L)}{t^2}\,\frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}} \;=$$
$$\frac{8 \cdot (d+1)^2 \cdot (d+2)^2}{\lambda^{d(d+1)+4}} \sum_{i=0}^{d+1} \int_{U_\lambda(B(x,t))} \left(\frac{\mathrm{dist}(x_i,L)}{2 \cdot t}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}}\,. \tag{4.22}$$

Let $P_i$ be the projection of $H^{d+2}$ onto its $i^{\text{th}}$ coordinate, $0 \leq i \leq d+1$. We trivially have the inclusion

$$P_i(U_\lambda(B(x,t))) \subseteq B(x,t). \tag{4.23}$$

Hence, fixing $0 \leq i \leq d+1$ and applying equation (4.23) and Fubini's Theorem to the corresponding term on the RHS of equation (4.22), we get the inequality

$$\int_{U_\lambda(B(x,t))} \left(\frac{\mathrm{dist}(x_i, L)}{2 \cdot t}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}} \leq \left(\frac{\mu(B(x,t))}{t^d}\right)^{d+1} \int_{B(x,t)} \left(\frac{\mathrm{dist}(x_i, L)}{2 \cdot t}\right)^2 \mathrm{d}\mu(x_i) = \left(\frac{\mu(B(x,t))}{t^d}\right)^{d+1} \cdot \beta_2^2(x,t,L) \cdot \mu(B(x,t)). \quad (4.24)$$

Summing the RHS of equation (4.24) over $0 \leq i \leq d+1$ and applying the $d$-regularity of $\mu$, we obtain that the RHS of equation (4.22) has the following bound

$$\int_{U_\lambda(B(x,t))} c_{MT}^2(X)\, \mathrm{d}\mu^{d+2}(X) \leq \frac{8 \cdot (d+1)^2 \cdot (d+2)^3}{\lambda^{d(d+1)+4}} \cdot C_\mu^{d+1} \cdot \beta_2^2(x,t,L) \cdot \mu(B(x,t)). \quad (4.25)$$

Since $L$ is an arbitrary $d$-plane, taking the infimum over all $d$-planes $L$ on the RHS of equation (4.25) proves the proposition. $\square$

## 4.5 A Proof of Proposition 4.0.3 via Geometric Multipoles

For an arbitrary ball $Q \subseteq H$, we note that the set $W_\lambda(Q)$ is invariant under permutations of the coordinates, and thus by the same argument producing equation (3.14), we have that

$$\int_{W_\lambda(Q)} c_{MT}^2(X)\, \mathrm{d}\mu^{d+2}(X) = \int_{W_\lambda(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X). \quad (4.26)$$

By applying equations (4.5) and (4.26), Proposition 4.0.3 can be deduced from the following stronger proposition.

**Proposition 4.5.1.** *There exists a constant $C_8 = C_8(d, C_\mu, \alpha_0)$ such that*

$$\int_{S_\lambda(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X) \leq \frac{C_8}{\lambda^2} \cdot J_d^{\mathcal{D}}\left(\mu|_Q\right) \quad (4.27)$$

*for all $\lambda > 0$ and all balls $Q \subseteq H$.*

### 4.5.1 Proof of Proposition 4.5.1

We first decompose the integral on the LHS of equation (4.27) into a countable sum of integrals on more simple regions according to specific scales and locations in $Q \cap \operatorname{supp}(\mu)$. Then we control each such integral by $\beta_2^2(B) \cdot \mu(B)$ for some $B \in \mathcal{D}(Q)$. For fixed $m \geq m(Q)$ (see equation (4.17)), we define

$$S_\lambda(m) = \left\{X \in S_\lambda : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}, \tag{4.28}$$

and we let $S_\lambda(m)(Q)$ denote the restriction of $S_\lambda(m)$ to the set $Q^{d+2}$. We note that the family $\{S_\lambda(m)(Q)\}_{m\in\mathbb{Z}}$ partitions $S_\lambda(Q)$. Therefore, since $\max_{x_0}(X) \leq \operatorname{diam}(Q)$, we have that

$$\int_{S_\lambda(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X) = \sum_{m \geq m(Q)} \int_{S_\lambda(m)(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X). \tag{4.29}$$

Next, we partition each $S_\lambda(m)(Q)$ according to location in $\operatorname{supp}(\mu)$ determined by the partition $\mathcal{P}_m$. For fixed $m \geq m(Q)$, $j \in \Lambda_m$, and $P_{m,j}$ as in Lemma (4.3.1), let

$$\widehat{P}_{m,j} = \{X \in S_\lambda(m)(Q) : x_0 \in P_{m,j}\}. \tag{4.30}$$

We thus obtain the inequality

$$\int_{S_\lambda(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X) \leq \sum_{m \geq m(Q)} \left[\sum_{j \in \Lambda_m(Q)} \int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X)\right].$$

Fixing $m \geq m(Q)$ and $j \in \Lambda_m(Q)$, we take the ball $B_{m,j} \in \mathcal{B}_m(Q)$ such that $\frac{1}{4} \cdot B_{m,j} \cap \operatorname{supp}(\mu) \subseteq P_{m,j} \subseteq \frac{3}{4} \cdot B_{m,j}$ as in Lemma 4.3.1. We will then show that

$$\int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\, \mathrm{d}\mu^{d+2}(X) \leq \frac{C_6}{\lambda^2} \cdot \beta_2^2(\beta_{m,j}) \cdot \mu(B_{m,j}). \tag{4.31}$$

Finally, summing over $j \in \Lambda_m(Q)$ and $m \geq m(Q)$ will conclude the proposition.

We establish equation (4.31) by following the basic argument behind Proposition 4.0.2. We first note that for all $X \in \widehat{P}_{m,j} \subseteq S_\lambda(m)(Q)$

$$\frac{\alpha_0}{8} \cdot \operatorname{diam}\left(B_{m,j}\right) = \alpha_0^{m+1} < \max_{x_0}(X) \leq \operatorname{diam}(X)\,. \tag{4.32}$$

Then, let $L$ be any $d$-plane in $H$. Combining Proposition 4.2.1 with equation (4.32) we obtain the following inequality for all $X \in \widehat{P}_{m,j}$

$$\frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \leq \frac{2 \cdot 8^2 \cdot (d+1)^2 \cdot (d+2)^2}{\lambda^2 \cdot \alpha_0^2} \cdot \frac{D_2^2\left(X, L\right)}{\operatorname{diam}^2\left(B_{m,j}\right)} \cdot \frac{1}{\left(\alpha_0^{d(m+1)}\right)^{d+1}}. \tag{4.33}$$

Furthermore, if $X = (x_0, \ldots, x_{d+1}) \in \widehat{P}_{m,j}$, then $x_0 \in \frac{3}{4} \cdot B_{m,j}$, and $\|x_i - x_0\| \leq \alpha_0^m = \operatorname{diam}(B_{m,j})/4$ for all $1 \leq i \leq d+1$. Consequently, $x_i \in B_{m,j}$ for all $0 \leq i \leq d+1$, and thus

$$\widehat{P}_{m,j} \subseteq \left(B_{m,j}\right)^{d+2}. \tag{4.34}$$

Combining equations (4.33) and (4.34) we obtain the bound

$$\int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) \leq \frac{2 \cdot 8^2 \cdot (d+1)^2 \cdot (d+2)^2}{\lambda^2 \cdot \alpha_0^2} \int_{(B_{m,j})^{d+2}} \frac{D_2^2\left(X, L\right)}{\operatorname{diam}^2\left(B_{m,j}\right)}\, \frac{\mathrm{d}\mu^{d+2}(X)}{\left(\alpha_0^{d(m+1)}\right)^{d+1}}. \tag{4.35}$$

Finally, applying the same types of computations leading to equations (4.24) and (4.25) and then taking the infimum over all $d$-planes $L$, we obtain equation (4.31) and hence the proposition. $\square$

### 4.5.2 Geometric Multipoles

The proof given above of Proposition 4.5.1 generated approximate decompositions of the Menger-type curvature of $\mu$ according to goodness of approximations by $d$-planes at different scales and locations. We refer to this strategy as geometric multipoles and see it as a geometric analog of the decomposition of special potentials by near-field interactions at different locations, as applied in the fast multipoles algorithm [20]. Unlike fast multipoles, which considers interactions between pairs of points, geometric multipoles takes into account simultaneous interactions between $d+2$ points. While fast multipoles neglects terms of distant interactions, $d$-dimensional geometric multipoles may neglect locations and scales well-approximated by $d$-planes.

## 4.6 The Reduction of Theorem 4.0.2

Our reduction of Theorem 4.0.2 relies on the decomposition of the set of simplices $S$ defined in Subsection 4.2.2 and the resulting decomposition of the multivariate integral $c^2_{MT}(\mu|_Q)$. Specifically, for an arbitrary ball $Q \subseteq H$, by applying equation (3.14) and using the sets $S(Q)$, $\widehat{S}(Q)$, and $S_{k,1}(Q)$ of equation (4.8) for $k \geq 3$, we can express the Menger-type curvature of $\mu|_Q$ as an integral over the well-scaled simplices plus a sum of integrals over the poorly scaled simplices, that is,

$$\begin{aligned} c^2_{MT}(\mu|_Q) &= \int_{Q^{d+2}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) = \int_{S(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) \\ &= \int_{\widehat{S}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) + \sum_{k\geq 3} \int_{S_{k,1}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X). \end{aligned} \tag{4.36}$$

The integral over the well-scaled simplices, i.e., the first term on the RHS of equation (4.36), is easily bounded by using Proposition 4.5.1 with $\lambda = \alpha_0^3$. We control the integrals over the poorly-scaled simplices, i.e., the integrals on the RHS of equation (4.36), in the following way.

**Proposition 4.6.1.** *If $k \geq 3$ and $p = 1$, then there exists a constant $C_9 = C_9(d, C_\mu, \alpha_0)$*

*such that for all balls $Q \subseteq H$*

$$\int_{S_{k,1}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2} \leq C_9 \cdot (k \cdot d + 1)^3 \cdot \left(\alpha_0 \cdot C_{\mathrm{p}}^2\right)^{k \cdot d} \cdot J_d^{\mathcal{D}}(\mu|_Q). \tag{4.37}$$

In Subsection 4.6.1 we categorize the elements of the sets $S_{k,p}$ , for $p = 1, 2$, according to configuration. We consider the more general $p$ in order to make the discussion applicable to Section 4.8. In Subsection 4.6.2 we further reduce Proposition 4.6.1 into two propositions using those configurations. Finally, in Subsection 4.6.3 we show that it is impossible to directly apply geometric multipoles to poorly scaled simplices. Throughout the rest of this section, we arbitrarily fix $k \geq 3$.

### 4.6.1 The Decomposition of $S_{k,p}$ According to Configuration

We partition $S_{k,p}$, $p = 1, 2$, according to the number of handles in the elements $X = (x_0, \ldots, x_{d+1})$ at $x_0$, i.e., according to the number of vertices $x_i$, $1 \leq i \leq d+1$, satisfying equation (4.9). We recall that this number can assume any value between 1 and $d$. Formally, for $1 \leq n \leq d$, we define the sets:

$$S_{k,p}^n = \left\{ X = (x_0, \ldots, x_{d+1}) \in S_{k,p} : \frac{\|x_i - x_0\|}{\max_{x_0}(X)} > \alpha_0^k \text{ for exactly } n \text{ vertices } x_i \right\}. \tag{4.38}$$

We note that

$$S_{k,p} = \bigcup_{n=1}^{d} S_{k,p}^n, \quad \text{and} \quad S_{k,p}^n \cap S_{k,p}^{n'} = \emptyset, \quad \text{for} \quad 1 \leq n \neq n' \leq d. \tag{4.39}$$

In this fashion we decompose $S_{k,p}$ into $n$-handled rakes for $1 \leq n \leq d$.

In turn, with the aid of multi-indices we partition each $S_{k,p}^n$ according to which vertices, $x_j$ for $1 \leq j \leq d+1$, are handles. Fix $1 \leq n \leq d$, and let $\eta \in \{0,1\}^{d+1}$ be a multi-index such that $|\eta| = n$ (i.e., $\eta$ has $n$ ones and $d+1-n$ zeros), and let $(\eta)_\ell$ , $1 \leq \ell \leq d+1$, denote the $\ell^{\text{th}}$ index of $\eta$. Let $S^n_{k,p}(\eta)$ denote the subset of $S^n_{k,p}$ whose $n$ handles occur at indices specified by $\eta$, that is,

$$S^n_{k,p}(\eta) = \left\{ X \in S^n_{k,p} : \frac{\|(X)_\ell - x_0\|}{\max_{x_0}(X)} > \alpha_0^k \ \text{ for all } \ 1 \le \ell \le d+1 \ \text{ such that } \ (\eta)_\ell = 1 \right\}.$$

The set $S^n_{k,p}$ is partitioned by the sets $S^n_{k,p}(\eta)$, where $|\eta| = n$, that is

$$S^n_{k,p} = \bigcup_{\{\eta:\ |\eta|=n\}} S^n_{k,p}(\eta), \quad \text{and} \quad S^n_{k,p}(\eta) \cap S^n_{k,p}(\eta') = \emptyset \ \text{ for } \ \eta \neq \eta'. \tag{4.40}$$

Given a ball $Q$ in $H$ we denote the restrictions of the above sets to $Q^{d+2}$ by $S^n_{k,p}(Q)$ and $S^n_{k,p}(\eta)(Q)$ respectively.

The sets $S^n_{k,p}(\eta)$ classify simplices by three basic characteristics, the first being the degeneracy of scale at $x_0$, which is indicated by the double subscript $k, p$. The second is the number of handles, which is indicated by the superscript $n$. The third is the configuration of the handles, which is indicated by the multi-index $\eta$. For example (see Figure 4.2.2), if $d = 3$ then for the multi-index $(0, 0, 0, 1)$ and any element

$$X = (x_0, x_1, x_2, x_3, x_4) \in S^2_{3,2}((0, 0, 0, 1, 1)),$$

we see that $X$ has exactly two handles located at the last two coordinates, $x_3$ and $x_4$, and that

$$\alpha_0^5 \le \frac{\min_{x_0}(X)}{\max_{x_0}(X)} \le \alpha^3.$$

Using these sets in conjunction with the permutation invariance of $\mathrm{p}_d\mathrm{sin}_{x_0}(X)$ (see equation (3.13)) reduces the computation of the integrals on the LHS of equation (4.37) in the following way. For fixed $1 \le n \le d$, let $\widetilde{\eta} = \widetilde{\eta}(n) \in \{0,1\}^{d+1}$ denote the multi-index whose first $n$ entries are ones and its last $d+1-n$ entries are zeros. Given any multi-index $\eta \in \{0,1\}^{d+1}$ with exactly $n$ nonzero elements, we let $\sigma$ be the permutation fixing 0 such that $\sigma(\eta)=\widetilde{\eta}$. Then, equation (3.13) implies the following equality for any ball $Q \in H$

$$\int_{S^n_{k,p}(\eta)(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) =$$
$$\int_{S^n_{k,p}(\sigma(\eta))(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(\sigma(X))}{\operatorname{diam}(\sigma(X))^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) = \int_{S^n_{k,p}(\widetilde{\eta})(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X). \quad (4.41)$$

Combining this observation with equation (4.40) results in the equality

$$\int_{S^n_{k,p}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) = \binom{d+1}{n} \int_{S^n_{k,p}(\widetilde{\eta})(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X). \quad (4.42)$$

Finally, equations (4.39) and (4.42) imply that

$$\int_{S_{k,p}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2} = \sum_{n=1}^{d} \binom{d+1}{n} \int_{S^n_{k,p}(\widetilde{\eta})(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}. \quad (4.43)$$

In order to simplify notation, let

$$\mathbf{S}^n_{k,p} = S^n_{k,p}(\widetilde{\eta}). \quad (4.44)$$

For a ball $Q \subseteq H$, let $\mathbf{S}^n_{k,p}(Q)$ denote the restriction of $\mathbf{S}^n_{k,p}$ to $Q^{d+2}$.

### 4.6.2 Reduction of Proposition 4.6.1

In estimating the integrals on the RHS of equation (4.43) we distinguish between the cases: $n=1$ and $1<n\le d$. For each of these cases we establish a corresponding proposition as follows, while assuming that $p=1$.

**Proposition 4.6.2.** *There exists a constant $C_{10}=C_{10}(d,C_\mu,\alpha_0)$ such that for any ball $Q$ in $H$*

$$\int_{\mathbf{S}^1_{k,1}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2} \le C_{10}\cdot(k\cdot d+1)\cdot\left(\alpha_0^d\cdot C_{\mathrm{p}}^2\right)^{k\cdot d}\cdot J^{\mathcal{D}}_d(\mu|_Q)\,.$$

**Proposition 4.6.3.** *If $1 < n \leq d$, then there exists a constant $C_{11} = C_{11}(d, C_\mu, \alpha_0)$ such that for any ball $Q$ in $H$*

$$\int_{\mathbf{S}^n_{k,1}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2} \leq C_{11} \cdot (k \cdot d + 1)^3 \cdot \left(\alpha_0 \cdot C_{\mathrm{p}}^2\right)^{k \cdot d} \cdot J_d^{\mathcal{D}}(\mu|_Q)\,.$$

Propositions 4.6.2 and 4.6.3 combined with equation (4.43) clearly establish Proposition 4.6.1. We prove those propositions in Sections 4.7 and 4.8 respectively.

### 4.6.3 The Insufficiency of Geometric Multipoles for Poorly-Scaled Simplices

We note that the decomposition of equation (4.36) forces us to look for an inequality of the form

$$\int_{S_{k,1}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X) \leq \gamma_k \cdot J_d^{\mathcal{D}}(\mu|_Q), \tag{4.45}$$

where the coefficients $\gamma_k$ are such that

$$\sum_{k \geq 3} \gamma_k < \infty. \tag{4.46}$$

While Proposition 4.2.1 is the most obvious avenue for accomplishing this, its direct application results in coefficients $\gamma_k$ that do not satisfy equation (4.46).

In order to establish this claim, we first note that calculating a bound of the form in equation (4.45) is equivalent to calculating a bound of the same form for each term on the RHS of equation (4.43). Furthermore, the coefficients $\gamma_k$ of equation (4.45) will be the sum of the coefficients calculated for each of the individual terms on the RHS of equation (4.43). We thus only focus on the term on the RHS of equation (4.43) corresponding to $n = d$, and we show that applying geometric multipoles to this term while trying to bound it by a constant times $J_d^{\mathcal{D}}(\mu|_Q)$, forces us to conclude that the $\gamma_k$ is larger than $\alpha_0^{-2(k+1)}$.

For simplicity, we assume that $Q = H$. Following the method of geometric multipoles, we form the following regions for any $k \geq 3$, $m \in \mathbb{Z}$ and $j \in \Lambda_m$ :

$$\mathbf{S}^d_{k,1}(m) = \left\{X \in \mathbf{S}^d_{k,1} : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}, \tag{4.47}$$

and

$$\widetilde{P}_{m,j} = \left\{X \in \mathbf{S}^d_{k,1}(m) : x_0 \in P_{m,j}\right\}.$$

From this we obtain the decomposition

$$\int_{\mathbf{S}^d_{k,1}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2} = \sum_{m\in\mathbb{Z}} \left[\sum_{j\in\Lambda_m} \int_{\widetilde{P}_{m,j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}\right], \tag{4.48}$$

and we then apply Proposition 4.2.1 to each of the individual terms on the RHS of equation (4.48).

Specifically, for such a fixed term and any $d$-plane $L$, by Proposition 4.2.1 and equation (4.47) we obtain the inequality

$$\int_{\widetilde{P}_{m,j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2} \leq \\ \frac{2 \cdot 8^2 \cdot (d+1)^2 (d+2)^2}{\alpha_0^{2+2(k+1)}} \sum_{i=0}^{d+1} \int_{\widetilde{P}_{m,j}} \frac{\operatorname{dist}^2(x_i, L)}{\operatorname{diam}^2(B_{m,j})} \frac{\mathrm{d}\mu^{d+2}}{\left(\alpha_0^{d(m+1)}\right)^{d+1}}. \tag{4.49}$$

Then, we notice that

$$\widetilde{P}_{m,j} = \bigcup_{x_0 \in P_{m,j}} \left[\{x_0\} \times A_0(x_0, \alpha_0^m)^d \times A_k(x_0, \alpha_0^m)\right] \subseteq \\ \bigcup_{x_0 \in P_{m,j}} \left[\{x_0\} \times B(x_0, \alpha_0^m)^d \times B(x_0, \alpha_0^{m+k})\right], \tag{4.50}$$

where this is essentially the "tightest fit" on $\widetilde{P}_{m,j}$ in terms of a cover of products of balls parameterized by $x_0 \in P_{m,j} \subseteq \operatorname{supp}(\mu)$. Hence, by the $d$-regularity of $\mu$ we can at best obtain the bounds

$$\int_{\widetilde{P}_{m,j}} \frac{\operatorname{dist}^2(x_i, L)}{\operatorname{diam}^2(B_{m,j})} \frac{\mathrm{d}\mu^{d+2}}{\left(\alpha_0^{d(m+1)}\right)^{d+1}} \;\leq\; \begin{cases} \frac{C_\mu^{d+1} \cdot \alpha_0^{k \cdot d}}{\alpha_0^{d(d+1)}} \cdot \beta_2^2(B_{m,j}, L) \cdot \mu(B_{m,j}), & \text{if } i = 0; \\ \frac{3^d \cdot C_\mu^{d+1} \cdot \alpha_0^{k \cdot d}}{\alpha_0^{d(d+1)}} \cdot \beta_2^2(B_{m,j}, L) \cdot \mu(B_{m,j}), & \text{if } 1 \leq i \leq d; \\ \frac{3^d \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)}} \cdot \beta_2^2(B_{m,j}, L) \cdot \mu(B_{m,j}), & \text{if } i = d+1. \end{cases}$$

Combining these bounds with equations (4.43) and (4.49), we see that

$$\gamma_k \;\geq\; (d+1) \cdot \alpha_0^{(d-2) \cdot k} + \alpha_0^{-2(k+1)}, \tag{4.51}$$

and thus $\lim\limits_{k \to \infty} \gamma_k = \infty$.

## 4.7 Multiscale Inequalities for the Polar Sine and a Proof of Proposition 4.6.2

Here we develop a multiscale inequality for the polar sine that avoids the problem outlined in Subsection 4.6.3. The idea is to use concentration inequalities for the polar sine to decompose a given rake into a sequence of well-scaled simplices satisfying a discrete inequality for the polar sine (see Lemma 4.7.2 below). This inequality then becomes an integral inequality whose terms are amenable to geometric multipoles.

In order to clearly formulate these inequalities we require a substantial bit of development. Subsection 4.7.1 develops the notion of well-scaled sequences and a related discrete multiscale inequality for the polar sine. Subsection 4.7.2 turns this inequality into a multiscale integral inequality. Finally, in Subsection 4.7.3 we prove Proposition 4.6.2 using geometric multipoles. Throughout this section we arbitrarily fix $k \geq 3$ and take $p = 1$ or $p = 2$.

### 4.7.1 From Rakes to Well-Scaled Sequences

We explain here how to decompose elements of $\mathbf{S}^1_{k,p}$ into sequences of well-scaled simplices. We recall that $\mathbf{S}^1_{k,p}$ is the set of rakes whose single handles are obtained at their first coordinate, such that $\alpha_0^{k+p} < \text{scale}_{x_0}(X) \leq \alpha_0^k$.

#### Well-Scaled Pieces and Augmented Elements

We define a *well-scaled piece* for $X \in \mathbf{S}^1_{k,p}$ to be a $(k \cdot d)$-tuple of the form

$$Y_X = (y_1, \ldots, y_{k \cdot d}) \in \prod_{q=1}^{k \cdot d} A_{k - \lceil \frac{q}{d} \rceil}(x_0, \max_{x_0}(X)) \,. \tag{4.52}$$

The coordinates of $Y_X$ are grouped into $k$ distinct clusters of $d$ points, with each individual cluster lying in a distinct annulus centered at $x_0$.

For fixed $X \in \mathbf{S}^1_{k,p}$ and an arbitrary well-scaled piece for $X$, $Y_X$, we define the corresponding *augmentation of* $X$ *by* $Y_X$, denoted by $\underline{X}$, to be the element

$$\underline{X} = X \times Y_X = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d}) \in \mathbf{S}^1_{k,p} \times H^{k \cdot d}. \tag{4.53}$$

For a fixed augmented element $\underline{X}$, we define two sequences of elements in $H^{d+2}$, the *auxiliary sequence*, $\widetilde{\Phi}_k(\underline{X}) = \left\{ \widetilde{X}_q \right\}_{q=0}^{k \cdot d}$, and the *well-scaled sequence* $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$. They will be used to formulate a multiscale inequality for the polar sine function.

#### The Auxiliary Sequence

If $a \in \mathbb{Z}$, then let $\overline{a} \in \{2, \ldots, d+1\}$ denote the unique integer such that $\overline{a} = a \bmod d$. We form the *auxiliary sequence* $\left\{ \widetilde{X}_q \right\}_{q=0}^{k \cdot d}$ recursively as follows.

**Definition 4.7.1.** *If* $X = (x_0, \ldots, x_{d+1}) \in \mathbf{S}^1_{k,p}$ *and* $\underline{X} = X \times Y_X = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d})$, *then let* $\widetilde{\Phi}_k(\underline{X}) = \left\{ \widetilde{X}_q \right\}_{q=0}^{k \cdot d}$ *be the sequence of elements in* $H^{d+2}$ *defined recursively as fol-*

*lows:*

$$\widetilde{X}_0 = X,$$

*and*

$$\widetilde{X}_q = \widetilde{X}_{q-1}\left(y_q, \overline{q+1}\right) \quad \text{for} \quad 1 \leq q \leq k \cdot d. \tag{4.54}$$

For example, if $d = 3$, then

$$\widetilde{X}_1 = (x_0, x_1, y_1, x_3, x_4), \quad \widetilde{X}_2 = (x_0, x_1, y_1, y_2, x_4), \quad \widetilde{X}_3 = (x_0, x_1, y_1, y_2, y_3);$$
$$\widetilde{X}_4 = (x_0, x_1, y_4, y_2, y_3), \quad \widetilde{X}_5 = (x_0, x_1, y_4, y_5, y_3), \quad \widetilde{X}_6 = (x_0, x_1, y_4, y_5, y_6).$$

In general, we note that the elements $\widetilde{X}_q$ have the following form:

$$\widetilde{X}_q = \begin{cases} (x_0, x_1, y_1, \ldots, y_q, x_{q+2}, \ldots, x_{d+1}), & \text{if } 1 \leq q \leq d-1; \\ (x_0, x_1, y_{j \cdot d+1}, \ldots, y_q, y_{q-d+1}, \ldots, y_{j \cdot d}), & \text{if } j \cdot d < q < (j+1) \cdot d \ \text{ for } 1 \leq j \leq k-1; \\ (x_0, x_1, y_{(j-1) \cdot d+1}, \ldots, y_{j \cdot d}), & \text{if } q = j \cdot d \ \text{ for } 1 \leq j \leq k. \end{cases} \tag{4.55}$$

In the special case where $d = 1$, the first two cases of equation (4.55) are meaningless and $\widetilde{X}_q = (x_0, x_1, y_q)$ for all $1 \leq q \leq k$.

### The Well-Scaled Sequence

We derive the *well-scaled* sequence $\Phi_k(\underline{X})$ from the auxiliary sequence $\widetilde{\Phi}_k(\underline{X})$ as follows.

**Definition 4.7.2.** *If $X \in \mathbf{S}^1_{k,p}$ and $\underline{X} = X \times Y_X = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d})$, then let $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ be the sequence of elements in $H^{d+2}$ such that*

$$X_q = \begin{cases} \widetilde{X}_{q-1}(y_q, 1), & \text{if } 1 \leq q \leq k \cdot d; \\ \widetilde{X}_{k \cdot d}, & \text{if } q = k \cdot d + 1. \end{cases} \tag{4.56}$$

For example, if $d = 3$, then the first six elements of the sequence are

$$X_1 = (x_0, y_1, x_2, x_3, x_4), \quad X_2 = (x_0, y_2, y_1, x_3, x_4), \quad X_3 = (x_0, y_3, y_1, y_2, x_4);$$

$$X_4 = (x_0, y_4, y_1, y_2, y_3), \quad X_5 = (x_0, y_5, y_4, y_2, y_3), \quad X_6 = (x_0, y_6, y_4, y_5, y_3).$$

We also note that in the very special case where $d = 1$, then

$X_1 = (x_0, y_1, x_2)$, $X_q = (x_0, y_q, y_{q-1})$, $1 < q \leq k \cdot d + 1$ and $X_{k \cdot d+1} = (x_0, x_1, y_{k \cdot d})$.

The following lemma shows that the elements $X_q \in \Phi_k\left(\underline{X}\right)$ , $1 \leq q \leq k \cdot d + 1$, are indeed well-scaled at the vertex $x_0$. We provide the straightforward details of the proof in Appendix 5.2.4.

**Lemma 4.7.1.** *If $X \in \mathbf{S}^1_{k,p}$ and $\underline{X} = X \times Y_X$, then each term of the sequence $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ is well-scaled at $x_0$ and we have the following estimates:*

$$\alpha_0^{k+1-\left\lceil \frac{q}{d} \right\rceil} \cdot \max_{x_0}(X) < \max_{x_0}(X_q) \leq \alpha_0^{k-\left\lceil \frac{q}{d} \right\rceil} \cdot \max_{x_0}(X), \quad \text{if} \;\; 1 \leq q \leq k \cdot d, \tag{4.57}$$

*and*

$$\alpha_0 \cdot \max_{x_0}(X) < \min_{x_0}(X_q) \leq \max_{x_0}(X_q) = \max_{x_0}(X), \quad \text{if} \;\; q = k \cdot d + 1. \tag{4.58}$$

### Augmented Sets and a Discrete Multiscale Inequality

Here we formulate a discrete multiscale inequality for the polar sine on $\mathbf{S}^1_{k,p}$ in terms of the well-scaled sequence, $\Phi_k(\underline{X})$. Using the constant $C_{\mathrm{p}}$ of equation (1.20), we form the *set augmentation* of the set $\mathbf{S}^1_{k,p}$, denoted by $\underline{\mathbf{S}^1_{k,p}}$, as follows:

$$\underline{\mathbf{S}^1_{k,p}} = \Bigg\{ \underline{X} \in \mathbf{S}^1_{k,p} \times [\mathrm{supp}(\mu)]^{k \cdot d} : \text{ the sequences } \widetilde{\Phi}_k(\underline{X}) \text{ and } \Phi_k(\underline{X}) \text{ satisfy the inequality}$$
$$\mathrm{p}_d\mathrm{sin}\left(\widetilde{X}_q\right) \leq C_{\mathrm{p}} \cdot \left(\mathrm{p}_d\mathrm{sin}\left(X_{q+1}\right) + \mathrm{p}_d\mathrm{sin}\left(\widetilde{X}_{q+1}\right)\right), \quad \text{for all} \;\; 0 \leq q < k \cdot d. \Bigg\} \tag{4.59}$$

The motivation for the above definition will not be entirely clear until Subsection 4.7.3. The augmented sets $\underline{\mathbf{S}^1_{k,p}}$ give rise to the following multiscale inequality, whose proof is included in Appendix 5.2.5.

**Lemma 4.7.2.** *If $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$, then the elements of the corresponding parameterized well-scaled sequence $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ satisfy the inequality*

$$\mathrm{p}_d \sin^2_{x_0}(X) \leq (k \cdot d + 1) \cdot C_{\mathrm{p}}^{2 \cdot k \cdot d} \sum_{q=1}^{k \cdot d+1} \mathrm{p}_d \sin^2_{x_0}(X_q).$$

### 4.7.2 Turning Discrete Inequalities into Integral Inequalities

In this subsection, we turn Lemma 4.7.2 into an integral inequality by bounding the integral over $\mathbf{S}^1_{k,p}$ by a related integral over its set augmentation $\underline{\mathbf{S}^1_{k,p}}$. Subsection 4.7.2 defines the preliminary notation, and in Subsection 4.7.2 we establish an integral identity relating integrals over $\mathbf{S}^1_{k,p}$ to certain integrals over $\underline{\mathbf{S}^1_{k,p}}$. We then apply the discrete inequality of Lemma 4.7.2, thus obtaining the desired multiscale integral inequality.

#### Truncations and Projections

We fix $0 \leq q \leq k \cdot d$. If $\underline{X} = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, then we define the $q^{th}$ *truncation of* $\underline{X}$ to be the function $T_q : \underline{\mathbf{S}^1_{k,p}} \to H^{d+2+q}$, where

$$T_q\left(\underline{X}\right) = \begin{cases} X, & \text{if } q = 0; \\ (x_0, \ldots, x_{d+1}, y_1, \ldots, y_q), & \text{if } 1 \leq q \leq k \cdot d. \end{cases} \tag{4.60}$$

The $(d+2+q)$-tuple $T_q\left(\underline{X}\right)$ is not to be confused with the projection $(\underline{X})_q \in H$. If $A \subseteq \underline{\mathbf{S}^1_{k,p}}$, then we denote the image of $A$ by $T_q(A)$.

For $\underline{X} = (x_0, \ldots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, we denote the pre-image of $T_q\left(\underline{X}\right) = (x_0, \ldots, y_q)$ by

$$T_q^{-1}(x_0, \ldots, y_q) = \left\{ \underline{X}' \in \underline{\mathbf{S}^1_{k,p}} : \; T_q\left(\underline{X}'\right) = T_q\left(\underline{X}\right) = (x_0, \ldots, y_q) \right\}, \tag{4.61}$$

where $(x_0, \ldots, y_q)$ is taken to mean $X$ if $q = 0$.

Now, fixing $1 \leq q \leq k \cdot d$, we define the $q^{th}$ projection of $\underline{X}$ onto $H$. For $\underline{X} = (x_0, \ldots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, let

$$\pi_q(\underline{X}) = y_q = (\underline{X})_{d+2+q} \,.$$

The set $\pi_q\left(T^{-1}_{q-1}\left(x_0, \ldots, y_{q-1}\right)\right)$ is composed of all possible $q^{th}$ coordinates of the well-scaled pieces $Y_X = (y_1, \ldots, y_{k \cdot d})$ such that $\underline{X'} = X \times Y_X \in T_q^{-1}(x_0, \ldots, y_{q-1})$.

### An Integral Identity and a Resulting Integral Inequality

Here we show how to replace integrals over $\mathbf{S}^1_{k,p}$ by certain integrals over $\underline{\mathbf{S}^1_{k,p}}$, and we then use the corresponding identity to obtain an integral inequality.

For any $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and all $1 \leq q \leq k \cdot d$, we define the functions $g^1_{k,q}$ as

$$g^1_{k,q}\left(\underline{X}\right) = \mu\left(\pi_q\left(T^{-1}_{q-1}\left(T_{q-1}\left(\underline{X}\right)\right)\right)\right). \tag{4.62}$$

Furthermore, we define the function

$$f^1_k\left(\underline{X}\right) = \prod_{q=1}^{k \cdot d} g^1_{k,q}\left(\underline{X}\right). \tag{4.63}$$

We note that the double subscript in this case is not intended to evoke the index $p = 1, 2$. For $1 \leq q \leq k \cdot d$, the following proposition establishes that the functions $g^1_{k,q}$, and hence $f^1_k$, are positive on $\underline{\mathbf{S}^1_{k,p}}$. The proof of the following estimate is in Appendix 5.2.6.

**Proposition 4.7.1.** *If $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and $1 \leq q \leq k \cdot d$, then*

$$\mu\left(B(x_0, \alpha_0^{k - \lceil \frac{q}{d} \rceil} \cdot \max\nolimits_{x_0}(X))\right) \geq g^1_{k,q}\left(\underline{X}\right) \geq \frac{1}{2} \cdot \mu\left(B(x_0, \alpha_0^{k - \lceil \frac{q}{d} \rceil} \cdot \max\nolimits_{x_0}(X))\right). \tag{4.64}$$

We can thus use $f^1_k$ in order to rescale the integration over $\underline{\mathbf{S}^1_{k,p}}$ as in the following proposition. We use the notation

$$N_k = (k+1) \cdot d + 2.$$

**Proposition 4.7.2.** *If $Q$ is a ball in $H$, then*

$$\int_{\mathbf{S}^1_{k,p}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) = \int_{\underline{\mathbf{S}^1_{k,p}(Q)}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k\,(\underline{X})}. \tag{4.65}$$

*Proof of Proposition 4.7.3.* For simplification, we let $Q = H$, however, our technique easily applies to any ball $Q$ in $H$. First, using Fubini's Theorem we obtain

$$\begin{aligned} &\int_{\underline{\mathbf{S}^1_{k,p}}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k\,(\underline{X})} \\ &\qquad = \int_{\mathbf{S}^1_{k,p}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \left( \int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} \right) \mathrm{d}\mu^{d+1}(X). \end{aligned} \tag{4.66}$$

Then, we iterate the inner integral on the RHS of equation (4.66) and get

$$\begin{aligned} &\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} \\ &= \int_{\pi_1\left(T_0^{-1}(X)\right)} \cdots \int_{\pi_q\left(T^{-1}_{q-1}(x_0,\ldots,y_{q-1})\right)} \cdots \int_{\pi_{k\cdot d}\left(T^{-1}_{k\cdot d-1}(x_0,\ldots,y_{k\cdot d-1})\right)} \frac{\mathrm{d}\mu\,(y_{k\cdot d}) \cdots \mathrm{d}\mu(y_1)}{\prod_{q=1}^{k\cdot d} g^1_{k,q}\,(\underline{X})}, \end{aligned} \tag{4.67}$$

where the sets $\pi_q\left(T^{-1}_{q-1}(x_0,\ldots,y_{q-1})\right)$ are conditionally defined given $y_{q-1} \in \pi_{q-1}\left(T^{-1}_{q-2}(x_0,\ldots,y_{q-2})\right)$, for $2 \le q \le k \cdot d$.

We note that for any fixed $(x_0,\ldots,y_{q-1}) \in H^{(d+1+q)}$, by the definition of $g^1_{k,q}$ we have the equality

$$\int_{\pi_q\left(T^{-1}_{q-1}(x_0,\ldots,y_{q-1})\right)} \frac{\mathrm{d}\mu(y_q)}{g^1_{k,q}\,(\underline{X})} = \int_{\pi_q\left(T^{-1}_{q-1}(x_0,\ldots,y_{q-1})\right)} \frac{\mathrm{d}\mu(y_q)}{\mu\left(\pi_q\left(T^{-1}_{q-1}(x_0,\ldots,y_{q-1})\right)\right)} = 1. \tag{4.68}$$

Applying this to the iterated integral on the RHS of equation (4.67) we obtain

$$\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} = 1, \text{ for all } X \in \mathbf{S}^1_{k,p}. \tag{4.69}$$

Combining equations (4.66) and (4.69), we conclude equation (4.65) and the proposition. □

The following multiscale integral inequality is a direct consequence of Lemma 4.7.2 and Proposition 4.7.2.

**Proposition 4.7.3.** *If $Q$ is a ball in $H$, $k \geq 3$, and $p = 1, 2$, then*

$$\int_{\mathbf{S}^1_{k,p}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2} \leq (k \cdot d + 1) \cdot C_{\mathrm{p}}^{2 \cdot k \cdot d} \sum_{q=1}^{k \cdot d+1} \int_{\underline{\mathbf{S}^1_{k,p}(Q)}} \frac{\mathrm{p}_d\mathrm{sin}^2(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})}.$$

**Remark 4.7.1.** *Since we do not take the time to establish the measurability of $f_k^1$, one can instead follow an alternative strategy: Proposition 4.7.1 implies that*

$$f_k^1 \approx \prod_{q=1}^{k \cdot d} \mu\left(B(x_0, \alpha_0^{k - \lceil \frac{q}{d} \rceil} \cdot \mathrm{max}_{x_0}(X))\right),$$

*where the constant of comparability is at worst $2^{k \cdot d}$. The latter function is clearly measurable in $\underline{X}$, and one can thus use it instead of $f_k^1$ for Propositions 4.7.2 and 4.7.3. Nevertheless such a strategy will increase the estimate on the constant $C_7$ of Theorem 4.0.2, and requires a slight change in the choice of $\alpha_0$ in equation (1.21).*

### 4.7.3 Concluding the Proof of Proposition 4.6.2

Here we prove the following proposition for $p = 1$, taking time to clearly explain the computations since they are the basis for a similar set of computations done in the proof of Proposition 4.6.3.

**Proposition 4.7.4.** *If $1 \leq q \leq k \cdot d + 1$, then there exists a constant $C_{12}$ such that*

$$\int_{\underline{\mathbf{S}^1_{k,1}(Q)}} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})} \leq C_{12} \cdot \alpha_0^{k \cdot d^2} \cdot J_d^{\mathcal{D}}(\mu|_Q) \tag{4.70}$$

*for any ball $Q \subseteq H$.*

We note that Proposition 4.6.2 is then a direct consequence of Propositions 4.7.3 and 4.7.4.

*Proof of Proposition 4.7.4.* We partition the sets $\underline{\mathbf{S}^1_{k,1}}(Q)$ by the size of $\mathrm{max}_{x_0}(X)$. If $m \geq m(Q)$, then let

$$\underline{\mathbf{S}^1_{k,1}(m)(Q)} = \left\{\underline{X} \in \underline{S_1^{k,1}(\widetilde{\eta})(Q)} : \mathrm{max}_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}. \tag{4.71}$$

Throughout the rest of the proof we fix $m \geq m(Q)$ and $1 \leq q \leq k \cdot d$, and we further partition the set $\underline{\mathbf{S}_{k,p}^1(m)(Q)}$ according to location in $\mathrm{supp}(\mu)$ in order to reflect the quantity $\max_{x_0}(X_q)$. Specifically, we let

$$\mathrm{sc}(m,q) = \begin{cases} m + k - \left\lceil \frac{q}{d} \right\rceil, & \text{if } 1 \leq q \leq k \cdot d; \\ \\ m, & \text{if } q = k \cdot d + 1. \end{cases} \tag{4.72}$$

The exponent $\mathrm{sc}(m,q)$ is called the *scale exponent of $m$ and $q$*, and indicates the correct scale for the decomposition of the set $\underline{\mathbf{S}_{k,1}^1(m)(Q)}$. Specifically, according to the estimates of Lemma 4.7.1, we have that

$$\max_{x_0}(X_q) \in \left(\alpha_0^{\mathrm{sc}(m,q)+2}, \alpha_0^{\mathrm{sc}(m,q)}\right].$$

Thus, for $j \in \Lambda_{\mathrm{sc}(m,q)}(Q)$, we let

$$\underline{P_{\mathrm{sc}(m,q),j}} = \left\{\underline{X} \in \underline{\mathbf{S}_{k,1}^1(m)} : x_0 \in P_{\mathrm{sc}(m,q),j}\right\}, \tag{4.73}$$

and obtain the following cover of $\underline{\mathbf{S}_{k,1}^1(m)(Q)}$ :

$$\underline{\mathcal{P}_{\mathrm{sc}(m,q)}}(Q) = \left\{\underline{P_{\mathrm{sc}(m,q),j}}\right\}_{j \in \Lambda_{\mathrm{sc}(m,q)}(Q)}. \tag{4.74}$$

By letting $m \geq m(Q)$ and $j$ vary, we note that the integral on the LHS of equation (4.70) satisfies the following inequality:

$$\int_{\underline{\mathbf{S}_{k,1}^1}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})} \leq$$
$$\sum_{m \geq m(Q)} \left[\sum_{j \in \Lambda_{\mathrm{sc}(m,q)}(Q)} \int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})}\right]. \tag{4.75}$$

Fixing an arbitrary $j \in \Lambda_{\mathrm{sc}(m,q)}(Q)$, in addition to the fixed $k$, $m$ and $q$, we will establish that

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})} \leq C_7 \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}\right) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \tag{4.76}$$

Equations (4.75) and (4.76) will directly imply Proposition 4.7.4.

We prove equation (4.76) as follows. Fix an arbitrary $d$-plane $L$. Since the elements $\{X_q\}_{q=1}^{k\cdot d+1}$ are well-scaled at $x_0$, by Proposition 4.2.1 and equation (4.7) we have the following bound on the LHS of equation (4.76):

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \mathrm{p}_d \sin^2_{x_0}(X_q) \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{\mathrm{diam}(X)^{d(d+1)} \cdot f_k^1(\underline{X})} \leq$$
$$\frac{2\cdot(d+1)^2\cdot(d+2)^2}{\alpha_0^6} \int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2^2(X_q, L)}{\mathrm{diam}(X_q)^2} \cdot \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{\mathrm{diam}(X)^{d(d+1)} \cdot f_k^1(\underline{X})}. \quad (4.77)$$

To bound the RHS of equation (4.77) we focus on the individual terms of

$$\frac{D_2^2\left(X_q, L\right)}{\mathrm{diam}^2(X_q)} = \sum_{s=0}^{d+1} \frac{\mathrm{dist}^2\left((X_q)_s, L\right)}{\mathrm{diam}^2(X_q)}.$$

We arbitrarily fix $0 \leq s \leq d+1$ and note the following cases of possible values of $(X_q)_s$:

Case 1: $(X_q)_s = x_0$. In this case $q$ has no restriction, that is, $1 \leq q \leq k\cdot d+1$.

Case 2: $(X_q)_s = x_1$. In this case $q = k\cdot d+1$ (see equations (4.55)-(4.56)) and thus $\mathrm{sc}(m,q) = m$.

Case 3: $(X_q)_s = x_i$, where $2 \leq i \leq d+1$. In this case $1 \leq q \leq d$ (see equations (4.55)-(4.56)) and thus $\mathrm{sc}(m,q) = m+k-1$.

Case 4: $(X_q)_s = y_\ell$, where $1 \leq \ell \leq k\cdot d$. In this case for each $1 \leq q \leq k\cdot d+1$, we have the following restriction on $\ell$: $\max\{1, q-d\} \leq \ell \leq q$ and thus

$$\max\left\{1, \left\lceil \frac{q}{d} \right\rceil - 1\right\} \leq \left\lceil \frac{\ell}{d} \right\rceil \leq \left\lceil \frac{q}{d} \right\rceil. \quad (4.78)$$

The calculation of the upper bound varies slightly according to which case we consider.

Considering the first three cases simultaneously, we let $0 \leq i \leq d+1$ and examine the integrals decomposing the RHS of equation (4.77), each of the form

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \mathrm{diam}(X)^{d(d+1)}}.$$

Per Fubini's Theorem we obtain the equality

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \mathrm{diam}(X)^{d(d+1)}} =$$
$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \left(\int_{\left\{Y_X : X \times Y_X \in \underline{P_{\mathrm{sc}(m,q),j}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{\mathrm{diam}^2(X_q) \cdot f_k^1(\underline{X})}\right) \mathrm{dist}^2(x_i, L)\, \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,. \tag{4.79}$$

To bound the inner integral on the RHS of equation (4.79) we first note that

$$\mathrm{diam}(X) \geq \max\nolimits_{x_0}(X) = \|x_0 - x_1\| > \alpha_0^{m+1} \quad \text{for all} \quad X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right). \tag{4.80}$$

Combining Lemma 4.7.1 with equations (4.72) and (4.80) we see that

$$\mathrm{diam}(X_q) \geq \max\nolimits_{x_0}(X_q) > \alpha_0^{\mathrm{sc}(m,q)+1}. \tag{4.81}$$

Since $\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right) = 8 \cdot \alpha_0^{\mathrm{sc}(m,q)}$, we rewrite equation (4.81) as follows:

$$\mathrm{diam}(X_q) \geq \frac{\alpha_0}{8} \cdot \mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right). \tag{4.82}$$

Combining equations (4.69) and (4.82), we obtain that if $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$, then

$$\int_{\left\{Y_X : X \times Y_X \in \underline{P_{\mathrm{sc}(m,q),j}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f_k^1(\underline{X}) \cdot \mathrm{diam}^2(X_q)} \leq \frac{64}{\alpha_0^2} \cdot \frac{1}{\mathrm{diam}^2(B_{\mathrm{sc}(m,q),j})}\,. \tag{4.83}$$

By the definition of $\underline{P_{\mathrm{sc}(m,q),j}}$, $\underline{\mathbf{S}^1_{k,1}(m)}$ and $\mathbf{S}^1_{k,1}(m)$ we have the equality

$$T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right) = \bigcup_{x_0 \in P_{\mathrm{sc}(m,q),j}} \left[\bigcup_{x_1 \in A_m(x_0,1)} \{(x_0, x_1)\} \times [A_k(x_0, \|x_1 - x_0\|)]^d\right].$$

From this we trivially obtain the inclusion

$$T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right) \subseteq \bigcup_{x_0 \in P_{\mathrm{sc}(m,q),j}} \{x_0\} \times B(x_0, \alpha_0^m) \times \left[B(x_0, \alpha_0^{m+k})\right]^d.$$

Applying this together with the inequalities of equations (4.80) and (4.83) to the RHS of equation (4.79) gives the following inequality for all $0 \leq i \leq d+1$:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \mathrm{diam}(X)^{d(d+1)}}$$
$$\leq \frac{64}{\alpha_0^{d(d+1)+2}} \int_{P_{\mathrm{sc}(m,q),j}} \int_{B(x_0,\alpha_0^m)} \int_{\left[B(x_0,\alpha_0^{m+k})\right]^d} \left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{[\alpha_0^m]^{d(d+1)}}. \quad (4.84)$$

Assume Case 1, that is, $i = 0$. Then, after iterating the integral on the RHS of equation (4.84), applying the defining property of $d$-regular measure and the inclusion $P_{\mathrm{sc}(m,q),j} \subseteq B_{\mathrm{sc}(m,q),j}$, we see that the term on the RHS of equation (4.84) has the bound

$$\frac{64 \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+2}} \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.85)$$

Assume Case 2, that is, $i = 1$, and recall that in this case $q = k \cdot d + 1$ and $\mathrm{sc}(m,q) = m$. Thus we have the inclusion

$$B(x_0, \alpha_0^m) \subseteq B_{\mathrm{sc}(m,q),j}, \ \text{ for all } x_0 \in P_{\mathrm{sc}(m,q),j}.$$

Hence, iterating the integral on the RHS of equation (4.84) and then applying similar arguments to Case 1, we obtain the following bound for the LHS of equation (4.84):

$$\frac{64 \cdot 4^d \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+2}} \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2\left(B_{m,j}, L\right) \cdot \mu\left(B_{m,j}\right). \quad (4.86)$$

Next, assume Case 3, that is, $2 \leq i \leq d+1$ and recall that in this case $1 \leq q \leq d$ and $\mathrm{sc}(m,q) = m+k-1$. Using the fact that $P_{\mathrm{sc}(m,q),j} \subseteq \frac{3}{4} \cdot B_{\mathrm{sc}(m,q),j}$, and the defining property of $d$-regular measures we have the inequality

$$\mu\left(P_{\mathrm{sc}(m,q),j}\right) \leq \mu\left(\frac{3}{4} \cdot B_{\mathrm{sc}(m,q),j}\right) \leq C_\mu \cdot \left(3 \cdot \alpha_0^{m+k-1}\right)^d.$$

Furthermore, we have the inclusion

$$B\left(x_0, \alpha_0^{m+k}\right) \subseteq B_{\mathrm{sc}(m,q),j}, \ \text{ for all } x_0 \in P_{\mathrm{sc}(m,q),j}.$$

Iterating the integral as in the previous calculations, the LHS of equation (4.84) is bounded by

$$\frac{64 \cdot 3^{d} \cdot C_{\mu}^{d+1}}{\alpha_0^{d(d+1)+d+2}} \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2 \left(B_{\mathrm{sc}(m,q),j}, L\right) \cdot \mu \left(B_{\mathrm{sc}(m,q),j}\right). \tag{4.87}$$

Therefore, taking the maximal coefficient from equations (4.85), (4.86) and (4.87), the LHS of equation (4.84) has the following uniform bound for all $0 \leq i \leq d+1$:

$$\frac{3^{d} \cdot 2^{7} \cdot C_{\mu}^{d+1}}{\alpha_0^{d(d+1)+d+2}} \cdot \beta_2^2 \left(B_{\mathrm{sc}(m,q),j}, L\right) \cdot \mu \left(B_{\mathrm{sc}(m,q),j}\right). \tag{4.88}$$

At last we consider Case 4, where the terms in the sum comprising the RHS of equation (4.77) are of the form:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \operatorname{diam}(X)^{d(d+1)}},$$

where $1 \leq l \leq k \cdot d$.

Iterating the integral and applying equation (4.82), we obtain

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \operatorname{diam}(X)^{d(d+1)}} \leq$$
$$\frac{64}{\alpha_0^2} \int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \left( \int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}_{k,p}^1}\right\}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k \cdot d}(Y_X)}{f_k^1(\underline{X})} \right) \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}}. \tag{4.89}$$

In order to bound the RHS of equation (4.89), we first calculate a uniform bound in $1 \leq l \leq k \cdot d$ for the interior integral. Then, completing the integration with respect to $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$ will give the desired bound in terms of the corresponding $\beta_2$ number.

For fixed $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$ and $1 \leq \ell \leq k \cdot d$, after iterating the interior integral on the RHS of equation (4.89) and applying equation (4.68) we have that

$$\int_{\left\{Y_X : X\times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \left(\frac{\operatorname{dist}\left(y_\ell, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f_k^1(\underline{X})} =$$

$$\int_{\pi_1\left(T_0^{-1}(X)\right)} \cdots \int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)} \left(\frac{\operatorname{dist}\left(y_\ell, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu\left(y_\ell\right)\cdots \mathrm{d}\mu(y_1)}{\prod_{s=1}^{\ell} g_{k,s}^d}. \quad (4.90)$$

Given $\underline{X} \in \underline{P_{\mathrm{sc}(m,q),j}}$ we fix $(x_0, \ldots, y_{\ell-1}) = T_{\ell-1}(\underline{X})$ and calculate a bound for the integral

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)} \left(\frac{\operatorname{dist}\left(y_\ell, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu\left(y_\ell\right)}{g_{k,\ell}^d}.$$

We first obtain an upper bound for

$$\frac{1}{g_{k,\ell}^d} = \frac{1}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)\right)},$$

and then complete the integration.

To obtain that bound, we apply Proposition 4.7.1 to get that for all $1 \le \ell \le k \cdot d$:

$$\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)\right) \ge \frac{1}{2} \cdot \mu\left(B\left(x_0, \alpha_0^{k-\left\lceil \frac{\ell}{d}\right\rceil} \cdot \max\nolimits_{x_0}(X)\right)\right). \quad (4.91)$$

Next, applying equations (4.72), (4.78) and (4.80) as well as the fact that $\alpha_0 < 1$, we note that

$$\alpha_0^{k-\left\lceil \frac{\ell}{d}\right\rceil} \cdot \max\nolimits_{x_0}(X) \ge \alpha_0^{k-\left\lceil \frac{\ell}{d}\right\rceil+m+1} = \alpha_0^{k-\left\lceil \frac{q}{d}\right\rceil+m+2} = \alpha_0^{\mathrm{sc}(m,q)+2}. \quad (4.92)$$

We now use Lemma 1.4.1 to obtain the following bound

$$\mu\left(B\left(x_0, \alpha_0^{\mathrm{sc}(m,q)+2}\right)\right) \ge \frac{1}{C_\mu^2} \cdot \left(\frac{\alpha_0^2}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.93)$$

Finally combining equations (4.91), (4.92) and (4.93) we conclude that

$$\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)\right) \ge \frac{1}{2\cdot C_\mu^2} \cdot \left(\frac{\alpha_0^2}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.94)$$

Noting that $\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right) \subseteq B_{\mathrm{sc}(m,q),j}$ and applying equation (4.94), we have the inequality

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \cdot \frac{\mathrm{d}\mu(y_s)}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)\right)} \leq \frac{2\cdot 4^d \cdot C_\mu^2}{\alpha_0^{2\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.95)$$

Then, applying this and equation (4.68) to the RHS of equation (4.90), we have the following inequality for all $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$:

$$\int_{\left\{Y_X : X\times Y_X \in \underline{\mathbf{S}_{k,p}^1}\right\}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f_k^1(\underline{X})} \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{2\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.96)$$

Furthermore, noting that

$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}} \leq \frac{C_\mu^{d+1}}{\alpha_0^{d(d+1)}} \cdot \alpha_0^{k\cdot d^2} \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right),$$

per equations (4.89) and (4.96), we have the following uniform bound for all $1 \leq l \leq k\cdot d$:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot \operatorname{diam}(X)^{d(d+1)}} \leq \frac{128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2} \cdot \beta_2^2(B_{\mathrm{sc}(m,q),j}) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \quad (4.97)$$

Finally, taking the larger of the coefficients from equations (4.88) and (4.97), we have the bound

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2(X_q, L)}{\operatorname{diam}^2(X_q)} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot\operatorname{diam}(X)^{d(d+1)}} \leq \frac{(d+2)\cdot 128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2}\cdot \beta_2^2(B_{\mathrm{sc}(m,q),j}, L)\cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right).$$

Therefore, taking the infimum over all such $d$-planes $L$ we obtain the bound

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2(X_q, L)}{\operatorname{diam}^2(X_q)} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot\operatorname{diam}(X)^{d(d+1)}} \leq \frac{(d+2)\cdot 128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2}\cdot \beta_2^2(B_{\mathrm{sc}(m,q),j})\cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right).$$

Combining this with equation (4.77) establishes the conclusion of equation (4.76). □

## 4.8 The Proof of Proposition 4.6.3

Here we develop analogous methods to those of Section 4.7 for integration over the regions composed of multi-handled rakes, $\mathbf{S}^n_{k,1}$ for $1 < n \leq d$. Again, we use the concentration inequality of Proposition **??** to decompose a given multi-handled rake into a sequence of simplices satisfying a relaxed simplex-type inequality, and then turn this into an integral inequality. Finally, geometric multipoles yields the desired bounds in terms of the squared $\beta_2$ numbers.

We prove Proposition 4.6.3 in four parts. Subsection 4.8.1 develops the notion of rake sequences, as well as a discrete pre-multiscale inequality for the polar sine on $\mathbf{S}^n_{k,1}$. Subsection 4.8.2 turns this inequality into an integral inequality, and in Subsection 4.8.3 this becomes a true multiscale integral inequality. Finally, in Subsection 4.8.4 we state the desired bounds in terms of the squared $\beta_2$ numbers.

Due to the similarity of the proofs and computations with those of Section 4.7, this current section consists only of statements. The proofs and computations are catalogued in the appendix for completeness. We fix $k \geq 3$ and $1 < n \leq d$. We denote

$$N_n = 2^{n-1} - 1$$

and

$$M_n = d + 2 + N_n = d + 1 + 2^{n-1}.$$

We remark that $N_n$ and $N_k$ (defined in Subsection 4.7.2) are two different constants.

### 4.8.1 Rake-Sequences and a "Pre-Multiscale" Inequality

We recall that $\mathbf{S}^n_{k,1}$ is the set of multi-handled rakes whose handles occur at their first $n$ coordinates and whose tines occur at their last $d+1-n$ coordinates. More precisely, for all $X \in \mathbf{S}^n_{k,1}$

$$\max_{x_0}(X) = \max_{1 \le i \le n} \|x_i - x_0\| \quad \text{and} \quad \min_{1 \le i \le n} \frac{\|x_i - x_0\|}{\max_{x_0}(X)} > \alpha_0^k.$$

**Short-Scale Pieces and Rake Sequences for $X \in \mathbf{S}^n_{k,1}$**

We define a *short-scale piece* for $X \in \mathbf{S}^n_{k,1}$ to be an $N_n$-tuple of the form

$$Z_X = (z_1, \ldots, z_{N_n}) \in \left[A_k(x_0, \max_{x_0}(X))\right]^{N_n}. \tag{4.98}$$

At times we index the coordinates of $Z_X$ as $z_{2^j+m-1}$ with $0 \le j < n-1$ and $1 \le m \le 2^j$.

For fixed $X \in \mathbf{S}^n_{k,1}$ and an arbitrary short-scale piece for $X$, $Z_X$, we define the corresponding augmentation of $X$ by $Z_X$, denoted by $\overline{X}$, to be the element

$$\overline{X} = X \times Z_X = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}) \in \mathbf{S}^n_{k,1} \times H^{N_n}. \tag{4.99}$$

We note that $\overline{X} \in H^{M_n}$.

For a fixed augmented element $\overline{X}$ we define two sequences of elements in $H^{d+2}$ that will be used to formulate a discrete inequality for the polar sine on $\mathbf{S}^n_{k,1}$.

**Definition 4.8.1.** *If $X \in \mathbf{S}^n_{k,1}$ and $\overline{X} = X \times Z_X = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n})$, then let $\widetilde{\Psi}_k\left(\overline{X}\right) = \left\{Z^j_m\right\}_{\substack{0 \le j \le n-1 \\ 1 \le m \le 2^j}}$ be the doubly indexed sequence of elements of $H^{d+2}$ defined recursively as follows:*

$$Z^0_1 = X,$$

*and for* $0 \leq j < n-1$ *and* $\sigma_j$ *denoting the transposition of* $n-j-1$ *and* $n-j$ *(acting on* $Z_m^j$ *by replacing its coordinates at those indices)*

$$Z_{2m-1}^{j+1} = Z_m^j \left(n-j, z_{2^j+(m-1)}\right), \tag{4.100}$$

*and*

$$Z_{2m}^{j+1} = \sigma_j \left(Z_m^j \left(n-j-1, z_{2^j+(m-1)}\right)\right). \tag{4.101}$$

Using the $(n-1)^{\text{th}}$-generation of elements of the auxiliary sequence $\widetilde{\Psi}_k\left(\overline{X}\right)$, we define the *rake sequence* $\Psi_k\left(\overline{X}\right)$ as follows.

**Definition 4.8.2.** *If* $X \in \mathbf{S}_{k,1}^n$, $\overline{X} = X \times Z_X$, *and* $\widetilde{\Psi}\left(\overline{X}\right)$ *as above, then let* $\Psi_k\left(\overline{X}\right) = \{X^s\}_{s=1}^{2^{n-1}}$ *be the sequence of elements in* $H^{d+2}$ *such that*

$$X^s = Z_s^{n-1}, \text{ for } 1 \leq s \leq 2^{n-1}. \tag{4.102}$$

For example, if $d = 3$ and $n = 3$, then the sequences $\widetilde{\Psi}_k\left(\overline{X}\right)$ and $\Psi_k\left(\overline{X}\right)$ fit into the following tree:

$$\begin{tikzcd}[column sep=small]
 & & Z_1^2 = (x_0, x_1, z_2, z_1, x_4) = X^1 \\
 & Z_1^1 = (x_0, x_1, x_2, z_1, x_4) \arrow[ur] \arrow[dr] & \\
 & & Z_2^2 = (x_0, x_2, z_2, z_1, x_4) = X^2 \\
X = (x_0, x_1, x_2, x_3, x_4) \arrow[uur] \arrow[ddr] & & \\
 & & Z_3^2 = (x_0, x_1, z_3, z_1, x_4) = X^3 \\
 & Z_2^1 = (x_0, x_1, x_3, z_1, x_4) \arrow[ur] \arrow[dr] & \\
 & & Z_4^2 = (x_0, x_3, z_3, z_1, x_4) = X^4
\end{tikzcd}$$

The following lemma establishes that all elements of $\Psi_k\left(\overline{X}\right)$ are single-handled rakes. We include the details of its proof in Appendix 5.2.7.

**Lemma 4.8.1.** *If $X \in \mathbf{S}^n_{k,1}$, $\overline{X} = X \times Z_X$, and $X^s \in \Psi_k\left(\overline{X}\right)$ with $1 \le s \le 2^{n-1}$, then there exists an integer $\widetilde{k} = \widetilde{k}(X,s)$ with $0 \le \widetilde{k} \le k-1$ such that*

$$\max\nolimits_{x_0}(X^s) = \|x_0 - (X^s)_1\| \in \left(\alpha_0^{\widetilde{k}+1} \cdot \max\nolimits_{x_0}(X), \alpha_0^{\widetilde{k}} \cdot \max\nolimits_{x_0}(X)\right]. \tag{4.103}$$

*Furthermore, the $X^s$ have the following form:*

$$(X^s)_i = \begin{cases} (X)_i, & \text{for } i = 0 \text{ or } n+1 \le i \le d+1; \\ (X)_{i_s}, & \text{for } i = 1 \text{ and } 1 \le i_s \le n; \\ (Z)_{\ell_i(s)}, & \text{for } 2 \le i \le n \text{ and } 1 \le \ell_i(s) \le N_n - 1. \end{cases} \tag{4.104}$$

*Consequently, each $X^s$ is a rake, and for $0 \le k' = k - \widetilde{k} - 1 \le k-1$, we have that*

$$X^s \in S^1_{k',2}(\widetilde{\eta}). \tag{4.105}$$

**Augmented Sets and a Discrete "Pre-Multiscale" Inequality for the Polar Sine**

Using the constant $C_{\mathrm{p}}$ of equation (1.20), we define the set augmentation of $\mathbf{S}^n_{k,1}$, denoted by $\overline{\mathbf{S}^n_{k,1}}$, as

$$\overline{\mathbf{S}^n_{k,1}} = \Big\{ \overline{X} \in \mathbf{S}^n_{k,1} \times [\mathrm{supp}(\mu)]^{N_n} : \text{ for all } 0 \le j < n-1 \text{ and } 1 \le m \le 2^j, \text{ the sequence}$$
$$\widetilde{\Psi}_k\left(\overline{X}\right) \text{ satisfies: } \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^j_m\right) \le C_{\mathrm{p}}\left[\mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m-1}\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m}\right)\right] \Big\}. \tag{4.106}$$

If $Q$ is a ball in $H$, then we let $\overline{\mathbf{S}^n_{k,1}(Q)} = \left\{\overline{X} \in \overline{\mathbf{S}^n_{k,1}} : X \in \mathbf{S}^n_{k,1}(Q)\right\}$.

The sequence $\Psi_k\left(\overline{X}\right)$ gives rise to the following pre-multiscale inequality.

**Lemma 4.8.2.** *If $\overline{X} \in \overline{\mathbf{S}^n_{k,1}}$, then the elements $X^s \in \Psi_k(\overline{X}) = \{X^s\}_{s=1}^{2^{n-1}}$ satisfy the inequality*

$$\mathrm{p}_d\mathrm{sin}^2_{x_0}(X) \le 2^{n-1} \cdot C_{\mathrm{p}}^{2\cdot(n-1)} \sum_{s=1}^{2^{n-1}} \mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s).$$

The details of its proof are in Appendix 5.2.8.

### 4.8.2 Forming a Pre-multiscale Integral Inequality

Here we formulate an integral inequality following the basic ideas of Subsection 4.7.1.

#### Truncations and Projections of $\overline{X}$

Fix $0 \leq s \leq N_n$. If $\overline{X} = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}) \in \overline{\mathbf{S}^n_{k,1}}$, then let the $s^{\text{th}}$ truncation of $\overline{X}$ be the function

$$T_s\left(\overline{X}\right) = \begin{cases} X, & \text{if } s = 0; \\ (x_0, \ldots, x_{d+1}, z_1, \ldots, z_s), & \text{if } 1 \leq s \leq N_n. \end{cases} \tag{4.107}$$

We denote the pre-image of $T_s\left(\overline{X}\right) = (x_0, \ldots, z_s)$ by

$$T_s^{-1}(x_0, \ldots, z_s) = \left\{ \overline{X}' \in \overline{\mathbf{S}^n_{k,1}} : \; T_s\left(\overline{X}'\right) = (x_0, \ldots, z_s) \right\}, \tag{4.108}$$

where if $s = 0$, the symbol $(x_0, \ldots, z_s)$ is taken to mean $X$.

For $\overline{X} = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}) \in \overline{\mathbf{S}^n_{k,1}}$ and $1 \leq s \leq N_n$, we define the $s^{th}$-projection of $\overline{X}$ onto $H$ by the formula

$$\pi_s\left(\overline{X}\right) = z_s. \tag{4.109}$$

This notation is distinguished from the previous notation of Subsection 4.7.2 by the use of the index $s$ and the overbar, as opposed to the index $q$ and the underbar. We will frequently use the sets $\pi_s\left(T_{s-1}^{-1}(x_0, \ldots, z_{s-1})\right)$, where $1 \leq s \leq N_n$. Such sets comprise all possible $s^{\text{th}}$ coordinates of the short-scale pieces $Z_X$ such that $\overline{X}' = X \times Z_X \in T_{s-1}^{-1}(x_0, \ldots, z_{s-1})$.

#### A Pre-Multiscale Integral Inequality

Just as in Subsection 4.7.2 we define the functions

$$g^n_{k,s}\left(\overline{X}\right) = \mu\left(\pi_s\left(T_{s-1}^{-1}\left(T_{s-1}(\overline{X})\right)\right)\right), \text{ for all } 1 \leq s \leq N_n, \tag{4.110}$$

and then define

$$f_k^n\left(\overline{X}\right) = \prod_{s=1}^{N_n} g_{k,s}^n\left(\overline{X}\right). \tag{4.111}$$

For completeness we include the proof of the following estimate in Appendix 5.2.9.

**Proposition 4.8.1.** *If $\overline{X} \in \overline{\mathbf{S}_{k,1}^n}$ and $1 \leq s \leq N_n$, then*

$$\mu\left(B(x_0, \alpha_0^k \cdot \max_{x_0}(X))\right) \geq g_{k,s}^n\left(\overline{X}\right) \geq \frac{1}{2} \cdot \mu\left(B(x_0, \alpha_0^k \cdot \max_{x_0}(X))\right). \tag{4.112}$$

Again, the function $f_k^n$ is measurable, and similar to Proposition 4.7.3 we have the following bound.

**Proposition 4.8.2.** *If $Q$ is a ball in $H$, then*

$$\int_{\mathbf{S}_{k,1}^n(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X)}{\operatorname{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X) \leq 2^{n-1} \cdot C_{\mathrm{p}}^{2\cdot(n-1)} \sum_{s=1}^{N_n} \int_{\overline{\mathbf{S}_{k,1}^n(Q)}} \frac{\mathrm{p}_d \sin_{x_0}^2(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}.$$

Furthermore, if one does not take the measurability of $f_k^n$ for granted, then it is possible to use a similar strategy to that given in Remark 4.7.1.

### 4.8.3 Generating Multiscale Discrete and Integral Inequalities

The individual integrals on the RHS of Proposition 4.8.2 are similar to the integral on the LHS of Proposition 4.6.2, mainly because the argument $X^s$ is a rake. In principle, one would like to change variables in order to directly apply Proposition 4.6.2 to these integrals. We avoid this because the region $\overline{\mathbf{S}_{k,1}^n}$ is complicated, and any change of variables would be further obstructed by our normalization using the function $f_k^n$. We find it more straightforward to adapt the methods of Section 4.7 to the individual terms on the RHS of Proposition 4.8.2.

Throughout the rest of this section, we fix $1 < n \leq d-1$, $k \geq 3$, and $1 \leq s \leq N_n$.

### The Decomposition of $\overline{\mathbf{S}^n_{k,1}}$ According to the Configuration of $X^s$

We first decompose $\overline{\mathbf{S}^n_{k,1}}$ according to the various configurations of the rake $X^s$, the only such variation being length of the handle at the coordinate $(X^s)_1 = x_{i_s}$, i.e., the quantity $\|x_{i_s} - x_0\|$. Let

$$\widehat{R}^s = \left\{\overline{X} \in \overline{\mathbf{S}^n_{k,1}} : X^s \text{ is well-scaled at } x_0\right\},$$

and, if $2 \leq k' \leq k-1$, then let

$$R^s_{k'} = \left\{\overline{X} \in \overline{\mathbf{S}^n_{k,1}} : X^s \in \mathbf{S}^1_{k',2}\right\}.$$

Furthermore, if $Q$ is a ball in $H$, then we localize these sets as usual by taking $\widehat{R}^s(Q)$ and $R^s_{k'}(Q)$. We note the following set equality

$$\overline{\mathbf{S}^n_{k,1}(Q)} = \widehat{R}^s(Q) \bigcup \left[\bigcup_{k'=2}^{k-1} R^s_{k'}(Q)\right]. \tag{4.113}$$

This decomposition yields the following integral inequality

$$\int_{\overline{\mathbf{S}^n_{k,1}(Q)}} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} \leq$$
$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} + \sum_{k'=2}^{k-1} \int_{R^s_{k'}(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)}. \tag{4.114}$$

The first term on the RHS of equation (4.114) can be controlled by $J^{\mathcal{D}}_d(\mu|_Q)$ via geometric multipoles (see Proposition 4.8.4 below). The sum on the RHS of equation (4.114) requires further analysis before such an application. We develop this in the rest of the section.

### Doubly-Augmented Elements and Another Discrete Multiscale Inequality

We fix $2 \leq k' \leq k-1$ and concentrate on the set $R^s_{k'}$. The integral over the augmented region $R^s_{k'}$ can be exchanged for yet another augmented integral, but in order to construct the appropriate augmented integral, we must perform two different types of augmentations. The first element is defined as follows. If $\overline{X} \in R^s_{k'}$, then we take a well-scaled piece for the rake $X^s \in S^1_{k',2}(\widetilde{\eta})$,

$$Y_{X^s} = (y_1, \ldots, y_{k' \cdot d}) \in \prod_{q=1}^{k' \cdot d} A_{k' - \lceil \frac{q}{d} \rceil}(x_0, \max\nolimits_{x_0}(X^s)),$$

and we form the "doubly-augmented" element

$$\overline{X} \times Y_{X^s} = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}, y_1, \ldots, y_{k' \cdot d}) \in R^s_{k'} \times H^{k' \cdot d}.$$

We also use another type of augmentation. If $\overline{X} \in R^s_{k'}$ and $Y_{X^s}$ is a well-scaled piece for $X^s$, then we form the augmented element

$$\underline{X^s} = X^s \times Y_{X^s} \in S^1_{k',2}(\widetilde{\eta}) \times H^{k' \cdot d}.$$

We note that $\underline{X^s}$ is an augmentation of the type introduced in Subsection 4.7.1, and is an orthogonal projection of the doubly-augmented element $\overline{X} \times Y_{X^s}$. For an element $\underline{X^s}$, we form the parameterized sequences

$$\widetilde{\Phi}_{k'}(\underline{X^s}) = \left\{ \widetilde{X}^s_q \right\}_{q=0}^{k' \cdot d} \quad \text{and} \quad \Phi_{k'}(\underline{X^s}) = \left\{ X^s_q \right\}_{q=1}^{k' \cdot d + 1}$$

as given in Definitions 4.7.1 and 4.7.2 of Subsection 4.7.1. We note that these sequences are functions of the element $\overline{X} \times Y_{X^s}$, but only depend on the element $\underline{X^s}$. We define the set augmentation

$$\underline{R^s_{k'}} := \Bigg\{ \overline{X} \times Y_{X^s} : \overline{X} \in R^s_{k'} \text{ and the sequences } \widetilde{\Phi}_{k'}(\underline{X^s}) \text{ and } \Phi_{k'}(\underline{X^s}) \text{ satisfy the inequality}$$
$$\mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}^s_q\right) \leq C_{\mathrm{p}} \cdot \left[\mathrm{p}_d\mathrm{sin}_{x_0}\left(X^s_{q+1}\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}^s_{q+1}\right)\right] \text{ for all } 0 \leq q < k' \cdot d \Bigg\}. \quad (4.115)$$

We have the following inequality which is a direct application of Lemma 4.7.2.

**Lemma 4.8.3.** *If $\overline{X} \times Y_{X^s} \in \underline{R^s_{k'}}$, then the well-scaled sequence $\Phi_{k'}(\underline{X^s}) = \left\{X^s_q\right\}_{q=1}^{k'\cdot d+1}$ satisfies the inequality*

$$\mathrm{p}_d\sin^2_{x_0}(X^s) \leq (k' \cdot d + 1) \cdot C_{\mathrm{p}}^{2\cdot k'\cdot d} \sum_{q=1}^{k'\cdot d+1} \mathrm{p}_d\sin^2_{x_0}\left(X^s_q\right).$$

### A Final Multiscale Integral Inequality

Here we formulate another multiscale integral inequality. First, we clarify notation in order to apply the methods of Subsection 4.7.2 to Lemma 4.8.3. Since we use two types of augmented elements, $\overline{X} \times Y_{X^s}$ and $\underline{X^s}$, we must distinguish between the different types of truncations and projections.

If $0 \leq i \leq N_n$, then let

$$T_i\left(\overline{X} \times Y_{X^s}\right) = T_i\left(\overline{X}\right), \tag{4.116}$$

where $T_i\left(\overline{X}\right)$ is defined in equation (4.107). The truncation $T_i$ is independent of $s$. Furthermore, if $1 \leq i \leq N_n$, then let $\pi_i\left(\overline{X} \times Y_{X^s}\right) = z_i$.

If $0 \leq q \leq k' \cdot d$ and $\overline{X} \times Y_{X^s} = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}, y_1, \ldots, y_{k'\cdot d})$, then let

$$T_{s,q}\left(\overline{X} \times Y_{X^s}\right) = \begin{cases} \overline{X}, & \text{if } q = 0; \\ (x_0, \ldots, z_{N_n}, y_1, \ldots, y_q), & \text{if } 1 \leq q \leq k' \cdot d. \end{cases} \tag{4.117}$$

If $1 \leq q \leq k' \cdot d$, then let $\pi_{s,q}\left(\overline{X} \times Y_{X^s}\right) = y_q$. Recalling the truncation and projection, $T_q$ and $\pi_q$, defined in equations (4.107) and (4.109), we note that the following set equality is trivially satisfied for all $1 \leq q \leq k' \cdot d$ and $\overline{X} \times Y_{X^s} \in \underline{R^s_{k'}}$:

$$\pi_{s,q}\left(T^{-1}_{s,q-1}\left(T_{s,q-1}\left(\overline{X} \times Y_{X^s}\right)\right)\right) = \pi_q\left(T^{-1}_{q-1}\left(T_{q-1}\left(\underline{X^s}\right)\right)\right). \tag{4.118}$$

Using equation (4.118), it is natural to extend the function $g^d_{k',q}$ of equation (4.62) as follows:

$$g^d_{k',q}\left(\overline{X} \times Y_{X^s}\right) = g^d_{k',q}\left(\underline{X^s}\right) = \mu\left(\pi_q\left(T^{-1}_{q-1}\left(T_{q-1}\left(\underline{X^s}\right)\right)\right)\right), \text{ for all } 1 \leq q \leq k' \cdot d, \tag{4.119}$$

and similarly extend the function $f^d_{k'}$ of equation (4.63) in the following way

$$f^d_{k'}\left(\overline{X}\times Y_{X^s}\right) = f^d_{k'}\left(\underline{X^s}\right) = \prod_{q=1}^{k'\cdot d} g^d_{k',q}\left(\underline{X^s}\right). \tag{4.120}$$

Since $\underline{X^s}$ is a continuous function of $\overline{X}\times Y_{X^s}$, $f^d_{k'}\left(\underline{X^s}\right)$ is a positive measurable function on $\underline{R^s_{k'}}$.

With these definitions, we formulate our ultimate multiscale integral inequality whose proof is hardly different from the proof of Proposition 4.8.2. As such we don't show it.

**Lemma 4.8.4.** *If $Q$ is a ball in $H$ and $X^s_q \in \Psi_{k'}\left(\underline{X^s}\right)$, then*

$$\int_{R^s_{k'}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} \leq$$
$$C_{\mathrm{p}}^{2k'\cdot d}\cdot(k'\cdot d+1)\sum_{q=1}^{k'\cdot d+1}\int_{\underline{R^s_{k'}(Q)}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)}.$$

Combining Lemma 4.8.4 with equation (4.114) directly implies the following bound.

**Proposition 4.8.3.** *If $Q$ is a ball in $H$, then*

$$\int_{\underline{\mathbf{S}^n_{k,1}(Q)}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) \;\leq\; \int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} +$$
$$\sum_{k'=2}^{k-1}\left[(k'\cdot d+1)\cdot C_{\mathrm{p}}^{2\cdot k'\cdot d}\sum_{q=1}^{k'\cdot d+1}\int_{\underline{R^s_{k'}(Q)}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)}\right]. \tag{4.121}$$

### 4.8.4 Concluding the Proof of Proposition 4.6.3

Applying geometric multipoles to the individual terms on the RHS of equation (4.121) results in the following bounds, whose combination with Proposition 4.8.2 and Lemma 4.8.4 establish Proposition 4.6.3. For completeness, we include the details of proof in Appendices 5.2.10 and 5.2.11.

**Proposition 4.8.4.** *If $Q$ is a ball in $H$, then there exists a constant $C_{13} = C_{13}(d, C_\mu, \alpha_0)$ such that*

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq C_{13} \cdot \alpha_0^{k \cdot d \cdot (d-n+2)} \cdot J_d^{\mathcal{D}}(\mu|_Q)\,. \tag{4.122}$$

**Proposition 4.8.5.** *If $Q$ is a ball in $H$ and $2 \leq k' \leq k-1$, then there exists a constant $C_{14} = C_{14}(d, C_\mu, \alpha_0)$ such that*

$$\int_{\underline{R^s_{k'}(Q)}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n + k' \cdot d}(\overline{X} \times Y_{X^s})}{f_k^n\left(\overline{X}\right) \cdot f_{k'}^d\left(\underline{X^s}\right)} \leq C_{14} \cdot \alpha_0^{k \cdot d \cdot (d-n+1)} \cdot J_d^{\mathcal{D}}(\mu|_Q)\,. \tag{4.123}$$

# Chapter 5

# Appendix

## 5.1 Appendix for Chapter 2

### 5.1.1 Proof of Proposition 2.2.1

For $\dim(H) > d+1$, the content functions $M_d$ and $M_{d+1}$, and the norm $\|\cdot\|$ are orthogonally invariant and thus $\mathrm{p}_d\mathrm{sin}_0$ and $\mathrm{g}_d\mathrm{sin}_0$ are orthogonally invariant. Moreover, in this case, $M_d$ and $M_{d+1}$ as well as the norm $\|\cdot\|$ scale linearly. That is, for all $1 \leq j \leq d+1$ and $\{\beta_i\}_{i=1}^{d+1}$ such that $\beta_i \neq 0$, where $1 \leq i \leq d+1$:

$$M_d(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1}) = \prod_{i \neq j} |\beta_i| \cdot M_d\left(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d+1}\right),$$

$$M_{d+1}(\beta_1 v_1, \ldots, \beta_{d+1} v_{d+1}) = \prod_{i=1}^{d+1} |\beta_i| \cdot M_{d+1}(v_1, \ldots, v_{d+1}),$$

and $\|\beta_i v_i\| = |\beta_i| \cdot \|v_i\|$. One can then observe that both the numerator and denominator of $|\,\mathrm{p}_d\mathrm{sin}_0\,|$ and $|\,\mathrm{g}_d\mathrm{sin}_0\,|$ scale similarly and thus the latter functions are invariant under nonzero dilations. Similarly, the proposition is satisfied when $\dim(H) = d+1$. $\square$

### 5.1.2 On the positivity of the coefficients $\{Q_i\}_{i=1}^{d+1}$ defined by equation (2.22)

We show here that the numerators and denominators of the terms $Q_i$, $1 \leq i \leq d+1$, defined by equation (2.22), have the same signs and thus conclude that these terms are positive.

For $1 \leq i \neq j \leq d+1$ we have that

$$\begin{aligned}
&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, 0, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})] \\
&\quad = \mathrm{sign}[\det(\beta_1 v_1 - \tilde{u}, \ldots, \beta_{j-1} v_{j-1} - \tilde{u}, -\tilde{u}, \beta_{j+1} v_{j+1} - \tilde{u}, \ldots, \beta_{d+1} v_{d+1} - \tilde{u})] \\
&\quad = -\,\mathrm{sign}[\det(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})]\,.
\end{aligned}$$

By the same calculation we also see that

$$\begin{aligned}
&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{i-1} v_{i-1}, 0, \beta_{i+1} v_{i+1}, \ldots, \beta_{d+1} v_{d+1})] \\
&\quad = -\,\mathrm{sign}[\det(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})]\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\tilde{u}}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, 0, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})] \\
&= \mathrm{sign}[\mathrm{g}_d\mathrm{sin}_{\beta_i v_i}(\beta_1 v_1, \ldots, \beta_{j-1} v_{j-1}, \tilde{u}, \beta_{j+1} v_{j+1}, \ldots, \beta_{i-1} v_{i-1}, 0, \beta_{j+1} v_{j+1}, \ldots, \beta_{d+1} v_{d+1})]\,,
\end{aligned}$$

and the claim is concluded.

### 5.1.3 Proofs of Proposition 2.5.1 and Corollary 2.5.1

We verify here Proposition 2.5.1 and Corollary 2.5.1. We first notice that Corollary 2.5.1 is an immediate consequence of Proposition 2.5.1 since whenever $x \in L$ we have that

$$\mathrm{C}_{\mathrm{one}}(\theta, L, x) \subseteq \mathrm{T}_{\mathrm{ube}}(L, \sin(\theta) \cdot r)\,.$$

Proposition 2.5.1 can be concluded from the following lemma:

**Lemma 5.1.1.** *The set* $\operatorname{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)$ *can be covered by* $N$ *balls of radius* $2 \cdot \sqrt{2} \cdot \epsilon \cdot r$, *such that*

$$N \leq \frac{(1+\epsilon)^m}{\epsilon^m} \leq \frac{2^m}{\epsilon^m} \,. \tag{5.1}$$

*Proof.* We choose a set $\{y_i\}_{i=1}^N$ in $\operatorname{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r)$, which is maximally separated by distances $2 \cdot \sqrt{2} \cdot \epsilon \cdot r$. That is,

$$\{y_i\}_{i=1}^N \subseteq \operatorname{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r) \,, \tag{5.2}$$

$$\|y_i - y_j\| > 2 \cdot \sqrt{2} \cdot \epsilon \cdot r \,, \quad \text{for } 1 \leq i < j \leq N \,, \tag{5.3}$$

and

$$\operatorname{supp}(\mu) \cap \mathrm{T}_{\mathrm{ube}}(L, \epsilon \cdot r) \cap B(x, r) \subseteq \bigcup_{i=1}^N B(y_i, 2 \cdot \sqrt{2} \cdot \epsilon \cdot r) \,. \tag{5.4}$$

We denote $z_i := P_L(y_i)$, $i = 1, \ldots, N$, that is, $z_i$ is the projection of the point $y_i$ onto the $m$-dimensional affine plane $L$. Equations (5.2) and (5.3) imply that $\{z_i\}_{i=1}^N$ are separated by distances $2 \cdot \epsilon \cdot r$. Consequently, the balls $\{B(z_i, \epsilon \cdot r)\}_{i=1}^N$ are disjoint and $\{z_i\}_{i=1}^N \subseteq L \cap B(x, r)$.

We denote by $\mathcal{H}_m$ the $m$-dimensional Hausdorff measure restricted to $L$, and recall that in our case $\mathcal{H}_m$ is a scaled Lebesgue measure on $L$, such that for any ball $B \subseteq L$, $\mathcal{H}_m(B) = (\operatorname{diam}(B))^m$. We thus obtain that

$$\begin{aligned} N \cdot (2 \cdot r \cdot \epsilon)^m &= \sum_{i=1}^N \mathcal{H}_m\left(B\left(z_i, \epsilon \cdot r\right)\right) = \mathcal{H}_m\left(\bigcup_{i=1}^N B\left(z_i, \epsilon \cdot r\right)\right) \\ &\leq \mathcal{H}_m\left(B\left(x, (1+\epsilon) \cdot r\right)\right) = 2^m \cdot (1+\epsilon)^m \cdot r^m \,. \end{aligned} \tag{5.5}$$

Equation (5.1) follows directly from equation (5.5) and thus the lemma is concluded. ☐

In order to conclude Proposition 2.5.1 we note that equation (5.4) and the definition of an Ahlfors regular measure imply that

$$\mu\left(\mathrm{T_{ube}}(L, \epsilon \cdot r) \cap B(x, r)\right) \;\leq\; \sum_{i=1}^{N} \mu\left(B(y_i, 2 \cdot \sqrt{2} \cdot \epsilon \cdot r)\right) \;\leq\; C_\mu \cdot N \cdot 2^{\frac{3\gamma}{2}} \cdot \epsilon^\gamma \cdot r^\gamma. \quad (5.6)$$

Then, combining equations (5.1) and (5.6), we conclude equation (2.49).

## 5.2 Appendix for Chapter 4

### 5.2.1 Proof of Proposition 4.2.1

Proposition 4.2.1 follows from the two observations:

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq \frac{2 \cdot (d+1)}{\mathrm{scale}_{x_0}(X)} \cdot \frac{h(X)}{\mathrm{diam}(X)}\,, \quad (5.7)$$

and

$$h(X) \leq \sqrt{2} \cdot \left\lceil \frac{d+1}{2} \right\rceil \cdot D_2(X, L), \quad \text{for any } d\text{-plane } L. \quad (5.8)$$

The two equations follow from elementary geometric estimates as follows.

**Proof of Equation** (5.7)

We first note that

$$\max_{0 \leq i \leq d+1} \mathrm{M}_d(X(i)) \leq\; (d+1) \cdot \max_{1 \leq i \leq d+1} \mathrm{M}_d(X(i)).$$

Indeed, if the maximum on the LHS of the above equation is obtained at $1 \leq i \leq d+1$, then the above inequality is trivial. If on the other hand this maximum is obtained at $i = 0$, then the inequality follows from the fact that the $d$-content of a face of a $(d+1)$-simplex is less than the sum of the $d$-contents of the other faces (this is since the $d$-content does not increase under projections and is subadditive on $\mathbb{R}^d$).

Then, using the fact that the product of any height of a $(d+1)$-simplex with the $d$-content of the opposite side is a constant (proportional to the $(d+1)$-content of the simplex), we obtain that

$$\min_{1\leq i\leq d+1} h_{x_i}(X) \cdot \max_{1\leq i\leq d+1} \mathrm{M}_d(X(i)) = h(X) \cdot \max_{0\leq i\leq d+1} \mathrm{M}_d(X(i)).$$

Combining the last two equations we deduce the inequality

$$\min_{1\leq i\leq d+1} h_{x_i}(X) \leq (d+1) \cdot h(X)\,. \tag{5.9}$$

Next, by equation (**??**), Proposition 3.1.1, and also equation (**??**) we obtain that

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq \min_{1\leq i\leq d+1} \frac{h_{x_i}(X)}{\|x_i - x_0\|} \leq \frac{\min_{1\leq i\leq d+1} h_{x_i}(X)}{\min_{x_0}(X)}.$$

Applying equation (5.9) to the RHS above, we have that

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq (d+1) \cdot \frac{h(X)}{\min_{x_0}(X)}.$$

Finally, applying the definition of $\mathrm{scale}_{x_0}(X)$ as well as the bound: $\mathrm{diam}(X) \leq 2 \cdot \max_{x_0}(X)$ to the latter equation establishes equation (5.7), and consequently the current proposition.

**Proof of Equation** (5.8)

We may assume that $X$ is non-degenerate, because otherwise $h(X) = 0$ and the bound holds trivially. Furthermore, since orthogonal projection decreases distances and reduces dimension of subspaces, we may assume that $\dim(H) = d+1$, in particular, $H = \mathbb{R}^{d+1}$. We first establish the bound

$$w(X) \leq \sqrt{2} \cdot D_2(X, L) \;\text{ for an arbitrary } d\text{-plane } L\,, \tag{5.10}$$

where $w(X)$ is the width of $S[X]$ defined in equation (4.12). We then apply the inequality on the RHS of equation (4.13) to conclude equation (5.8).

For a given $d$-plane $L$, let $L_1$ and $L_2$ be the two unique translates of $L$ supporting the simplex $S[X]$ and let $w_L(X)$ denote the distance between $L_1$ and $L_2$. Furthermore, let $x_{L_1}$ be a vertex of $X$ contained in $L_1$ and $x_{L_2}$ be a vertex contained in $L_2$. The $d$-planes $L_1$ and $L_2$ separate $\mathbb{R}^{d+1}$ into three regions, being the two disjoint half spaces of $\mathbb{R}^{d+1}$ and the intermediate region bounded by the $d$-planes whose closure contains $S[X]$.

If $L$ is contained in one of the disjoint half spaces described above, then we may assume that $L_1$ sits between $L$ and $L_2$. We establish equation (5.10) in this case as follows:

$$w(X) \leq w_L(X) = \operatorname{dist}(x_{L_2}, L_1) \leq \operatorname{dist}(x_{L_2}, L) \leq D_1(X, L).$$

In the second case, where the plane $L$ is contained in the intermediate region, we obtain equation (5.10) in the following way.

$$w(X) \leq w_L(X) = \operatorname{dist}(x_{L_1}, L) + \operatorname{dist}(x_{L_2}, L) \leq$$
$$\sqrt{2} \cdot \left(\operatorname{dist}^2(x_{L_1}, L) + \operatorname{dist}^2(x_{L_2}, L)\right)^{1/2} \leq \sqrt{2} \cdot D_2(X, L).$$

### 5.2.2 Proof of Lemma 4.3.1

The idea is to categorize the elements $B' \in \frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n]$ according to the first element of $\frac{1}{4} \cdot \mathcal{B}_n$ they intersect, and then use this to take the "appropriate" part of $B'$. Then, once this is done, for each $j \in \Lambda_n$ the element $P_{n,j}$ is formed by adding these appropriate pieces to the corresponding ball $\frac{1}{4} \cdot B_{n,j}$. We clarify this as follows.

If $\frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n] = \emptyset$, then we take the partition $\mathcal{P}_n = \{P_{n,j}\}_{j \in \Lambda_m}$, where for fixed $j \in \Lambda_n$

$$P_{n,j} = \operatorname{supp}(\mu) \cap \frac{1}{4} \cdot B_{n,j}, \text{ for } B_{n,j} \in \mathcal{B}_n.$$

We thus assume that $\frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n] \neq \emptyset$, and we index the elements of $\frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n]$ by the set $\Omega_n = \{1, 2, \ldots\}$, which is either finite or $\mathbb{N}$, i.e., $\frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n] = \{B'_m\}_{m \in \Omega_n}$. From this set of balls, we then recursively form the following sets. For $m=1$, let

$$\bar{B}'_1 = B'_1 \cap \left( \bigcup \frac{1}{4} \cdot \mathcal{B}_n \right)^c,$$

and for $m \geq 2$, let

$$\bar{B}'_m = B'_m \bigcap \left( \left[ \bigcup \frac{1}{4} \cdot \mathcal{B}_n \right] \bigcup \left[ \bigcup_{i=1}^{m-1} \bar{B}'_i \right] \right)^c.$$

Note that the elements of $\left\{ \bar{B}'_m \right\}_{m \in \Omega_n}$ are mutually disjoint, and that $\operatorname{supp}(\mu)$ is covered by the collection of sets

$$\frac{1}{4} \cdot \mathcal{B}_n \bigcup \left\{ \bar{B}'_m \right\}_{m \in \Omega}.$$

Let the function $g_n : \frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n] \to \Lambda_n$ be defined as follows:

$$g_n \left( B' \right) = \min \left\{ j \in \Lambda_n : \frac{1}{4} \cdot B_{n,j} \cap B' \neq \emptyset \right\}. \tag{5.11}$$

It follows from the maximality of $\frac{1}{4} \cdot \mathcal{B}_n$ that for every $B' \in \frac{1}{4} \cdot \mathcal{B}'_n$, there exists a $B \in \frac{1}{4} \cdot \mathcal{B}_n$ such that $B \cap B' \neq \emptyset$. Consequently, the minimum of equation (5.11) is obtained at an element of $\Lambda_n$, and taking

$$P_{n,j} = \operatorname{supp}(\mu) \bigcap \left( \frac{1}{4} \cdot B_{n,j} \bigcup \bigcup_{g_n(B'_m)=j} \bar{B}'_m \right), \tag{5.12}$$

we note that the sets $P_{n,j}$ are disjoint and cover $\operatorname{supp}(\mu)$. The desired set inclusions follow from the definition of $P_{n,j}$ and observing that $B' \subseteq \frac{3}{4} \cdot B_{n,j}$ for any $B' \in \frac{1}{4} \cdot [\mathcal{B}'_n \setminus \mathcal{B}_n]$ such that $g_n \left( B' \right) = j$.

### 5.2.3 Proof of Proposition 4.3.1

We show that for $n \geq m(Q)$, where $m(Q)$ is defined in equation (4.17), the following inequality holds:

$$\sum_{B \in \mathcal{B}_n(Q)} \beta_2^2(B) \cdot \mu(B) \leq C_1 \int_{5 \cdot \alpha_0^n}^{6 \cdot \alpha_0^n} \int_{6 \cdot Q} \beta_2^2(x,t) \, \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t}, \tag{5.13}$$

where $\mathcal{B}_n(Q)$ is defined in equation (4.19). From this we will conclude the proposition.

We fix $n \geq m(Q)$ and assume the collection $\mathcal{B}_n(Q)$ is non-empty, since otherwise equation (5.13) holds trivially. We then fix $B \in \mathcal{B}_n(Q)$ and note the set inclusion

$$B \subseteq B(x,t) \text{ for all } (x,t) \in \frac{1}{4} \cdot B \times (5 \cdot \alpha_0^n, 6 \cdot \alpha_0^n]\,.$$

Combining this equation with Lemma 1.4.1 and the definition of the beta numbers, we obtain that for all $t \in (5 \cdot \alpha_0^n, 6 \cdot \alpha_0^n]$ and for a.e. $x \in \frac{1}{4} \cdot B$:

$$\beta_2^2(B) \leq C_\mu^2 \cdot \left(\frac{3}{2}\right)^{d+2} \cdot \; \beta_2^2(x,t). \tag{5.14}$$

We want to turn the bound above on $\beta_2^2(B)$ into a bound on $\sum_{B \in \mathcal{B}_n(Q)} \beta_2^2(B) \cdot \mu(B)$. In order to do this, we note that the family $\frac{1}{4} \cdot \mathcal{B}_n(Q)$ is disjoint and furthermore

$$\bigcup \frac{1}{4} \cdot \mathcal{B}_n(Q) \subseteq 6 \cdot Q. \tag{5.15}$$

Combining equations (5.14) and (5.15), we obtain the following inequality for all $t \in (5 \cdot \alpha_0^n, 6 \cdot \alpha_0^n]$:

$$\sum_{B \in \mathcal{B}_n(Q)} \beta_2^2(B) \cdot \mu\left(\frac{1}{4} \cdot B\right) \leq C_\mu^2 \cdot \left(\frac{3}{2}\right)^{d+2} \int_{6 \cdot Q} \beta_2^2(x,t)\, \mathrm{d}\mu(x). \tag{5.16}$$

Furthermore, integrating in $t$ and applying Lemma 1.4.1 we obtain the bound of equation (5.13).

Finally, noting that $6 \cdot \alpha_0^{n+1} < 5 \cdot \alpha_0^n$ and $6 \cdot \alpha_0^n \leq 6 \cdot \operatorname{diam}(Q)$, for $n \geq m(Q)$, we have that the intervals of $t$ in equation (5.13) are disjoint and contained in $(0, 6 \cdot \operatorname{diam}(Q)]$. Therefore summing over $n \geq m(Q)$ establishes the desired bound.

### 5.2.4 Proof of Lemma 4.7.1

In order to show that the elements $X_q$ are well-scaled, we note that it is sufficient to prove the estimates of equations (4.57) and (4.58) in addition to the following two estimates:

$$\alpha_0^{k+p} \cdot \max_{x_0}(X) < \min_{x_0}(X_q) \leq \alpha_0^k \cdot \max_{x_0}(X), \quad \text{if} \;\; 1 \leq q \leq d, \tag{5.17}$$

and

$$\alpha_0^{k+2-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X) < \min_{x_0}(X_q) \leq \alpha_0^{k+1-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X), \quad \text{if} \;\; d+1 \leq q \leq k \cdot d. \tag{5.18}$$

Indeed, combining equations (4.57), (5.17) and (5.18), as well as the assumption that $p = 1, 2$, we have that

$$\alpha_0^3 < \frac{\min_{x_0}(X_q)}{\max_{x_0}(X_q)} \leq 1, \quad \text{for all} \;\; 1 \leq q \leq k \cdot d.$$

Furthermore equation (4.58) yields the estimate

$$\alpha_0 < \frac{\min_{x_0}(X_{k \cdot d+1})}{\max_{x_0}(X_{k \cdot d+1})} \leq 1.$$

Therefore, all of the elements $X_q$, $1 \leq q \leq k \cdot d + 1$, are all well-scaled at $x_0$.

We start by establishing the estimate of equation (4.58) since it is the most apparent of all of the estimates. By equations (4.52), (4.55), and (4.56) we have that

$$(X_{k \cdot d+1})_j = (Y_X)_{j-1+(k-1) \cdot d} \in A_0(x_0, \max_{x_0}(X)) \;\; \text{for all} \;\; 2 \leq j \leq d+1.$$

Since $(X_{k \cdot d+1})_1 = x_1$ and $\max_{x_0}(X) = \|x_0 - x_1\|$, equation (4.58) follows immediately.

Next we establish equation (5.17). Assume that $1 \leq q \leq d$. Since $X \in \mathbf{S}^1_{k,p}$ we have the estimate

$$\min_{x_0}(X_q) = \min_{q+1 \leq j \leq d+1} \|x_0 - x_j\| \in \left(\alpha_0^{k+p} \cdot \max_{x_0}(X), \alpha_0^k \cdot \max_{x_0}(X)\right], \tag{5.19}$$

which is equivalent to equation (5.17).

In order to conclude the proof, we verify the estimates of equations (4.57) and (5.18). If $1 \leq q \leq k \cdot d$ and $1 \leq j \leq \min\{q, d+1\}$, then we define the index function $I(q,j)$ such that

$$(Y_X)_{I(q,j)} = (X_q)_j \, .$$

We note that this index function is implicitly defined by equations (4.55)-(4.56), and that for fixed $q$, $1 \leq q \leq k \cdot d$, the function

$$I(q,\cdot) : [1, \min\{q, d+1\}] \cap \mathbb{N} \to [\max\{1, q-d\}, q] \cap \mathbb{N}$$

is surjective. Therefore, for fixed $q$ the function

$$\left\lceil \frac{I(q,\cdot)}{d} \right\rceil : [1, \min\{q, d+1\}] \cap \mathbb{N} \to \left\{ \max\left\{1, \left\lceil \frac{q}{d} \right\rceil - 1\right\}, \left\lceil \frac{q}{d} \right\rceil \right\} \tag{5.20}$$

is surjective. Furthermore, if $1 \leq q \leq k \cdot d$, then by equation (4.52) we have that

$$(Y_X)_{I(q,j)} \in A_{k - \left\lceil \frac{I(q,j)}{d} \right\rceil} \left(x_0, \max_{x_0}(X)\right), \quad \text{for all} \;\; 1 \leq j \leq \min\{q, d+1\}. \tag{5.21}$$

By combining equations (5.20) and (5.21), we get that for all $1 \leq q \leq k \cdot d$:

$$\max_{x_0}(X_q) = \max_{1 \leq j \leq d+1} \|x_0 - (Y_x)_{i(q,j)}\| \in (\alpha_0^{k+1-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X),\; \alpha_0^{k-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X)),$$

i.e., equation (4.57) holds.

Similarly, to establish equation (5.18), we note that if $d+1 \leq q \leq k \cdot d$, then by equations (5.20) and (5.21) we have the estimate

$$\begin{aligned} &\min_{x_0}(X_q) = \\ &\min_{1 \leq j \leq d+1} \|x_0 - (Y_X)_{I(q,j)}\| \in \left( \alpha_0^{k+1-(\lceil \frac{q}{d} \rceil - 1)} \cdot \max_{x_0}(X), \alpha_0^{k+1-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X) \right], \end{aligned} \tag{5.22}$$

which is equivalent to equation (5.18). $\square$

### 5.2.5 Proof of Lemma 4.7.2

We use induction to show that if $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$, then the corresponding auxiliary and well-scaled sequences satisfy the following inequality for any $1 \le q \le k \cdot d$:

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \le \sum_{j=1}^{q} C_{\mathrm{p}}^{j} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}(X_j) + C_{\mathrm{p}}^{q} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_q)\,. \tag{5.23}$$

By setting $q = k \cdot d$ and recalling that $\widetilde{X}_{k \cdot d} = X_{k \cdot d+1}$, we will then have the bound

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \le \sum_{q=1}^{k \cdot d} C_{\mathrm{p}}^{q \cdot d} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}(X_q) + C_{\mathrm{p}}^{k \cdot d} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}\left(X_{k \cdot d+1}\right). \tag{5.24}$$

Noting that $C_{\mathrm{p}} \ge 1$ and then applying the Cauchy-Schwartz inequality to the RHS of equation (5.24) establishes the lemma as a consequence of equation (5.23).

In order to prove equation (5.23), we use the definition of the set $\underline{\mathbf{S}^1_{k,p}}$ to see that for all $0 \le q < k \cdot d$:

$$\mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}_q\right) \le C_{\mathrm{p}}\left(\mathrm{p}_d\mathrm{sin}_{x_0}(X_{q+1}) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}_{q+1}\right)\right), \tag{5.25}$$

and apply this inequality recursively as follows. If $q = 0$, then equation (5.23) follows from equation (5.25) and the fact that $\widetilde{X}_0 = X$. We assume by induction that equation (5.23) is satisfied for some $0 \le q < k \cdot d$. Applying equation (5.25) to the last term on the RHS of equation (5.23), we get

$$\begin{aligned}\mathrm{p}_d\mathrm{sin}_{x_0}(X) &\le \sum_{j=1}^{q} C_{\mathrm{p}}^{j} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}(X_j) + C_{\mathrm{p}}^{q+1} \cdot \left(\mathrm{p}_d\mathrm{sin}_{x_0}(X_{q+1}) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}_{q+1}\right)\right) \\ &= \sum_{j=1}^{q+1} C_{\mathrm{p}}^{j} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}(X_j) + C_{\mathrm{p}}^{q+1} \cdot \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}_{q+1}\right).\end{aligned} \tag{5.26}$$

Hence equation (5.23) and thus the current lemma are proved.

### 5.2.6 Proof of Proposition 4.7.1

First, for any $X \in \mathbf{S}^1_{k,p}$, along with $Z \in [\mathrm{supp}(\mu)]^{d+2}$ such that $(Z)_0 = (X)_0 = x_0$, as well as $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$ and $1 \leq j \leq k \cdot d$, the following estimate holds

$$\mu\left(B(x_0, r)\right) \geq \mu\left(U_{C_{\mathrm{p}}}(Z, 1, \overline{j+1}) \bigcap A_0(x_0, r)\right) \geq \frac{1}{2} \cdot \mu\left(B(x_0, r)\right) > 0. \tag{5.27}$$

In other words, there is a sufficient amount of $\mathrm{supp}(\mu)$ in the annulus $A_0(x_0, r)$ satisfying the relaxed two term inequality defined by the set $U_{C_{\mathrm{p}}}(Z, 1, \overline{j+1})$. Furthermore, for such $X$ we have the inequality $\max_{x_0}(X) \leq \mathrm{diam}(X) \leq \mathrm{diam}(\mathrm{supp}(\mu))$, and we note that the radius

$$r = \alpha_0^{k - \lceil \frac{j}{d} \rceil} \cdot \max_{x_0}(X) \tag{5.28}$$

is in the above range for all $1 \leq j \leq k \cdot d$, i.e., $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$. Substituting this choice of radius into equation (5.27) we obtain the estimate

$$\mu\left(B(x_0, \alpha_0^{k - \lceil \frac{j}{d} \rceil} \cdot \max_{x_0}(X))\right) \geq \mu\left(U_{C_{\mathrm{p}}}\left(Z, 1, \overline{j+1}\right) \bigcap A_{k - \lceil \frac{j}{d} \rceil}(x_0, \max_{x_0}(X))\right) \geq \frac{1}{2} \cdot \mu\left(B(x_0, \alpha_0^{k - \lceil \frac{j}{d} \rceil} \cdot \max_{x_0}(X))\right). \tag{5.29}$$

Next, we will also show that the following set equality holds for any $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and $\widetilde{X}_{q-1}$, $1 \leq q \leq k \cdot d$, of the sequence $\widetilde{\Phi}_k(\underline{X})$:

$$U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1}, 1, \overline{q+1}\right) \bigcap A_{k - \lceil \frac{q}{d} \rceil}(x_0, \max_{x_0}(X)) = \pi_q\left(T^{-1}_{q-1}\left(x_0, \ldots, y_{q-1}\right)\right). \tag{5.30}$$

By taking $Z = \widetilde{X}_{q-1}$ in equation (5.29) and combining it with equation (5.30) we establish the proposition.

We start by proving equation (5.27). The inequality on the LHS of equation (5.27) is trivial since $A_0(x_0, r) \subseteq B(x_0, r)$ by definition. To prove the inequality on the RHS of equation (5.27) we note that

$$\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap A_0(x_0,r)\right)=\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right)+$$
$$\mu\left(A_0(x_0,r)\right)-\mu\left(\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right]\bigcup A_0(x_0,r)\right). \quad (5.31)$$

By formulating lower bounds for the terms on the RHS of the above equation we can then establish the inequality on the RHS of equation (5.27).

With the above assumptions on $X$, $Z$, and $r$, by Proposition **??** we have the inequality

$$\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\right)\cap B(x_0,r)\right)\geq\frac{3}{4}\cdot\mu(B(x_0,r)). \quad (5.32)$$

Furthermore, by equation (**??**) we have the inequality

$$\mu\left(A_0(x_0,r)\right)\geq\frac{3}{4}\cdot\mu(B(x_0,r)). \quad (5.33)$$

Noting the inclusion $\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right]\bigcup A_0(x_0,r)\subseteq B(x_0,r)$, we obtain the inequality

$$\mu\left(\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right]\bigcup A_0(x_0,r)\right)\leq\mu(B(x_0,r)).$$

Finally, applying this and equations (5.32) and (5.33) to the RHS of equation (5.31) we obtain

$$\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap A_0(x_0,r)\right)\geq\frac{1}{2}\cdot\mu(B(x_0,r)), \quad (5.34)$$

and thus conclude equation (5.27).

Next, for $\underline{X}\in\underline{\mathbf{S}^1_{k,p}}$ and a fixed $(x_0,\ldots,y_{q-1})=T_{q-1}(\underline{X})$, we establish equation (5.30) by proving the inclusion

$$U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1},1,\overline{q+1}\right)\bigcap A_{k-\lceil\frac{q}{d}\rceil}(x_0,\max_{x_0}(X))\subseteq\pi_q\left(T^{-1}_{q-1}\left(x_0,\ldots,y_{q-1}\right)\right). \quad (5.35)$$

The opposite inclusion follows directly from the definitions of the sets $U_{C_{\mathrm{p}}}(\widetilde{X}_{q-1},1,\overline{q+1})$, $1\leq q\leq k\cdot d$, and $\underline{\mathbf{S}^1_{k,p}}$ (see equations (**??**) and (4.59)).

Our approach to proving equation (5.35) is to fix $1 \leq q \leq k \cdot d$ and take an arbitrary point

$$y'_q \in U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1}, 1, \overline{q+1}\right) \bigcap A_{k-\lceil \frac{q}{d} \rceil}(x_0, \max_{x_0}(X)) \,. \tag{5.36}$$

We then iteratively use the inequality of equation (5.27) to construct an element

$$\underline{X}' = (x_0, \ldots, y_{q-1}, y'_q, \ldots, y'_{k \cdot d}) \in T_{q-1}^{-1}(x_0, \ldots, y_{q-1}) \,,$$

so that $y'_q = \pi_q(\underline{X}') \in T_q\left(\pi_{q-1}^{-1}(\pi_{q-1}(\underline{X}))\right)$.

Fixing $1 \leq q \leq k \cdot d$ and $y'_q$ satisfying equation (5.36), we recursively form the sequence $\{y'_i\}_{i=q+1}^{k \cdot d}$ together with additional elements of an auxiliary sequence $\left\{\widetilde{X}'_i\right\}_{i=q}^{k \cdot d}$ as follows. First we initialize the auxiliary sequence by defining

$$\widetilde{X}'_q = \widetilde{X}_{q-1}(y'_q, \overline{q+1}) \,.$$

Next, given $q+1 \leq i \leq k \cdot d$ and assuming that $\{y'_i\}_{i=q}^{i-1}$ and $\left\{\widetilde{X}'_i\right\}_{i=q}^{i-1}$ have already been defined, we fix arbitrarily

$$y'_i \in U_{C_{\mathrm{p}}}(\widetilde{X}'_{i-1}, 1, \overline{i+1}) \cap A_{k-\lceil \frac{i}{d} \rceil}(x_0, \max_{x_0}(X)) \,, \tag{5.37}$$

and form

$$\widetilde{X}'_i = \widetilde{X}'_{i-1}(y'_i, \overline{i+1}) \,.$$

We remark that this procedure is well defined since equation (5.27) implies that for each $q+1 \leq j \leq k \cdot d$:

$$\mu\left(U_{C_{\mathrm{p}}}\left(\widetilde{X}'_{j-1}, 1, \overline{j+1}\right) \bigcap A_{k-\lceil \frac{j}{d} \rceil}(x_0, \max_{x_0}(X))\right) > 0 \,.$$

That is, the right term of equation (5.37) is never empty.

Finally, forming

$$\underline{X}' = (x_0, \ldots, y_{q-1}, y'_q, \ldots, y'_{k\cdot d}) = X \times Y'_X \in H^{(k+1)\cdot d+2}\,,$$

we note that $Y'_X$ is a well-scaled element for $X$ and the elements of the sequences $\widetilde{\Phi}_k(\underline{X}') = \left\{\widetilde{X}_q\right\}_{q=0}^{k\cdot d}$ and $\Phi_k(\underline{X}') = \{X_q\}_{q=1}^{k\cdot d+1}$ satisfy the inequality

$$\mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_{j-1}) \leq C_{\mathrm{p}} \left(\mathrm{p}_d\mathrm{sin}_{x_0}\left(X_j\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}_j\right)\right).$$

Therefore, $\underline{X}' \in \underline{\mathbf{S}^1_{k,p}}$. Furthermore, $T_{q-1}(\underline{X}') = (x_0, \ldots, y_{q-1})$, and thus

$$\underline{X}' \in \pi^{-1}_{q-1}(\pi_{q-1}(x_0, \ldots, y_{q-1})).$$

Since $\pi_q(\underline{X}') = y'_q$, equation (5.35) and consequently equation (5.30) are now established.

### 5.2.7 Proof of Lemma 4.8.1

We will prove the following statement by induction.

If $X \in \mathbf{S}^n_{k,1}$ and $\overline{X} = X \times Z_X = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n})$, then for any $1 \leq j \leq n-1$, $1 \leq m \leq 2^j$, and $Z^j_m \in \widetilde{\Psi}_k\left(\overline{X}\right)$, we have that

$$Z^j_m = \left(x_0, x_{i_{n-j}}, \ldots, x_{i_1}, z_{\ell_1}, \ldots, \ldots, z_{\ell_j}, x_{n+1}, \ldots, x_{d+1}\right), \tag{5.38}$$

where the indices $i_b = i_b(j,m)$ and $\ell_t = \ell_t(j,m)$ are unique in $1 \leq b \leq n-j$ and $1 \leq t \leq j$, with ranges $1 \leq i_p \leq n$ and $1 \leq \ell_t \leq 2^{j-1} + \left(\left\lceil \frac{m}{2} \right\rceil - 1\right)$. We note our convention of taking $m = s$ when $j = n-1$. In particular, when $j = n-1$ we define

$$i_1(n-1, s) = i_s, \tag{5.39}$$

and we have that

$$X^s = (x_0, x_{i_s}, z_{\ell_1}, \ldots, z_{\ell_n}, x_{n+1}, \ldots, x_{d+1}). \tag{5.40}$$

We note that equation (4.103) follows from equation (5.40) and the definition of $\mathbf{S}^n_{k,1}$. Indeed,

$$\max_{x_0}(X^s) = \|x_0 - x_{i_s}\| \geq \min_{1\leq j\leq n} \|x_j - x_0\| > \alpha_0^k \cdot \max_{x_0}(X).$$

In order to obtain equation (4.105) we note that equation (5.40), together with the definition of the short-scale piece $Z_X$, and the definition of $S^n_{k,p}(\widetilde{\eta})$ give the following estimates:

$$\|x_0 - (X^s)_j\| \in \left(\alpha_0^{k+1} \cdot \max_{x_0}(X), \alpha_0^k \cdot \max_{x_0}(X)\right] \text{ for all } 2 \leq j \leq d+1.$$

Combining this with equation (4.103) we obtain equation (4.105).

We now establish equation (5.38) and the ranges for the indices $i_b$ and $\ell_t$. If $j = 1$, then according to Definition 4.8.1 we have that $Z^1_1 = X(n-1, z_1)$ and $Z^1_2 = \sigma_1\left(X(n-2, z_1)\right)$, and thus both equation (5.38) and the ranges for the indices follow by inspection.

Then, we assume that equation (5.38) and the ranges for the indices hold for some $1 \leq j < n-1$ and all $1 \leq m \leq 2^j$. Inspecting the pair of elements

$$Z^{j+1}_{2m-1} = Z^j_m(n-j, z_{2^j+(m-1)}), \text{ and } Z^{j+1}_{2m} = \sigma_j\left(Z^j_m(n-j-1, z_{2^j+(m-1)})\right)$$

from the $(j+1)^{st}$-generation of the sequence $\widetilde{\Psi}_k\left(\overline{X}\right)$, we see that these elements have exactly $j+1$ coordinates drawn from the short-scale piece $Z_X$ occupying the positions (within these two elements) with indices $n-j \leq l \leq n$. Furthermore, the highest index within the short-scale piece $Z_X$ of such coordinates is $2^j + (m-1)$, because by induction those of $Z^j_m$ have highest index $2^{j-1} + \left(\left\lceil \frac{m}{2} \right\rceil - 1\right)$. The other claims follow by inspecting the form of equation (5.38) for $Z^j_m$. Consequently, equation (5.38) holds for all $1 \leq j \leq n-1$ and $1 \leq m \leq 2^j$ by induction.

### 5.2.8 Proof of Lemma 4.8.2

We use induction to show that for any $\overline{X} \in \overline{S^n_{k,p}(\widetilde{\eta})}$ and $0 \le j \le n-1$, the $j^{th}$-generation of the auxiliary sequence $\widetilde{\Psi}_k\left(\overline{X}\right)$ satisfies the inequality:

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \le C_{\mathrm{p}}^j \sum_{m=1}^{2^j} \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^j_m\right). \tag{5.41}$$

Setting $j = n-1$ and applying the Cauchy-Schwartz inequality to the RHS of equation (5.41) will establish the lemma.

By the definition of $\overline{S^n_{k,p}(\widetilde{\eta})}$, for all $0 \le j < n-1$ and $1 \le m \le 2^j$ we have that

$$\mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^j_m\right) \le C_{\mathrm{p}}\left(\mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m-1}\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m}\right)\right). \tag{5.42}$$

If $j = 0$, then equation (5.41) holds trivially since $X = Z^0_1$. We assume by induction that equation (5.41) holds for some $0 \le j < n-1$. In this case, applying equation (5.42) to each of the terms on the RHS of equation (5.41) we obtain

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \le C_{\mathrm{p}}^{j+1} \sum_{m'=1}^{2^{j+1}} \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{m'}\right).$$

Therefore, by induction we conclude equation (5.41).

### 5.2.9 Proof of Proposition 4.8.1

We prove the inequality in exactly the same manner as we proved Proposition 4.7.1.

We rely on the fact (demonstrated in the proof of Proposition 4.7.1) that for any $Z \in H^{d+2}$ such that $(Z)_0 = x_0 \in \mathrm{supp}(\mu)$, as well as any $0 < r \le \mathrm{diam}(\mathrm{supp}(\mu))$, the following inequality holds:

$$\mu(B(x_0,r)) \ge \mu\left(U_{C_{\mathrm{p}}}(Z,t,\ell) \cap A_0(x_0,r)\right) \ge \frac{1}{2}\cdot\mu(B(x_0,r)), \text{ for all } 1 \le t < \ell \le d{+}1. \tag{5.43}$$

Then, we use this to show that

$$\pi_s\left(T_{s-1}^{-1}\left(T_{s-1}\left(\overline{X}\right)\right)\right) = U_{C_\mathrm{p}}\left(Z_m^j, n-j-1, n-j\right)\bigcap A_k(x_0, \max_{x_0}(X)). \tag{5.44}$$

Finally, combining equations (5.43) and (5.44) we will establish the proposition.

We note that one inclusion of equation (5.44) follows directly from the definition of the augmented set $\overline{\mathbf{S}_{k,1}^n}$. Now, to establish the opposite inclusion, we do exactly as we did in Proposition 4.7.1. By applying equation (5.43), we note that for any $Z_{m'}^{j'} \in \widetilde{\Psi}_{k,d}\left(\overline{X}\right)$ we have that

$$U_{C_\mathrm{p}}\left(Z_{m'}^{j'}, n-j-1, n-j\right)\bigcap A_k(x_0, \max_{x_0}(X)) \neq \emptyset. \tag{5.45}$$

We arbitrarily fix $(x_0, \ldots, z_{s-1}) = T_{s-1}\left(\overline{X}\right)$, $j$ and $m$ such that $2^j + (m-1) = s$, and

$$z_s' \in U_{C_\mathrm{p}}\left(Z_m^j, n-j-1, n-j\right)\bigcap A_k(x_0, \max_{x_0}).$$

Then, by the virtually the same construction done in Proposition 4.7.1 we iteratively construct a sequence of points $z_i' \in A_k(x_0, \max_{x_0}(X)) \cap \operatorname{supp}(\mu)$ such that

$$\overline{X}' = (x_0, \ldots, z_{s-1}, z_s', \ldots, z_i', \ldots, z_{N_n}') \in T_{s-1}^{-1}(x_0, \ldots, z_{s-1}).$$

This establishes the desired inclusion and hence the proposition.

### 5.2.10 Proof of Proposition 4.8.4

For fixed $m \geq m(Q)$ we define

$$\widehat{R}^s(m)(Q) = \left\{\overline{X} \in \widehat{R}^s(Q) : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}. \tag{5.46}$$

This gives the following decomposition of the integral

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} = \sum_{m \geq m(Q)} \int_{\widehat{R}^s(m)(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}. \tag{5.47}$$

We fix $m \geq m(Q)$ and we partition $\widehat{R}^s(m)(Q)$ to reflect the quantity $\max_{x_0}(X^s)$. In order to determine a suitable scale exponent (e.g., equation (4.72)), we do the following.

If $\overline{X} \in \widehat{R}^s(m)$, then Lemma 4.8.1 and the fact that $X^s$ is well-scaled implies that

$$\max_{x_0}(X^s) = \|x_0 - x_{i_s}\| \in \left(\alpha_0^k \cdot \max_{x_0}(X), \alpha_0^{k-3} \cdot \max_{x_0}(X)\right]. \tag{5.48}$$

According to equations (5.46) and (5.48), we use the following scale exponent:

$$\mathrm{sc}(m,k) = m + k - 3 \geq m(Q). \tag{5.49}$$

The family $\left\{P_{\mathrm{sc}(m,k),j}\right\}_{j \in \Lambda_{\mathrm{sc}(m,k)}(Q)}$ covers the product $[Q \cap \mathrm{supp}(\mu)]^{d+2}$, and we take the induced cover of $\widehat{R}^s(m)(Q)$ consisting of

$$\widehat{P}_{\mathrm{sc}(m,k),j} = \left\{\overline{X} \in \widehat{R}^s(m) : x_0 \in P_{\mathrm{sc}(m,k),j}\right\}, \text{ for all } j \in \Lambda_{\mathrm{sc}(m,k)(Q)}. \tag{5.50}$$

Using these covers, we obtain the inequality

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq \sum_{m \geq m(Q)} \left[\sum_{j \in \Lambda_{\mathrm{sc}(m,k)}(Q)} \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}\right]. \tag{5.51}$$

Fixing arbitrary $m \geq m(Q)$ and $j \in \Lambda_{\mathrm{sc}(m,k)}(Q)$ we will show that

$$\int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq C_{13} \cdot \alpha_o^{k \cdot d \cdot (d-n+2)} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k),j}\right) \cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right). \tag{5.52}$$

Equations (5.51) and (5.52) then imply the proposition.

We fix an arbitrary $d$-plane $L$. If $\overline{X} \in \widehat{P}_{\mathrm{sc}(m,k),j}$, then by the estimate of equation (5.48) coupled with the estimate on $\max_{x_0}(X)$, we obtain

$$\mathrm{diam}(X^s) \geq \alpha_0^{m+k+1} = \frac{\alpha_0^4}{8} \cdot \mathrm{diam}(B_{\mathrm{sc}(m,k),j}).$$

Applying this and Proposition 4.2.1 we get that

$$\int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} \leq$$

$$\frac{2^7 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha_0^{14}} \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{D_2^2\left(X^s, L\right)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right) \cdot \operatorname{diam}(X)^{d(d+1)}}. \tag{5.53}$$

To bound the RHS of equation (5.53) we focus on the individual terms of

$$\frac{D_2^2\left(X^s, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k),j}\right)} = \sum_{t=0}^{d+1} \left(\frac{\operatorname{dist}\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k),j}\right)}\right)^2.$$

We arbitrarily fix $0 \leq t \leq d+1$ and note the cases for the possible values of $\left(X^s\right)_t$. Per Lemma 4.8.1 we have the following cases.

Case 1: $(X^s)_t = x_0$, $x_{i_s}$, (see equation (4.104)), or $x_i$, where $n+1 \leq i \leq d+1$.

Case 2: $(X^s)_t = z_\ell$, where $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$.

Assume Case 1. After applying Fubini's theorem we obtain that the corresponding terms on the right hand side of equation (5.53) have the form

$$\int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right) \cdot \operatorname{diam}(X)^{d(d+1)}} =$$

$$\int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}}, \tag{5.54}$$

where $T_0$ is the truncation of equation (4.107). The definition of the set $\widehat{P}_{\mathrm{sc}(m,k),j}$, along with equation (5.48) and Lemma 4.8.1 imply the set inclusion

$$T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right) \subseteq \bigcup_{x_0 \in P_{\mathrm{sc}(m,k),j}} \left[\{x_0\} \times \prod_{i=1}^{d+1} B(x_0, \alpha_0^{p_i})\right], \tag{5.55}$$

where

$$p_i = \begin{cases} m+k-3, & \text{if } i = i_s, \text{ where } i_s \text{ is defined in equation (4.104);} \\ m+k, & \text{if } n+1 \leq i \leq d+1; \\ m, & \text{otherwise.} \end{cases}$$

Equation (5.55) then yields the following bound on the RHS of equation (5.54)

$$\int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2(B)} \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}} \leq$$
$$\frac{1}{\alpha_0^{d(d+1)}} \int_{P_{\mathrm{sc}(m,k),j}} \int_{\prod_{i=1}^{d+1} B(x_0,\alpha_0^{p_i})} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu(x_{d+1})\cdots\mathrm{d}\mu(x_0)}{[\alpha_0^m]^{d(d+1)}}. \quad (5.56)$$

Applying the usual calculations (see the proofs of Propositions 4.5.1 and 4.6.2) to the RHS of equation (5.56), we obtain the following bound on the RHS of equation (5.54)

$$\frac{3^d \cdot C_\mu^{d+1} \cdot \alpha_0^{k\cdot d\cdot(d-n+2)}}{\alpha_0^{d(d+1)+3\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right). \quad (5.57)$$

For Case 2, where the terms are of the form

$$\int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)},$$

for $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$, we iterate the integral and obtain the equality

$$\int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} =$$
$$\int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \left(\int_{\left\{Z_X : X\times Z_X \in \widehat{P}_{\mathrm{sc}(m,k),j}\right\}} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{N_n}(Z_X)}{f_k^n\left(\overline{X}\right)}\right) \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}}. \quad (5.58)$$

To bound the RHS of equation (5.58) we calculate a uniform bound for the inner integral, and then complete the integration over $T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)$. We arbitrarily fix $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$. The set inclusion of equation (5.55) implies that it is sufficient to bound

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu(z_\ell)}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right)},$$

where $(x_0,\ldots,z_{\ell-1}) \in T_{\ell-1}\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)$. Applying Proposition 4.8.1, equation (5.46), and then Lemma 1.4.1, we obtain

$$\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right) \geq \frac{1}{2}\cdot\mu\left(B(x_0, \alpha_0^k \cdot \max\nolimits_{x_0}(X))\right) \geq \frac{1}{2\cdot C_\mu^2}\cdot\left(\frac{\alpha_0^4}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right). \quad (5.59)$$

Furthermore, $\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\dots,z_{\ell-1})\right)\subseteq B_{\mathrm{sc}(m,k),j}$, and thus

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\dots,z_{\ell-1})\right)}\frac{\mathrm{dist}^2(z_\ell,L)}{\mathrm{diam}^2(B_{\mathrm{sc}(m,k),j})}\frac{\mathrm{d}\mu(z_{\ell_i})}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\dots,z_{\ell-1})\right)\right)}\le \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{4\cdot d}}\cdot\beta_2^2(B_{\mathrm{sc}(m,k),j},L). \quad (5.60)$$

Therefore the inner integral on the RHS of equation (5.58) is bounded by

$$\int_{\left\{Z_X:X\times Z_X\in\widehat{P}_{\mathrm{sc}(m,k),j}\right\}}\frac{\mathrm{dist}^2(z_{\ell_i},L)}{\mathrm{diam}^2(B_{\mathrm{sc}(m,k),j})}\frac{\mathrm{d}\mu^{N_n}(Z_X)}{f_k^n\left(\overline{X}\right)}\le\frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{4\cdot d}}\cdot\beta_2^2(B_{\mathrm{sc}(m,k),j}). \quad (5.61)$$

This yields the following upper bound for the RHS of equation (5.58):

$$\frac{2\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+6\cdot d}}\cdot\alpha_0^{k\cdot d\cdot(d-n+2)}\cdot\beta_2^2(B_{\mathrm{sc}(m,k),j},L)\cdot\mu\left(B_{\mathrm{sc}(m,k),j}\right). \quad (5.62)$$

Applying equations (5.57) and (5.62) to the RHS of equation (5.53) and then taking the infimum over all $d$-planes $L$ establishes equation (5.52), and hence the proposition.

### 5.2.11 Proof of Proposition 4.8.5

Throughout this proof we fix $2\le k'\le k-1$ and $1\le q\le k'\cdot d+1$. For fixed $m\ge m(Q)$, we define

$$\underline{R_{k'}^s}(m)(Q)=\left\{\overline{X}\times Y_{X^s}\in\underline{R_{k'}^s(Q)}:\max\nolimits_{x_0}(X)\in(\alpha_0^{m+1},\alpha_0^m]\right\}. \quad (5.63)$$

This gives the following decomposition of the integral on the LHS of equation (4.122)

$$\int_{\underline{R_{k'}^s}(Q)}\frac{\mathrm{p}_d\sin_{x_0}^2(X_q^s)}{\mathrm{diam}(X)^{d(d+1)}}\frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)}=\sum_{m\ge m(Q)}\int_{\underline{R_{k'}^s}(m)(Q)}\frac{\mathrm{p}_d\sin_{x_0}^2(X_q^s)}{\mathrm{diam}(X)^{d(d+1)}}\frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)}. \quad (5.64)$$

We now fix $m\ge m(Q)$ and partition $\underline{R_{k'}^s}(m)(Q)$ to reflect the quantity $\max_{x_0}\left(X_q^s\right)$. In order to pick a scale exponent, we do the following. By Lemma 4.8.1 and the definition of $\underline{R_{k'}^s}$ we have that

$$\alpha_0^{m+k-k'+2}\le\max\nolimits_{x_0}(X^s)<\alpha_0^{m+k-k'-2}. \quad (5.65)$$

We note that if $k' = k - 1$, then the upper bound on the RHS of equation (5.65) is actually too large, since we always have the bound $\max_{x_0}(X^s) \leq \alpha_0^m < \alpha_0^{m-1}$. As such, we amend the estimate of equation (5.65) in the following way. Let

$$e(m, k') = \begin{cases} m + k - k' - 2, & \text{if } 2 \leq k' \leq k - 2; \\ \\ m, & \text{if } k' = k - 1. \end{cases} \tag{5.66}$$

We note that $e(m, k') \geq m$ and

$$\alpha_0^{e(m,k')+4} \leq \max_{x_0}(X^s) \leq \alpha_0^{e(m,k')}. \tag{5.67}$$

Now, combining Lemma 4.7.1 and equation (5.67) we have the following estimate

$$\max_{x_0}\left(X_q^s\right) \in \begin{cases} \left(\alpha_0^{k'+e(m,k')-\lceil \frac{q}{d} \rceil + 5}, \alpha_0^{k'+e(m,k')-\lceil \frac{q}{d} \rceil}\right], & \text{if } 1 \leq q \leq k' \cdot d; \\ \left(\alpha_0^{e(m,k')+1}, \alpha_0^{e(m,k')}\right], & \text{if } q = k' \cdot d + 1. \end{cases} \tag{5.68}$$

Hence we define the scale exponent as follows

$$\mathrm{sc}(m, k', q) = \mathrm{sc}(m, k, k', q) = \begin{cases} k' + e(m, k') - \left\lceil \dfrac{q}{d} \right\rceil, & \text{if } 1 \leq q \leq k' \cdot d; \\ \\ e(m, k'), & \text{if } q = k' \cdot d + 1. \end{cases} \tag{5.69}$$

We note that the scale exponent is independent of $s$, and furthermore, we have the inequality

$$\mathrm{sc}(m, k', q) \; \geq \; e(m, k'), \text{ for all } 1 \leq q \leq k' \cdot d + 1. \tag{5.70}$$

The family $\left\{P_{\mathrm{sc}(m,k',q),j}\right\}_{j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)}$ covers $[Q \cap \mathrm{supp}(\mu), ]^{d+2}$ and we take the induced cover of $\underline{R_{k'}^s}(m)(Q)$ consisting of

$$\underline{P_{\mathrm{sc}(m,k',q),j}} = \left\{\overline{X} \times Y_{X^s} \in \underline{R_{k'}^s}(m) : x_0 \in P_{\mathrm{sc}(m,k',q),j}\right\}, \text{ for } j \in \Lambda_{\mathrm{sc}(m,k',q)(Q)}. \tag{5.71}$$

Letting $m \geq m(Q)$ and $j$ vary we obtain the inequality

$$\int_{\underline{R^s_{k'}}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)} \leq$$
$$\sum_{m \geq m(Q)} \left[ \sum_{j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)} \int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)} \right]. \quad (5.72)$$

Fixing arbitrary $m \geq m(Q)$ and $j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)$, we will establish the following inequality:

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)} \leq$$
$$C_{14} \cdot \alpha_0^{k\cdot d\cdot(d-n+1)} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}\right) \cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \quad (5.73)$$

Equations (5.72) and (5.73) then imply the proposition.

Let $L$ be an arbitrary $d$-plane. If $\overline{X} \times Y_{X^s} \in \underline{P_{\mathrm{sc}(m,k',q),j}}$, then by equations (5.68) and (5.69)

$$\operatorname{diam}\left({X^s}_q\right) \geq \alpha_0^{\mathrm{sc}(m,k',q)+5} = \frac{\alpha_0^5}{8} \cdot \operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right).$$

Applying this and Proposition 4.2.1 we obtain the inequality

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \mathrm{p}_d \sin^2_{x_0}\left(X^s_q\right) \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{\operatorname{diam}(X)^{d(d+1)} \cdot f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)}$$
$$\leq \frac{2^7 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha_0^{16}} \int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{D_2^2\left(X^s_q, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k',q),j}\right)} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{\operatorname{diam}(X)^{d(d+1)} \cdot f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)}. \quad (5.74)$$

To bound the RHS of equation (5.74), we focus on the individual terms of

$$\frac{D_2^2\left(X^s_q, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k',q)}\right)} = \sum_{t=0}^{d+1} \frac{\operatorname{dist}^2\left(\left(X^s_q\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k',q)}\right)}.$$

We arbitrarily fix $0 \leq t \leq d+1$ and per equations (4.55)-(4.56) and Lemma 4.8.1 we note the following cases for the possible values of $\left(X^s_q\right)_t$:

Case 1: $\left(X^s_q\right)_t = x_0$. In this case $q$ has no restriction, that is, $1 \leq q \leq k' \cdot d$.

Case 2: $\left(X_q^s\right)_t = x_{i_s}$, where the index $i_s$ is defined in equation (4.104). In this case $q = k' \cdot d + 1$ and $\mathrm{sc}(m, k', q) = k' + e(m, k')$ by equations (4.55)-(4.56) and (5.38).

Case 3: $\left(X_q^s\right)_t = x_i$, where $n+1 \leq i \leq d+1$. In this case $1 \leq q \leq d$ by equations (4.55)-(4.56) and (5.38).

Case 4: $\left(X_q^s\right)_t = z_\ell$, where $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$. In this case, we again have $1 \leq q \leq d$ by equations (4.55)-(4.56) and (5.38).

Case 5: $\left(X_q^s\right)_t = y_i$, where $1 \leq i \leq k' \cdot d$. In this case, for each $1 \leq q \leq k' \cdot d + 1$, we have the following restriction on the quantity $\left\lceil \frac{i}{d} \right\rceil$, just as in equation (4.78):

$$\max\left\{1, \left\lceil \frac{q}{d} \right\rceil - 1\right\} \leq \left\lceil \frac{i}{d} \right\rceil \leq \left\lceil \frac{q}{d} \right\rceil. \tag{5.75}$$

For the first three cases, after applying Fubini's theorem we obtain that the corresponding terms on the RHS of equation (5.74) reduce to

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{\mathrm{dist}^2\left(\left(X_q^s\right)_t, L\right)}{\mathrm{diam}^2\left(B_{\mathrm{sc}(m,k',q),j}\right)} \frac{\mathrm{d}\mu^{M_n + k' \cdot d}(\overline{X} \times Y_{X^s})}{\mathrm{diam}(X)^{d(d+1)} \cdot f_k^n\left(\overline{X}\right) \cdot f_{k'}^d\left(\underline{X^s}\right)} =$$

$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)} \frac{\mathrm{dist}^2\left(\left(X_q^s\right)_t, L\right)}{\mathrm{diam}^2\left(B_{\mathrm{sc}(m,k',q),j}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}}, \tag{5.76}$$

where $T_0$ is the truncation of equation (4.116). By the definition of the set $\underline{P_{\mathrm{sc}(m,k',q),j}}$, equation (5.65), and Lemma 4.8.1, we have the following set inclusion

$$T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right) \subseteq \bigcup_{x_0 \in P_{\mathrm{sc}(m,k',q),j}} \left[\{x_0\} \times \prod_{i=1}^{d+2} B(x_0, \alpha_0^{p_i})\right], \tag{5.77}$$

where

$$p_i = \begin{cases} m + k, & \text{if } n+1 \leq i \leq d+1; \\ e(m, k'), & \text{if } i = i_s; \\ m, & \text{otherwise.} \end{cases} \tag{5.78}$$

Applying the usual computations and taking care to note the values of $\mathrm{sc}(m,k',q)$ and $e(m,k')$ we obtain that the RHS of equation (5.76) has the bound

$$\frac{3^d\cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+3\cdot d}}\cdot\alpha_0^{k\cdot d\cdot(d-n+1)}\cdot\beta_2^2\left(B_{\mathrm{sc}(m,k',q),j},L\right)\cdot\mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \tag{5.79}$$

Assume Case 4. Fixing $1\le\ell\le 2^{n-2}+\left\lceil\frac{s}{2}\right\rceil-1$ and iterating the integral we obtain the equality

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}}\left(\frac{\mathrm{dist}\left(z_\ell,L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2\frac{\mathrm{d}\mu^{M_n+k'\cdot d}\left(\overline{X}\times Y_{X^s}\right)}{\mathrm{diam}(X)^{d(d+1)}\cdot f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)}=$$
$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)}\left(\int_{\left\{Z_X:X\times Z_X\in R_{k'}^s\right\}}\left(\frac{\mathrm{dist}\left(z_\ell,L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2\frac{\mathrm{d}\mu^{N_n}\left(Z_X\right)}{f_k^n(\overline{X})}\right)\frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}}. \tag{5.80}$$

Following an argument familiar from the proof of Proposition 4.7.4, we iterate the inner integral on the RHS of equation (5.80), obtaining

$$\int_{\left\{Z_X:X\times Z_X\in R_{k'}^s\right\}}\left(\frac{\mathrm{dist}\left(z_\ell,L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2\frac{\mathrm{d}\mu^{N_n}\left(Z_X\right)}{f_k^1(\overline{X})}=\int_{\pi_1\left(T_0^{-1}(x_0,\ldots,x_{d+1})\right)}\cdots$$
$$\cdots\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)}\left(\frac{\mathrm{dist}\left(z_\ell,L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2\frac{\mathrm{d}\mu(z_\ell)\cdots\mathrm{d}\mu(z_1)}{\prod_{l=1}^{\ell}\mu\left(\pi_l\left(T_{l-1}^{-1}(x_0,\ldots,z_{l-1})\right)\right)}. \tag{5.81}$$

Furthermore, since $1\le q\le d$, by the definitions of $\mathrm{sc}(m,k',q)$ and $e(m,k')$ we have that

$$m+k-3\le\mathrm{sc}(m,k',q)\le m+k-2.$$

Hence, Proposition 4.8.1 and the $d$-regularity of $\mu$ imply the following inequality for all $1\le\ell\le 2^{n-2}+\left\lceil\frac{s}{2}\right\rceil-1$:

$$\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right)\ge\frac{1}{2}\cdot\mu\left(B(x_0,\alpha_0^k\cdot\max_{x_0}(X))\right)\ge\frac{\alpha_0^{4\cdot d}}{2\cdot 4^d\cdot C_\mu^2}\cdot\mu\left(B_{\mathrm{sc}(m,k',q),j}\right) \tag{5.82}$$

Moreover, for fixed $(x_0,\ldots,z_{\ell-1})\in T_{\ell-1}\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$, we have that

$$\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\subseteq B\left(x_0,\alpha_0^k\cdot\max_{x_0}(X)\right)\subseteq\frac{3}{4}\cdot B_{\mathrm{sc}(m,k',q),j}. \tag{5.83}$$

Combining equations (5.82) and (5.83) we obtain the following inequality for fixed $(x_0, \ldots, z_{\ell-1})$ in $T_{\ell-1}\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$:

$$\int_{\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\ldots,z_{\ell-1})\right)} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu(z_\ell)}{\mu\left(\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\ldots,z_\ell-1)\right)\right)} \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{4\cdot d}}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right). \quad (5.84)$$

Applying this to the RHS of equation (5.81), we get that for each $X \in T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$:

$$\int_{\left\{Z_X : X\times Z_X\in R^s_{k'}\right\}} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{N_n}(Z_X)}{f^1_k(\overline{X})} \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{4\cdot d}}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right).$$

Finally, applying this and the set inclusion of equation (5.77) to the RHS of equation (5.80), results in the inequality:

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\times Y_{X^s}\right)}{\operatorname{diam}(X)^{d(d+1)}\cdot f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)} \leq \frac{2\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+4\cdot d}}\cdot \alpha_0^{k\cdot d\cdot(d-n+1)}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right)\cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \quad (5.85)$$

At last we consider Case 5. In this case, it is sufficient to calculate the following bound in $1 \leq i \leq k'\cdot d$ and $\overline{X} \in T_{s,0}\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$:

$$\int_{\left\{Y_{X^s} : \overline{X}\times Y_{X^s}\in \underline{P_{\mathrm{sc}(m,k',q),j}}\right\}} \left(\frac{\operatorname{dist}(y_i, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k'\cdot d}\left(Y_{X^s}\right)}{f^d_{k'}\left(\underline{X^s}\right)} \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{6\cdot d}}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right). \quad (5.86)$$

Indeed, if this is the case, then iterating the integral over $\underline{P_{\mathrm{sc}(m,k',q),j}}$ gives the inequality

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \left(\frac{\operatorname{dist}(y_i, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\times Y_{X^s}\right)}{\operatorname{diam}(X)^{d(d+1)}\cdot f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)} \leq \frac{2\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+6\cdot d}}\cdot \alpha_0^{k\cdot d\cdot(d-n+1)}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right)\cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \quad (5.87)$$

But, equation (5.86) follows easily from the usual computations.

Finally, applying equations (5.79), (5.85), and (5.87), to the RHS of equation (5.74) and then taking the infimum over all $d$-planes $L$, equation (5.73) holds and the proposition is proved.

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