Combinatorial Topological Field Theory

A Dissertation
Submitted to The Faculty of the Graduate School
of the
University of Minnesota
by

Alexander Louis Hanhart

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

Advised by Professor Alexander Voronov

July 2009

Acknowledgments

Many thanks to my family, friends, and colleagues who have supported me over the past years. In particular I'd like to thank my parents, Elma and Richard Hanhart, for making sure I made it home every Thanksgiving. A special thanks also goes to my advisor, Alexander Voronov, for shaping my graduate years and for answering all of my questions, with patience and a smile. Finally, I'd like to thank Matthias Gobbert, for introducing me to research in mathematics and encouraging me to travel to Minneapolis to finish what I started in Baltimore years ago.

Abstract

This work contains the construction of a topological quantum field theory (TQFT, or TFT) based on combinatorial information which consists of directed metric graphs with vertices labeled by metric ribbon graphs. A canonical map between such objects and smooth Riemann surfaces is established using the theory of quadratic differentials investigated by Strebel and others. The surfaces derived have natural decomposition into finite and infinite length cylinders enumerated by the edges of the directed metric graph. Moreover, the surfaces have a gluing operation which agrees with a natural connecting operation on the level of graphs. Finally, the cylindrical decomposition gives the surfaces the structure of a model surface originally investigated by M. Schwartz. He offers a functor on the category of such surfaces which satisfies the properties of a TFT. Combining this functor with the combinatorial information gives the construction presented herein.

Contents

List of Figures

CHAPTER 1

Introduction

The relationship between string topology and Floer theory has been of interest in recent years [**18**] [**17**] [**1**] [**4**]. In particular, let M be a compact manifold and $HF_*(T^*M)$ stand for the Floer homology of the cotangent bundle of M. Then the following conjecture is proposed by R. Cohen in [**5**]:

CONJECTURE 1.1. Given a ribbon graph Γ with p "in" and "q 'out" boundary components, there exists an operation

$$\Theta_\Gamma : (HF_*(T^*M))^{\otimes p} \to (HF_*(T^*M))^{\otimes q}$$

with the properties that

- The maps Θ have the structure of a Topological Field Theory (TFT) with positive boundary.
- With respect to the Salamon-Weber isomorphism

  $$HF_*(T^*M) \simeq H_*(LM)$$

  the Floer theory operations Θ_Γ agree with the operations of string topology.

Such a theorem will give a relationship between Gromov-Witten invariants on T^*M and invariants related to string topology.

The goal of this work is to give some first steps in the investigation of such results. The idea is to replace the information of the ribbon graph Γ with a more general class of combinatorial data, $\mathbf{\Gamma}$, which consists of a directed metric graph with vertices labeled by ribbon graphs. We then give a canonical method for constructing a smooth Riemann surface based on the information $\mathbf{\Gamma}$. It turns out that such a surface fits into the framework of a TFT construction based on the Floer cohomology of a closed symplectic manifold M given by M.Schwarz in [**21**]. Therefore we introduce an operation

$$\Theta_{\mathbf{\Gamma}} : (HF^*(M))^{\otimes p} \to (HF^*(M))^{\otimes q}$$

based on the combinatorial information $\mathbf{\Gamma}$.

1. Outline of Methods and Results

A topological field theory was first described by Atiyah in [**3**] and based on the work of Segal and Witten. It can be thought of most succinctly as a functor between the (symmetric, monoidal) categories of cobordisms and vector spaces. In [**21**], M. Schwarz produces such an object for a particular class of *model* cobordisms:

$$Z(\Sigma) : HF^{*\otimes a} \to HF^{*\otimes b}$$

where Σ is a surface with $a + b$ boundary components, each isomorphic to S^1. The vector spaces under consideration are tensor products of the Floer cohomology of a fixed closed symplectic manifold. The operation above is created by generalizing the analysis of Floer in [**8**] [**7**] [**9**] on Morse flows in the loop space, $\mathcal{L}M$, as follows: Consider a compact, oriented surface with boundary isomorphic to a disjoint collection of copies of S^1. Suppose further that for the ith S^1 component of the boundary, there exists a cylindrical embedding

$$Z^i = [0, \infty) \times S^1 \hookrightarrow \Sigma$$

so that as $s \to \infty$ in the local coordinate structure of the cylindrical embedding, one converges to the S^1 boundary. Now suppose that there exists a function $F : \Sigma\mathcal{L}M \to$

$\mathbb{R}$ so that when restricted to Z^i we have a collection

$$F_i := F_{Z^i} : \mathcal{L}M \to \mathbb{R}$$

which are Morse functions on $\mathcal{L}M$ independent of the surface Σ. Now consider the space of solutions $u : \Sigma \to \mathcal{L}M$ subject to the condition that each $u_i(s,t) = u_{Z_i}$ satisfies the Morse flow equation

$$\frac{du_i}{ds} = \nabla F_i(u_i(s)).$$

The results of Schwartz provide a framework for the examination of Conj. 1.1 by replacing the cobordism Σ with a combinatorial object Γ which consists of several graphs. This is motivated by the well known relationship between surfaces and graphs which has been established by several authors, *e.g.* [**15**], [**11**], and [**23**]. Let Σ be a smooth oriented surface with genus g and n marked points, thought of as punctures. By means of a deformation retract, one can consider a subset $X \subset \Sigma$ which is the injective image of a combinatorial graph $\Gamma \to X \hookrightarrow \Sigma$. The graph Γ can be given the additional structure of an ordering of the edge relations on each vertex induced by the orientation of the surface Σ. The orientation given to the graph Γ allows for a well defined notion of boundary components, the number of which corresponds exactly to the number n of marked points. Such combinatorial objects are called *ribbon* or *fat*graphs. There is a fundamental relationship between *decorated* surfaces and ribbon graphs:

$$\Sigma \times (\mathbb{R}^+)^n \leftrightarrow \Gamma. \tag{1}$$

One method of describing the relationship between surfaces and graphs is by the theory of quadratic differentials. Given a metric ribbon graph, one can construct a smooth Riemann surface whose local coordinate structure is determined by the vertical and horizontal foliations associated to a unique quadratic differential investigated by Jenkins [**12**] and Strebel [**22**]. This machinery has been extremely useful in approaching problems in Teichmüller theory and the theory of the moduli of Riemann surfaces.

However, there is some difficulty in simply replacing the surfaces Σ (used in Gromov-Witten theory, for example) by their corresponding graphs Γ. This is due to the fact that the establishment of the TFT properties for a functor on the space of appropriate surfaces relies on careful analysis of "gluing" operations. It is not well known how these operations interact on the level of the graphs. In order to surmount this difficulty, we generalize the class of combinatorial data to a list $\mathbf{\Gamma}$ which consists of a graph G with free half-edges (leaves) and a labeling of the vertices of G by metric ribbon graphs with a natural agreement between them. The main point is that there is a canonical map $\mathbf{\Gamma} \to \Sigma_{\mathbf{\Gamma}}$ so that each edge and leaf of G corresponds exactly to a cylindrical embedding discussed before. Moreover, there is a gluing operation on the level of the combinatorial data which agrees with the gluing operations. Finally, there exists a method to create a function F above, so that when restricted to each of the cylindrical components, we have a Morse flow. Hence we are able to apply the conclusions of [**21**] where we consider the data $\mathbf{\Gamma}$ to be a cobordism whose boundaries are the free leaves of G.

The main new results of this thesis can be summarized as follows. We present a construction of combinatorial surfaces out of a directed metric graph G with a vertex labeling of ribbon graphs Γ_v which satisfies an admissibility relation described in Chap. 2. We offer such a construction by means of a canonically defined quadratic differential given in Theorem. 2.15 which generalizes the unique differentials of Jenkins and Strebel to the setting of admissible graph systems. We then show that there exists an attaching operation on pairs of such admissible graph systems which agrees with a gluing of the associated combinatorial surfaces in Theorem 2.21. Next we show that the moduli space of pseudoholomorphic maps from such generalized combinatorial surfaces to a given symplectic manifold is an example of a more general class of moduli spaces. This allows us to build a TFT based on our combinatorial data described in Theorem 3.16.

The thesis is structured as follows: The remainder of this chapter gives preliminary material on the basic definitions and results associated to topological field theories and a description of the construction of Floer (co)homology for a symplectic manifold. In Chapter 2 we review the correspondence between the combinatorial concepts of ribbon graphs and Strebel differentials, offering the formal version of (1). We then introduce the generalized combinatorial data $\mathbf{\Gamma}$ and establish a generalization of (1). We conclude by introducing gluing operations on the level of the surface and the combinatorial information and verify that they agree. Chapter 3 introduces the settings and results of [**21**]. We verify that the combinatorial construction offered in Chap. 2 fits within this framework. By offering constructions based on $\mathbf{\Gamma}$ and proving their necessary properties, we finally arrive at a mapping

$$Z(\mathbf{\Gamma}) : V^{\otimes \#(H^+)} \to V^{\otimes \#(H^-)}$$

where $H^+ \cup H^-$ is the set of leaves of $G \in \mathbf{\Gamma}$ and V is the Floer cohomology of a compact symplectic manifold M.

2. Topological Field Theory

Topological (Quantum) Field Theories (TFTs or TQFTs) were introduced first by physicists and then later given an axiomatic definition by Atiyah [**3**] in 1988. We give the definition of an n-dimensional TFT, although we will restrict ourselves to the case of 2-dimensional TFTs in this work. We conclude with a classification theorem about such objects.

Definition 1.2. An *nD TFT* consists of two operations:

- To each closed, oriented $(n-1)$-dim manifold X we associate a module $V(X)$ over a ground ring. Examples include (co)chains, (co)homology groups, forms, etc. Typical rings include $\mathbb{Z}, \mathbb{Z}_2, \mathbb{R}, \mathbb{C}$.
- To each n-dim oriented cobordism (X_1, Σ, X_2) we associate a linear operator $V(\Sigma) : V(X_1) \to V(X_2)$.

The operations are well defined on diffeomorphism classes of their inputs and satisfy the following compatibility conditions:

- $V(X_1 \bigsqcup X_2) = V(X_1) \otimes V(X_2)$.
- $V(\Sigma_1 \#_X \Sigma_2) = V(\Sigma_2) \circ_{V(X)} V(\Sigma_1)$, where $\#_X$ represents a composition of cobordisms defined by gluing together a common submanifold X of their boundaries and $\circ_{V(X)}$ is a composition of linear operators through $V(X)$.

As stated in Sec. 1 a TFT can be considered as a tensor functor between the categories of cobordisms and vector spaces.

There is a standard classification theorem associated to the class of TFTs corresponding to surfaces (D= 2). Though attributed to many, the proof first appeared in [**2**] in the late 1990's

Theorem 1.3. *A 2D TFT is equivalent to a Frobenius algebra A over the ring associated to X.*

Recall that a Frobenius algebra over a ring, say $\mathbb{C}$, is a finite-dimensional, commutative, associative algebra A with a unit element and a trace map $\Theta : A \to \mathbb{C}$ which gives a nonsingular bilinear product $A \otimes A \to \mathbb{C}$ by $(a, b) \mapsto \Theta(ab)$.

In this work the vector spaces under consideration are tensor products of the Floer (co)homology of a fixed closed symplectic manifolds, which is the target of the maps u described in Sec. 1. We describe these objects in detail next.

3. Floer Theory

Floer theory is often times characterized as an infinite dimensional Morse theory applied to the loop space of a manifold. We will review the basic constructions and theorems associated to Floer theory since they form an important instance of the general construction presented in this work.

Let (M,ω) be a compact symplectic manifold. Let $H_t = H_{t+a} : M \to \mathbb{R}$ be a smooth, time-dependent, a-periodic family of Hamiltonian functions. Recall that since the form ω determines an isomorphism $I_\omega : T^*M \to TM$, we can generate the Hamiltonian vector field on M by $\iota(X_{H_t})\omega = dH$. We note that $X_{H_t} : M \to TM$ and gives us a Hamiltonian differential equation

$$x'(t) = X_{H_t}(x(t)). \tag{2}$$

Define

$$\mathcal{C}(H) = \{x : \mathbb{R}/a\mathbb{Z} \to M | x'(t) = X_{H_t}(x(t))\} \tag{3}$$

as the set of a-periodic solutions of (2). There are two key conditions that we require for such solutions. The first is that such solutions are contractible in M. The second is that these solutions be nondegenerate. That is, if we consider the family of symplectomorphisms $\phi_t : M \to M$ by

$$\frac{d}{dt}\phi_t = X_{H_t} \circ \phi_t, \quad \phi_0 = id,$$

then we have that the fixed points of the map ϕ_a are in correspondence to the elements of $\mathcal{P}(H)$. Nondegeneracy of the elements x of $\mathcal{C}(H)$ is then the condition

$$\det(id - d\phi_a(x(0))) \neq 0. \tag{4}$$

These conditions are necessary to ensure that $\mathcal{C}(H)$ describe the critical points for a Morse flow given by the symplectic action functional on the universal cover of the loop space of the manifold.

DEFINITION 1.4. A Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ for which all $x \in \mathcal{C}(H)$ satisfy condition (4) is called *regular*.

This condition is not restrictive since it is known that it holds for a generic set of H [**19**].

An *almost complex* structure on M is a vector bundle map $J : TM \to TM$ with $J^2 = -I$, where I is the identity map. The structure is *compatible* with ω provided that

$$g(V, W) = \omega(V, JW)$$

gives a Riemannian metric on M. It is well known that the space of complex structures J compatible with (M, ω) is contractible [**14**]. Therefore, the first Chern class $c_1(TM, J)$ does not depend on the choice of J. We will assume the following condition on M: There exists a fixed real constant $\tau > 0$ so that

$$\phi_{c_1}(v) := \int_{S^1} v^* c_1 = \phi_\omega(v) := \tau \int_{S^1} v^* \omega \tag{5}$$

for all smooth $v : S^1 \to M$. Such a symplectic manifold is called *monotone*.

Define $\mathcal{L}(M)$ to be the space of smooth contractible loops in M. For the following discussion we will normalize the period of H_t to be $a = 1$. Hence $x \in \mathcal{L}(M)$ is a smooth map $x : S^1 \simeq \mathbb{R}/\mathbb{Z} \to M$. We consider all such loops to be contractible and so there must exist smooth map $u : B \to M$ where $B = \{z \in \mathbb{C} | \, |z| \leq 1\}$ is the unit disk and u satisfies the boundary condition $U(\mathrm{e}^{2\pi i t}) = x(t)$. Two such maps u_1 and u_2 are equivalent if the boundary sum $u_1 \#(-u_2)$ is homologous to zero in $H_2(M, \mathbb{Z})$. Hence we can create the space $\widetilde{\mathcal{L}(M)}$ of equivalence classes of pairs $[x, u]$. This will be the appropriate space for us to do Morse theory on. Note that $\widetilde{\mathcal{L}(M)}$ is a covering space of $\mathcal{L}(M)$. Moreover, it is unique in the sense that its deck transformations are given exactly by the image of $\pi_2(M) \xrightarrow{\sim} H_2^S(M) \subset H_2(M, \mathbb{Z})$ under the Hurewicz homomorphism.

Our Morse function will be given by the symplectic action functional defined by $\mathcal{A}_H : \widetilde{\mathcal{L}(M)} \to \mathbb{R}$, which depends on a given 1-periodic Hamiltonian, and is defined in this case by

$$\mathcal{A}_H([x,u]) = -\int_B u^*\omega - \int_0^1 H_t(x(t))\ \mathrm{d}t. \tag{6}$$

This is a well defined functional on $\widetilde{\mathcal{L}(M)}$ due to the fact that for any element $A \in H_2^S(M) \subset H_2(M)$ we have that $\mathcal{A}_H(A\#[x,u]) = \mathcal{A}_H([x,u]) - \omega(A)$. Finally, any two equivalent elements in $\widetilde{\mathcal{L}(M)}$ differ by a class equivalent to zero.

It can be shown directly that the critical points of $\mathcal{A}_H$ are exactly the elements $[x,u] \in \widetilde{\mathcal{L}(M)}$ for which $x \in \mathcal{C}(H)$. The condition of nondegeneracy ensures that $\mathcal{A}_H$ is a Morse function on $\widetilde{\mathcal{L}(M)}$ and that the set $\mathcal{C}(H)$ is finite. Floer was the first to apply Morse theory in this context in order to solve the Arnold conjecture.

In order to do Morse theory, we need to understand the gradient flow lines of the symplectic action functional. Note that the tangent space of $\mathcal{L}M$ at a fixed loop x can be given by $T_x\mathcal{L}M = C^\infty(\mathbb{R}/\mathbb{Z}, x^*TM)$. A metric on the loop space will be given by a fixed smooth 1-periodic family of ω-compatible almost complex structures $J_t \in \mathcal{J}(M,\omega)$ by defining $\langle \xi, \eta \rangle_t = \omega(\xi, J_t\eta)$ for $\xi, \nu \in TM$. Extend to $\xi(t), \eta(t) \in T_x\mathcal{L}M$ by setting

$$\langle \xi, \eta \rangle = \int_0^1 \langle \xi(t), \eta(t) \rangle_t\ \mathrm{d}t.$$

Now we have by a direct computation that

$$d\mathcal{A}_H([x,u])\xi = \int_0^1 \omega(\dot{x} - X_{H_t}(x), \xi)\,\mathrm{d}t$$

and so therefore, on $\mathcal{L}M$, we may write that

$$\operatorname{grad} \mathcal{A}_H(x) = J_t(x)(\dot{X} - X_{H_t}(x))$$

by applying the above inner product to the differential. Finally we arrive at the equation for the downward gradient flow lines for the symplectic action functional

$$\bar{\partial}_{J,H} u = \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0. \tag{7}$$

This partial differential equation for smooth maps $u : \mathbb{R} \times S^1 \to M$ is often called the Floer equation.

Next we establish the moduli spaces of flows on which the Floer (co)homology is constructed. Define the energy of a solution to (7) by

$$E(u) = \frac{1}{2}\int_{S^1}\int_{\mathbb{R}} \left(|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_{H_t}(u)|^2 \right) \, \mathrm{d}s \, \mathrm{d}t.$$

With the energy given above, we have the following result:

THEOREM 1.5. *Let $u : \mathbb{R} \times S^1 \to M$ be a solution to* (7)*. Then the following are equivalent.*

i $E(u) < \infty$

ii *There exist loops $x^{\pm} \in \mathcal{P}(H)$ such that*

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t) \tag{8}$$

and

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s,t) = 0$$

uniformly in t.

iii *There exist constants $\delta > 0$ and $c > 0$ such that*

$$|\frac{\partial u}{\partial s}(s,t)| \leq c\mathrm{e}^{-\delta|s|}$$

for all $s, t \in \mathbb{R}$.

Thm. 1.5 gives important properties for the solution space of Eq. (7) and (8). Let

$$W_{x,y}(J,H) = \left\{u \in C^\infty(\mathbb{R} \times S^1, M) | \bar{\partial}_{J,H}(u) = 0, u(-\infty) = x, u(\infty) = y\right\}$$

to be solutions of (7) with prescribed boundary conditions $x, y \in \mathcal{P}(H)$ in the sense of (8). For a generic Hamiltonian, these spaces are finite dimensional manifolds of dimension

$$\dim W_{x,y}(J,H) = \mu_{CZ}(y) - \mu_{CZ}(x) < \infty$$

where $\mu_{CZ} : \widetilde{\mathcal{C}(H)} \to \mathbb{R}$ is the Conley-Zehnder index. This is a version of the Maslov index and is defined by examining the path of symplectic matrices generated by the linearized Hamiltonian flow along $x(t)$ after first trivializing the tangent bundle over the disc $u(B)$ for an element $[x, u] \in \widetilde{\mathcal{C}(H)}$, see [**16**].

There is a natural action of elements $\tau \in \mathbb{R}$ on elements $u \in W_{x,y}(J,H)$ given by translation in the s direction: $u(s + \tau, t) \in W_{x,y}(J,H)$. Define the moduli space of equivalences by dividing by this free action

$$\mathcal{M}_{x,y}(J,H) = W_{x,y}(J,H)/\mathbb{R}.$$

In the case that J and H have been fixed in the discussion, we simplify notation to $W(x,y) := W_{x,y}(J,H)$ and likewise for $\mathcal{M}(x,y)$. Also, we may write that $\mathcal{M}^k(x,y)$ is the moduli space for which $\dim W(x,y) = k$.

Now let $u \in W(x,y)$ and consider two classes $[x, u_x]$ and $[y, u_y]$ with $u_y = u_x \# u$. Then it is easy to show that

$$E(u) = \mathcal{A}_H(x, u_x) - \mathcal{A}_H(y, u_y).$$

Using this energy equality, one can show that in the space $\mathcal{M}^1(x, y)$ is 0-dimensional and compact, hence finite. For the analysis necessary to show this, see [**14**], as well as Floer's original papers: [**8**], [**7**], and [**9**].

Now, under the assumption that (M, ω) is monotone and H is a Hamiltonian of generic type, define the kth Floer chains as

$$CF_k(H) = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x) = k}} \mathbb{F} < x >$$

where typically $\mathbb{F} = \mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{Q}$, but in general is a principle ideal domain. The idea is to define a boundary operator which counts, taking orientation into consideration, the flow lines which connect critical points in $\mathcal{P}(H)$ and to use this operator to establish a homology theory. Floer and Hofer [**10**] showed that there exists a (nonuniquely defined) system of coherent orientations on the moduli spaces $\mathcal{M}(x, y)$ under which a certain (Floer) gluing map is orientation preserving. Let $\varepsilon(u) \in \{\pm 1\}$ indicate the orientation of $u \in \mathcal{M}^1$ relative to the flow orientation given by (7). Then we have the Floer boundary operator $\partial : CF_k(H) \to CF_{k-1}(H)$ defined by:

$$\partial < y > = \sum_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x) = k-1}} \sum_{[u] \in \mathcal{M}^1(y,x)} \varepsilon(u) < x > . \tag{9}$$

The necessary result that follows is that

Theorem 1.6 (Floer). *If (M, ω) is monotone and H is generic then $\partial \circ \partial = 0$.*

The immediate result is that one can construct a homology on the chain complex $(CF_*(H), \partial)$ by

$$HF_*(M, \omega, H, J; \mathbb{F}) = \frac{\ker \partial}{\operatorname{im} \partial}.$$

An important component in the proof of 1.6 is to establish that there exists a gluing map

$$\mathcal{M}^1(z,y) \times \mathcal{M}^1(y,x) \times (R_0,\infty) \to \mathcal{M}^2(z,x). \tag{10}$$

This map is defined only on compact subsets of $\mathcal{M}^1(z,y) \times \mathcal{M}^1(y,x)$ and in this case is a diffeomorphism onto an open subset of $\mathcal{M}^2(z,x)$.

The homology groups are in fact an invariant and do not depend on H or J:

THEOREM 1.7. *Let (M,ω) be a monotone symplectic manifold. For two pairs (H^1,J^1) and (H^2,J^2) which satisfy the requirements for the definition of Floer homology, there exists a natural isomorphism*

$$\Phi^{2,1} : HF_*(M,\omega,H^1,J^1) \to HF_*(M,\omega,H^2,J^2)$$

This theorem is established by looking at a homotopy family which connects (H^1,J^1) to (H^2,J^2).

CHAPTER 2

Generalized Combinatorial Surfaces

The basic object under investigation in this section is the *combinatorial surface*. Given combinatorial information $\mathbf{\Gamma}$, we will give a canonical construction of a surface $\Sigma_{\mathbf{\Gamma}}$ based on the topological type of the graph. The construction given will also yield a quadratic differential which will give $\Sigma_{\mathbf{\Gamma}}$ the structure of a smooth Riemann surface. This surface will be used to construct the Floer-style moduli spaces used in a field theory construction.

We begin by reviewing the standard definitions and results concerning ribbon graphs and quadratic differentials, culminating in the relationship between isomorphism classes of metric ribbon graphs and the moduli space of Riemann surfaces by means of the quadratic Strebel differential. Then we explain how to modify this construction to the case of interest to us: constructing Riemann surfaces from a general class of combinatorial information given to us by multiple metric ribbon graphs. Finally, we show that elementary surgery constructions on the level of Riemann surfaces can be viewed as a surgery construction on the combinatorial level. Therefore, the combinatorial information completely determines the important properties of the Riemann surface necessary for the analysis of Chap. 3

1. Metric Ribbon Graphs

We begin by describing a graph, which is a finite collection of points and line segments. A ribbon graph is a graph that is "drawn" on an orientable surface. We will want to consider isomorphisms between such objects and finally restrict our attention to isomorphism class of the objects. We note that the idea of relating combinatorial graphs to surfaces originated with Grothendiek's *child's drawings*, described in [**20**].

We use the following suitable definition of a graph. The basic building block is a *half-edge*:

DEFINITION 2.1. A *graph* $\Gamma = (\mathcal{H}, \mathcal{E} : \mathcal{H} \to \mathcal{H}, \mathcal{V} : \mathcal{H} \to V)$ is a finite set $\mathcal{H} = \{h_k\}_{k=1}^{\nu}$ of *half edges* together with two maps. The first is the *edge map* $\mathcal{E}$ which is an involution of the half edges. The *edge set* E of a graph is the orbit set of elements of $\mathcal{H}$ under $\mathcal{E}$:

$$E := \{\mathcal{E}(\{h\}) \,|\, h \in \mathcal{H}\}\,.$$

If a half edge is a fixed point of $\mathcal{E}$ then we call it a *leaf.* The second map takes the set of half edges onto a finite set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ which is the *vertex set* of the graph. This allows for a simple definition of the *degree* of a vertex

$$\deg(v_i) = \#(\mathcal{V}^{-1}(v_i)).$$

In general we will not consider graphs with vertices of degree less than 3, or graphs that are not *connected* in the obvious sense. A graph with no leaves is *compact.* We will assume all graphs are compact for this subsection. An example of construction of a graph is given in Figure 1.

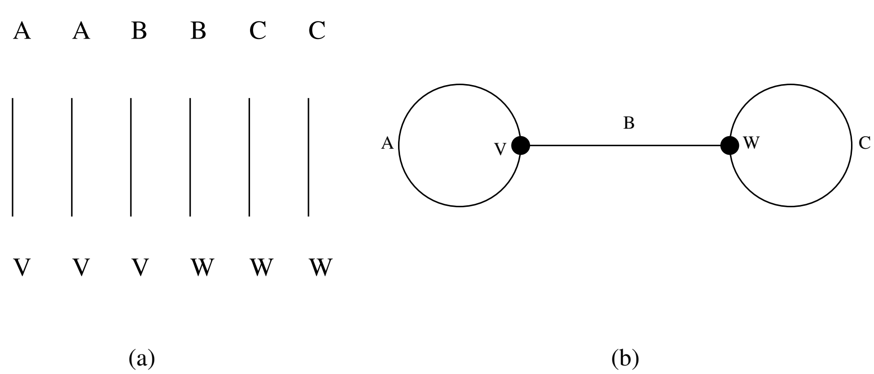


FIGURE 1. (a) A collection of six half edges. The letters A, B, C indicate the edges given by the involution map $\mathcal{E}$. The letters V and W indicate the vertices given by the map $\mathcal{V}$. (b) The resulting graph with two vertices and three edges. The vertices are all of degree three.

Let Γ and Γ' be two graphs with a bijective map $\alpha : \mathcal{H} \to \mathcal{H}'$ between the half edges.

DEFINITION 2.2. The map α above is a *graph isomorphism* if the following two conditions hold: (1) following diagram commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{E}} & \mathcal{H} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathcal{H}' & \xrightarrow{\mathcal{E}'} & \mathcal{H}' \end{array}$$

in which case we say that the map preserves the edge relation and (2) for any two half edges $a, b \in \mathcal{H}$ we must have

$$\mathcal{V}(a) = \mathcal{V}(b) \Leftrightarrow \mathcal{V}'(\alpha(a)) = \mathcal{V}'(\alpha(b))$$

in which case we say the map preserves the vertex relation.

Now we can introduce ribbon graphs and their automorphisms.

DEFINITION 2.3. A *ribbon graph* (or *fatgraph*) is a graph Γ together with a cyclic ordering on the set of half–edges incident to each vertex $v \in V$, e.g. it is a pair (Γ, ρ) where $\rho : v \mapsto \rho(v) \in S_{\mathcal{H}(v)}$, where $S_{\mathcal{H}(v)}$ is the symmetric group on the set of half-edges incident to V. An *automorphism* of a ribbon graph will be an element of the automorphism group of Γ in the sense of (2.2) which also preserves the cyclic ordering on the half edges. Figure 2 gives two examples of non-isomorphic ribbon graphs.

Notationally, we suppress the choice of ρ_Γ. For the rest of this chapter we will always consider Γ to be a ribbon graph with underlying graph denoted by $\underline{\Gamma}$.

A vertex V along with the half edges incident to it can be represented in the plane with standard orientation. In this way, one can imagine gluing oriented strips to the half edges which meet at the vertex, as in Fig. 3. If the strips are glued by

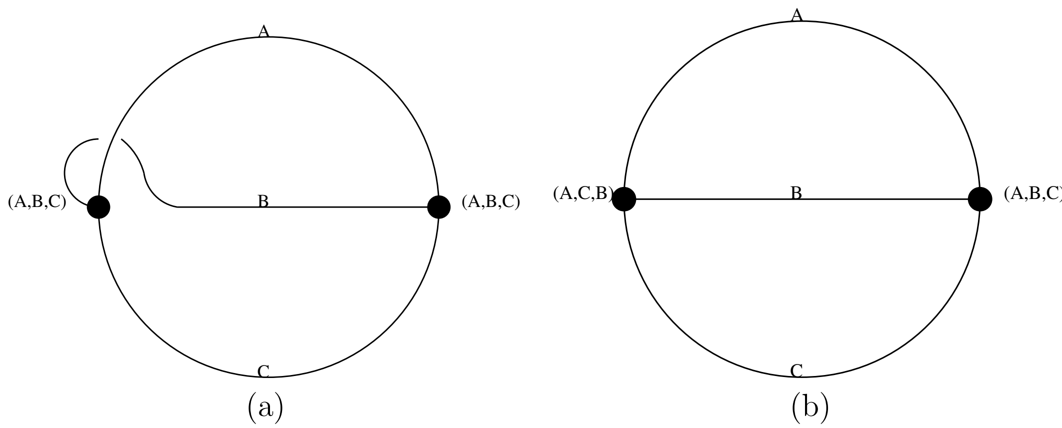


FIGURE 2. Non-isomorphic ribbon graphs.

following the orientation of their boundaries, then we can consider each ribbon graph to be represented by an orientable surface. We can consider each ribbon graph Γ to be an oriented surface σ_Γ called a "skinny surface" by Penner [**15**]. The surface σ_Γ is a surface with boundary. It will be compact if the original graph was.

DEFINITION 2.4. The boundary components of a ribbon graph Γ are the edge cycles which correspond to the boundary components of σ_Γ. Additionally, define

$$\mathrm{Aut}_\delta(\Gamma) \tag{11}$$

to be the subgroup of graph automorphisms which preserve the labeling of the boundary components.

This construction allows us to go further. To the compact skinny surface obtained from attaching "ribbons" as above, now attach an oriented disk to each boundary component of σ_Γ. The new space $C(\Gamma)$ is a compact topological surface. We note the following two basic results of this construction. The first is that the underlying graph $\underline{\Gamma}$ gives rise to a cellular decomposition of $C(\Gamma)$. The second is that by letting $v(\Gamma)$, $e(\Gamma)$, and $b(\Gamma)$ be the number of vertices, edges, and boundary components

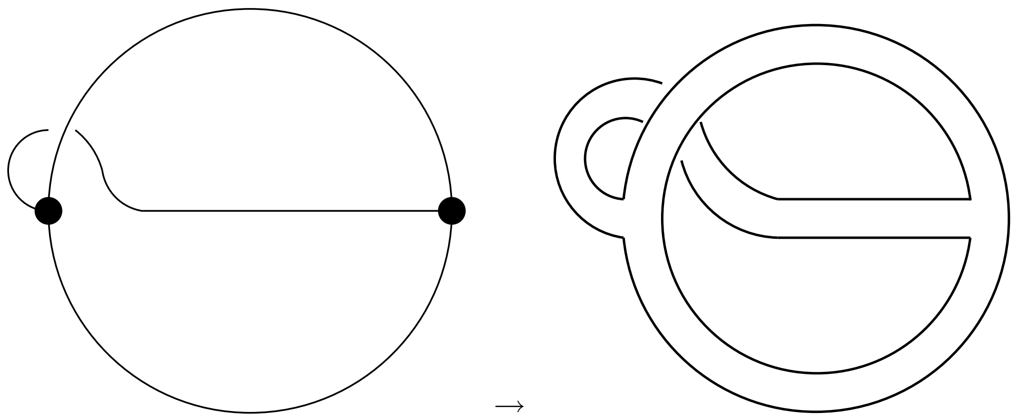


FIGURE 3. Construction of an oriented "skinny surface."

respectively, then the genus g of $C(\Gamma)$ is given by

$$v(\Gamma) - e(\Gamma) + b(\Gamma) = 2 - 2g(C(\Gamma)).$$

We want to restrict our attention to specific types of ribbon graphs, namely those which can be written as *spines* (i.e. inclusion is a strong deformation retraction) of a fixed reference surface of type (g, n). Therefore we introduce the following notation so as to index graph types by pairs (g, n).

DEFINITION 2.5. By $RG_{g,n}$ we mean the set of all isomorphism classes of connected compact ribbon graphs Γ with no vertex of degree less than three such that

$$\begin{cases} g \geq 0 \\ \chi(\Gamma) = v(\Gamma) - e(\Gamma) = 2 - 2g - n < 0 \\ b(\Gamma) = n \geq 1, \end{cases}$$

where v, e, b represent the number of vertices, edges, and boundary components of Γ respectively.

A *positive metric* μ on a compact ribbon graph Γ is a function $\mu : \mathcal{E} \to \mathbb{R}_+$. If $b = (e_1, \ldots, e_n)$ is a closed edge–path on Γ, we can define the length $\ell(b) = \sum \mu(e_i)$ where the sum is taken over all of the edges e_i that comprise b counting multiplicity.

This definition is necessary since we can describe the length of each of the boundary components by writing each component as a closed edge–path. We write M_Γ for the set of positive metrics on Γ and the pair (Γ, M_Γ) is the set of metric graphs with Γ as the underlying graph. Hence a *metric ribbon graph* will be a choice of a ribbon graph as well as a metric to go with it. Geometrically, the set is an orbifold:

$$\frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)},$$

where the action of $\mathrm{Aut}(\Gamma)$ on $\mathbb{R}_+^{e(\Gamma)}$ is given by the homomorphism $\phi : \mathrm{Aut}(\Gamma) \to S_{E(\Gamma)}$. Note that the metric is pushed forward by the automorphism $p \in \mathrm{Aut}(\Gamma)$, that is the length associated to an edge of $p(\Gamma)$ will be the length of that edge before the action.

We can classify compact metric ribbon graphs by the topological conditions given in (2.5). Define the set of isomorphism classes of metric ribbon graphs by

$$RG_{g,n}^{\mathrm{met}} = \coprod_{\Gamma \in RG_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)}.$$

Each of these components are called *rational cells* of $RG_{g,n}^{\mathrm{met}}$. This definition extends to metric ribbon graphs with labeled boundary components:

$$BRG_{g,n}^{\mathrm{met}} = \coprod_{\Gamma \in BRG_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}_\delta(\Gamma)}. \tag{12}$$

This space also has the structure of an orbifold with a cellular decomposition given by the rational cells above.

2. Quadratic Differentials

The theory of quadratic differentials allow us to move between Riemann surfaces and metric ribbon graphs: Given a Riemann surface we can find a metric graph which encodes it's topological information be means of a "Strebel differential." On the other hand, given a metric ribbon graph, there is a canonical construction for a Riemann surface with metric given by a particular quadratic differential. It is this latter construction which we will modify for our particular class of combinatorial surfaces, fundamental to the field theory constructed directly from the combinatorial information included with a metric ribbon graph.

Let $F_{g,n}$ be a compact Riemann surface of genus g with $\{x_i\}$, $i = 1, \ldots, n$ marked points. Let K_F be the canonical sheaf of $F_{g,n}$, which is defined to be the $\dim(F_{g,n})$ exterior product of the cotangent bundle. We can think of it as the sheaf of holomorphic 2-forms. Therefore, $K_F^{\otimes 2}$ can denote the symmetric quadratic tensor product of the canonical sheaf. We define a holomorphic *quadratic differential* on F as an element of the set $H^0(F, K_F^{\otimes 2})$. In local coordinate z of f, an element $q \in H^0(F, K_F^{\otimes 2})$ is represented simply as $q = f(z)(\,\mathrm{d}z)^2$. This definition extends to meromorphic functions by replacing F with $F - \{x_i\}$, in which case f above may have a pole. Note that if $w(z), z$ are both local coordinates in a neighborhood of F then we have the following change-of-variables formula:

$$f(z) = g(w(z)) \left(\frac{\mathrm{d}w(z)}{\mathrm{d}z} \right)^2. \tag{13}$$

Let $\gamma : (a, b) \to F$ be a curve and q be a quadratic differential. If

$$(f \circ \gamma)(t) \left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \right)^2 \tag{14}$$

is positive for all $t \in (a, b)$ then we say γ is a *horizontal leaf* (= *horizontal trajectory*). If (14) is negative, then it is a *vertical leaf.* The collection of all horizontal or vertical leaves is a *real codimension* 1 *foliation* on F less the singular points and zeros of q.

Before giving an example of horizontal and vertical leaves we introduce the *canonical coordinate*. Take a local representation for q defined above and consider a point

z_0 so that q is non zero there. Then define

$$w(z) = \int_{z_0}^{z} \sqrt{q} = \int_{z_0}^{z} \sqrt{f(z)}\, dz. \tag{15}$$

It should be clear that this definition of a coordinate is not unique, but it is unique up to sign change and additive constant. With this coordinate it can be shown through direct calculation that:

THEOREM 2.6. *For an open Riemann surface S with quadratic differential q and a given point $z \in S$ there is a unique horizontal and vertical leaf through z. Additionally, these curves intersect at right angles.*

We wish to classify the behavior of a quadratic differential q in neighborhoods of (a) regular (non-singular, non-zero) values, (b) zeros of any order, and (c) poles of order two. Behavior of q in these neighborhoods will give the geometry for the Riemann surface to be constructed from a given metric ribbon graph.

EXAMPLE 2.7 (Foliation at a regular point, z_0, of q). Using the canonical coordinate, we may assume $q = (\mathrm{d}z)^2$. In this case horizontal leaves are of the form

$$t \mapsto t + ci, t \in \mathbb{R}$$

and vertical leaves are

$$t \mapsto it + c, t \in \mathbb{R}.$$

Since c is arbitrary, this gives a simple foliation of $\mathbb{C}$ consisting of vertical and horizontal lines.

EXAMPLE 2.8 (Foliation near a zero). Consider $f(z) = z^m$ as an example of a function with a zero of order m. The quadratic differential $q(z) = f(z)(\mathrm{d}z)^2$ will have, according to 15, horizontal leaves

$$t \to t \exp\left(\frac{2\pi ik}{m+2}\right), \quad t \in \mathbb{R}^+$$

and vertical leaves

$$t \to t \exp\left(\frac{\pi i + 2\pi i k}{m+2}\right), \quad t \in \mathbb{R}^+.$$

This gives a total of $2m+4$ rays ($k = 0, \ldots, m+1$) emanating from the origin.

The important point here is that both horizontal and vertical trajectories have possible endpoints at zeros of a quadratic differential. Next, consider the foliation at a quadratic pole:

EXAMPLE 2.9 (Foliation about a quadratic pole). Consider $f(z) = -z^{-2}$ as a general example of a function with a quadratic pole. Then the differential we investigate is given by $q(z) = f(z)(\,\mathrm{d}z)^2$. We wish to find curves γ so that equation 15 is either strictly positive or strictly negative. The horizontal leaves are given by $\gamma_h(t) = z_0 \mathrm{e}^t$ where $z_0 \in \mathbb{C}$ and the vertical leaves are given by $\gamma_v(t) = r\mathrm{e}^{it}$ where r is a positive real constant. Hence vertical leaves are concentric circles which surround the pole and horizontal leaves are rays which emanate from the pole. Figure 4 indicates the plot of vertical leaves in a more general case. Note that the foliation fills the space about the pole and converges to a polygonal shape whose vertices are zeros of q, as described in 2.8.

Now there is a famous result of Strebel [**22**], which may be worded as follows:

THEOREM 2.10 (Strebel). *Let $F_{g,n}$ be a smooth Riemann surface of genus $g \geq 0$ with $n \geq 1$ marked points given by $(p_1, \ldots, p_n)$ so that $2-2g-n < 0$. Let $a_1, \ldots, a_n) \in \mathbb{R}^+$ be a list of n positive real numbers. There exists a unique meromorphic quadratic differential q on F so that:*

(1) *q is holomorphic on $F - \{p_i\}$.*
(2) *q has a double pole at each p_i.*
(3) *The union of all non-compact horizontal leaves forms a closed subset of F with measure zero.*

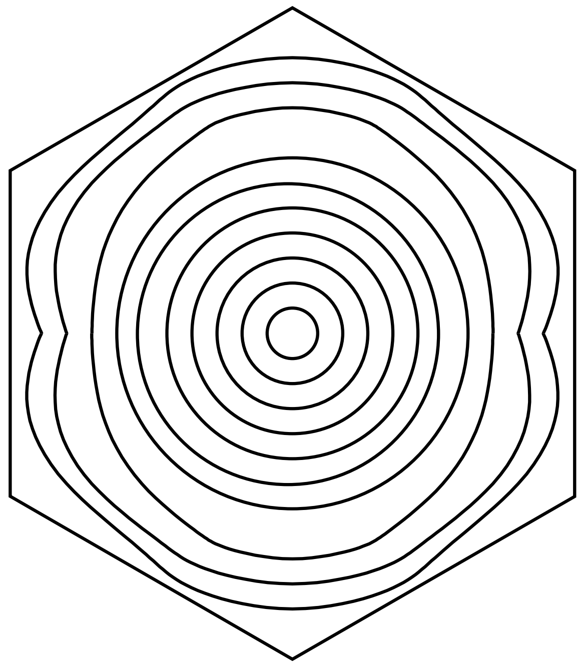


FIGURE 4. A sketch of representative curves for a foliation about a general quadratic pole. The curves converge to a polygon, the vertices of which are zeros of the differential.

(4) *Every compact horizontal leaf α is a simple loop surrounding one of the p_i's. In this case we have*

$$a_i = \oint_\alpha \sqrt{q},$$

where we take a positive branch of the square root so that the integral has positive value with respect to the orientation of α given by the complex structure of F.

This unique quadratic differential is called the *Strebel differential*.

A few comments are in order: Note that the properties of the Strebel differential come from the fact that its poles are quadratic and therefore the general behavior of its horizontal and vertical leaves has been discussed in Examples 2.8 and 2.9. The point is that the non-compact horizontal leaves described in result 3 of Theorem 2.10 will give the edges of a fatgraph which describes the surface F. Since the horizontal leaves form concentric circles and the vertical leaves are rays, they can be described as a product $S^1 \times \mathbb{R}^+$. The uniqueness of the Strebel differential will therefore allow us to associate to a given metric fatgraph a smooth Riemann surface with a local coordinate structure which will agree with the framework of cylindrical Floer flows.

Theorem 2.10 gives us a map

$$\left\{ \begin{array}{c} \text{Riemann surfaces with genus } g \\ \text{and } n \text{ marked points} \end{array} \right\} \times \mathbb{R}^n_+ \to \left\{ \begin{array}{c} \text{Metric ribbon graphs} \\ \text{of type } g, n \end{array} \right\}.$$

A major result is that this map descends to a map

$$\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \to BRG^{\text{met}}_{g,n}, \tag{16}$$

where $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces of genus g with n marked points. In fact:

Theorem 2.11 (Harer-Mumford-Thurston). *The map* (16) *is a homeomorphism.*

This result based on a different construction (which gives a different homeomorphism) is due to Penner, [**15**].

We describe some of the constructions involved in the proof of Theorem 2.11 which will be of interest later when we generalize Strebel's result to a generalized class of combinatorial surfaces.

Lemma 2.12. Given a metric ribbon graph Γ of type g, n there exists a Riemann surface Σ_Γ of type (g, n) and a unique quadratic differential q on Σ_Γ so that

(1) q has quadratic poles at each of the n marked points of Σ_Γ
(2) The edges of the graph Γ can be identified with the critical horizontal trajectories and the vertices with the zeros of q.

Proof. Consider a metric ribbon graph Γ with labeled boundary components $b_1, b_2, ..., b_n$. Recall that the boundary components are related to the boundary components given to us by the oriented skinny surface. By choosing an orientation for this surface, we must have oriented boundaries as well. Therefore we can partition the b_i's into two classes: "in" and "out." To each boundary component associate a half infinite cylinder

$$Z_i = \begin{cases} S_i^1 \times (-\infty, 0] & b_i \text{ is an "in" boundary} \\ S_i^1 \times [0, \infty) & b_i \text{ is an "out" boundary} \end{cases}$$

We also require that the circumference of $S_i^1 = l_i$ where l_i is the length of the b_i. Hence there exists a canonical map $\alpha_i : b_i \to Z_i$ and we can construct a surface:

$$\Sigma_\Gamma = \left(\sqcup_{i=1}^n Z_i\right) / \sim$$

where

$$\alpha_i(x) \sim \alpha_j(x) \Leftrightarrow x \in b_i \cap b_j.$$

Such a surface has a natural map $\alpha : \Gamma \to \Sigma_\Gamma$ and away from the image $\alpha(\Gamma)$ inherits the natural cylindrical coordinates $z = s + it$ given to it by the cylinders. Figure 5 illustrates the construction of such a *simple surface* from a given graph.

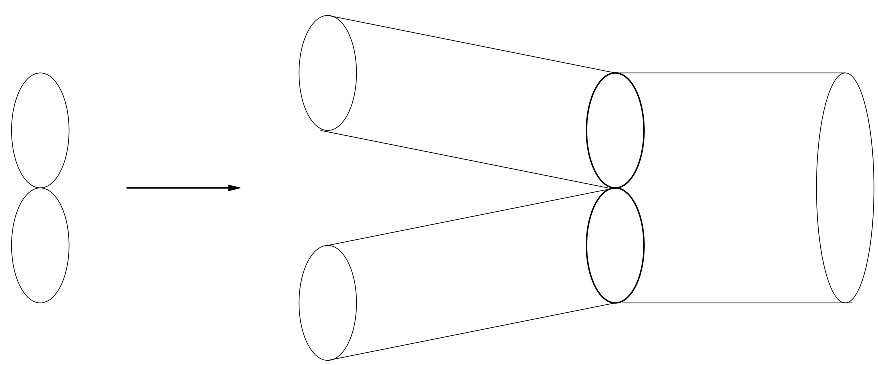


FIGURE 5. The construction of a simple surface from a figure eight graph.

Next we describe how to give such a surface a system of complex coordinates so that it has the structure of a Riemann surface. We give the charts explicitly and relate them to the natural cylindrical coordinates of the simple surface.

First, let Z_b be the half infinite cylinder corresponding to boundary component b. We use the coordinates $(s,t) \in S^1 \times (0,\infty)$ and identify them with $s + it = z \in \mathbb{C}$.

Next, each edge e of Γ has a direction given by the orientation of the boundary components. To this edge, consider a set

$$U_e = \{v = a + ib \in \mathbb{C} \,|\, 0 \leq a < \mu(e)\}$$

where μ is the metric on the graph. Let e_i be an ordered list of edges corresponding to an "out" boundary component b. There is a map

$$U_{e_n} \cap \{w \geq 0\} \rightarrow Z_b \tag{17}$$

$$v \mapsto z = v + \sum_{i=1}^{n-1} \mu(e_i). \tag{18}$$

A similar map exists for "in" cylinders and the lower part of each strip.

Next, to each vertex v of degree m, we have an ordered list of m edges $e_1, \ldots, e_m$ given to us by the cyclic ordering information of the ribbon graph. Consider the disk

$$U_v = \{w \in \mathbb{C} \,|\, |w| < L_v\}$$

where $L_v = \frac{1}{2} \min \{\mu(e_j)\}$ is half of the smallest edge length incident to the vertex V. Then for each $j = 1, \ldots, m$ we have a map

$$U_{e_j} \rightarrow U_v \tag{19}$$

$$v_j \mapsto w = \mathrm{e}^{2\pi i(j-1)/m} v_j^{2/m} \tag{20}$$

which attaches each edge strip to a vertex.

From (17) and (19) give the appropriate conformal coordinate changes for an atlas of maps

$$\bigcup_{v \in V} U_v \cup \bigcup_{e \in E} U_e \cup \bigcup_{b \in B} Z_b \rightarrow \Sigma_\Gamma \tag{21}$$

where V, E, and B are the vertex, edge, and boundary component sets of the graph Γ.

Next we describe the quadratic differential q with canonical coordinate that agrees with the atlas given. On each cylinder the local expression for q is simply $(\mathrm{d}z)^2$. Also, on each strip, $q = (\mathrm{d}v)^2 = (\mathrm{d}z)^2$ from (17). On U_V, (19) gives the local expression

$$q = (dz)^2 = \frac{m^2}{4} w^{m-2} (\mathrm{d}w)^2. \tag{22}$$

Note that this gives a zero for the quadratic differential for each point of the surface which corresponds to a vertex of the graph, as desired. Moreover, the quadratic differential q is clearly holomorphic on the surface.

In order to complete the lemma, we replace the cylindrical components with a unit disc for each boundary component. Consider the map

$$Z_b \quad \rightarrow \quad U_b = \{|u| \leq 1\} \tag{23}$$

$$z \quad \mapsto \quad u = \mathrm{e}^{\frac{2\pi i}{\mu(b)} z}. \tag{24}$$

Replacing Z_B with U_B in (21) gives a compact surface (essentially we have filled in the cylinder's "holes."). The quadratic differential now has the form

$$q = (\mathrm{d}z)^2 = -\frac{\mu(b)^2}{4\pi^2} \frac{(\mathrm{d}u)^2}{u^2} \tag{25}$$

and therefore has poles at n points where n is the number of boundary components of Γ. Each of the n points is labeled by a fixed value $\mu(b)$ where b is the corresponding boundary component. Hence the quadratic differential q has exactly the properties of the Strebel differential in Theorem 2.10 and must therefore be unique. The final point is that the critical horizontal trajectories corresponds exactly to the boundaries of the discs U_b and are connected together at the zeros of q according to the behavior of q on U_v. Therefore the critical horizontal trajectories are the image of the graph Γ, as required. $\square$

The point is that for a given metric graph, the coordinates given to us by the construction of a simple combinatorial surface are in complete agreement with a smooth Riemann structure. This greatly simplifies the requisite analysis in the next chapter.

3. Generalized Combinatorial Surfaces and Their Smooth Structure

We will now introduce the novel concept of the *generalized combinatorial surface*, which can be described as a surface obtained by connecting several simple combinatorial surfaces together. As a result we can associate to such a surface a canonical quadratic differential, based on the Strebel differentials of each simple surface. In this manner we extend the result of Lemma 2.12 to give a simple version of Theorem 2.11 for this new context.

The generalized combinatorial surface will rely on combinatorial information in two ways. The first is a collection of ribbon graphs which describe the simple combinatorial surfaces. The second is a directed metric graph which indicates how these surfaces will be attached together. We describe the second object now.

Consider a graph G. Recall that if G is not compact, then it will have some leaves. Let E and H be the set of edges and leaves, respectively, of G. A metric on a non-compact graph will be an assignment

$$\mu : E \cup H \to \mathbb{R}^{+} \cup \{\infty\}$$

with the requirement that the value of μ is ∞ if and only if it is evaluated on a leaf. Under this definition, we can think of the non-compact metric graphs as limit points for the space of compact metric graphs. That is, if we select an edge of a metric graph and ask that its length goes to infinity, we may say that the limit object is a non-compact metric fatgraph where the edge has been removed and two infinite length half edges have been attached to the graph, one at each vertex at which the original edge was incident.

A *directed* graph is a pair (G, Θ) where G is a graph with half edges $\mathcal{H}$ and

$$\Theta : \mathcal{H} \to \{\pm 1\},$$

is a partition of the half edges of G under the condition that for all non-leaves $h \in \mathcal{H} - H$ we have

$$\Theta(h)\Theta(\mathcal{E}(h)) = -1. \tag{26}$$

The partition of the half edges of a directed graph gives two sets

$$\mathcal{H}^{\pm} = \Theta^{-1}(\pm 1)$$

and so (26) gives a requirement that any edge of a directed graph consists of an element of $\mathcal{H}^-$ and an element of $\mathcal{H}^+$. We can think of this as an "out" half edge (element of $\mathcal{H}^+$) attached to an "in" half edge (element of $\mathcal{H}^-$) giving the direction, see Figure 6.

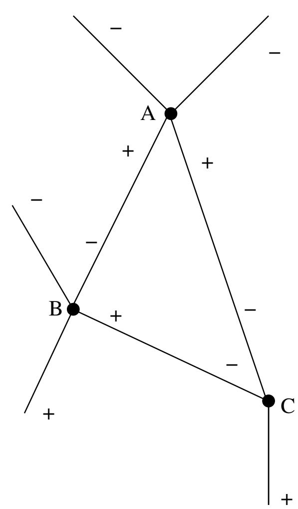


FIGURE 6. An example of a directed graph. Half edges are labeled by $\pm$ to indicate the direction. Note that an edge is composed of two half edges and therefore must have a '+' and a '$-$' label.

DEFINITION 2.13. An *A-graph* G is any directed metric graph.

An A-graph G will give us the basic (A=)assembly data for our generalized surface. Finite length edges will correspond to finite length cylinders, infinite length leaves will correspond to infinite cylinders. Hence each vertex of G will have to describe how each cylinder is to be attached. Therefore we will label each vertex v of G with a compact metric fatgraph Γ_v. Recall that for a given compact metric fatgraph, the boundary components can be partitioned into two disjoint classes: $\partial(\Gamma_v) = \partial^+(\Gamma_v) \sqcup \partial^-(\Gamma_v)$. Likewise, for each vertex of a graph, the half edges incident to that vertex, $\mathcal{H}_v$, can be partitioned into two disjoint sets:

$$\mathcal{H}_v^\pm = \mathcal{H}_v \cap \mathcal{H}^\pm.$$

We may now define:

DEFINITION 2.14. An *admissible graph system* is a triple

$$\mathbf{\Gamma} = (G, \{\Gamma_v\}, \gamma)$$

consisting of a directed metric graph G, a set of compact metric ribbon graphs $\{\Gamma_v\}$ and a map

$$\gamma : \bigsqcup_{v \in V(G)} (\{v\} \times \mathcal{H}_v) \to \bigsqcup_{v \in V(G)} \partial(\Gamma_v)$$

so that the following conditions hold:

(1) *The set $\{\Gamma_v\}$ is a labeling of the vertex set V of G*: The map $V \to \{\Gamma_v\}$ by $v \mapsto \Gamma_v$ is bijective.

(2) *γ maps "in" and "out" half-edges to "in" and "out" boundary components*: For each $v \in V$ the two restrictions of γ at v given by

$$\gamma_v : \mathcal{H}_v^+ \leftrightarrow \partial^+(\Gamma_v)$$

and

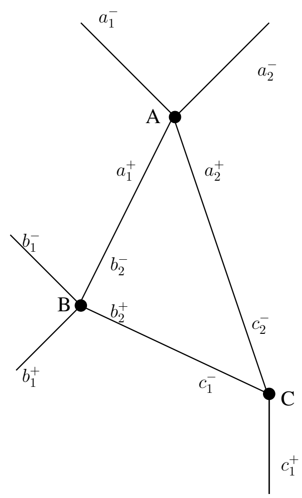
$$\gamma_v : \mathcal{H}_v^- \leftrightarrow \partial^-(\Gamma_v)$$

are bijections.

(3) *Corresponding boundary components must be of the same length*: Any edge $e \in E$ of G consists of an "in" and an "out"half edge, $e = \{h^+, h^-\}$. Let $v^\pm$ be the vertex that $h^\pm$ is incident to. Then if $\mu_{v^\pm}$ is the metric for the graph $\Gamma_{v^\pm}$ we require that

$$\mu_{v^+}(\gamma_{v^+}(h^+)) = \mu_{v^-}(\gamma_{v^-}(h^-)). \tag{27}$$

In order for the above definition to be well defined, we may require that the $\pm$ labeling of the boundary components to be additional information given to a metric graph, so that the choice of labels for the partitions $\partial^\pm$ of the boundary components of a given compact metric graph is not arbitrary. Figures 7 and 8 give an example of an admissible graph system.



$$\begin{cases} a_1^+ = b_2^- \\ a_2^+ = c_2^- \\ b_2^+ = c_1^-. \end{cases}$$

FIGURE 7. An A-graph G for an admissible graph system, compare with Fig. 6. The metric labels for the edges of G have been suppressed. The half edges for G have been labeled by $x_i^\pm$ where x indicates to which vertex the half edge is incident, $\pm$ indicates the direction of the half edge, and i enumerates the number of half edges with a given direction incident to a particular vertex.

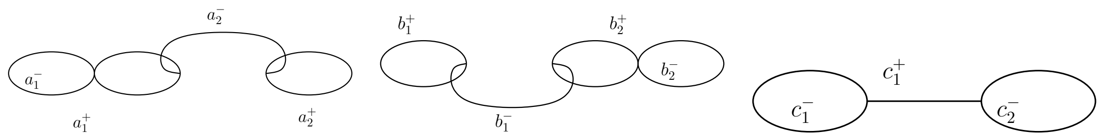


FIGURE 8. Admissible vertex graphs for the D-graph in Fig. 7. The boundary component labeling corresponds directly to the labeling of the half edges in Fig. 7. "Out" boundary components labeled '+' are those traced by moving counterclockwise and following the "right hand rule" at each vertex. Those labeled '−' are the "in" boundary components and are traced counterclockwise and follow the "left hand rule" at each vertex.

In addition to the metric μ_G on the A-graph G, once G is incorporated into an admissible graph system, we get a new labeling on the edges and leaves of G due to property (27):

$$\mu_\partial : E_G \cup H_G \to \mathbb{R}^+$$

given by

$$\mu_\partial(h) = \mu_v(\gamma_v(h)),$$

for any leaf $h \in H_G$ incident to a vertex $v \in V_G$; and

$$\mu_\partial(e) = \mu_{v^\pm}(\gamma_{v^\pm}(h^\pm)),$$

where the edge e is identified with its half-edges ($e \leftrightarrow \{h^\pm\}$) and $v^\pm$ are the vertices to which $h^\pm$ is incident. This is well defined, since boundary components of Γ_v's related by an edge of G must be the same length.

An admissible graph system is the essential combinatorial information that will form the base for the TFT construction. The first step in the construction is an assignment

$$\{\text{admissible graph system}\} \to \{\text{smooth Riemann surface}\}. \tag{28}$$

We proceed in direct analogue to the construction of simple combinatorial surfaces in Lemma 2.12.

THEOREM 2.15. *Given admissible graph data* $\mathbf{\Gamma} = (G, \{\Gamma_v\}, \gamma)$ *there exists a smooth Riemann surface* $\Sigma_{\mathbf{\Gamma}}$ *with a canonical quadratic differential* q *so that*

(1) q *is meromorphic on* $\Sigma_{\mathbf{\Gamma}}$. *The singularities of* q *are all quadratic poles. There is a one to one correspondence between the poles of* q *and the leaves of* G.

(2) *The critical horizontal trajectories of* q *on* $\Sigma_{\mathbf{\Gamma}}$ *is a set of measure zero and is isomorphic to* $\sqcup_v \Gamma_v$.

(3) *The critical horizontal trajectories of* q *partition* $\Sigma_{\mathbf{\Gamma}}$ *into a finite disjoint collection of* fundamental domains, $Z = \Sigma_{\mathbf{\Gamma}} - \mathrm{image}\,(\sqcup_v \Gamma_v)$. *There is a one-to-one correspondence between the components of* Z *and the set* $E_G \cup H_G$ *of edges and leaves of* G.

(4) *If* Z_l *is a component of* Z *corresponding to a leaf* l *at vertex* v *of* G *then, in the canonical coordinate given by* q, *it has a local representation as a half-infinite cylinder:*

$$Z_h = \begin{cases} Z^-_{\mu_v(\gamma_v(h))} = (\mu_v(\gamma_v(h))/2\pi) \cdot S^1 \times (-\infty, 0) & h \in H^- \\ Z^+_{\mu_v(\gamma_v(h))} = (\mu_v(\gamma_v(h))/2\pi) \cdot S^1 \times (0, \infty) & h \in H^+, \end{cases}$$

(5) *If* Z_e *is a component of* $\Sigma_{\mathbf{\Gamma}}$ *corresponding to an edge* e *of* G, *then, then in the canonical coordinate given by* q *it has a local representation of a finite length cylinder:*

$$Z_e = Z^{\mu_G(e)}_{\mu_\partial(e)} = (\mu_\partial(e)/2\pi) \cdot S^1 \times (-\mu_G(e)/2, \mu_G(e)/2)$$

A few comments before the proof: Theorem 2.15 essentially tells us that we can assemble a smooth Riemann surface from a collection of finite and infinite length cylinders. The type and number of these cylinders is given by a canonical quadratic differential q. Therefore we can think of q as the analytic manifestation of the combinatorial information $\mathbf{\Gamma}$. The cylindrical composition of $\Sigma_{\mathbf{\Gamma}}$ is important, since cylinders are the natural topological type for the Floer-style flows we will investigate in the Chapter 3.

PROOF. We give an explicit construction for the surface $\Sigma_{\mathbf{\Gamma}}$ first as a welded collection of finite and infinite cylinders. Next, we give it a conformal structure by describing it as open sets in $\mathbb{C}$ with complex-smooth transition maps, as in Lemma 2.12. Finally, we will give a representation of q in local coordinates given by each neighborhood and then verify the listed properties.

We begin by identifying the cylindrical components. For each "in" leaf $h^- \in H^-$ of G associate a complex set

$$U_{h^-} = \left\{ z_{h^-} = a + ib \in \mathbb{C} \,|\, 0 \leq a\mu_\partial h^+, -\infty < b \leq 0 \right\}.$$

Set $Z_{h^-} = U_{h^-}/\sim$ where $ib \sim \mu_\partial(h^-)$. Hence Z_{h^-} takes the form of a half infinite cylinder with circumference given by $\mu_\partial(h^-)$. Similarly we can construct a half infinite cylinder $Z_{h^+} \simeq S^1 \times [0, \infty)$ for any "out" leaf $h^+ \in H^+$. For any proper edge $e \in E$ of G associate a complex set

$$U_e = \left\{ z_e = a + ib \in \mathbb{C} \,|\, 0 \leq a \leq \mu_\partial(e), -\frac{1}{2}\mu_G(e) \leq b \leq \frac{1}{2}\mu_G(e) \right\}.$$

Again, construct a cylinder $Z_e = U_e/\sim$ where $ib \sim \mu_\partial(e) + ib$.

Next, we wish to assemble these cylindrical components together by welding them to the vertex graphs Γ_v according to the map γ_v. Let β_v be a boundary component of Γ_v for some $v \in V$. There exists a canonical map

$$\left\{ z = \mathrm{e}^{2\pi i s/\mu_v(\beta_v)} \in \mathbb{C} \,|\, s \in [0, \mu_v(\beta_v)] \right\} \rightarrow \beta_v$$

which we will describe as $s \mapsto \beta_v(s)$. β_v is the image of some half edge h of G under the map γ_v. There are several cases to consider. If $h = h^-$ is an"in" leaf of G then we will identify the boundary component β to the boundary of Z_{h^-} via

$$\beta_v(s) \sim (s, 0) \in Z_{h^-}. \tag{29}$$

If $h = h^+$ is an "out" leaf, then use

$$\beta_v(s) \sim (s, 0) \in Z_{h^+}. \tag{30}$$

In the case that h is not a leaf, then let e be the edge of G which is the orbit of h under the edge involution map. Then identify the boundary component β_v to a boundary of the finite cylinder Z_e by

$$\beta_v(s) \sim (s, -\Theta(h)\frac{1}{2}\mu_G(e)) \in Z_e. \tag{31}$$

Under these identifications, we can construct our formal combinatorial surface:

$$\left(\bigsqcup_{v\in V} \Gamma_v \sqcup \bigsqcup_{e\in E} Z_e \sqcup \bigsqcup_{h^\pm \in H^\pm} Z_{h^\pm}\right) / \sim$$

where the identification $\sim$ was described in (29) to (31). Note that the interiors of the cylinders Z_{h^+}, Z_{h^-}, Z_e will form the cylindrical decomposition of the surface $\Sigma_{\mathbf{\Gamma}}$ demanded in the theorem. See Figure 9 for an example of this construction.

Next, we will give a complete smooth conformal structure to the surface. Again we focus on a particular vertex of G. Let Γ_v be the ribbon graph associated to a vertex v of G. Let E_v, V_v, and B_v be the set of edges, vertices and boundary components of Γ_v, respectively. Let ε be an edge of E_v. Then ε is a member of at most two boundary components $\partial_\varepsilon^\pm \in B_v$. Each boundary component is related to a half edge $h_\varepsilon^\pm = \gamma_v^{-1}(\partial_\varepsilon^\pm)$, which is in the half-edge set of G. Fix a value

$$\ell_\varepsilon = \min\left\{\mu_G(h_\varepsilon^\pm)\right\}$$

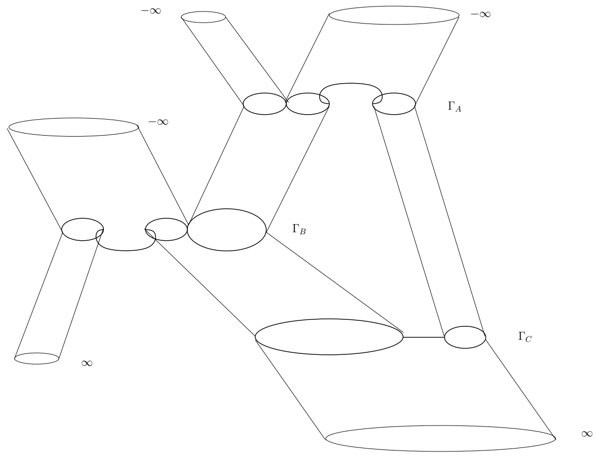


FIGURE 9. An example of a general combinatorial surface constructed from the admissible combinatorial data $(G, \{\Gamma_A, \Gamma_B, \Gamma_C\}, \gamma)$. The cylindrical components correspond precisely to the edges and leaves of the A-Graph in Figure 7. Those labeled by $\pm\infty$ are half infinite in length and correspond to the leaves of G. The other cylinders are of finite length given by the metric on G. The Γ's are taken from taken from Figure 8. Their shapes have been distorted slightly to indicate the agreement in lengths of corresponding boundary components.

where $\mu_G(h) = \mu_G(\{h, \mathcal{E}(h)\})$ is the length of the full edge associated to h under the edge involution $\mathcal{E}$ of G. Fix a second value

$$w_\varepsilon = \mu_{\Gamma_v}(\varepsilon),$$

the length of the edge under the metric for the vertex graph Γ_v. Now associate an open set

$$U_\varepsilon = \{z_\varepsilon = a + ib \in \mathbb{C} \,|\, 0 < a < w_\varepsilon, -\ell_\varepsilon < b < \ell_\varepsilon\}\,. \tag{32}$$

Next we focus on a vertex $\varphi \in V_v$. Let m be the degree of the vertex and number $\varepsilon_1, \ldots, \varepsilon_m$ be an ordering of the edges given by the ribbon information associated to the vertex φ. Define

$$R_\varphi = \frac{1}{2} \min_i \{\mu_{\Gamma_v}(\varepsilon_i)\}$$

to be half the length of the smallest edge incident to the vertex φ. Now consider the open disk

$$U_\varphi = \{w_\varphi \in \mathbb{C} \,|\, |w| < R_\varphi\}\,. \tag{33}$$

Finally, we complete the patch description by associating to each boundary component $\beta \in B_v$ an annulus (possibly a punctured disc) depending on the map γ_v. There are two cases to consider. First, $\gamma_v^{-1}(\beta)$ is a leaf. Then set

$$U_\beta = \{z_\beta \in \mathbb{C} \,|\, 0 < |z| < 1\}\,. \tag{34}$$

If $h = \gamma_v^{-1}(\beta)$ is not a leaf, let e be the edge of G described by the orbit of h under the edge involution map of G. Then set

$$U_\beta = \left\{ z_\beta \in \mathbb{C} \,|\, \mathrm{e}^{\frac{-2\pi\mu_G(e)}{\mu_\partial(e)}} < z_\beta < 1 \right\}. \tag{35}$$

Now we have a collection of open sets $U_\varepsilon, U_\varphi, U_\beta$ for each vertex v of G. The next step is to how to weld each of these open sets together and to the cylinders described previously.

Again we work at a fixed vertex v of G. Let φ be a degree m vertex of Γ_v with an ordering of incident edges $\varepsilon_1, \ldots, \varepsilon_m$ given by the ribbon graph structure of Γ_v. We conformally map neighborhoods of $U_{\varepsilon_j} \to U_\varphi$ by

$$z_{\varepsilon_j} \mapsto z_\varphi = \mathrm{e}^{\frac{2\pi i(j-1)}{m}} z_{\varepsilon_j}^{2/m}. \tag{36}$$

Next we map edge and boundary patches to the cylinders. We proceed in the following order, first we will map the patches (32) which correspond to the edges to the cylinders Z_e. Then we will map these cylinders to the boundary patch annuli (34) or (35).

Let $\varepsilon_1, \ldots, \varepsilon_p$ be an ordering of the edges of Γ_v which correspond to a boundary component β of Γ_v. If $\gamma^{-1}(\beta)$ is an "in" half edge of the type h^- then consider the sets

$$\widehat{U}_{\varepsilon_j} = U_{\varepsilon_j} \cap \mathrm{LHP},$$

where LHP is the open lower half plane of $\mathbb{C}$. Now we can map $\widehat{U}_{\varepsilon_j} \to Z_{h^-}$:

$$z_{\varepsilon_j} \mapsto z_{h^-} = \sum_{k=1}^{j-1} \mu_{\Gamma_v}(\varepsilon_k) + z_{\varepsilon_j}. \tag{37}$$

Note that an edge may be listed twice in a boundary component, so it may be true that $\varepsilon_{j_1} = \varepsilon_{j_2}$ for some $j_2 \neq j_1$. In this case, assume without loss of generality that $j_2 > j_1$ and replace the LHP with the upper half-plane (UHP) in the definition of $\widehat{U}_{\varepsilon_{j_2}}$. Then the map (37) is replaced with

$$z_{\varepsilon_{j_2}} \mapsto z_{h^-} = \sum_{k=1}^{j_2} \mu_{\Gamma_v}(\varepsilon_k) - z_{\varepsilon_{j_2}}.$$

If $\gamma^{-1}(\beta)$ is an "out" boundary component of the type h^+ then we repeat the above paragraph with UHP in place of LHP (and vice-versa) in the description. However, the form of the map (37) is exactly the same.

As a final step in defining the conformal structure on $\Sigma_{\mathbf{\Gamma}}$ we map each cylinder component to an annuli: $Z_{\gamma_v^{-1}(\beta)} \to U_\beta$. Again, focusing on a vertex v of G we consider a boundary component β and the corresponding half edge h of G given by γ_v. If h is a leaf, the we have a map

$$z_h \in Z_h \to z_\beta = \mathrm{e}^{\Theta(h)2\pi i z/\mu_{\Gamma_v}(\beta)} \in U_\beta.$$

If h is not a leaf , let e be the edge given as the orbit of h under the edge involution map of G. Then we have the map

$$z_e \in Z_e \to z_\beta = \mathrm{e}^{\frac{\Theta(h)2\pi i\left(z+\Theta(h)\frac{1}{2}\mu_G(e)\right)}{\mu_{\Gamma_v}(\beta)}}.$$

Combining sets U_ε, U_φ, and U_β gives us a local conformal structure in a neighborhood of the graph Γ_v for each vertex of G. Using the above relation between the boundary sets U_β and the cylinders Z_e and $Z_{h^\pm}$ we have a complete conformal atlas for $\Sigma_{\mathbf{\Gamma}}$ with the given smooth transition maps above. Hence $\Sigma_{\mathbf{\Gamma}}$ is a smooth Riemann surface, as required.

To finish the theorem, we need to give a quadratic differential, q, on the surface $\Sigma_{\mathbf{\Gamma}}$. We require that the canonical coordinate of q on each neighborhood of the atlas is agrees exactly with the natural coordinate given by the patch:

$$q = (\,\mathrm{d}z)^2.$$

In this case, we may appeal directly to formulas (25) and (22) in Lemma 2.12 to give the behavior of q on each of the coordinate patches. In particular, this gives all of the required properties of the theorem. $\square$

3.1. Surface Surgery: Extending the Combinatorial Construction. A fundamental feature of a topological field theory is the relation between topological surgery and operator composition. To describe this behavior in the context of combinatorial surfaces, we need to produce an analogous surgery construction on the level of admissible graph systems, $\mathbf{\Gamma}$. It turns out that the surgery constructions on the combinatorial and analytic levels will commute in the sense that

$$\Sigma_{\mathbf{\Gamma}^1} \widehat{\#}_\rho \Sigma_{\mathbf{\Gamma}^2} = \Sigma_{\mathbf{\Gamma}^1 \#_\rho \mathbf{\Gamma}^2},$$

where $\widehat{\#}_\rho, \#_\rho$ represent the analytic and combinatorial constructions, respectively.

We begin by considering a pair of model combinatorial surfaces and describe how a new surface can be constructed from these. We assume that each surface has corresponding half-infinite cylinders of the same circumference. Then the construction is straightforward: Take the semi-infinite "out" and "in" cylinders which are to be attached. Remove all but a finite length of each cylinder, giving two truncated surfaces. Attach these truncated surfaces together and reparametrize the new finite length cylinder according to the remaining lengths. Finally, repeat this process for every pair of cylinders to be attached.

We formalize the above description as follows. Consider a given combinatorial surface $\Sigma_{\mathbf{\Gamma}}$. Then there exist a number of "in" and "out" half-cylinders.

Definition 2.16. Two combinatorial surfaces are *gluable* provided that there exists an "in" cylinder from one surface and an "out" cylinder from the other each with the same circumference.

We may perform the surgery operation on two gluable surfaces at more than one place. Let o_n and i_n be a list of "out" and "in" cylinders so that the circumference of o_n and i_n are equal. We give *gluing data* $\rho = (\rho_1, \ldots, \rho_N)$, where $\rho_n = (o_n, i_n, R_n)$. We ask that the values $R_N \geq 0$. Then we have the so-called *linear gluing operation* described by Schwartz [**21**]

Definition 2.17. Given two gluable surfaces Σ^1, Σ^2 with gluing data ρ_n we give a map

$$(\Sigma^1, \Sigma^2) \mapsto \Sigma^1 \widehat{\#}_\rho \Sigma^2$$

defined as follows. Let

$$Z_n^a = Z_{R_n}^{a_n} \simeq \begin{cases} S^1 \times [R_n/2, \infty) & a = o \\ S^1 \times (-\infty, -R_n/2] & a = i \end{cases}$$

give the cylindrical parametrizations of open sets of either surface to be glued. Consider the truncated surfaces

$$\Sigma_\rho^k = \overline{\Sigma^k \setminus \bigcup_{Z_n^a \subset \Sigma^k} \mathring{Z}_n^a},$$

with $k = 1, 2$, and a map $f = (f_1, \ldots, f_N)$, with

$$f_n : S^1 \times \{R_n\} \subset Z_n^o \xrightarrow{\sim} S^1 \times \{-R_n\} \subset Z_n^i$$

an isomorphism. Then we may define

$$\Sigma^1 \widehat{\#}_\rho \Sigma^2 := \Sigma^1_\rho \cup_f \Sigma^2_\rho. \tag{38}$$

Definition 2.17 gives an appropriate surgery construction on the level of surfaces. It is clear that there is a smooth reparametrizations in a neighborhood of the glued circle in $\Sigma^1 \widehat{\#}_\rho \Sigma^2$ which will agree with the conformal structure of the original surfaces and so (38) gives a new Riemann surface. Moreover, since the definition relies solely on the existence of cylindrical coordinates in a neighborhood of "gluable" extremities of Riemann surfaces, it applies to our generalized class of combinatorial surfaces as well. We now describe the surgery operation on the combinatorial level.

Formally, we interpret Def. 2.17 as a manipulation of the admissible graph data for both surfaces to be glued. It is this interpretation that we choose in future applications of the operation. Recall from 2.14 that the data for a surface $\Sigma_{\mathbf{\Gamma}}$ is $\mathbf{\Gamma} = (G, \{\Gamma_n\}, \gamma)$, were Γ_n are metric ribbon graphs which label the vertices of the directed metric graph G and γ is a map which identifies the boundary components of the Γ_n to the half edges of G in an appropriate way. Essentially, the gluing of two surfaces will correspond to attaching corresponding half edges for two dual attaching graphs and labeling the new edge with a finite value. We provide the definition for gluing at a given pair of cylinders.

Consider a pair of admissible graph systems $\mathbf{\Gamma}^k = (G^k, \{\Gamma^k_n\}, \gamma^k)$, $k = 1, 2$. Let $h^+(1)$ be an isolated "out"-directed half-edge of G^1 and $h^-(2)$ be an isolated "in"-directed half edge of G^2. Given a triple $\rho = (h^+(1), h^-(2), R)$, consider the new graph $G^1 \#_\rho G^2$ defined by taking copies of both G's, attaching the specified half edges together, and considering a new metric which leaves the lengths of all unmanipulated edges and half edges alone, but defines the length of the new edge created as R. Formally: let G^k be two directed metric graphs (A-graphs). Define a new set of half edges

$$\widetilde{\mathcal{H}} = \mathcal{H}(G^1) \cup \mathcal{H}(G^2)$$

as the union of the half-edges of both graphs. Also, let

$$\widetilde{H}^{\pm} = H \pm (G^1) \cup H^{\pm}(G^2)$$

be the unions of the sets of "out" and "in" leaves. Choose a triple $\rho = (h^+, h^-, R)$ of $h^{\pm} \in \widetilde{H}^{\pm}$ and $R > 0$. We define a new metric directed graph $G^1 \#_\rho G^2$ on the set $\widetilde{\mathcal{H}}$ as follows:

- If $\mathcal{E}^k$, $k = 1, 2$ are the edge involution maps for the G^k, $k = 1, 2$, respectively, then we have a new edge involution $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{H}}$ by

$$\widetilde{\mathcal{E}}(h) = \begin{cases} h^- & h = h^+ \\ h^+ & h = h^- \\ \mathcal{E}^k(h) & h \in \mathcal{H}^k, \quad k = 1, 2. \end{cases}$$

  We will use the notation $\tilde{E}$ to represent the new set of edges created by this involution map. We point out that

$$\tilde{E} \simeq E^1 \sqcup E^2 \sqcup \{\tilde{e}\}$$

  where E^1 and E^2 are the edges of G^1 and G^2; and $\tilde{e}$ is the new half edge created by $\widetilde{\mathcal{E}}$.
- If $\mathcal{V}^k : \mathcal{H}^k \to V^k$, $k = 1, 2$ are the vertex maps for the G^k, set

$$\widetilde{V} = V^1 \sqcup V^2$$

  and without loss of generality we can identify $\widetilde{V}$ with a set

$$\left\{v_1, \ldots, v_{\#(V^1)+\#(V^2)}\right\}.$$

Give a new vertex map $\widetilde{\mathcal{V}} : \widetilde{\mathcal{H}} \to \widetilde{V}$ by

$$\widetilde{\mathcal{V}}(h) = \begin{cases} \mathcal{V}^1(h) & h \in \mathcal{H}^1 \\ \mathcal{V}^2(h) & h \in \mathcal{H}^2. \end{cases} \tag{39}$$

Therefore we have a description

$$\tilde{G} = (\widetilde{\mathcal{H}}, \widetilde{\mathcal{E}}, \widetilde{\mathcal{V}}) \tag{40}$$

of a proper graph.

We give $\tilde{G}$ the structure of a directed graph by the direction map $\tilde{\Phi} : \widetilde{\mathcal{H}} \to \{\pm 1\}$:

$$\tilde{\Phi}(h) = \Phi^k(h); \quad h \in \mathcal{H}^k,$$

for $k = 1, 2$ and ϕ^k the direction map on G^k. This is a well defined direction map since the new edge created, $\tilde{e}$, will be composed of an "out" and "in" half-edge.

We also give it a metric structure $\tilde{\mu} : \tilde{E} \cup \tilde{H} \to \mathbb{R}^+ \cup \{\infty\}$ on the set of edges $\tilde{E}$ and leaves $\tilde{H}^{\pm}$:

$$\tilde{\mu}(e) = \begin{cases} \mu^k(e) & e \in E^k \cup H^k \\ R & e = \tilde{e}. \end{cases}$$

Summarizing the above gives

Definition 2.18. Given a triple ρ described above, we have a gluing operation on the space of A-graphs given by

$$G^1 \#_\rho G^2 := (\tilde{G}, \tilde{\mu}, \tilde{\Phi}),$$

It is clear that $G^1 \#_\rho G^2$ has the structure of an A-graph.

Finally we extend the graph gluing operation to the full description of an admissible graph system so that he underlying A-Graph is given in Def. 2.18. The only restriction is the following:

Definition 2.19. Two admissible graph systems $\Gamma^k = (G^k, \{\Gamma^k_v\}, \gamma^k),\ k = 1, 2$ are *gluable* if there exists $h^\pm \in \tilde{H}^\pm$ so that if $h^\pm$ is incident to vertex $v^\pm$ then

$$\mu_{\Gamma^{k^+}_{v^+}}(\gamma^{k^+}_{v^+}(h^+)) = \mu_{\Gamma^{k^-}_{v^-}}(\gamma^{k^-}_{v^-}(h^-)),$$

where $k^\pm = 1, 2$ as required. In this case, the triple (h^+, h^-, R) for $R > 0$ is called an *admissible gluing triple.*

The point is that gluable systems must agree in terms of the metric lengths of boundary components which may be attached through the described gluing procedure.

Given two gluable systems we can create a vertex labeling by metric ribbon graphs will given by the set $\{\tilde{\Gamma}_n\} = \{\Gamma^1_{n^1}\} \sqcup \{\Gamma^2_{n^2}\}$ where we ask that the vertex labeling agree with the vertex map (39). Then, a new set bijection between the half edges of $G^1 \#_\rho G^2$ and the boundary components of the graphs in $\{\tilde{\Gamma}_n\}$ is given canonically by the the γ^k's:

$$\tilde{\gamma}_v(h) = \gamma^k_v(h)$$

where $v \in V^k$ and $h \in \mathcal{H}^k$. With these identifications, we have the

Definition 2.20.

$$\mathbf{\Gamma}^1 \#_\rho \mathbf{\Gamma}^2 := (G^1 \#_\rho G^2, \{\tilde{\Gamma}_n\}, \tilde{\gamma}).$$

Clearly, $\#_\rho$ gives an admissible graph system, provided $\mathbf{\Gamma}^k$ are both admissible systems themselves and ρ is an admissible triple of gluing data. Fig. 3.1 gives an elementary example graph gluing on the level of A-graphs.

In general, the gluing data will be a triple of the type $\rho = (\mathbf{h}^+(1), \mathbf{h}^-(2), \mathbf{R})$ where $\mathbf{h}^+(1)$ is a finite ordered list of "out" half edges of G^1, $\mathbf{h}^-(2)$ are "in" half edges for G^2, and $\mathbf{R}$ are positive values. Each list will have the same number of elements; their ordering naturally indicates which half edges are to be connected and what the metric value for the new edge should be. Hence, pairs of admissible graph systems can be glued at many "out"-"in" boundary pairs by repeating Definition 2.20 for each triple

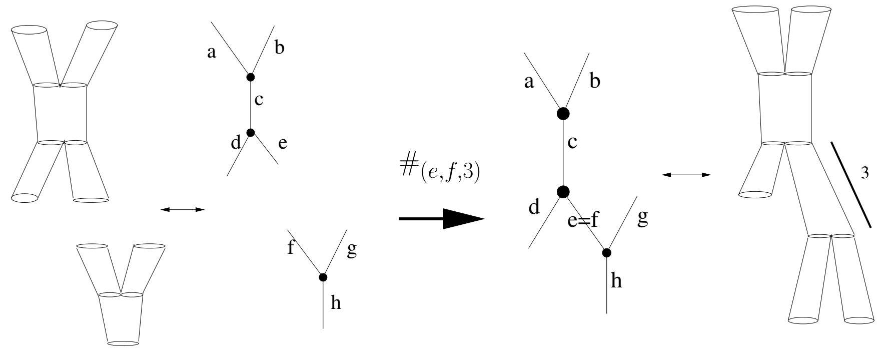


FIGURE 10. Attaching of two A-graphs and the corresponding gluing of two surfaces. In this case the gluing happens only at one pair of boundaries.

$\rho_n = (h_n^+(1), h_n^-(2), R_n) \in \rho$. It is understood that each triple ρ_k is admissible in the sense of Definition 2.19.

Since $\mathbf{\Gamma}^1 \#_\rho \mathbf{\Gamma}^2$ is an admissible system we can construct $\Sigma_{\mathbf{\Gamma}^1 \#_\rho \mathbf{\Gamma}^2}$ according to Theorem 2.15. The question is how does this construction compare to the linear gluing operation given in Def. 2.17. This is answered by the following:

THEOREM 2.21. *Let Γ^k, $k = 1, 2$ be two gluable graph systems with gluing data ρ. Then*

$$\Sigma_{\mathbf{\Gamma}^1 \#_\rho \mathbf{\Gamma}^2} \simeq \Sigma_{\mathbf{\Gamma}^1} \widehat{\#}_\rho \Sigma_{\mathbf{\Gamma}^2}.$$

The understanding is that the o_n and i_n labeling of cylindrical sets agree exactly with "out" and "in" leaves.

PROOF. The Theorem is true by the constructions described in Def. 2.17 and Def. 2.20. For completeness, we will point out new coordinate charts in neighborhoods of the surgery region of $\Sigma_{\mathbf{\Gamma}_1} \widehat{\#}_\rho \Sigma_{\mathbf{\Gamma}^2}$ which agree conformally with the original charts on $\Sigma_{\mathbf{\Gamma}_1}$ and $\Sigma_{\mathbf{\Gamma}_2}$. This chart is exactly the image of a finite length cylinder which would be prescribed by the proof of Thm. 2.15. Since all other regions of the surface or combinatorial data are not affected by the gluing, we need not worry about them.

For simplicity, let us assume that the gluing data $\rho = (o \sim h^+, i \sim h^-, R)$ is a single triple. In the general case, the argument given would be repeated for each triple.

The point is that the truncated surfaces, before gluing, have coordinate embeddings which are finite length cylinders of length $R/2$. Hence, after surgery, two such cylinders will give one of length R. Let $Z^{\pm}_\rho \subset \Sigma^k_\rho$ be the cylindrical components of the truncated surfaces of Def. 2.17, so

$$Z^*_\rho = \begin{cases} S^1 \times [0, R/2] & * = + \\ S^1 \times [-R/2, 0] & * = -. \end{cases}$$

We wish to show that $Z_\rho = S^1 \times [-R/2, R/2]$ is a closed neighborhood of the circle $S^1 \subset \Sigma_{\mathbf{\Gamma}_1} \widehat{\#}_\rho \Sigma_{\mathbf{\Gamma}^2}$ used by the surgery operation (38). This is clear by the embeddings:

$$\begin{aligned} Z^+_\rho &\to Z_\rho, \quad \text{by} \\ (s,t) &\mapsto (s - R/2, t), \end{aligned}$$

and

$$\begin{aligned} Z^-_\rho &\to Z_\rho, \quad \text{by} \\ (s,t) &\mapsto (s + R/2, t). \end{aligned}$$

In other words, there is a parametrization for the glued cylinder given by

$$\psi : [-\rho_k/2, \rho_k/2] \times S^1 \to \Sigma^1 \widehat{\#}_\rho \Sigma^2. \tag{41}$$

So we have that in a neighborhood of a glued circle, we have exactly the coordinates of a finite length cylinder Z_ρ. But this is exactly the local coordinate structure given by the construction of the surface $\Sigma_{\mathbf{\Gamma}^1\#_\rho\mathbf{\Gamma}^2}$for the cylindrical component assigned to the new edge of $G^1\#_\rho G^2$of metric length R given by Def. 2.20 and the proof of Thm. 2.15. □

4. Summary

This chapter introduces one of the basic objects of study in this work: a *generalized combinatorial surface* which can be constructed form an admissible graph system of graphs. The useful properties of such a surface relevant to the remainder of this work can be summarized in the following corollary of Theorem 2.15:

Corollary 2.22. The assignment $\mathbf{\Gamma} \mapsto \Sigma_{\mathbf{\Gamma}}$ gives a smooth surface which can be partitioned into a finite collection of open sets so that each set is a flat cylinder. The flat metric structure is given by the real coordinate structure $(s,t) \in S^1 \times I$ where I is an open, connected interval in $\mathbb{R}$. The cylinders can be chosen so that their number is equal to the number of edges and leaves of $G \in \mathbf{\Gamma}$. In this case, the length and circumference of each cylinder is given by the metric μ_G and the labeling μ_∂ on the graph G.

Note also that the cylindrical decomposition agrees completely with the coordinates used for a surface surgery in which pairs of infinite length cylinders are truncated and then glued together to create new surfaces with finite length cylindrical components. Therefore, by Thm. 2.21, the surgery can be interpreted as an operation on the combinatorial information given in an admissible graph system. This fact, and Cor. 2.22 will be important in the analytical work of the next chapter.

Since the eventual goal is the establishment of a functor Z which takes data $\mathbf{\Gamma}$ to operations on the Floer (co)homology of a symplectic manifold M which satisfy the properties of a TFT, we point out that the A-graph $G \in \mathbf{\Gamma}$ plays the role of the cobordism (X_1, Σ, X_2) of Def. 1.2. Let $H^{\pm} \subset \mathcal{H}$ be the collection of "in" and "out" directed leaves of G. Then the object (X_1, Σ, X_2) will be replaced by the triple $(H^-, \mathbf{\Gamma}, H^+)$.

The combinatorial surface construction described offers a number of interesting avenues for future research, which all are related to the idea of simplifying the combinatorial information to give a more elegant surface construction. Perhaps the best place to begin would be to characterize equivalences between $\mathbf{\Gamma}$s in order to give an analogue of Thm. 2.11. The theory of quadratic differentials is the appropriate way to approach such a result. Typically, one chooses to reduce a Riemann surface into either a collection of punctured discs (= semi-infinite cylinders) or a collection of annular regions (= finite length cylinders) and in either case there exists a unique quadratic differential so that the critical trajectories give embedded graphs. However, the approach of mixing the decompositions, which would be the case of the canonical quadratic differential of Thm. 2.15, is not known to the author.

Another idea in simplifying the construction would be to reduce the combinatorial information to an A-graph G with a vertex labeling by ribbon graphs which give only pairs of pants, instead of general surfaces. In particular this would give a simplify a definition for the concept of a *combinatorial bordism* (H^-, G, H^+) introduced earlier.

The other approach would be to attempt to eliminate the extra information of the A-graph altogether. In order to do this one would need to understand well how the surface surgery given in Def. 2.17 acts on ribbon graphs. That is, let Σ_1, Σ_2 be two Riemann surfaces with punctures decorated by positive real values. Then to each surface one has a unique metric ribbon graphs Γ_1, Γ_2 given by the unique Strebel differentials on those surfaces. For admissible gluing data ρ, construct a new graph $\Gamma(\rho)$ which is given by the unique Strebel differential on $\Sigma_1 \widehat{\#}_\rho \Sigma_2$. Therefore the exists a map:

$$\rho : BRG^{\mathrm{met}}_{g_1,n_1} \times BRG^{\mathrm{met}}_{g_2,n_2} \to BRG^{\mathrm{met}}_{g(\rho),n(\rho)} \tag{42}$$

given by $(\Gamma_1, \Gamma_2) \mapsto \Gamma(\rho)$. Understanding the construction of $\Gamma(\rho)$ directly from the pair (Γ_1, Γ_2) would allow us to replace much of the framework of Schwarz in the next Chapter with an explicitly combinatorial one.

CHAPTER 3

Construction of the Topological Field Theory

There has been much interest in studying the mathematical structure of cylindrical flows based on a single ribbon graph ($\mathcal{M}$) under gradient flows on a symplectic manifold. Their interest is to give a relationship between geometric invariants (i.e. Gromov-Witten invariants) and combinatorial invariants of the type given by string topological constructions. The major aim of this thesis is to provide some first steps in this direction by offering a topological field theoretic construction for $\mathcal{M}$ based on combinatorial data, at both the chain and (co)homology level, for a symplectic manifold. To do so, we construct a version of the spaces $\mathcal{M}$ using the semi-infinite dimensional Morse theory established by Andreas Floer. These spaces are modeled on the general combinatorial surfaces given in Chapter 2 and so will rely only on the combinatorial data $\mathbf{\Gamma}$.

This Chapter is divided into three parts. The first is a summary of the work of M. Schwarz and others on a 2-D TFT based on smooth surfaces, which is constructed by means of generalized Floer operators whose zero sets satisfy a gluing theorem similar to (10). Next, we define $\mathcal{M} = \mathcal{M}_{\mathbf{\Gamma}}$, the moduli space of cylindrical Floer flows based on data $\mathbf{\Gamma}$, and give a differential operator $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}$ so that $\mathcal{M}_{\mathbf{\Gamma}}$ is the kernel of this operator. We then verify that this operator is of the same class studied by Schwartz and that the corresponding gluing theorem agrees with the graph attachment operation of Def. 2.20. Finally we extend the results of Schwarz to conclude the existence of a map on the space of admissible graph systems which satisfies the properties of a 2-D TFT.

1. The Generalized Floer Operator on Smooth Surfaces

For the discussion of this section, we work with a *model* surface Σ which is smooth, connected, compact, and oriented so that each component of the boundary, $\partial\Sigma$, is homeomorphic to a copy of S^1. We presume that $\pi_0(\partial\Sigma)$ is finite. Let $\partial\Sigma_i$ denote the ith element of $\partial\Sigma$. We further require that for each $\partial\Sigma_i$ there exists a smooth embedding

$$\psi_i : Z^i \hookrightarrow \Sigma$$

where $Z^i = [0, \pm\infty) \times S^1$ with sign determined by the orientation of that particular boundary component. Note that each of the $1 \leq n \leq N$ cylinders $Z^n \hookrightarrow \Sigma$ can be identified with a subset of $\mathbb{R} \times S^1 \simeq \mathbb{C}/i\mathbb{Z}$ and hence the subset $\psi_n(Z^n) \subset \Sigma$ gains the conformal structure given by i. It is presumed that

- $\psi_i(Z^i) \cap \phi_j(Z^j)$ is empty if $i \neq j$.
- $\psi_i(s, \cdot) \to \psi_{i,\infty} : S^1 \xrightarrow{\sim} \partial\Sigma_i$ as $s \to \pm\infty$ in $C^0(S^1, \Sigma)$.
- The interior of the surface comes with a conformal structure j which agrees with i on the cylindrical ends. That is

$$j(\phi_n(s,t)) \circ \mathrm{d}\phi_n(s,t) = \mathrm{d}\phi_n(s,t) \circ i$$

  for each $1 \leq n \leq N$, $(s,t) \in Z^n$.

We use the notation $\Sigma_Z = \cup\phi_i(Z^i)$.

The following proposition is important and establishes that the combinatorial construction of Sec. 3 gives such a model surface.

Proposition 3.1. Given admissible graph data $\mathbf{\Gamma}$, there exist a surface $\bar{\Sigma}_{\mathbf{\Gamma}}$ with the property of being a model surface such that $\mathring{\Sigma}_{\mathbf{\Gamma}} = \Sigma_{\mathbf{\Gamma}}$.

Proof. This is a corollary of Thm. 2.15. We must compactify the surface by adding boundary components for each of the cylindrical components which correspond to a leaf of G. Recall that conformally, each of these cylinders is of the type $[0, \infty) \times$

$S^1 \simeq \widehat{D}$, where $\widehat{D}$ is the punctured open unit disc. Therefore, we cannot compactify the surface in a manner that agrees with this topology, since this would correspond to "filling the hole" in the punctured unit disc and there would not be an cobordism with S^1 boundaries. Instead we simply consider a surface $\bar{\Sigma}_{\mathbf{\Gamma}}$, by gluing copies of S^1 to cylindrical ends. Hence $\Sigma = \bar{\Sigma}_{\mathbf{\Gamma}}$ is a real manifold with boundary satisfying the properties of a model surface whose open interior has the structure of a Riemann surface. $\square$

For the remainder of this work we will also use a symplectic manifold with ω-compatible almost complex structure: (M, ω, J) as introduced in Chap. 1. The class of almost complex structures used in the discussion below is defined on the space $\Sigma \times M$, that is, it will be allowed to vary with $z \in \Sigma$. To wit, using the projection $\pi : \Sigma \times M$ we ask that $J \in C^\infty_\Sigma(\pi^* \operatorname{End}(TM))$, along with the property that $J^2 = -I$ and $\omega(\cdot, J(z, \cdot))$ is a Riemann metric on TM for each $z \in \Sigma$.

1.1. Fundamental Mapping Spaces. Before introducing the generalized Floer operator, care must be given in defining the appropriate mapping spaces.

DEFINITION 3.2. For a model surface Σ, a map $f : \Sigma \to \mathbb{R}$ is C^k_Σ*-smooth* if $f \in C^k(\mathring{\Sigma}, \mathbb{R})$ and extends smoothly to the boundary in the sense that

$$(f \circ \psi_i)\left(\frac{\pm s}{\sqrt{1-s^2}}, t\right) = \phi_{i|[0,1)}(s,t)$$

for a function $\phi_i \in C^k([0,1] \times S^1, \mathbb{R})$.

This definition extends to maps $\Sigma \to N$, for M a smooth manifold by using the local atlas of M.

If Σ has $1, \ldots, N$ boundary components, let

$$\mathbf{x} = (x_i) \in \prod_{n=1}^{N} C^\infty(S^1, M).$$

Such a list is a *boundary labeling* of Σ by $C^\infty(S^1, M)$. We define

$$C^\infty_{\mathbf{x}}(\Sigma, M) = \{h \in C^\infty_\Sigma(M) \,|\, \phi_i(1,\cdot) = x_i\}\,, \tag{43}$$

which forms the basic space upon which the rest will be built.

In the original work of Floer, the operator acts on the space u^*TM for a map $u : Z \to M$. We build the analogous spaces next, with care in establishing the appropriate Sobelev structures. Let $\mathrm{Vec}^*_{C^\infty}(\Sigma)$ be the set of vector bundles which are C^∞_Σ-smooth with trivial restrictions to $\partial\Sigma$. Let $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ with smooth trivializations ϕ_i of $\xi_{|\psi_i(Z^i)}$. The following spaces of sections with Sobolev structure will be of interest:

$$H^{k,p}_\Sigma(\xi) = \left\{ s \in H^{k,p}_{\mathrm{loc}}(\mathring{\Sigma}, \xi) \,|\, \phi_i(s \circ \psi_i) \in H^{k,p}(Z^i, \mathbb{R}^n) \right\}, \tag{44}$$

with $p > 2$. Now there is a completion of the the space of curves given by (43) to the Banach manifold on which the generalized Floer operator will act. The space will be modeled on the Banach spaces given in (44).

Recall the important property that $g := \omega(\cdot, J\cdot) : TM^2 \to \mathbb{R}$ is a smooth Riemannian structure. Therefore we have a unique exponential map (exp) associated to the Levi-Cevita connection for g. For the tangent bundle $\pi : TM \to M$ there exists an associated set $\mathcal{D}$ called the *injectivity neighborhood* of exp so that

$$\begin{aligned} \mathcal{D} &\hookrightarrow M \times M \\ v &\mapsto (\pi(v), \exp(v)) \end{aligned}$$

is a diffeomorphism onto a neighborhood of the diagonal $\Delta \subset M \times M$. Now, suppose $u \in C^\infty_\Sigma(M)$ so that $u_{|\partial\Sigma} \subset L_0 M$. Then the pullback space

$$u^*TM = \{(z, V) \in \Sigma \times TM \,|\, u(z) = \pi(V)\}$$

is an element of $\mathrm{Vec}^*_{C^\infty}(\Sigma)$ and we can define the following:

Definition 3.3. Fix a model surface Σ and a boundary labeling $\mathbf{x} \subset L_0 M$. Then set

$$\mathcal{P}^{1,p}_{\mathbf{x}}(\Sigma, M) := \left\{ \exp_u v \in C^0(\Sigma, M) \,|\, v \in H^{1,p}_\Sigma(u^*\mathcal{D}), u \in C^\infty_{\mathbf{x}}(\Sigma, M) \right\}.$$

It is this set $\mathcal{P} = \mathcal{P}^{1,p}_{\mathbf{x}}(\Sigma, M)$ that will be the domain of the generalized Floer operator. We note that there is a canonical association

$$T\mathcal{P} \simeq H^{1,p}_\Sigma(\mathcal{P}^* TM),$$

where

$$H^{k,p}_\Sigma(\mathcal{P}^* TM) := \cup_{u \in \mathcal{P}} H^{1,p}_\Sigma(u^* TM).$$

1.2. The Generalized Floer Operator. Schwartz' generalized Floer operator is defined as a smooth section of a Banach vector bundle over the Banach manifold $\mathcal{P} = \mathcal{P}^{1,p}_{\mathbf{x}}(\Sigma, M)$ described in Def. 3.3. Consider the product space $\Sigma \times M$ and bundles

$$\begin{aligned} L &= T^*\Sigma \otimes TM \text{ with fiber} \\ L_{z,m} &= Hom(T_z\Sigma, T_m M) \end{aligned}$$

and

$$\begin{aligned} X^J &= T^{0,1}\Sigma \circ_J TM \subset L \text{ with fiber} \\ X^J_{z,m} &= \{\phi \in L_{z,m} \,|\, \phi \circ j(z) = -J(z,m) \circ \phi\}. \end{aligned}$$

The bundle X^J is the bundle of (j, J)-antiholomorphic homomorphisms and so there is a map

$$\begin{aligned} \Lambda_J : C^\infty_\Sigma(L) &\to C^\infty_\Sigma(X^J) \\ \phi &\mapsto \phi + J \circ \phi \circ j. \end{aligned} \tag{45}$$

The Banach vector bundles we wish to examine are pullbacks of L, X^J by $\mathcal{P}$:

$$L^p_\Sigma(\mathcal{P}^*E) = \Cup_{u\in\mathcal{P}}\{u\} \times L^p_\Sigma(u^*E) \tag{46}$$

where $E = L$ or X^J. In Appendix A of [**21**], Schwartz establishes that 3.3 has the structure of an infinite dimension separable Banach manifold. Furthermore the spaces (46) form smooth Banach space bundles over $\mathcal{P}$. These facts are established using techniques of Eliasson in [**6**]. Note also that the map (45) extends to a map $L^p_\Sigma(\mathcal{P}^*L) \to L^p_\Sigma(\mathcal{P}^*X^J)$.

We are now prepared to introduce the generalized Floer operator, modeled on the Floer operator (7):

$$\bar\partial_{J,H} := \partial_s + J\partial_t + \nabla H_t.$$

Let us fix the almost complex structure J.

Definition 3.4. Fix an almost complex structure J. Recall the notation

$$Z^n_T = [T,\infty) \times S^1$$

for $1 \le n \le N$ truncated cylindrical components of Σ. A function $k \in C^\infty_\Sigma(L)$ is called *admissible* if there exists a positive value of $T \in \mathbb{R}$ so that when restricted to the nth cylinder Z^n_T, k depends only on t and has the form

$$k_{|Z^n_T}(t,m) = -\,\mathrm{d}t \otimes X_{H^n_t}(m) \tag{47}$$

for all $(t,m) \in S^1 \times M$ where $H^n : S^1 \times M$ is a regular Hamiltonian in the sense of Def. 1.4. For a given admissible k, a set $H = \{H^n\}$ for which (47) holds is a *Hamiltonian labeling* of k.

Recall the definition of the set $\mathcal{C}(H)$ of critical loops of a Hamiltonian H given in (3). If $x_i \in \mathcal{C}(H^i)$ then for an admissible k we have the property that

$$\lim_{s\to\infty} k(\phi_i(s,t), x_i(t) = -\,\mathrm{d}t \otimes \frac{dx_i}{dt}(t).$$

Now, fix a model surface Σ with boundary components enumerated by $1 \leq n \leq N$. Given an admissible k with Hamiltonian label H and a list $\mathbf{x} = (x_n)$ with $x_n \in \mathcal{P}(H^n)$, we consider the operator

$$\begin{array}{rcl} C^\infty_{\mathbf{x}}(\Sigma, M) & \to & C^\infty_\Sigma(u^*L) \\ u & \mapsto & \mathrm{d}u + k(u). \end{array} \tag{48}$$

Schwarz establishes that the operator (48) extends uniquely to the level of the Banach manifold $\mathcal{P}$ and associated bundle $L_\Sigma(\mathcal{P}^*L)$, finally giving the following definition for the generalized Floer operator:

$$\bar{\partial}_{J,k} = \Lambda_J \circ (d + k) : \mathcal{P} \to L^p_\Sigma(\mathcal{P}^* X^J) \tag{49}$$

as a smooth section in a the vector bundle $L^p_\Sigma(\mathcal{P}^*X^J)$ over $\mathcal{P}$.

An important result concerning the $\bar{\partial}_{J,k}$ operator is the following restatement of Thm. 4.1.2 in [**20**], which gives a complete characterization of the zero set of the operator.

Theorem 3.5. *Let k be admissible with a Hamiltonian label $H = (H^1, H^2, \ldots)$. Fix a list $\mathbf{x} = (x_n)$ with $x_n \in \mathcal{C}(H^n)$. Then every solution $u \in \mathcal{P}$ to the equation $\bar{\partial}_{J,k}(u) = 0$ is an element of $C^\infty_\Sigma(M)$. Moreover, such a u has the property that its restriction to the nth cylinder, $u_n = u_{|Z^n}$, has the property*

$$\lim_{s\to\infty} u_i(\pm s, t) \to x_n(t) \text{ in } C^1(S^1)$$

where the sign is chosen appropriately with relation to the parametrization of the cylinder Z^n.

Thm. 3.5 is established by applying the techniques originally used by Floer in [**8**], [**7**], [**9**] to the generalized operator: Smoothness of elements in the zero set is given by the ellipticity of the generalized operator and applying a bootstrapping argument, see McDuff and Salamon [**14**]. Convergence of boundaries to fixed critical loops is a result of establishing a bound on the energy of the solutions and applying Thm. 1.5 to the restrictions u_n.

A final point of interest is the fact that the linearization of the operator $\bar{\partial}_{J,k}$ at a zero solution u is Fredholm and therefore has a well defined index (Compare to Sec. 3. We will use the following definition in the description of the TFT constructions:

Definition 3.6. The *local dimension* of the zero set $\{\bar{\partial}_{J,k} u = 0\}$ at u is

$$\dim_{\mathrm{loc}} u = \operatorname{ind} D_u$$

where D_u is the linearization of $\bar{\partial}_{J,k}$ at u.

1.3. Gluing Zero Sets of Generalized Operators. We use the notation

$$\mathcal{M}_{\mathbf{y}}^{\mathbf{x}}(\Sigma, J, k(H)) = \left\{ u \in C_{\Sigma}^{\infty}(M) \,|\, \bar{\partial}_{J,k}(u) = 0 \right\}$$

where $\mathbf{x} \cup \mathbf{y} \subset \mathcal{P}(H)$ are the critical “in” and “out” loops corresponding to the boundary components of Σ. We will introduce a gluing theorem analogous to Eq. 10. Stated roughly, the general rule is that elements of moduli spaces can be glued together provided the “out” boundary components on one agree with the “in” boundary components of the other. We formalize this idea by examining a pair of spaces:

$$\mathcal{M}^i = \mathcal{M}_{\mathbf{y}^i}^{\mathbf{x}^i}(\Sigma^i, J^i, k^i(H^i)) \quad i = 1, 2.$$

DEFINITION 3.7. Two kernel spaces $\mathcal{M}^{1,2}$ are said to be *gluable* provided the set $\mathbf{y}^1 \cap \mathbf{x}^2 \neq \emptyset$ (alternatively $\mathbf{y}^2 \cap \mathbf{x}^1 \neq \emptyset$) and that for any element $x \in \mathbf{y}^1 \cap \mathbf{x}^2$ ($\in \mathbf{y}^2 \cap \mathbf{x}^1$) we have the following condition: Let Z^i, $i = 1, 2$ be the boundary cylinder corresponding to x in Σ^1 and Σ^2 respectively, with restrictions Z^i_T begin that part of the cylinder for which $s > T$. Then there exists $T_0 > 0$ so that

$$k^i(H^i)_{|Z^i_T} = H^i \in C^\infty(S^1 \times M, \mathbb{R})$$

for any $T > T_0$. We require that

$$(J_{|Z^1_T}, H^1) = (J_{|Z^2_T}, H^2)$$

are equal.

Without loss of generality, we will always assume that $\mathcal{M}^{1,2}$ are gluable and that the "out" labeling of $\mathcal{M}^1$ will be glued to the "in" labeling of $\mathcal{M}^2$.

Now, it is clear that if $u \in \mathcal{M}^1$ and $v \in \mathcal{M}^2$, then there exists a new function

$$u\#v : \Sigma \to M$$

for some model surface Σ. However, the interesting point is that with the addition of an additional parameter, ρ, which controls the gluing at each boundary component, we can construct a map,

$$u\#_\rho v : \Sigma^1 \widehat{\#}_\rho \Sigma^2 \to M$$

with $\widehat{\#}_\rho$ the linear gluing map of Def. 2.17, so that $u\#_\rho v$ is in the zero set of some generalized Floer operator based on the surface $\Sigma^1 \widehat{\#}_\rho \Sigma^2$. We construct this operator by specifying, in addition to the surface on which it is based, the almost complex structure J, the admissible k, and the specific lists of critical loops $\mathbf{x}, \mathbf{y}$.

DEFINITION 3.8. Suppose $\mathcal{M}^1$ and $\mathcal{M}^2$ are gluable at a single critical loop x. Label the "out" cylinder of Σ^1 whose boundary corresponds to x by o. Likewise, label the appropriate "in" cylinder of Σ^2 by i. Let $R > 0$ be large enough that the functions k are admissible in the sense of Def. 3.4. Then for a single triple of gluing data, $\rho = (o, i, R)$ as described by Def. 2.17, we have the following definitions:

- $\mathbf{x} = \mathbf{x}^1 \#_\rho \mathbf{x}^2 = \mathbf{x}^1 \cup \mathbf{x}^2 \setminus \{x\}$ gives a new list of "in" loops. $\mathbf{y} = \mathbf{y}^1 \#_\rho \mathbf{y}^2 = \mathbf{y}^1 \cup \mathbf{y}^2 \setminus \{x\}$ gives a new list of "out" loops.
- There is a canonical almost complex structure

$$J = J^1 \#_\rho J^2 \in C^\infty_{\Sigma^1 \widehat{\#}_\rho \Sigma^2}(\pi^* \operatorname{End}(TM))$$

  and a canonical $k(H) = k(H^1) \#_\rho k(H^2)$ which are given by the pairwise relationship in Def. 3.7.

It is clear that the definitions extend if there are many "in"-"out" pairs by repeating the above at every gluing triple.

With this notation fixed, we offer the following restatement of Thm. 4.4.22 in [**21**].

Theorem 3.9. *Consider compact subsets of gluable kernel spaces*

$$K_1 \times K_2 \subset \mathcal{M}^1 \times \mathcal{M}^2,$$

with a labeling of the gluable boundary loops by $(\mathbf{o}, \mathbf{i})$*. We assume that the number of pairs to be glued is* N*. Then there exists a constant* $\rho_o = \rho_o(K_1, K_2)$ *and a family of smooth embeddings parametrized by* $\mathbf{R} \in [\rho_0, \infty)^N$*:*

$$\begin{aligned} \#_{(\mathbf{o},\mathbf{i},\mathbf{R})} : K^1 \times K^2 &\to \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^1 \widehat{\#}_{(\mathbf{o},\mathbf{i},\mathbf{R})} \Sigma^2, J, k(H)) \\ (u, v) &\mapsto u \#_{(\mathbf{o},\mathbf{i},\mathbf{R})} v \end{aligned}$$

with $\mathbf{x}, \mathbf{y}, J, k(H)$ *given by Def. 3.8. The map* $\#_{(\mathbf{o},\mathbf{i},\mathbf{R})}$ *has the following properties:*

(1) $\#_{(\mathbf{o},\mathbf{i},\mathbf{R})}$ *is a local diffeomorphism.*

(2) *If* $\mathbf{R}_k$ *is a sequence so that* $\mathbf{R}_{n,k} \overset{n\to\infty}{\longrightarrow} \infty$ *for* $1 \leq n \leq N$ *then*

$$u\#_{(\mathbf{o},\mathbf{i},\mathbf{R}_n)}v \rightharpoonup (u,v).$$

(3) *If*

$$w_n \in \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^1 \widehat{\#}_{(\mathbf{o},\mathbf{i},\mathbf{R}_n)}\Sigma^2, J, k(H))$$

with $w_n \rightharpoonup (u,v)$ *as* $n \to \infty$ *then there exist*

$$(u_n, v_n) \in \mathcal{M}^{\mathbf{x}^1}_{\mathbf{y}^1}(\Sigma^1, J^1, k^1) \times \mathcal{M}^{\mathbf{x}^2}_{\mathbf{y}^2}(\Sigma^2, J^2, k^2)$$

so that $u_n \to u$ *and* $v_n \to v$ *uniformly on compact sets and there is* $N \in \mathbb{Z}^+$ *so that if* $n \geq N$ *then* $w_n = u_n\#_{(\mathbf{o},\mathbf{i},\mathbf{R}_n)}v_n$.

The notion of convergence ($\rightharpoonup$) used in (2) and (3) of Thm. 3.9 is *geometric convergence of first degree*:

Definition 3.10. With the notation established in Thm. 3.9

$$w_n \in \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^1 \widehat{\#}_{(\mathbf{o},\mathbf{i},\mathbf{R}_n)}\Sigma^2, J, k(H))$$

converges geometrically of first degree to a broken solution

$$(u, v) \in \mathcal{M}^{\mathbf{x}^1}_{\mathbf{y}^1}(\Sigma^1, J^1, k^1) \times \mathcal{M}^{\mathbf{x}^2}_{\mathbf{y}^2}(\Sigma^2, J^2, k^2)$$

for $n \to \infty$ if:

(1) $R_{n,k} \to \infty$ for each $1 \leq n \leq N$.

(2) When restricted to the truncated surfaces (see Def. 2.17), the sequences

$$\begin{aligned} w_{n|\Sigma^1_{(\mathbf{o},\mathbf{i},\widehat{\mathbf{R}})}} &\to u_{|\Sigma^1_{(\mathbf{o},\mathbf{i},\widehat{\mathbf{R}})}} \\ w_{n|\Sigma^2_{(\mathbf{o},\mathbf{i},\widehat{\mathbf{R}})}} &\to v_{|\Sigma^2_{(\mathbf{o},\mathbf{i},\widehat{\mathbf{R}})}} \end{aligned}$$

both converge uniformly on compact sets for any $\widehat{\mathbf{R}} \in (\mathbb{R}^+)^N$. See Fig. 1.

(3) On the the kth glued cylinder with parametrization ψ_k given by Eq. (41) we have

$$w_n \circ \psi_k \longrightarrow x_k$$

in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1, M)$ for each $1 \leq k \leq N$, where x_k is the critical loop at which the two moduli agree. See Fig. 2.

Note that the definition of geometric convergence is valid only under the condition that the two zero sets are gluable.

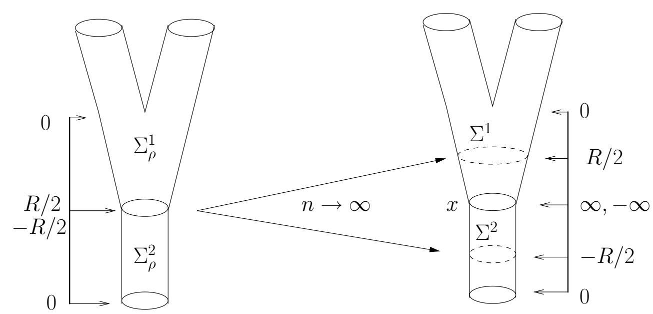


FIGURE 1. An illustration of property (2) of the definition of geometric convergence. The surfaces used are a pair of pants (Σ^1) and a single cylinder (Σ^2). The left figure is the image of the truncated surfaces under w_n. The right figured is the image of both surfaces under u and v meeting at a critical loop. In both cases the coordinates corresponding to the truncation are used.

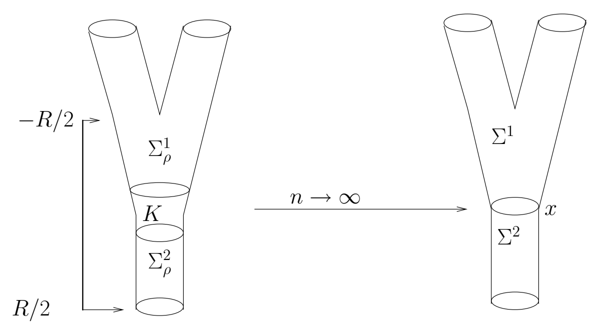


FIGURE 2. An illustration of property (3) of the definition of geometric convergence. The surfaces used are a pair of pants (Σ^1) and a single cylinder (Σ^2). The left figure is the image of the truncated surfaces under w_n. The right figured is the image of both surfaces under u and v meeting at a critical loop. The image of a compact set K of the glued surface converges to the critical loop x.

1.4. TFT Construction. We now give a brief description of the cohomological operations on tensor products of the Floer cohomology of a symplectic manifold M with $\mathbb{Z}_2$ coefficients (see Sec. 3), which is the major result of Schwarz in [**21**]. For a model surface Σ, these operations can be thought of as the image of a functor Z which satisfies the conditions of a TFT (see Sec. 2.

The framework is the following: Consider a model surface Σ with conformal structure j which agrees with the induced conformal structure of the cylindrical ends. For a closed symplectic manifold (M, ω) consider a compatible almost complex structure on the family $\Sigma \times M \to M$. Let $k \in C^{\infty}_{\Sigma}$ be admissible. We require that the pair (J, k) satisfy the properties:

- The linearization of $\bar{\partial}_{J,k}$ at a zero is an onto operator.
- The restriction of the pair (J, k) to the n-th cylindrical end gives a regular pair indexed by n: (J^n, H^n).

We also restrict to the case of a strictly monotone symplectic manifold, that is, M satisfies the property that the symplectic volume of spheres is exactly zero, and so the value of the first Chern class on spheres is also zero. See the discussion in Sec. 3.

Let Σ have a "in" and b "out" cylindrical components. We will define an operation

$$Z(\Sigma, j, J, k) : HF(M)^{\otimes a} \to HF(M)^{\otimes b}$$

and give the necessary results which indicate that Z behaves as a functor, is independent of the information j, J, k; depending only on the topological information of Σ, and has the properties of a topological quantum field theory.

First we need a well-defined "count" on the number of elements of the zero sets $\mathcal{M}$ which connect critical loops together. We require the following Thm. 4.3.19 in [**21**], which is a result of the gluing property described by Thm. 3.9.

THEOREM 3.11. *Let $\pi_2(M) \to H_2^S(M) \subset H_2(M, \mathbb{Z})$ by the Hurewicz homomorphism. Fix a collection of critical loops $\mathbf{x}$, $\mathbf{y}$ with allowable choices for J, k. Then the* *set*

$$\left\{ u \in \mathcal{M}_{\mathbf{x}}^{\mathbf{y}}(\Sigma, j, J, k) \,|\, \{\bar{u}\} \in H_2^S(M), \dim_{\text{loc}} u = 0 \right\}$$

is finite. The notation $\{\bar{u}\}$ indicates the associated homology class of the image of the map u.

The theorem is established by verifying a dimension additivity property for glued moduli spaces when restricted to fixed homology classes of zero solutions.

Consider the following Floer cochains of M:

$$CF_i^* = CF^*(J^i, H^i)$$

with J^i, H^i the restrictions of J, k to the ith boundary cylinder. For simplicity, these chains are graded $\mathbb{Z}_2$ vector spaces

$$CF_i^k = \operatorname{span}_{\mathbb{Z}_2} \left\{ x_i \in \mathcal{P}(H^i) \,|\, \mu_{CZ}(x_i) = k \right\}$$

where μ_{CZ} is the Conley-Zhender index of Sec. 3. The tensor product gives the following complexes with the indicated grading:

$$(CF_1^* \otimes \cdots \otimes CF_a^*)^k = \bigoplus_{\sum_{i=1}^a k_i = k} CF_1^{k_1} \otimes \cdots \otimes CF_a^{k_a}$$

The operations $\delta_i : CF_i^* \to CF_i^{*+1}$ of Floer cohomology induce a coboundary operation on these tensor product spaces in an obvious way:

$$\delta = \bigoplus_{i=1}^{a} I \otimes \cdots \otimes I \otimes \delta_i \otimes I \otimes \cdots \otimes I$$

where δ_i is in the ith position of the tensor product and I is the identity map.

Let $\mathbf{x}$ and $\mathbf{y}$ be critical loops labeling the boundary components of the surface Σ which agrees with the Hamiltonian labeling of k. Thm. 3.11 then gives us the following well defined count:

Definition 3.12. Define the $\mathbb{Z}_2$-*number*

$$\langle \mathbf{x}; \mathbf{y} \rangle_{\Sigma,j,J,k} := \#\left\{u \in \mathcal{M}_{\mathbf{x}}^{\mathbf{y}}(\Sigma, j, J, k) \,|\, \{\bar{u}\} \in H_2^S(M), \dim_{\mathrm{loc}} u = 0\right\} \quad \bmod 2.$$

This count is used to establish a linear operator between tensor complexes of the type considered above. Label the "in" cylinders by $1, \ldots, a$ and the "out" cylinders by $a+1, \ldots, a+b$. Then there is a map

$$\mathcal{O}(\Sigma, j, J, k) : CF_1^* \otimes \cdots \otimes CF_a^* \to CF_{a+1}^* \otimes \cdots \otimes C_{a+b}^*$$

by

$$x_1 \otimes \cdots \otimes x_a \mapsto \sum_{\{(y_1,\ldots,y_b \,|\, y_i \in \mathcal{P}(H^{i+a})\}} \langle x_1, \ldots, x_a; y_1, \ldots, y_b \rangle_{\Sigma,j,J,k}\, y_1 \otimes \cdots \otimes y_b$$

which can be extended via linearity.

It can be verified that the operator $\mathcal{O}$ commutes with the coboundary operation $\mathcal{O} \circ \delta = \delta \circ \mathcal{O}$ by following the steps of the proof that the relation $\partial \circ \partial = 0$ in Floer's original work, replacing the needed information about attaching cylindrical solutions with the zero set gluing statement Thm. 3.9. Therefore, if we use the Künneth identification

$$H^*(CF_1^* \otimes \cdots \otimes CF_a^*; \delta) \simeq H^*(C_1^*; \delta_1) \otimes \cdots \otimes H^*(CF_a^*; \delta_a),$$

we have the operation $Z(\Sigma, j, J, k) = \mathcal{O}^*$ which maps

$$HF^*(H^1, J^1) \otimes \cdots \otimes HF^*(H^a, J^a) \to HF^*(H^{1+a}, J^{1+a}) \otimes \cdots \otimes HF^*(H^{a+b}, J^{a+b}). \tag{50}$$

The operation Z will be the essential component of our topological field theory, the idea being to replace the data Σ, j, J, H with a type derived to agree with our combinatorial description in Chap. 2.

However, our goal is for the field theoretic construction to rely solely on the combinatorial data $\mathbf{\Gamma}$. This is possible, since in fact, the operation Z depends only on the surface Σ and NOT on any data which can be varied continuously while leaving the asymptotic data fixed (ref. Thms. 5.2.8, and 5.3.1 in [**21**]. In particular, this refers to the choice of conformal extension j, the almost complex structure J on $\sigma \times TM$, and the admissible k. These results are established by examining the invariance of the operation Z under homotopies of the families (j, J, k), which is the idea used by Floer to establish the independence of the cohomology classes HF^* on the choice of J, H. Combining all of these result together we have an operation

$$Z(\Sigma) : \bigotimes_{i=1}^{a} HF^*(M, \mathbb{Z}_2) \to \bigotimes_{i=1}^{b} HF^*(M, \mathbb{Z}_2). \tag{51}$$

The point is that as long as the data restricted to the boundary cylinders is fixed, any choice of j, J, k will do. This will be of use to us, since we will give specific values to these items based on a labeling of the data $\mathbf{\Gamma}$.

We restate for completeness Thm. 5.4.11 in [**21**] which establishes the TQFT properties of Z.

THEOREM 3.13. *Let (M, ω) be a closed symplectic manifold with the properties that $\phi_\omega, \phi_c = 0$, then there is a functor Z with the following properties:*

(1) *$S \mapsto Z(S)$ where S is a compact oriented 1-dimensional submanifold of M and $Z(S)$ is a $\mathbb{Z}$-graded $\mathbb{Z}_2$-vector space. Specifically, S consists of a copies of S^1 and*

$$Z(A) = HF^{*\otimes a}.$$

(2) *For any cobordism (A, Σ, B) which consists of a compact oriented surface Σ with $\partial\Sigma = A \cup B$, then $\Sigma \mapsto Z(\Sigma) : Z(A) \to Z(B)$ is given by Eq. 51.*

This Z satisfies the axioms of a TQFT given in Def. 1.2.

2. The Moduli Spaces of Cylindrical Graph Flows

We now return to the combinatorial structures considered in Chap. 2. Let $\mathbf{\Gamma}$ be an admissible graph system, as described in Def. 2.14. Let $\Sigma_{\mathbf{\Gamma}} = \bar{\Sigma}_{\mathbf{\Gamma}}$ as described in Prop. 3.1. We introduce the following notation for well well defined mapping spaces:

$$\begin{aligned} C^k_{\mathbf{\Gamma}}(M) &:= C^k_{\Sigma_{\mathbf{\Gamma}}}(M) \text{ and} \\ H^{k,p}_{\mathbf{\Gamma}}(M) &:= H^{k,p}_{\Sigma_{\mathbf{\Gamma}}}(M). \end{aligned}$$

By Cor. 2.22 in Chap. 2, a map u in either space above can be thought of as a collection of maps of appropriate type:

$$u_e = u_{|Z_e} \in H^{k,p}(Z^e, M) \text{ (or } C^k(Z^e, M)),$$

where $Z^e \subset \Sigma_{\mathbf{\Gamma}}$ is the finite (or semi-infinite) length cylinder corresponding to the edge (or leaf) e of $G \in \mathbf{\Gamma}$. We will describe an analogue of Floer's moduli spaces described in Section 3 by asking that each of the u_e's above satisfy the flow of equation (7) for a family of periodic Hamiltonians (H^e).

Specifically, let $\mathrm{Ham}(S^1 \times M, \mathbb{R})$ be the set of Hamiltonian functions with non-degenerate periodic solutions to (2). For an admissible system $\mathbf{\Gamma}$, let $\widehat{E} = E \cup H$ be the union of the edges and leaves of the A-graph $G \in \mathbf{\Gamma}$. Consider a labeling

$$H^{\mathbf{\Gamma}} = \left\{ e \mapsto H^e \in \mathrm{Ham}(S^1 \times M, \mathbb{R}) \,|\, e \in \widehat{E} \right\}$$

of the set $\widehat{E}$ by Hamiltonians. Without loss of generality, we assume that the period of the Hamiltonian related to (half-)edge e is $\mu_\partial(e)$, that is, the period is the circumference of the cylinder $Z_e \subset \Sigma_{\mathbf{\Gamma}}$. We can consider the set $H^{\mathbf{\Gamma}}$ as a single piecewise function, also called $H^{\mathbf{\Gamma}}$, on the product space $\Sigma_{\mathbf{\Gamma}} \times M$ by

$$H^{\mathbf{\Gamma}}(z,m) := \begin{cases} H_t^e(w) & s+it \leftrightarrow z \in \mathring{Z}_e \\ 0 & \text{otherwise.} \end{cases}$$

Note that $H^{\mathbf{\Gamma}} = 0$ exactly on the subset of $\Sigma_{\mathbf{\Gamma}}$ which corresponds to the union of the vertex ribbon graphs Γ_v of $\mathbf{\Gamma}$.

Now, $H^{\mathbf{\Gamma}}$ gives us a list of Hamiltonian functions to use in mapping the cylindrical components of a surface into the manifold M. However, they must agree, in some sense, in neighborhoods of the intersection of the cylinders: the graphs $\Gamma_v \in \mathbf{\Gamma}$. Let $R_0 \subset R_1$ be two non-equal neighborhoods of $\cup_v \Gamma_v \subset \Sigma_{\mathbf{\Gamma}}$ so that there exists a function $\phi_{R_0,R_1} \in C^\infty(\Sigma_{\mathbf{\Gamma}} \times M, [0,1])$ which is constantly zero and one in R_0 and outside of R_1 respectively. Let $\mathbf{H} = \mathbf{H}_{(\mathbf{\Gamma},R_0,R_1)} \in C^\infty(\Sigma_{\mathbf{\Gamma}} \times M, \mathbb{R})$ be a smooth real valued function defined by

$$\mathbf{H}(z,w) = \phi(z) H^{\mathbf{\Gamma}}(z,w). \tag{52}$$

We use the notation $\mathbf{H}^e = \mathbf{H}_{|Z_e}$ and point out that $\mathbf{H}^e = H^e$ away from $\cup_v \Gamma_v$.

We are now ready to give a formal definition to the moduli space of cylindrical flows. Let $h \in H$ be a leaf of G. These label the "in" and "out" boundary components of the combinatorial data $\mathbf{\Gamma}$. Consider the set $\mathbf{x} = \{x_h\}$ with

$$x_h \in \mathcal{C}(H^h).$$

Definition 3.14. Let $\mathbf{\Gamma}$ be an admissible graph system with a Hamiltonian label, $\mathbf{H}$, and a critical loop list, $\mathbf{x}$, described above. Then the *moduli space of cylindrical flows* for $(\mathbf{\Gamma}, \mathbf{H}, \mathbf{x})$ is a set

$$\mathcal{M}_{\mathbf{\Gamma}}^{\mathbf{x}}(J, \mathbf{H}) \subset C_{\mathbf{\Gamma}}^\infty(M)$$

is the space of smooth maps from $\Sigma_{\mathbf{\Gamma}} \to M$ so that for each $u \in \mathcal{M}$ the restrictions $u_e = u_{|Z_e}$ satisfy the following:

(1) For each $e \in \widehat{E}$

$$\frac{\partial u_e}{\partial s} + J_t(u_e)\frac{\partial u}{\partial t} + \nabla \mathbf{H}^e_t(u_e) = 0.$$

(2) For each $h^{\pm} in H^{\pm}$

$$\lim_{s\to\pm\infty} u_{h^{\pm}}(s,t) = x_{h^{\pm}} \text{ in } C^1(S^1).$$

We now prove

Proposition 3.15. Let $(\mathbf{\Gamma}, \mathbf{H}, \mathbf{x})$ be given as in Def. 3.14. Then there exists an admissible $k(H^{\mathbf{\Gamma}}) \in C^{\infty}_{\mathbf{\Gamma}}(L)$ with Hamiltonian label $H^{\mathbf{\Gamma}}$ so that

$$\{\bar{\partial}_{J,k}(u) = 0\} = \mathcal{M}^{\mathbf{x}}_{\mathbf{\Gamma}}(J, \mathbf{H}).$$

The point is that the moduli space of cylindrical flows is exactly the kernel of a generalized Floer operator. This allows us to use the results of Sec. 1 to our moduli space of cylindrical flows.

Proof. This is a result of the Def. 52 of $\mathbf{H}$ and Thm. 3.5. Since $\mathbf{H}$ is smooth with respect to $\Sigma_{\mathbf{\Gamma}}$ we can define k locally by

$$k_{Z^e} = -\,\mathrm{d}t \otimes X_{\mathbf{H}_{Z^e}} \tag{53}$$

for the cylindrical coordinates on each cylinder Z^e. This gives a well defined definition for k on all of $\Sigma_{\mathbf{\Gamma}}$ since from the definition of $\mathbf{H}$ we have $k \equiv 0$ on the intersection of the cylinders (the images of $\Gamma_v \hookrightarrow \Sigma_{\mathbf{\Gamma}}$), which is a coordinate free expression.

In particular, let $U = Z^h$ for a leaf h of $G \in \mathbf{\Gamma}$. Recall that these half edges correspond exactly to the boundary cylinders of $\Sigma_{\mathbf{\Gamma}}$. There exists $0 < T_h\mathbb{R}$ with the property that $\mathbf{H}_{|Z^h_{T^h}} = H^h_{|Z^h_{T^h}}$. Let $T = \max_h T^h$ and we see that k is admissible. We use the notation $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}$ for $\bar{\partial}_{J,k}$ when $k(H^{\mathbf{\Gamma}})$ is given by (53).

Next we show that $\{\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}} = 0\} = \mathcal{M}^{\mathbf{x}}_{\mathbf{\Gamma}}(J, \mathbf{H})$. Let $u \in \ker \bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}$. Then by Thm. 3.5, u satisfies condition 2 of Def. 3.14 for $\mathcal{M}^{\mathbf{x}}_{\mathbf{\Gamma}}(J, \mathbf{H})$. Therefore, the final step is to show that $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) = 0$ is equivalent to satisfying the cylindrical flow equations in condition 1 of Def. 3.14. This is established through the following computation. Note that $X^J \xrightarrow{\sim} TM$ by evaluation at $\frac{\partial}{\partial s}$. Therefore in any coordinate neighborhood $s + it \mapsto z \in U \subset \Sigma_{\mathbf{\Gamma}}$

$$
\begin{aligned}
\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) \cdot \frac{\partial}{\partial s} &= \Lambda_J(\,\mathrm{d}u + k(\mathbf{H}) \cdot \frac{\partial}{\partial s} \\
&= \mathrm{d}u \cdot \frac{\partial}{\partial s} + k(\mathbf{H}) \cdot \frac{\partial}{\partial s} + J(z,u) \circ (\,\mathrm{d}u + k(\mathbf{H})) \circ j(z) \cdot \frac{\partial}{\partial s} \\
&= \frac{\partial u}{\partial s} + k(\mathbf{H}) \cdot \frac{\partial}{\partial s} + J(z,u) \circ (\,\mathrm{d}u + k(\mathbf{H})) \cdot \frac{\partial}{\partial t}.
\end{aligned}
$$

Restrict to the set $U = Z^e$, so k takes the form (53). Then we continue with

$$
\begin{aligned}
\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) \cdot \frac{\partial}{\partial s} &= \frac{\partial u}{\partial s} + (-\,\mathrm{d}t \otimes X_{\mathbf{H}^e_{s,t}}(u)) \cdot \frac{\partial}{\partial s} + J(z,u) \circ (\,\mathrm{d}u - \,\mathrm{d}t \otimes X_{\mathbf{H}^e_{s,t}}(u)) \cdot \frac{\partial}{\partial t} \\
&= \frac{\partial u}{\partial s} + J(z,u)\frac{\partial u}{\partial t} + \nabla \mathbf{H}^e_{s,t} u,
\end{aligned}
$$

Where we use the notation $\mathbf{H}^e = \mathbf{H}_{|Z^e}$.

From the above calculation, it is clear that if $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) = 0$ then condition 1 of Def. 3.14 holds. On the other hand, if the condition holds on the cylinders, then $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) \cdot \frac{\partial}{\partial s} = 0$. Since $-J \circ \Lambda_J = \Lambda_J \circ j$ we have $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) \cdot \frac{\partial}{\partial t}$ as well. Therefore $\bar{\partial}_{J,\mathbf{\Gamma},\mathbf{H}}(u) = 0$ in general. $\square$

3. Summary

The main result of this Chapter as follows:

THEOREM 3.16. *Let (M,ω) be a closed symplectic manifold with the property that the fundamental maps $\phi_\omega, \phi_{c_1} = 0$. Then there is a functor Z with the following properties:*

(1) *For a finite collection of leaves H with a common direction, we have $H \mapsto Z(H) = (HF^*)^{\otimes a}$, where a is the size of the set H.*
(2) *Let $\mathbf{\Gamma}$ be an admissible graph system. Let H^+, H^- be the free "in" and free "out" leaves of $G \in \mathbf{\Gamma}$. Then we have*

$$\mathbf{\Gamma} \mapsto Z(\mathbf{\Gamma}) : (HF^*)^{\#(H^+)} \to (HF^*)^{\#(H^-)}.$$

Moreover, the functor Z satisfies the properties of a TQFT given by Def. 1.2 when generalized to the graph setting in an obvious way.

PROOF. The theorem is a collection of the results of Thms. 2.21, Cor. 2.22 in Chapter 2, as well as Prop. 3.15 and Schwarz' result given as Thm. 3.13. $\square$

3.1. Outlook. There are several obvious refinements in the theory constructed in this chapter. The first would be an inclusion of Eq. 42 into the construction, which would simplify a great deal the combinatorial information. However, it is not clear how this would affect the local definitions of the generalized operator $\bar{\partial}_{J,k}$. Resolving this issue was a primary motivation for the generalized combinatorial surface construction of Chap. 2.

Also, one may wish to relax the continuity condition on k constructed in Prop. 3.15 in the following sense. Suppose to a given admissible system $\mathbf{\Gamma}$ one gave an additional labeling of Hamiltonians to the edges and leaves of $G \in \mathbf{\Gamma}$. Then it is possible to construct a moduli space of cylindrical flows without first creating a piecewise operator k based on the Hamiltonians. In this case, one should no longer expect that solutions to $\bar{\partial}_{J,k}(u) = 0$ to be smooth, or even continuous, in neighborhoods of the embedded graphs $\Gamma_v \hookrightarrow \Sigma_{\mathbf{\Gamma}}$. An important question would be to derive conditions on the Hamiltonians which would give some regularity (at least continuity) to weak solutions of $\bar{\partial}_{J,k}(u) = 0$ in neighborhoods of the ribbon graphs.

These questions and those of Sec. 4 are important in order to understand how to apply the major results of this work in investigating a combinatorial approach to the study of perturbed holomorphic curves. For example, in the application of investigating the relationship between Gromov-Witten invariant theory and string topology as in Conj. 1.1. However, a major result would be to derive a fully combinatorial version of Gromov-Witten theory as follows.

The geometric picture of Gromov-Witten theory according to Kontsevich and Manin in [**13**] can be summarized in the following manner. Let (M, ω, J) be a compact symplectic manifold with ω-tame almost complex structure J. Fix a homology class $A \in H_2(M;\mathbb{Z})$ and consider the compactified moduli space $\overline{mcM}_{g,k}(A;J)$ of j-holomorphic curves in M with k marked points and genus g which represents the homology class A. Using the natural maps of evaluation at the marked points, we have the following diagram:

$$\begin{array}{ccc} \bar{\mathcal{M}}_{g,k}(A;J) & \xrightarrow{\text{ev}} & M^k \\ {\scriptstyle\pi}\big\downarrow & & \\ \bar{\mathcal{M}}_{g,k} & & \end{array} \tag{54}$$

In the case that the three objects in 54 are compact smooth manifolds, we can define the Gromov-Witten invariant homomorphism

$$\mathrm{GW}^M_{g,k,A} : H^*(M;\mathbb{Q})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{g,k};\mathbb{Q}) \to \mathbb{Q}. \tag{55}$$

A combinatorial version would be to replace the cohomology of M with the Floer cohomology $HF^*(M,\mathbb{Q})$ and to replace the (g,k) moduli space by Thm. 2.11:

$$\mathrm{GWComb}^M_{g,k,A} : HF^*(M;\mathbb{Q})^{\otimes k} \otimes H_*(\overline{BRG}^{\mathrm{met}}_{g,k};\mathbb{Q}) \to \mathbb{Q}$$

Such a result would require a refinement of the results of this paper and a notion of the compactified space of ribbon graphs, which is related to the work of Zuniga in [**24**].

Bibliography

1. A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Communications on Pure and Applied Mathematics **59** (2006), no. 2, 254–316.
2. L. Abrams, *Two-dimensional topological quantum field theories and frobenius algebras*, J. Knot Theory Ramifications **5** (1996).
3. M. Atiyah, *Topological quantum field theories*, Publications Mathématiques de L'IHÉS **68** (1988), no. 1, 175–186.
4. K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory*, 2006, arXiv:0706.3284.
5. R.L. Cohen, K. Hess, and A.A. Voronov, *String topology and cyclic homology*, Birkhauser, 2006.
6. H. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry **1** (1967), 169–194.
7. A. Floer, *A relative Morse index for the symplectic action*, Communications on Pure and Applied Mathematics **41** (1988), no. 4, 393–407.
8. ———, *Morse theory for Lagrangian intersections*, J. Differential Geom **28** (1988), no. 3, 513–547.
9. ———, *The unregularized gradient flow of the symplectic action*, Communications on Pure and Applied Mathematics **41** (1988), no. 6, 775–813.
10. A. Floer and H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry*, Mathematische Zeitschrift **212** (1993), no. 1, 13–38.
11. J. Harer, *The cohomology of the moduli space of curves, Theory of Moduli, edited by E. Sernesi, LNM 1337*, 1988.
12. J.A. Jenkins, *On the existence of certain general extremal metrics*, Annals of Mathematics **3** (1957), 440–453.
13. M. Kontsevich and Y. Manin, *Gromov-witten classes, quantum cohomology, and enumerative geometry*, Communications in Mathematical Physics **164** (1994), no. 3, 525–562.
14. D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society, 2004.

15. R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Communications in Mathematical Physics **113** (1987), no. 2, 299–339.

16. D. Salamon, *Lectures on floer homology*, Symplectic Geometry and Topology (Y. Eliashberg and L. Traynor, eds.), IAS/Park City Mathematics Series, vol. 7, A.M.S., 1999, pp. 145–229.

17. D. Salamon and J. Weber, *Floer homology and the heat flow*, Geometric And Functional Analysis **16** (2006), no. 5, 1050–1138.

18. ———, *J-holomorphic curves in cotangent bundles and Morse theory on the loop spaceduke mathema*, Duke Mathematical Journal **133** (2006), no. 3, 527–568.

19. D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian equations and the Maslov index*, Comm. Pure Appl. Math **45** (1992), 1303–1360.

20. L. Schneps, *The Grothendieck theory of dèssins d'enfants*, London Mathematical Society Lecture Notes, vol. 200, Cambridge University Press, 1994.

21. M. Schwarz, *Cohomology operations from s^1-cobordisms in floer homology*, Ph.D. thesis, Swiss Federal Institute of Technology Zurich, 1995.

22. K. Strebel, *Quadratic differentials*, Springer, 1984.

23. W.P. Thurston, *Three-dimensional geometry and topology. Vol. 1, volume 35 of Princeton Mathematical Series*, 1997.

24. J. Zuniga, *Compactifications of moduli spaces and cellular decompositions*, Ph.D. thesis, University of Minnesota, 2007.