Asymptotic Properties of Positive Solutions of Parabolic Equations and Cooperative Systems with Dirichlet Boundary Data

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ABSTRACT

We study symmetry properties of non-negative bounded solutions of fully nonlinear parabolic equations on bounded reflectionally symmetric domains with Dirichlet boundary conditions. First we consider scalar case, and we propose sufficient conditions on the equation and domain, which guarantee asymptotic symmetry of solutions. Then we consider fully nonlinear weakly coupled systems of parabolic equations. Assuming the system is cooperative we prove the asymptotic symmetry of positive bounded solutions. To facilitate an application of the method of moving hyperplanes, we derive several estimates for linear parabolic equations and systems, such as maximum principle on small domains, Alexandrov- Krylov estimate and Harnack type estimates.

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Contents

Chapter 1

Introduction

A natural question arising in mathematics, physics, and many other disciplines is *whether a symmetry of a problem implies the symmetry of solutions*. If the solution is unique, then the answer to this question is yes. Indeed, if we transform a solution by a symmetry of the problem, then we obtain another solution, that, by the uniqueness, must be equal to the original one. Hence, the solution is invariant under the transformation and therefore it is symmetric.

However, there are many important problems that either do not possess the unique solution or the uniqueness is not known. For several of these problems, special methods proving symmetry of solutions were developed or counterexamples showing nonsymmetric solutions were constructed. Usually, an affirmative answer to our question provides a better understanding of the behavior of solutions, their qualitative properties and it often reduces the complexity of the problem.

In this work we restrict our attention to problems formulated in the form of parabolic partial differential equations. For such problems we investigate symmetry properties of positive solutions. Our techniques are based on the method of moving hyperplanes, maximum principle, Harnack inequalities, and constructions of suitable subsolutions and supersolutions.

There are other results using different methods to prove symmetry of solutions of parabolic problems as a consequence of other properties, such as stability [44, 45, 47] or being a minimizer for some variational problem [42], but these are not going to be discussed here.

The general setup in this work is the following. We consider a fully nonlinear parabolic problem of the form

$$\left.\begin{aligned} \partial_t u &= F(t,x,u,Du,D^2u), && (x,t)\in\Omega\times(0,\infty)\,, \\ u &= 0, && (x,t)\in\partial\Omega\times(0,\infty)\,, \\ u &\geq 0, && (x,t)\in\Omega\times(0,\infty)\,. \end{aligned}\right\} \tag{1.1}$$

Here, Dg and D^2g denote the gradient and Hess matrix of a function g. We assume that Ω is a symmetric convex domain, that is, we assume

(d1) $\Omega\subset\mathbb{R}^N$, $N\geq 1$, is a bounded domain, convex in x_1, and symmetric with respect to the hyperplane

$$H_0 := \{x=(x_1,\cdots,x_N) : x_1 = 0\}\,.$$

The non-linearity F satisfies regularity, ellipticity and symmetry conditions specified below. Our goal is to investigate symmetry and monotonicity properties of global solutions u (vector or scalar valued function) as $t\to\infty$. Observe, that we do not discuss the existence of global solutions satisfying (1.1), rather we investigate their properties once they exist.

To achieve our goal, we have to extend various linear results to serve our purposes. We believe, that some of the extensions might be of independent interest, such as maximum principle on small space-time domains or Alexandrov-Krylov estimate and full Harnack inequality for cooperative systems.

Most results in this work are included in [26] for scalar problems, and in [28] for systems.

1.1 History

The first symmetry results for positive solutions of elliptic equations date back to the celebrated paper of Gidas, Ni, Nirenberg [30]. They showed that if u is a positive classical solution of the problem

$$\begin{aligned} \Delta u = f(u), \qquad & x \in \Omega\,, \\ u = 0, \qquad & x \in \partial\Omega\,, \end{aligned} \tag{1.2}$$

with a smooth domain Ω satisfying (d1) and a Lipschitz function f, then u is even in x_1 and $\partial_{x_1} u < 0$ in

$$\Omega_0 := \{x = (x_1, \cdots, x_N) \in \Omega : x_1 > 0\}\,.$$

Of course, if Ω is also symmetric and convex in other directions, then so is the solution. In particular, positive solutions on a ball are radially symmetric. Gidas et al. also constructed nonnsymmetric positive solutions of (1.2), for Hölder continuous functions f. The two main mathematical tools used in the proof of the symmetry results were the maximum principle and the method of moving hyperplanes introduced by Alexandrov [1] and later developed by Serrin [53] ([53] also contains a related result on radial symmetry).

Similar symmetry results were later proved for fully nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$, first for smooth domains by Li [39], later for general bounded symmetric domains by Berestycki and Nirenberg [8] (see also [20]). Da Lio and Sirakov [41] considered an even more general class of elliptic equations, including equations involving Pucci operators, and they proved the symmetry of positive viscosity solutions. Many authors, starting again with Gidas, Ni and Nirenberg [31], established symmetry and monotonicity properties of positive solutions of elliptic equations on unbounded domains under various conditions on the equations and the solutions. For surveys of these theorems, as well as additional results on bounded domains, we refer the readers to [7, 35, 46].

Extensions of the symmetry results to cooperative elliptic systems were first made by Troy [56], then by Shaker [54] (see also [18]) who considered semilinear equations on smooth bounded domains. In [22], de Figueiredo removed the smoothness assumption on the domain in a similar way as Berestycki and Nirenberg [8] did for the scalar equation. For cooperative systems on the whole space, a general symmetry result was proved by Busca and Sirakov [11], an earlier more restrictive result can be found in [25]. The cooperative structure of the system assumed in all these references is in some sense unavoidable. Without it, neither is the maximum principle applicable nor do the symmetry result hold in general (see [12] and [54] for counterexamples).

The situation for parabolic problems is more complicated, since one cannot expect solutions to be symmetric, if the initial condition is not symmetric. However, it is possible that solutions 'symmetrize' as time approaches infinity, regardless of initial data. More precisely, we say that u is *asymptotically symmetric* if all functions in the ω-limit set of u:

$$\omega(u) := \{z : z = \lim_{k\to\infty} u(\cdot, t_k), \text{for some}\, t_k \to \infty\} \tag{1.3}$$

are even in x_1 and nonincreasing in Ω_0. The limit in (1.3) is in the supremum norm.

For parabolic equations, first symmetry results appeared in the nineties. In [21], Dancer and Hess proved the spatial symmetry of periodic solutions of time-periodic reaction diffusion equations. Then, in [34], Hess and Poláčik considered solutions of the problem

$$\begin{aligned} u_t - \Delta u &= f(t,u), && (x,t) \in \Omega \times (0,\infty)\,, \\ u &= 0, && (x,t) \in \partial\Omega \times (0,\infty)\,. \end{aligned}$$

where f is Hölder continuous in t and Lipschitz in u, and Ω is a smooth domain satisfying (d1). In addition, it was assumed that

$$\nu_1(x) > 0 \qquad (x = (x_1, x') \in \partial\Omega, x_1 > 0)\,, \tag{1.4}$$

where $(\nu_1(x), \nu'(x)) = \nu(x)$ is the exterior unit normal vector to $\partial\Omega$ at x. This geometric condition does not appear in the elliptic case but is essential in the parabolic one, as discussed below.

Independently to [34], Babin [4, 5] established asymptotic symmetry of solutions for fully nonlinear autonomous equations

$$u_t = F(x, u, Du, D^2u), \qquad x \in \Omega\,,$$

on a bounded domain Ω. He assumed that F satisfies assumptions (N1)–(N3) specified in Section 2.1. In addition he supposed that $F(x, 0, 0, 0) \geq 0$ for each $x \in \Omega$, $z > 0$ in Ω for each $z \in \omega(u)$, and the orbit $\{u(\cdot, \cdot + \tau)\}$ of the positive solution u is relatively compact in $C(\bar{\Omega} \times [0,1]) \cap C^{2,1}(\bar{D} \times [0,1])$, for each D with $\bar{D} \subset \Omega$. Babin and Sell [6] then extended these results to time-dependent fully nonlinear equations. Unlike in [34], no smoothness of the domain Ω was assumed in [4, 5, 6]. On the other hand, observe that, rather strong positivity hypotheses were made in these papers on the nonlinearity and the solution.

This and other shortcomings were removed in [52], where Poláčik considered classical bounded nonnegative solutions of the fully nonlinear problem $u_t = F(t, x, u, Du, D^2)$ on general bounded domains. If F satisfies (N1)–(N3) specified in Section 2.1, he showed that such solution is asymptotically symmetric if and only if either $\omega = \{0\}$ or there is $\phi \in \omega(u)$ with $\phi > 0$ in Ω. Poláčik in [52] also showed two explicit sufficient conditions for asymptotic symmetry - Ω being a ball or

$$\liminf_{t \to \infty} F(x, t, 0, 0, 0) \geq 0 \qquad (x \in \Omega)\,. \tag{1.5}$$

In [52] one can also find the following example, for which the asymptotic symmetry of solutions fails.

Example 1.1.1. Let $\Omega = [-1,1] \times [-1,1]$. There is a Lipschitz function f on $[-1,1] \times \mathbb{R}$ such that the problem

$$\begin{aligned} u_t &= \Delta u + f(y, u), \qquad && (x, y, t) \in \Omega \times (0, \infty), \\ u &= 0, && (x, y, t) \in \partial\Omega \times (0, \infty), \end{aligned}$$

has a global, bounded positive solution u with $\omega(u) = \{z\}$, where $z \in C_0(\bar{\Omega})$ is a nonnegative function satisfying $z(x,t) > 0$ if $(x, y) \in \Omega_0$ and $z(0, y) = 0$ for each $y \in [-1, 1]$. In particular, z is not monotone in $x > 0$.

Other directions of research include parabolic equations on unbounded domains [50, 51], asymptotically symmetric problems [27, 29]. For more detailed survey we refer the reader to [49].

1.2 Motivating problems

In this work we exting the existent results in two directions. First, we give a general explicit sufficient condition, that guarantees asymptotic symmetry of solutions. Next, we prove a symmetry result for solutions of cooperative parabolic systems.

To illustrate the results on a model problem, assume that Ω is a Lipschitz domain satisfying (d1) and the following assumption.

(d2) For any $\delta^* > 0$ there is $\varepsilon > 0$ and a unit vector $v \in \mathbb{R}^N \setminus \{e_1\}$ such that

$$\mathrm{Cone}_{x,\varepsilon}(e_1, v) \subset \bar{\Omega} \qquad (x \in \partial\Omega, x_1 \geq \delta^*)\,.$$

Here, $\mathrm{Cone}_{x,\varepsilon}(r, s)$ be the part of the cone spanned by $-r, -s$ with the tip at x, which lies inside the ball of radius ε centered at x:

$$\mathrm{Cone}_{x,\varepsilon}(r, s) := \{y \in \mathbb{R}^N : x - y = \alpha r + \beta s, \alpha, \beta \geq 0, |x - y| \leq \varepsilon\}\,. \tag{1.6}$$

Let $f : (0, \infty) \times [0, \infty) \to \mathbb{R}$ be a continuous function such that

(f1) $f : (t, u) \mapsto f(t, u)$ is Lipschitz continuous in u uniformly with respect to t, meaning that there is $\beta_0 > 0$ such that

$$\sup_{t>0} |f(t, u) - f(t, \bar{u})| \leq \beta_0 |u - \bar{u}| \qquad (u, \bar{u} \in [0, \infty))\,.$$

(f2) $f(\cdot, 0)$ is a bounded function.

Let u be a global, nonnegative, bounded, classical (that is, $u \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$) solution of the problem:

$$\begin{aligned} u_t - \Delta u &= f(t, u), && (x, t) \in \Omega \times (0, \infty)\,, \\ u &= 0, && (x, t) \in \partial\Omega \times (0, \infty)\,. \end{aligned} \tag{1.7}$$

As a result we obtain.

Theorem 1.2.1. *If a Lipschitz domain Ω satisfies (d1), (d2), a function f satisfies (f1), (f2), and u is a global, nonnegative, bounded, classical solution of* (1.7)*, then u is asymptotically symmetric, that is, for each $z \in \omega(u)$*

$$z(x_1, x') = z(-x_1, x') \qquad ((x_1, x') \in \Omega),$$

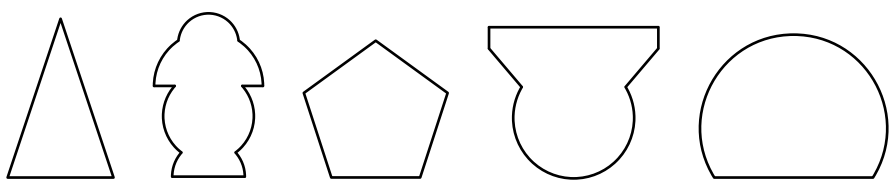
and either $z \equiv 0$ or z is strictly decreasing in Ω_0.

Examples of Lipschitz domains that satisfy (d1) and (d2), include (see the figures),

- symmetric domains, which are strictly convex in x_1, that is,

$$\alpha(x_1, x') + (1-\alpha)(x_1, y') \in \Omega \qquad (\alpha \in (0,1), (x_1, x'), (x_1, y') \in \partial\Omega)\,.$$

- some symmetric domains, which are not strictly convex in x_1 such as isosceles triangles, pentagons, pyramids, upper half balls, and so on.



Notice that a rectangle (cf. Example 1.1.1) does not satisfy (d2), but it is a 'borderline' case. Moreover, if Ω is a C^2 domain satisfying (d1), then (1.4) implies (d2). Hence, Theorem 1.2.1 is a generalization of results in [34].

The main contribution of our Theorem 1.2.1 and more general results in the next section, as compared to the results of [52], is that it gives a general, explicit, and easily verifiable condition, under which the asymptotic symmetry holds.

In Chapter 4 we extend Theorem 1.2.1 to fully nonlinear problems such as (1.1). That is, we formulate a sufficient condition for asymptotic symmetry only in terms of Ω and F. This condition covers a larger class of problems, compared to the explicit sufficient conditions from [52]. For example, if the F does not satisfy (1.5), then asymptotic symmetry of solutions of (1.1) was not discussed in [6], and [52] requires Ω to be a ball. For a general domain Ω, the asymptotic symmetry theorem of [52] applies only to solutions whose ω-limit set contains a positive function. We show that, if we in addition to (d1) and (N1)–(N3) assume (d2) and minor monotonicity assumptions on F, then the asymptotic symmetry holds.

As a by-product we obtain an improvement of the results in [15, 19, 23] on the question when a nonnegative, nontrivial solution of an elliptic problem is positive (cf. Corollary 2.1.5).

The second model problem illustrates our results for cooperative systems of parabolic equations. Consider the reaction-diffusion system

$$u_t = D(t)\Delta u + f(t,u), \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.8}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.9}$$

As before we assume that Ω satisfies (d1). Denoting by $\mathbb{R}^n_+$ the cone in $\mathbb{R}^n$ consisting of vectors with nonnegative components, we further assume that

(S1) $D(t) = \operatorname{diag}(d_1(t), \ldots, d_n(t))$ is a diagonal matrix, where $d_i : [0,\infty) \to \mathbb{R}$ are continuous functions satisfying $\alpha_0 \le d_i(t) \le \beta_0$ ($t \ge 0$, $i = 1, \ldots, n$) for some positive constants α_0 and β_0;

(S2) $f = (f_1, \ldots, f_n) : [0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is a continuous function which is locally Lipschitz in u uniformly with respect to t, that is, for each M there exists a constant $\beta_1 = \beta_1(M) > 0$ such that

$$|f(t,u) - f(t,v)| \le \beta_1 |u - v| \quad (t \ge 0,\ u, v \in \mathbb{R}^n_+, |u|, |v| \le M);$$

(S3) for each $i \ne j$ and $t \in [0,\infty)$ one has $\partial f_i(t,u)/\partial u_j \ge 0$ for each $u \in \mathbb{R}^n_+$ such that the derivative exists (which is for almost every u by (S2)).

The following strong cooperativity condition will allow us to relax the positivity assumptions on the considered solutions. In Chapter 2 we formulate a different condition, the irreducibility of the system, which together with (S3) can be used in place of (S4).

(S4) For each M there is a constant $\sigma = \sigma(M) > 0$ such that for all $i \neq j$ and $t \in [0, \infty)$ one has $\partial f_i(t, u)/\partial u_j \geq \sigma$ for each $u \in \mathbb{R}^n_+$ with $|u| \leq M$ whenever the derivative exists.

We consider a global classical solution u of (1.8), (1.9) which is nonnegative (by which we mean that all its components are nonnegative) and bounded: $\sup_{x\in\Omega, t\geq 0} |u(x, t)| < \infty$. Moreover, we require that u assume the Dirichlet boundary condition uniformly with respect to time:

$$\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} \sup_{t\geq 0} |u(x, t)| = 0. \tag{1.10}$$

As in the scalar case, we formulate the asymptotic symmetry of u in terms of its ω-limit set:

$$\omega(u) = \{z : z = \lim_{k\to\infty} u(\cdot, t_k) \text{ for some } t_k \to \infty\},$$

where the limit is taken in the space $E^n := (C(\bar{\Omega}))^n$ equipped with the supremum norm. It follows from standard parabolic interior estimates and the assumptions on u, specifically the boundedness and (1.10), that the orbit $\{u(\cdot, t) : t \geq 0\}$ is relatively compact in E^n. Therefore $\omega(u)$ is nonempty and compact in E^n and it attracts u in the sense that

$$\lim_{t\to\infty} \operatorname{dist}_E(u(\cdot, t), \omega(u)) = 0.$$

Theorem 1.2.2. *Assume (D1), (D2), (S1)–(S3) and let u be a bounded nonnegative global solution of* (1.8) *satisfying* (1.10). *Assume in addition that one of the following conditions holds:*

(i) there exists $\varphi = (\varphi_1, \ldots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \ldots, n\}$;

(ii) (S4) holds and there is $\varphi \in \omega(u)$ *such that* $\varphi_i > 0$ *in* Ω *for some* $i \in \{1, \dots, n\}$.

Then for each $z = (z_1, \dots, z_n) \in \omega(u) \setminus \{0\}$ *and each* $i = 1, \dots, n$, *the function* z_i *is even in* x_1 *and it is strictly decreasing in* x_1 *on* $\{x \in \Omega : x_1 > 0\}$.

The additional assumption in the theorem means that along a sequence of times all components of u (or at least some components, if (S4) is assumed) stay away from 0 at every $x \in \Omega$. Although this assumption can be relaxed somewhat, it cannot be removed completely even if u is required to be strictly positive and $n = 1$ (see Example 1.1.1). Without this assumption, a weaker symmetry theorem is valid if (S4) holds. Namely, the components of u symmetrize, as $t \to \infty$, around a hyperplane $\{x : x_1 = \mu\}$ with $\mu \geq 0$ possibly different from zero. The strong cooperativity condition (S4), or more general cooperativity and irreducibility conditions given in Chapter 2, are needed to guarantee that all the components symmetrize around the same hyperplane.

Theorem 1.2.2 follows from more general Theorem 2.2.1 given in Chapter 2. The equicontinuity condition (2.2) assumed in Theorem 2.2.1 is easily verified in the semilinear setting using (1.10), the boundedness of u, and standard parabolic interior estimates, as mentioned above. In Chapter 2 we also formulate sufficient conditions in terms of the nonlinearity and the domain, which guarantee that (i) or (ii) in the previous theorem holds true.

Chapter 2

Symmetry results

This chapter is divided into two sections. In the first one, we explain our results for scalar problems, while in the second one, we state our results for cooperative systems of equations.

First, let us introduce the following notation. Let $\mathcal{P}_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection in the hyperplane H_λ, where

$$H_\lambda := \{x = (x_1, x') \in \mathbb{R}^N : x_1 = \lambda\} \qquad (\lambda \in \mathbb{R}),$$

that is, $\mathcal{P}_\lambda(x) = (2\lambda - x_1, x')$ for any $x = (x_1, x') \in \mathbb{R}^N$. Next, for any subset Ω of $\mathbb{R}^N$ define

$$\Omega_\lambda := \{x = (x_1, x') \in \Omega : x_1 > \lambda\},$$

and

$$\ell := \sup\{x_1 : x \in \Omega\}.$$

2.1 Scalar problem

Consider the problem (1.1) and assume the following hypotheses.

(D1) Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), such that $\Omega'_\lambda := \mathcal{P}_\lambda(\Omega_\lambda) \subset \Omega$ for all $\lambda \geq 0$.

(D2) For each $\lambda > 0$ the set Ω_λ has only finitely many connected components.

(D3) Ω is symmetric with respect to the hyperplane H_0.

Notice, that (D1) and (D3) are equivalent to (d1) from the introduction. We formulate them differently here to have a unified setting for directions other than e_1, where we cannot require Ω to be symmetric.

The hypothesis (D2) occurred already in [52] and it is still unknown if it is just technical or not. Based on the proofs in this work it can be relaxed in several directions, although not completely removed. Observe that Lipschitz (and even Hölder) continuity of Ω implies (D2).

Let $\mathcal{T} \in \mathbb{R}^{N^2}$ be the matrix corresponding to $\mathcal{P}_0$, the reflection in the hyperplane H_0:

$$\mathcal{T}_{ij} := \delta_{ij} - 2\delta_{i1}\delta_{j1} \qquad (i, j = 1, \cdots, N),$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

We identify the space of $N \times N$ matrices with $\mathbb{R}^{N^2}$. We then assume that the real valued function $F : (t, x, u, p, q) \mapsto \mathbb{R}$ is defined on $[0, \infty) \times \bar{\Omega} \times \mathcal{O}$, where $\mathcal{O}$ is an open convex subset of $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ invariant under the transformation

$$Q : (u, p, q) \mapsto (u, p\mathcal{T}, \mathcal{T}q\mathcal{T}),$$

and it satisfies the following conditions.

(N1) *Regularity.* The function F is continuous, differentiable with respect to q and Lipschitz continuous in (u, p, q) uniformly with respect to $(x, t) \in \bar{\Omega} \times \mathbb{R}^+$. This means that there is $\beta_0 > 0$ such that

$$\sup_{x \in \Omega, t \geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta_0 |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$

$$((x, t) \in \bar{\Omega} \times \mathbb{R}^+, \, (u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{O}).$$

(N2) *Ellipticity.* There is a constant $\alpha_0 > 0$ such that for each $\xi \in \mathbb{R}^N$ and each $(t, x, u, p, q) \in [0, \infty) \times \bar{\Omega} \times \mathcal{O}$ one has

$$\frac{\partial F}{\partial q_{jk}}(t, x, u, p, q)\xi_j\xi_k \geq \alpha_0 |\xi|^2.$$

Here and also in the rest of this work we use the summation convention, that is, when an index appears twice in a single term, then we are summing over all its possible values, usually from 1 to N.

(N3) *Symmetry and monotonicity.* For each $(t,u,p,q) \in [0,\infty) \times \mathcal{O}$, and any $x = (x_1, x'), (\tilde{x}_1, x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$

$$F(t, \mathcal{T}x, Q(u,p,q)) = F(t,x,Q(u,p,q)) = F(t,x,u,p,q)\,,$$
$$F(t, x_1, x', u, p, q) \geq F(t, \tilde{x}_1, x', u, p, q)\,.$$

As an easy example or F that satisfies (N1) – (N3) is $F(t,x,u,p,q) = q_{ii} + f(t,u)$, where f satisfies (f1) and (f2).

We remark that although the hypotheses are formulated with fixed constants α_0, β_0, we really need them to be fixed on the range of (u, Du, D^2u) for each considered solution u. Thus, for example, if the solution in question is bounded and has bounded derivatives Du, D^2u, then the Lipschitz continuity can be replaced with the local Lipschitz continuity, and similarly one can relax the ellipticity and irreducibility (formulated below) conditions. Note, however, that we do not assume any boundedness of the derivatives of u.

As mentioned in the introduction, there are non-symmetric solutions of the problem (1.2), when f is merely Hölder continuous, thus we cannot relax (N1) in this direction. On the other hand, we remark that (N3) allows nontrivial generalizations. One can for example consider asymptotically symmetric or asymptotically monotone problems as in [27, 29].

By a solution of (1.1) we mean a classical solution, that is, a function $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$, with $(u, Du, D^2u) \in \mathcal{O}$, which satisfies (1.1) everywhere. We only consider bounded global solutions, that is,

$$\sup_{t \in [0,\infty)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty\,, \tag{2.1}$$

such that the family of functions $\{u(\cdot, \cdot + s)\}_{s\geq 1}$, is equicontinuous on $\bar{\Omega} \times [0,1]$:

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},t,\bar{t}\in[0,1],\\ |t-\bar{t}|,|x-\bar{x}|<h\\ s\geq 1}} |u(x,t+s) - u(\bar{x},\bar{t}+s)| = 0\,. \tag{2.2}$$

Under these assumptions the positive semi-orbit $\{u(\cdot,t) : t \geq 0\}$ of u is relatively compact in the space $E := C(\bar{\Omega})$. Then its ω-limit set (for the definition see (1.3)) is nonempty and compact in E and

$$\lim_{t\to\infty} \mathrm{dist}_E(u(\cdot,t), \omega(u)) = 0\,, \tag{2.3}$$

where dist_E denotes the distance in E.

A sufficient condition that implies (2.2) is formulated in the following proposition. For the proof see [52, Proposition 2.7].

Proposition 2.1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying the exterior cone condition and let (N1), (N2) hold. Further assume $(0,0,0) \in \mathcal{O}$ and the function $F(\cdot,\cdot,0,0,0)$ is bounded in $[0,\infty)\times\Omega$. Then for any global solution u of* (1.1) *satisfying* (2.1) *the following holds. There are constants $\alpha > 0$ and C such that*

$$\sup_{\substack{x,\bar{x}\in\bar{\Omega},x\neq\bar{x}\\ t,\bar{t}\in[s,s+1],s\neq\bar{s}\\ s>0}} \frac{|u(x,t)-u(\bar{x},\bar{t})|}{|x-\bar{x}|^{\alpha} + |t-\bar{t}|^{\frac{\alpha}{2}}} \leq C\,.$$

We remark, that the exterior cone condition holds true if Ω is Lipschitz continuous.

We are ready to formulate our first main result.

Theorem 2.1.2. *Assume (D1) – (D3), (N1) – (N3) and let u be a nonnegative global solution of* (1.1) *satisfying* (2.1) *and* (2.2)*. Then there exists $\lambda_0 \geq 0$ such that for each $z \in \omega(u)$ the following is true: z is monotone nonincreasing in x_1 on Ω_{λ_0} and there is a connected component U of Ω_{λ_0} such that*

$$z(x_1,x') = z(2\lambda_0 - x_1, x') \qquad ((x_1,x')\in U)\,. \tag{2.4}$$

Moreover if $\lambda_0 > 0$ *there is* $\tilde{z} \in \omega(u)$ *and a connected component* $\tilde{U}$ *of* Ω_{λ_0} *such that*

$$\tilde{z}(x_1, x') = \tilde{z}(2\lambda_0 - x_1, x') \qquad ((x_1, x') \in \tilde{U}), \tag{2.5}$$

$$\tilde{z}(x) > 0 \qquad (x \in \tilde{U}). \tag{2.6}$$

If Ω_{λ_0} *is connected then for each* $z \in \omega(u)$ *either* $z \equiv 0$ *in* Ω_{λ_0} *or* z *is strictly decreasing in* x_1 *in* Ω_{λ_0}. *The latter holds in the form* $z_{x_1} < 0$ *if* $z_{x_1} \in C(\Omega_{\lambda_0})$ *for some* $z \in \omega(u)$.

This theorem is an improvement of [52, Theorem 2.4], as it gives more precise characterization (property (2.6)) of the function $\tilde{z}$, if $\lambda_0 > 0$. Property (2.6) is important in the proof of the next theorem.

Next, we turn our attention to the question when λ_0 from the previous theorem is equal to 0, that is, when the solution is asymptotically symmetric. This is not always the case, even if u is strictly positive, as one can construct examples similar to Example 1.1.1, for which $\lambda_0 = \frac{n-1}{n}\ell$ with $n \in \mathbb{N}$, $n \leq n_0$ and n_0 depends on $\alpha_0, \beta_0, N, \operatorname{diam}\Omega$.

However, we show that the solution is asymptotically symmetric if Ω and F satisfy analogous symmetry assumptions in a direction $\hat{v} \neq e_1$, as they satisfy in the direction e_1. Define

$$\lambda^*(v) = \inf\{\mu : \Omega_{\mu,v} \subset \Omega_{0,e_1} \text{ and } \Omega'_{\lambda,v} \subset \Omega, \text{ for each } \lambda > \mu\}. \tag{2.7}$$

Here

$$\Omega'_{\lambda,v} := \mathcal{P}_{\lambda,v}(\Omega_{\lambda,v}), \quad \text{where} \quad \Omega_{\lambda,v} := \{x \in \Omega : x \cdot v \geq \lambda |v|\},$$

and $\mathcal{P}_{\lambda,v} : \mathbb{R}^N \to \mathbb{R}^N$ is the reflection in the hyperplane

$$H_{\lambda,v} := \{x \in \mathbb{R}^N : x \cdot v = \lambda\}.$$

We assume that $\lambda^*(v)$ is sufficiently small at least for one vector $v \neq e_1$. More precisely, we suppose that the following hypotheses hold for some $\delta^* > 0$. In our theorems we need δ^* less than or equal to a certain constant depending on N, α_0, β_0 and diam Ω (as specified in (4.18)).

(D4) There exists a unit vector $\hat{v} \in \mathbb{R}^N$ such that $0 < \hat{v} \cdot e_1 < 1$ and $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$.

(D5) $\Omega_{\lambda,v}$ has finitely many connected components for all vectors $v \in W := \{v \in \mathrm{Cone}_{0,1}(-e_1, -\hat{v}) : |v| = 1\}$ and all $\lambda \geq \lambda^*(v)$, where $\hat{v}$ is as in (D4) and $\mathrm{Cone}_{0,1}$ was defined in (1.6).

At the end of this section (in the proof of Theorem 1.2.1), we prove that Lipschitz continuity of Ω, (d1), and (d2) imply, that (D4) holds for each $\delta^* > 0$. Also, for Lipschitz domains $\Omega_{\lambda,v}$ has finitely many connected components for any v and λ, so that (D5) holds. However, even for Lipschitz domains satisfying (d1), the assumption (D4) is weaker than (d2) (consider for example n-gon with sufficiently large n).

In Lemma 4.2.1, we prove, that for any $v \in W$ sufficiently close to e_1, (D4) holds with $\hat{v}$ replaced by v. Hölder continuity of Ω provides a sufficient condition for (D5).

In addition to examples in Section 1.2, an example of a domain that satisfies (D1) – (D5), and bears all complications of a general domain, is the union of finitely many overlapping balls or upper half balls centered at H_0. Generally, such domain is neither convex nor rotationally symmetric.

Let us turn our attention to the assumption on the nonlinearity F. For any unit vector $v \in \mathbb{R}^N$ denote $\mathcal{T}^v : \mathbb{R}^N \to \mathbb{R}^N$ the matrix that represents the reflection in the hyperplane $H_{0,v}$:

$$\mathcal{T}^v_{ij} = \delta_{ij} - 2v_i v_j, \qquad (i, j \in \{1, \cdots N\}),$$

and let Q_v be the transformation

$$Q_v : (u, p, q) \mapsto (u, p\mathcal{T}^v, \mathcal{T}^v q \mathcal{T}^v).$$

For $\hat{v}$, already fixed in (D1), suppose that the set $\mathcal{O}$ (defined in the paragraph before (N1)), is invariant under $Q_{\hat{v}}$. An easy argument shows that $\mathcal{O}$ is then invariant under Q_v for any $v \in W$ as well.

(N4) For each $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, and any $x, \tilde{x} \in \Omega_{\lambda^*(\hat{v}),\hat{v}}$ with $\tilde{x} = x + \xi\hat{v}$, $\xi \geq 0$,

$$F(t, \mathcal{T}^{\hat{v}}x, Q_{\hat{v}}(u, p, q)) = F(t, x, Q_{\hat{v}}(u, p, q)) = F(t, x, u, p, q)\,,$$
$$F(t, x, u, p, q) \geq F(t, \tilde{x}, u, p, q)\,.$$

Using (N3) and (N4), it is easy to prove that for any $v \in W$

$$F(t, \mathcal{T}^{v}x, Q_{v}(u, p, q)) = F(t, x, Q_{v}(u, p, q)) = F(t, x, u, p, q)\,,$$
$$F(t, x, u, p, q) \geq F(t, \tilde{x}, u, p, q)\,,$$

where $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, and $x, \tilde{x} \in \Omega_{\lambda^*(v),v}$ with $\tilde{x} = x + \xi v$, $\xi \geq 0$.

The function $F(t, x, u, p, q) = q_{ii} + f(u)$ satisfies (N4). For more a complex example, suppose without loss of generality (or use a rotation preserving e_1) that $\hat{v}$ has the form $\hat{v} = \hat{\sigma}_1 e_1 + \hat{\sigma}_2 e_2$, where $\hat{\sigma}_1 \geq 0$ and $\tilde{\sigma}_2 > 0$. Then, F satisfies (N3) and (N4), if it depends only on

$$(t, |(x_1, x_2)|, x', u, p_1^2 + p_2^2, p_i, q_{11} + q_{22}, q_{ij}) \qquad (3 \leq i, j \leq n)\,.$$

We formulate the next main result.

Theorem 2.1.3. *There exists $\delta^* = \delta^*(N, \alpha_0, \beta_0, diam\,\Omega) > 0$ such that if (D1) – (D5), (N1) – (N4) hold, and u is a nonnegative global solution of* (1.1) *satisfying* (2.1), (2.2), *then for each $z \in \omega(u)$ the function z is nonincreasing in x_1 in Ω_0 and*

$$z(x_1, x') = z(-x_1, x') \qquad ((x_1, x') \in \Omega_0)\,.$$

Moreover, $z \equiv 0$ in Ω or z is strictly decreasing in x_1 in Ω_0. The latter holds in the form $z_{x_1} < 0$ if $z_{x_1} \in C(\Omega_0)$.

Corollary 2.1.4. *Under the assumptions of the previous theorem either $z \equiv 0$ or $z > 0$ in Ω, for any $z \in \omega(u)$.*

When the problem (1.1) is time independent and u is an equilibrium, we obtain, from the corollary, an improvement of results in [15, 19, 23] to the nonsmooth domain with space dependent nonlinearity.

Corollary 2.1.5. *There exists $\delta^* = \delta^*(N, \alpha_0, \beta_0, diam\,\Omega) > 0$ such that, if Ω is a domain satisfying (D1) – (D5) and $u : \Omega \to \mathbb{R}$ is a classical nonnegative solution of*

$$\begin{aligned} F(x, u, Du, D^2u) = 0, &\qquad x \in \Omega\,, \\ u = 0, &\qquad x \in \partial\Omega\,, \end{aligned}$$

with F satisfying (N1)–(N4), then either $u \equiv 0$ or $u > 0$ in Ω.

Let us prove that Theorem 1.2.1 follows from the previous theorem.

Proof of Theorem 1.2.1. It is easy to check that (D1) and (D3) are equivalent to (d1). As mentioned above, Lipschitz continuity of Ω implies (D2) and (D5). Moreover, for any $\delta^* > 0$, (d2) implies the existence of v and $\varepsilon > 0$ such that $\mathrm{Cone}_{x,\varepsilon}(e_1, v) \subset \bar{\Omega}$ for any $x \in \partial\Omega$ with $x_1 \geq \delta^*$. Then a perturbation argument, (D1), and (D2) yield that (D4) holds for a unit vector $\hat{v} \in \mathrm{span}\{e_1, v\}$, which is sufficiently close to e_1.

In problem (1.7), one has $F(t, x, u, p, q) = q_{ii} + f(t, u)$, where the function f satisfies (f1) and (f2). In this case (N1) – (N4) hold. Finally, by [52, Proposition 2.7], Lipschitz continuity of Ω, (N1), (N2), boundedness of u and (f2) imply (2.1) and (2.2). Therefore the assumptions of Theorem 2.1.3 are satisfied and Theorem 1.2.1 follows. $\square$

2.2 Cooperative systems

Let Ω be a bounded domain in $\mathbb{R}^N$ for some $N \geq 1$, that satisfied (D1), (D2) and (D3) from the previous section. Fix $n \in \mathbb{N}$, $n \geq 1$ and define

$$S := \{1, 2, \cdots n\}$$

We consider systems of nonlinear parabolic equations of the form

$$\partial_t u_i = F_i(t, x, u, Du_i, D^2u_i), \quad (x, t) \in \Omega \times (0, \infty),\ i \in S, \tag{2.8}$$

where $n \geq 1$ is an integer, $u := (u_1, \cdots, u_n)$, and Du_i, D^2u_i denote, respectively, the gradient and Hessian matrix of u_i with respect to x.

Observe that while the system (2.8) is fully nonlinear, it is still only weakly coupled in the sense that the arguments of F_i do not involve the derivatives of u_j for $j \neq i$.

Next, we formulate precisely the assumptions on the nonlinearities

$$F_i : (t, x, u, p, q) \mapsto F_i(t, x, u, p, q) \in \mathbb{R}^n .$$

The function F_i is defined on $[0, \infty) \times \bar{\Omega} \times \mathcal{O}_i$, where $\mathcal{O}_i$ is an open convex subset of $\mathbb{R}^{n+N+N^2}$ invariant under the transformation

$$Q : (u, p, q) \mapsto (u, -p_1, p_2, \ldots, p_N, \tilde{q}),$$
$$\tilde{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i, j \text{ equals } 1, \\ q_{ij} & \text{otherwise.} \end{cases}$$

We further assume that $F = (F_1, \ldots, F_n)$ satisfies the following hypotheses:

(F1) *Regularity.* For each $i \in S$ the function $F_i : [0, \infty) \times \bar{\Omega} \times \mathcal{O}_i \to \mathbb{R}^n$ is continuous, differentiable with respect to q and Lipschitz continuous in (u, p, q) uniformly with respect to $(x, t) \in \bar{\Omega} \times \mathbb{R}^+$. This means that there is $\beta_0 > 0$ such that

$$|F_i(t, x, u, p, q) - F_i(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta_0 |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})| \qquad ((x, t) \in \bar{\Omega} \times \mathbb{R}^+,\ (u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{O}_i). \tag{2.9}$$

(F2) *Ellipticity.* There is $\alpha_0 > 0$ such that for all $i \in S$, $(t, x, u, p, q) \in [0, \infty) \times \bar{\Omega} \times \mathcal{O}_i$, and $\xi \in \mathbb{R}^N$ one has

$$\sum_{j,k=1}^{N} \frac{\partial F_i}{\partial q_{jk}}(t, x, u, p, q)\xi_j\xi_k \geq \alpha_0 |\xi|^2.$$

(F3) *Symmetry and monotonicity.* For each $i \in S$, $(t,u,p,q) \in [0,\infty) \times \mathcal{O}_i$, and any $(x_1,x'), (\tilde{x}_1,x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$ one has

$$F_i(t,\pm x_1,x',Q(u,p,q)) = F_i(t,x_1,x',u,p,q) \geq F_i(t,\tilde{x}_1,x',u,p,q)\,.$$

(F4) *Cooperativity.* For all $i,j \in S$, $i \neq j$, $(t,x,u,p,q) \in [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$ one has

$$\frac{\partial F_i}{\partial u_j}(t,x,u,p,q) \geq 0,$$

whenever the derivative exists.

In some results we need to complement (F4) with the following condition.

(F5) *Irreducibility.* There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$ there exist $i \in I$, $j \in J$ such that

$$\frac{\partial F_i}{\partial u_j}(t,x,u,p,q) \geq \sigma$$

for all $(t,x,u,p,q) \in [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$ such the derivative exists.

Note that the derivatives in (F4) and (F5) exist almost everywhere by (F1).

If $n = 2$, condition (F5) is equivalent to $\partial F_i/\partial u_j \geq \sigma$ for all $i,j \in S$, $i \neq j$. For $n \geq 3$, the latter condition is stronger than (F5), for example, consider functions satisfying $\partial F_i/\partial u_j \geq \sigma$ for all $i,j \in S$, $|i-j| = 1$ and $\partial F_i/\partial u_j \equiv 0$ for all $i,j \in S$, $|i-j| > 1$. It is not hard to verify that, aside from the uniformity in all variables, condition (F5) is equivalent to other commonly used notions of irreducibility, see for example [2] and references therein.

Notice that the assumption (F3) implies the following condition

(F3 cor) For each $i \in S$, F_i is even in x_1 and for any $(t,u,p,q) \in [0,\infty) \times \mathcal{O}_i$, $\lambda > 0$ and $(x_1,x') \in \Omega_\lambda$, $\lambda > 0$, one has

$$F_i(t,2\lambda - x_1,x',Q(u,p,q)) \geq F_i(t,x_1,x',u,p,q).$$

This weaker condition is sufficient for some of our results but for simplicity and consistency we just assume (F3) in all our symmetry theorems.

By a *solution* of (2.8), (1.9) we mean a function $u = (u_1, \dots, u_n)$ such that $u_i \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$, $(u, Du_i, D^2 u_i) \in \mathcal{O}_i$ for all $i \in S$ and u satisfies (2.8), (1.9) everywhere. By a *nonnegative (positive) solution* we mean a solution with all components u_i nonnegative (positive) in $\Omega \times (0,\infty)$. All solutions of (2.8) considered in this work are assumed to be nonnegative, regardless of whether it is explicitly stated or not. We shall consider solutions such that

$$\sup_{t\in[0,\infty)} \max_{i\in S} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{2.10}$$

and the functions $u(\cdot, \cdot + s)$, $s \geq 1$, are equicontinuous on $\bar{\Omega} \times [0,1]$, that is,

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},\, t,\bar{t}\geq 1,\\ |t-\bar{t}|,|x-\bar{x}|<h}} |u(x,t) - u(\bar{x},\bar{t})| = 0. \tag{2.11}$$

We remark that, as in the scalar case, if for each $i \in S$ one has $(0,0,0) \in \mathcal{O}_i$, (F1), (F2) hold and the function $(x,t) \mapsto F_i(x,t,0,0,0)$ is bounded on $\Omega \times [0,\infty)$, then (2.11) holds if Ω satisfies the exterior cone condition and (2.10) holds (cf. Proposition 2.1.1).

The orbit $\{u(\cdot,t) : t \geq 0\}$ of a solution satisfying (2.10), (2.11) is relatively compact in the space $E = (C(\bar{\Omega}))^n$ and then, as noted in the introduction, the ω-limit set

$$\omega(u) = \{z : z = \lim_{k\to\infty} u(\cdot,t_k) \text{ for some } t_k \to \infty\},$$

is nonempty, compact in E and it attracts $u(\cdot,t)$ as $t \to \infty$.

We now state a more general version of Theorem 1.2.2.

Theorem 2.2.1. *Assume (D1), (D2), (D3), (F1) - (F4). Let u be a nonnegative solution of* (2.8), (1.9) *satisfying* (2.10) *and* (2.11)*. Assume in addition that one of the following conditions holds:*

(i) there exists $\varphi = (\varphi_1, \dots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \dots, n\}$;

(ii) (F5) holds and there is $\varphi \in \omega(u)$ such that $\varphi_i > 0$ in Ω for some $i \in \{1, \dots, n\}$.

Then for each $z = (z_1, \dots, z_n) \in \omega(u)$ and $i \in S$, the function z_i is even in x_1:

$$z_i(x_1, x') = z_i(-x_1, x') \quad ((x_1, x') \in \Omega_0), \tag{2.12}$$

and either $z_i \equiv 0$ on Ω or z_i is strictly decreasing in x_1 on Ω_0. The latter holds in the form $(z_i)_{x_1} < 0$ if $(z_i)_{x_1} \in C(\Omega_0)$.

The last condition, $(z_i)_{x_1} \in C(\Omega_0)$, is satisfied if $\{u_{x_1}(\cdot, t) : t \geq 1\}$ is relatively compact in $C(\bar{\Omega})$. This is the case if, for example, $|D^2 u|$ is bounded (which we do not assume).

Remark 2.2.2. We will prove (see the proof of Theorems 2.2.1 and 2.2.4 in Section 5.2) that if (N5) holds, in addition to all the other hypotheses of Theorem 2.2.1, then for any $\varphi \in \omega(u)$ the relation $\varphi_i(x) > 0$ holds *for all* $x \in \Omega$ and $i \in S$ as soon as it holds *for some* $x \in \Omega$ and $i \in S$. Hence either $\varphi \equiv 0$ or all its components are strictly positive in Ω. This of course may not be true if (N5) does not hold as can be seen on examples of decoupled systems.

The assumption (i) or (ii) in the previous theorem is somewhat implicit in the sense, that it requires some knowledge of the asymptotic behavior of the solution as $t \to \infty$. One can formulate various alternative more explicit conditions in terms of the nonlinearity or the domain. The next theorem, which extends [52, Theorem 2.4] to cooperative systems, shows that the asymptotic positivity of the nonlinearity implies that if (F5) holds then (ii) is satisfied unless u converges to zero as $t \to \infty$. Other conditions can be formulated in terms of regularity and geometry of Ω (see for example Theorem 2.2.5 below).

One can also, as in the scalar case, formulate general explicit conditions in terms of Ω and F_i and prove similar results. We do not present this approach, mainly due to a cumbersome notation.

Theorem 2.2.3. *Assume (D1), (D2), (D3), (F1), (F2), (F4). Further assume that for each $i \in S$ one has $(0,0,0) \in \mathcal{O}_i$ and*

$$\liminf_{t\to\infty,\, x\in\Omega} F_i(t,x,0,0,0) \geq 0. \tag{2.13}$$

Let u be a nonnegative solution of (2.8), (1.9) *satisfying* (2.10) *and* (2.11). *Then for each $i \in S$ either $\|u_i(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ or else there exists $\varphi \in \omega(u)$ with $\varphi_i > 0$ in Ω.*

Without assumptions (i), (ii), Theorem 2.2.1 is not valid in general, even for positive solutions of scalar equations (see Example 1.1.1). However, the elements of $\omega(u)$ still have some reflectional symmetry property, although the symmetry hyperplane may not be the canonical one. This is stated in the following theorem which extends [52, Theorem 2.4].

Theorem 2.2.4. *Assume (D1), (D2), (D3), (F1) - (F4). Let u be a nonnegative solution of* (2.8), (1.9) *satisfying* (2.10) *and* (2.11). *Then there exists $\lambda_0 \in [0,\ell)$ such that the following assertions hold for each $z \in \omega(u)$.*

(i) For every $j \in S$ the function z_j is nonincreasing in x_1 on Ω_{λ_0} and there are $i \in S$ and a connected component U of Ω_{λ_0} such that

$$z_i(x_1,x') = z_i(2\lambda_0 - x_1, x') \quad ((x_1,x') \in U). \tag{2.14}$$

(ii) If (N5) is satisfied, there is a connected component U of Ω_{λ_0} such that (2.4) *holds for each $i \in S$.*

(iii) If (N5) is satisfied and Ω_{λ_0} is connected, then (2.4) *holds with $U = \Omega_{\lambda_0}$ for all $i \in S$ and either $z \equiv 0$ on Ω_{λ_0} or else for each $i \in S$ the function z_i is strictly decreasing in x_1 on Ω_{λ_0}. The latter holds in the form $(z_i)_{x_1} < 0$ if $(z_i)_{x_1} \in C(\Omega_{\lambda_0})$.*

As usual, in many problems with rotational symmetry, one can use reflectional symmetries in different directions to prove the radial symmetry of solutions. We give only one such symmetry results, assuming Ω is a ball. Rotational symmetry in just some variables can be examined in a similar way. Notice that in the next theorem we do not need the assumption on the existence of a positive element of $\omega(u)$.

Assume that Ω is the unit ball centered at the origin and consider the problem

$$\left.\begin{aligned} \partial_t u_i &= F_i(t,|x|,u,|Du_i|,\Delta u_i), && (x,t)\in\Omega\times(0,\infty), \\ u_i &= 0, && (x,t)\in\partial\Omega\times(0,\infty), \end{aligned}\right\}\quad i=1,\dots,n. \tag{2.15}$$

The functions $F_i(t,r,u,\eta,\xi)$, $i\in S$, are defined on $[0,\infty)\times[0,1]\times\mathcal{B}$ where $\mathcal{B}$ is a ball in $\mathbb{R}^{n+2}$ centered at the origin and we make the following hypotheses:

$(F1)_{rad}$ For each $i\in S$ the function $F_i:[0,\infty)\times[0,1]\times\mathcal{B}\to\mathbb{R}^n$ is continuous in all variables, differentiable in ξ, and Lipschitz continuous in (u,η,ξ) uniformly with respect to $(r,t)\in[0,1]\times\mathbb{R}^+$.

$(F2)_{rad}$ There is a positive constant α_0 such that

$$\frac{\partial F_i}{\partial\xi}(t,r,u,\eta,\xi)\ge\alpha_0\quad((t,r,u,\eta,\xi)\in[0,\infty)\times[0,1]\times\mathcal{B},\ i\in S).$$

$(F3)_{rad}$ For each $i\in S$, F_i is nonincreasing in r.

$(F4)_{rad}$ For any $i,j\in S$ with $i\ne j$, and any $(t,r,u,\eta,\xi)\in[0,\infty)\times[0,1]\times\mathcal{O}_i$ one has

$$\frac{\partial F_i}{\partial u_j}(t,r,u,\eta,\xi)\ge 0$$

whenever the derivative exists.

$(F5)_{rad}$ There exists $\sigma>0$ such that for any nonempty subsets $I,J\subset S$ with $I\cap J=\emptyset$, $I\cup J=S$ there exist $i\in I$, $j\in J$ such that

$$\frac{\partial F_i}{\partial u_j}(t,r,u,\eta,\xi)\ge\sigma$$

for all $(t,r,u,\eta,\xi)\in[0,\infty)\times[0,1]\times\mathcal{O}_i$ such the derivative exists.

Theorem 2.2.5. *Let* Ω *be the unit ball and assume that* $(F1)_{rad} - (F5)_{rad}$ *hold. Let* u *be a nonnegative solution of* (2.15) *satisfying* (2.10) *and* (2.11). *Then for any* $z \in \omega(u) \setminus \{0\}$ *and* $i \in S$*, the function* z_i *is radially symmetric and strictly decreasing in* $r = |x|$. *The latter holds in the form* $(z_i)_r < 0$ *if* $(z_i)_r \in C(\Omega_0)$.

Chapter 3

Preliminaries

The proofs of our symmetry theorems use the method of moving hyperplanes and they depend on estimates of solutions of linear equations and systems. We prepare general techniques and estimates in this chapter. For more specific results concerning systems see Section 5.1.

Recall the following standard notation. For an open set $Q \subset \mathbb{R}^{N+1}$ we define the parabolic boundary $\partial_P Q$ (see [13, 36, 40]) to be the set of all points $(x_0, t_0) \in \partial Q$ (boundary as a subset of $\mathbb{R}^{N+1}$) for which there exist a positive number α and a continuous function $x(t) \in \mathbb{R}^N$ defined for $t \in [t_0 + \alpha, t_0]$ such that

$$x(t_0) = x_0 \quad \text{and} \quad (x(t), t) \in Q \qquad (t \in (t_0, t_0 + \alpha])\,.$$

Let

$$Q_M := \{(x, s) \in Q : s \in M\} \qquad (M \subset \mathbb{R}) \tag{3.1}$$

be a time cut of Q, and if $M = \{t\}$ we also write Q_t instead of $Q_{\{t\}}$.

For bounded sets U, U_1 in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\operatorname{diam} U$ stands for the diameter of U, and $|U|$ for its Lebesgue measure (if it is measurable). The open ball in $\mathbb{R}^N$ centered at x with radius r is denoted by $B(x, r)$. Symbols f^+ and f^- denote the positive and negative parts of a function f: $f^{\pm} := (|f| \pm f)/2 \geq 0$.

Recall that we already defined $x^\lambda = \mathcal{P}_\lambda(x) = (2\lambda - x_1, x')$ and

$$\begin{aligned}
H_\lambda &= \{x = (x_1, x') \in \mathbb{R}^N : x_1 = \lambda\} && (\lambda \in \mathbb{R}),\\
\Omega_\lambda &= \{x = (x_1, x') \in \Omega : x_1 > \lambda\} && (\lambda \in \mathbb{R}),\\
\Omega'_\lambda &= \mathcal{P}_\lambda(\Omega_\lambda) = \{x^\lambda : x \in \Omega_\lambda\} && (\lambda \in \mathbb{R}),\\
\ell &= \sup\{x_1 : x = (x_1, x') \in \Omega\}.
\end{aligned}$$

We shall use the following definition.

Definition 3.0.6. Given an open set $Q \subset \mathbb{R}^{N+1}$, and positive numbers α_0, β_0, we say that an operator L of the form

$$L(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k} + c(x,t) \tag{3.2}$$

belongs to $E(\alpha_0, \beta_0, Q)$ if its coefficients a_{km}, b_k and c are measurable functions defined on Q and they satisfy

$$\begin{aligned}
&|a_{km}|, |b_k|, |c| \le \beta_0 && (k, m = 1, \dots, N),\\
&a_{km}(x,t)\xi_k\xi_m \ge \alpha_0|\xi|^2 && ((x,t) \in Q,\ \xi \in \mathbb{R}^N).
\end{aligned}$$

Let us now recall how linear problems are obtained from (1.1) via reflections in hyperplanes.

3.1 Linearization via reflections

Assume that $\Omega \subset \mathbb{R}^N$ is a domain satisfying the symmetry hypothesis (D1) and let functions F satisfy (N1)–(N3). Let u be a nonnegative solution of (1.1), satisfying (2.1). Using the notation introduced in the previous section, let

$$\begin{aligned}
u^\lambda(x,t) &:= u(x^\lambda, t) && ((x,t) \in \Omega_\lambda \times (0,t), \lambda \in [0,\ell)),\\
w^\lambda(x,t) &:= u^\lambda(x,t) - u(x,t) && ((x,t) \in \Omega_\lambda \times (0,t), \lambda \in [0,\ell)).
\end{aligned}$$

By (N3), for each $x \in \Omega_\lambda$ and $t > 0$, one has

$$\partial_t u^\lambda \ge F(t, x, u(x^\lambda, t), Du(x^\lambda, t), D^2u(x^\lambda, t)), \qquad (x,t) \in \Omega_\lambda \times (0, \infty).$$

Hence

$$\begin{aligned}\partial_t w^\lambda(x,t) \geq F(t,x,u(x^\lambda,t),Du(x^\lambda,t),D^2u(x^\lambda,t))& \\ - F(t,x,u(x,t),Du(x,t),D^2u(x,t))& \\ = L^\lambda(x,t)w^\lambda + c(x,t)w^\lambda\,, \quad (x,t)\in\Omega_\lambda\times(0,\infty),&\end{aligned} \tag{3.3}$$

where

$$L^\lambda(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k}$$

and the λ-dependent coefficients a_{km}, b_k, c are obtained from the Hadamard formula. Specifically (omitting the argument (x,t) of u and u^λ),

$$c(x,t) = \begin{cases} \displaystyle\int_0^1 F_u(t,x,u+s(u^\lambda-u),u,Du,D^2u)\,ds, & \text{if } u^\lambda(x,t)\neq u(x,t), \\ 0, \quad \text{if } u^\lambda(x,t)=u(x,t), & \end{cases}$$

$$b_k(x,t) = \begin{cases} \displaystyle\int_0^1 F_{p_k}(t,x,u^\lambda,\dots,u^\lambda_{x_{k-1}},u_{x_k}+s(u^\lambda_{x_k}-u_{x_k}), & \\ \qquad\qquad u_{x_{k+1}},\dots,D^2u)\,ds, \quad \text{if } u^\lambda_{x_k}(x,t)\neq u_{x_k}(x,t), & \\ 0, \quad \text{if } u^\lambda_{x_k}(x,t)=u_{x_k}(x,t), & \end{cases}$$

$$a_{km}(x,t) = \int_0^1 F_{q_{km}}(t,x,u^\lambda,Du^\lambda,D^2u+s(D^2u^\lambda-D^2u))\,ds\,.$$

By (N1) the coefficients are well defined on $\Omega_\lambda\times(0,\infty)$ and they are measurable functions with absolute values bounded by β_0. This and (N2) imply that $L^\lambda \in E(\alpha_0,\beta_0,\Omega_\lambda\times(0,\infty))$ and $\|c\|_{L^\infty(\Omega_\lambda\times(0,\infty))}\leq\beta_0$.

The Dirichlet boundary condition and nonnegativity of u yield

$$w^\lambda(x,t)\geq 0 \qquad ((x,t)\in\partial\Omega_\lambda\times(0,\infty))\,. \tag{3.4}$$

3.2 Linear estimates

In this subsection, we derive several estimates for linear problems such as (3.3). Since the results might be of independent interest, we state them under more general assumptions than needed for symmetry theorems.

Let Q be an open subset (bounded or unbounded) of $\mathbb{R}^N \times (0,\infty)$ and let β_0, α_0 be positive constants. We consider a general linear parabolic inequality

$$v_t \geq L(x,t)v + f(x,t), \qquad (x,t) \in Q\,, \tag{3.5}$$

$$v \geq g(x,t), \qquad (x,t) \in \partial_P Q\,, \tag{3.6}$$

where $L \in E(\alpha_0, \beta_0, Q)$, $f \in L^{N+1}(Q)$ and $g \in C(\partial_P Q) \cap L^\infty(\partial_P Q)$. Denote by a_{ij}, b_i, c, $i,j \in \{1,2,\cdots,n\}$, the coefficients of L

$$L(x,t) := a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t) \qquad ((x,t) \in Q)\,,$$

and let

$$M(x,t) := L(x,t) - c(x,t) \qquad ((x,t) \in Q)\,. \tag{3.7}$$

We say that v is a solution of (3.5) (or that it satisfies (3.5)) if it is an element of the space $W^{2,1}_{N+1,loc}(Q)$ and (3.5) is satisfied almost everywhere. If (3.5) is complemented by (3.6), we also require the solution to be continuous on $\bar{Q}$ and to satisfy the boundary inequalities everywhere.

One of the key tools in our paper is the maximum principle. If we mention the maximum or comparison principle, we refer to the following theorem with $f \equiv 0$. The proof of Theorem 3.2.1 can be found in [13] see also [36, 40, 57]. Recall that Q can be unbounded.

Theorem 3.2.1. *If $Q \subset \mathbb{R}^N \times [T_1, T_2]$ for some $T_1 < T_2$, $v \in C(\bar{Q})$ is a bounded solution of* (3.5) *with $L \in E(\alpha_0, \beta_0, Q)$, $c \leq 0$ and $f \in L^{N+1}(Q)$, then*

$$\sup_Q v^- \leq \sup_{\partial_P Q} v^- + C\|f^-\|_{L^{N+1}(Q)}\,,$$

where C depends on $N, \alpha_0, \beta_0, T_2 - T_1$.

Corollary 3.2.2. *If $c \leq 0$ in the previous theorem is changed to $c \leq \beta$ for some $\beta > 0$, and all the other assumptions are retained, then*

$$\sup_Q v^- \leq e^{\beta(T_2 - T_1)}\left(\sup_{\partial_P Q} v^- + C\|f^-\|_{L^{N+1}(Q))}\right)\,.$$

where C depends on $N, \alpha_0, \beta_0, T_2 - T_1$.

Proof of Corollary 3.2.2. We see that the function $\tilde{v} := e^{-\beta t} v$ satisfies (3.5) with c replaced by $c - \beta$ and f changed to $e^{-\beta t} f$. Since $c - \beta \leq 0$ in Q, Theorem 3.2.1 yields

$$\sup_{Q} \tilde{v}^- \leq \sup_{\partial_P Q} \tilde{v}^- + C \|\tilde{f}^-\|_{L^{N+1}(Q)} ,$$

where $\tilde{f}(x,t) = e^{-kt} f(x,t)$ and C depends on $N, \alpha_0, \beta_0, T_2 - T_1$. Using the definition of $\tilde{v}$ we obtain the desired result. $\square$

The following lemma states a version of the maximum principle on small domains. It was originally proved in [8] in elliptic setting with $f = g \equiv 0$. For related results and extensions to elliptic systems see [4, 9, 10, 13, 22]. A generalization to parabolic problems on cylindrical domains was proved in [52] with $f = g \equiv 0$ and later in [29] for general f and g. Here, we present yet another extension to sets in space-time (not necessarily cylindrical). The proof is partly motivated by elliptic results in [13] and it is only based on Theorem 3.2.1. It does not rely on a construction of supersolutions as in [8, 52]. However, such construction is possible for general space-time sets by an application of parabolic Monge-Ampère equation, and we present this approach in the proof of Lemma 3.2.3 bellow.

Lemma 3.2.3. *Given any $k > 0$ there exists $\delta = \delta(\alpha_0, \beta_0, N, k) > 0$ such that $|Q_{[t,t+1]}| < \delta$ for any $t \in \mathbb{R}$ implies the following. If $v \in C(\bar{Q})$ is a solution of problem* (3.5), (3.6) *with $L \in E(\alpha_0, \beta_0, Q)$ then*

$$\begin{aligned} \|v^-\|_{L^\infty(Q_t)} \leq 2 \max\{ &\|v^-\|_{L^\infty(Q_\tau)} e^{-k(t-\tau)}, \|g^-\|_{L^\infty(\partial_P Q_{[\tau,t]} \setminus Q_\tau)}\} \\ &+ C \|f^-\|_{L^{N+1}(Q_{[\tau,t]})} \qquad (\tau < t) , \end{aligned} \tag{3.8}$$

where C depends on N, β_0, α_0.

Proof. In the proof the constant C can vary from step to step, but it only depends on N, β_0, α_0. To simplify the notation let

$$\partial_S Q_{[a,b]} := \partial_P Q_{[a,b]} \setminus Q_a \qquad (a < b) .$$

Corollary 3.2.2 implies, that on a time interval of length at most one we have

$$\begin{aligned} \|v^-\|_{L^\infty(Q_{(t,t+s)})} &\le e^{\beta_0 s}\max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ Ce^{\beta_0 s}\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \qquad (t\in[\tau,T-1], s\in[0,1]) \,. \end{aligned} \tag{3.9}$$

The function $w := e^{(k+\ln 2)t}v$ satisfies

$$\begin{aligned} w_t - (M(x,t)-c^-)w &\ge (c^+ + k + \ln 2)w + \tilde{f} \qquad &((x,t)\in Q)\,, \\ w(x,t) &\ge \tilde{g}(x,t) \qquad &((x,t)\in \partial_S Q)\,, \end{aligned}$$

where $\tilde{f}(x,t) := e^{(k+\ln 2)t}f(x,t)$, $\tilde{g}(x,t) := e^{(k+\ln 2)t}g(x,t)$ and M was defined in (3.7). Since $-c^- \le 0$, Theorem 3.2.1 yields

$$\begin{aligned} &\|v^-\|_{L^\infty(Q_{t+s})} = e^{-(k+\ln 2)(t+s)}\|w^-\|_{L^\infty(Q_{t+s})} \\ &\le e^{-(k+\ln 2)(t+s)}\max\{\|w^-\|_{L^\infty(Q_t)}, \|\tilde{g}^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ e^{-(k+\ln 2)(t+s)}C\|(c^+ + k + \ln 2)w^- + \tilde{f}^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ C\left[(\beta_0 + k + \ln 2)\delta^{\frac{1}{N+1}}\|v^-\|_{L^\infty(Q_{[t,t+s]})} + \|f^-\|_{L^{N+1}(Q_{[t,t+s]})}\right] \\ &\qquad\qquad\qquad\qquad (t\in[\tau,T-s], s\in[0,1]) \,. \end{aligned} \tag{3.10}$$

If we choose δ such that

$$C(\beta_0 + k + \ln 2)\delta^{\frac{1}{N+1}} \le \frac{e^{-k}}{2}e^{-\beta_0}\,,$$

then by (3.10) and (3.9)

$$\begin{aligned} \|v^-\|_{L^\infty(Q_{t+s})} &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &\quad + \frac{e^{-k}}{2}e^{-\beta_0}\|v^-\|_{L^\infty(Q_{[t,t+s]})} + C\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ \frac{e^{-k}}{2}\max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\qquad\qquad\qquad\qquad (t\in[\tau,T-1]) \,. \end{aligned} \tag{3.11}$$

In particular for $s = 1$

$$\begin{aligned}\|v^-\|_{L^\infty(Q_{t+1})} &\le \max\{\frac{e^{-k}}{2}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} \\ &\quad + \frac{e^{-k}}{2}\max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \\ &\le \max\{e^{-k}\|v^-\|_{L^\infty(Q_t)}, 2\|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \\ &\qquad (t \in [\tau, T-1])\,.\end{aligned}$$

Iterating the previous expression for any $j \in \mathbb{N}$ with $t + j \le T$ we obtain:

$$\begin{aligned}\|v^-\|_{L^\infty(Q_{t+j})} &\le \max\{e^{-kj}\|v^-\|_{L^\infty(Q_t)}, 2\|g^-\|_{L^\infty(\partial_S Q_{[t,t+j]})}\} \\ &\quad + C\|f^-\|_{L^{N+1}(Q_{[t,t+j]})} \qquad (t \in [\tau, T-j])\,.\end{aligned} \tag{3.12}$$

Since any $t \in [\tau, T]$ can be expressed in the form $t = \tau + j + s$ where $j \in \mathbb{N} \cup \{0\}$ and $s \in [0, 1)$, (3.11) and (3.12) imply

$$\begin{aligned}\|v^-\|_{L^\infty(Q_t)} &\le \max\{e^{-kj}\|v^-\|_{L^\infty(Q_{\tau+s})}, 2\|g^-\|_{L^\infty(\partial_S Q_{[\tau+s,t]})}\} + C\|f^-\|_{L^{N+1}(Q_{[\tau+s,t]})} \\ &\le 2\max\{e^{-k(t-\tau)}\|v^-\|_{L^\infty(Q_\tau)}, \|g^-\|_{L^\infty(\partial_S Q_{[\tau,t]})}\} + C\|f^-\|_{L^{N+1}(Q_{[\tau,t]})}\,. \quad \square\end{aligned}$$

Remark 3.2.4. From the proof of Lemma 3.2.3 one can see that (3.8) can be changed to

$$\begin{aligned}\|v^-\|_{L^\infty(Q_t)} &\le 2\max\{\|v^-\|_{L^\infty(Q_\tau)}e^{-k(t-\tau)}, \|g^-\|_{L^\infty(\partial_P Q_{[\tau,t]}\setminus Q_\tau)}\} \\ &\quad + C\frac{1}{1+e^{-k}}\sup_{t\in[\tau,T-1]}\|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \qquad (\tau < t)\,.\end{aligned}$$

For the reader's convenience and for an easier reference later on, we formulate the following lemma that was proved in [52].

Lemma 3.2.5. *For any $r > 0$ there exist a constant $\gamma = \gamma(r, N, \alpha_0, \beta_0) > 0$ and a smooth function h_r on $B(0, r)$ with*

$$h_r(x) > 0 \quad (x \in B(0,r)), \quad h_r(x) = 0 \quad (x \in \partial B(0,r)),$$

such that for any $x_0 \in \Omega$ *with* $B(x_0,r) \subset \Omega$ *and any* $L \in E(\alpha_0, \beta_0, B(x_0,r) \times (0,\infty))$*, the function* $\phi(x,t) = e^{-\gamma t} h_r(x-x_0)$ *satisfies*

$$\begin{aligned} \partial_t \phi - L(x,t)\phi < 0, \qquad & (x,t) \in B(x_0,r) \times (0,\infty)\,, \\ \phi = 0 \quad , \qquad & (x,t) \in \partial B(x_0,r) \times (0,\infty)\,. \end{aligned} \tag{3.13}$$

As a consequence we have the following result.

Corollary 3.2.6. *Given* $r > 0$*, let* $\gamma = \gamma(r, N, \alpha_0, \beta_0) > 0$ *be as in Lemma 3.2.5. For fixed* $x_0 \in \mathbb{R}^N$ *and* $\tau < T$ *set* $Q = B(x_0,r) \times (\tau, T)$*, and assume that* $v \in C(\bar{Q})$ *satisfies* (3.5), (3.6) *with* $g = f \equiv 0$*, and* $L \in E(\alpha_0, \beta_0, Q)$*. If* $v(\cdot,\tau) \geq q$ *in* $B(x_0,r)$ *for some* $q > 0$*, then*

$$v(x,t) \geq \tilde{c}_r q e^{-\gamma(t-\tau)} \qquad \left((x,t) \in B\left(x_0, \frac{r}{2}\right) \times [\tau, T)\right),$$

where $0 < \tilde{c}_r \leq 1$ *depends only on* N, α_0, β_0 *and* r*.*

Proof. For the given r consider γ, h_r and ϕ as in Lemma 3.2.5. Then

$$\begin{aligned} v_t - L(x,t)v \geq 0 > \phi_t - L(x,t)\phi, \qquad & (x,t) \in B(x_0,r) \times (\tau,T)\,, \\ v(x,t) \geq 0 = \phi(x,t), \qquad & (x,t) \in \partial B(x_0,r) \times [\tau,T]\,, \end{aligned}$$

and

$$v(x,\tau) \geq q \frac{\phi(x,\tau)}{\|\phi(\cdot,\tau)\|_{L^\infty(B(x_0,r))}} \qquad (x \in B(x_0,r))\,.$$

An application of the comparison principle for v and ϕ gives

$$\begin{aligned} v(x,t) \geq q \frac{\phi(x,t)}{\|\phi(\cdot,\tau)\|_{L^\infty(B(x_0,r))}} \geq q e^{-\gamma(t-\tau)} \frac{h_r(x-x_0)}{\|h_r(\cdot - x_0)\|_{L^\infty(B(x_0,r))}} \\ ((x,t) \in B(x_0,r) \times [\tau,T])\,. \end{aligned} \tag{3.14}$$

Since $h > 0$ in $B(x_0,r)$, we obtain the desired result for

$$\tilde{c}_r = \frac{\inf_{x \in B(x_0,r/2)} h_r(x-x_0)}{\|h_r(\cdot - x_0)\|_{L^\infty(B(x_0,r))}}\,. \qquad \square$$

The next lemma, proved in [52, Lemma 3.4.], is a version of Krylov-Safonov Harnack inequality [36, 37] for sign changing supersolutions of parabolic problems (see also [33, 40]). Its formulation needs the following notation. For any open bounded subset S of $\mathbb{R}^{n+1}$ and any bounded, continuous function $f : S \to \mathbb{R}$ define

$$[f]_{p,S} := \left(\frac{1}{|S|} \int_S |f|^p dx\, dt \right)^{\frac{1}{p}} \qquad (p > 0)\,. \tag{3.15}$$

Lemma 3.2.7. *Given $\Omega \subset \mathbb{R}^N$, $d > 0$, $\varepsilon > 0$, $\theta > 0$, there are positive constants κ, κ_1, p determined only by $N, \mathrm{diam}\,\Omega, \alpha_0, \beta_0, d, \varepsilon$ and θ with the following properties. Let D and U be domains in Ω with $D \subset\subset U$, $\mathrm{dist}(\bar{D}, \partial U) \geq d$, $|D| > \varepsilon$ and let $Q = U \times (\tau, \tau + 4\theta)$. Assume that $v \in C(\bar{Q})$ satisfies* (3.5) *with $L \in E(\alpha_0, \beta_0, Q)$ and $f \in L^{N+1}(Q)$. Then*

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v(x,t) \geq \kappa [v^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \sup_{\partial_p(U\times(\tau,\tau+4\theta))} e^{4M\theta} v^- - \kappa_1 \|f^-\|_{L^{N+1}(Q)}\,,$$

where $M = \sup_{U\times(\tau,\tau+4\theta)} c$. If the inequality in (3.5) *is replaced by equality, then the conclusion holds with $p = \infty$ and with κ, κ_1 independent of ε, but with f^- replaced by f.*

Remark 3.2.8. The proof for the case $p = \infty$, which was not includes in [52], was given in [48].

Chapter 4

Scalar problem

This chapter starts with auxiliary sections, where we show how to derive linear equations from the nonlinear ones, by the reflections in hyperplanes, with normal vectors different from e_1, and also we prove geometrical properties of Ω. Then, we prove our main results for scalar problems.

4.1 Reflections in hyperplanes

Fix a unit vector $v \in W$, where W was defined in (D5).

In Section 3.1, we showed how to derive linear problems via reflection in hyperplanes if $v = e_1$. Since the method of derivation is analogous if $v \neq e_1$, we only state the most important steps for later references.

Recall, that for any vector $v \neq 0$ and $\lambda \in \mathbb{R}$ we already defined $\mathcal{P}_{\lambda,v} : \mathbb{R} \to \mathbb{R}$ to be the reflection in the hyperplane

$$H_{\lambda,v} = \{x \in \mathbb{R}^N : x \cdot v = \lambda|v|\}\,.$$

We also set $x^{\lambda,v} = \mathcal{P}_{\lambda,v}(x)$ for any $x \in \mathbb{R}^n$ and we denote

$$\begin{aligned} \Omega_{\lambda,v} &= \{x \in \Omega : x \cdot v \geq \lambda|v|\} && (\lambda \in \mathbb{R}, v \in \mathbb{R}^N)\,, \\ \ell(v) &= \sup\left\{x \cdot \frac{v}{\|v\|} : x \in \Omega\right\} && (\lambda \in \mathbb{R}, v \in \mathbb{R}^N)\,. \end{aligned}$$

Fix $v \neq e_1$ and in addition to (D1) and (N1)–(N3) assume (D4) and (N4). Similarly as in the case $v = e_1$ we obtain that for any $\lambda \in (\lambda^*(v), \ell(v))$, the function $u^{\lambda,v}(x,t) := u(x^{\lambda,v}, t)$ satisfies

$$\partial_t u^{\lambda,v} \geq F(t, x, u^{\lambda,v}, Du^{\lambda,v}, D^2 u^{\lambda,v}), \qquad (x,t) \in \Omega_{\lambda,v} \times (0,\infty)\,,$$

and the function $w^{\lambda,v} : \bar{\Omega}_{\lambda,v} \times (0,\infty) \to \mathbb{R}$, $w^{\lambda,v} : (x,t) \mapsto u(x^{\lambda,v}, t) - u(x,t)$ satisfies

$$\partial_t w^{\lambda,v} \geq L^{\lambda,v}(x,t) w^{\lambda,v}, \qquad (x,t) \in \Omega_{\lambda,v} \times (0,\infty)\,, \tag{4.1}$$

$$w^{\lambda,v} \geq 0, \qquad (x,t) \in \partial\Omega_{\lambda,v} \times (0,\infty)\,, \tag{4.2}$$

where $L^{\lambda,v} \in E(\alpha_0, \beta_0, \Omega_{\lambda,v} \times (0,\infty))$.

Convention 4.1.1. If $v = e_1$ we omit the argument v in $H_{\lambda,v}$, $\Omega_{\lambda,v}$, $x^{\lambda,v}$, $\Omega'_{\lambda,v}$, $\ell(v)$ and we simply write H_λ, Ω_λ, x^λ, Ω'_λ, ℓ instead.

4.2 Properties of $\Omega_{\lambda,v}$

In this purely geometrical section we assume that Ω is a bounded domain satisfying (D1), (D3) and (D4). Let us start with a lemma that extends property (D4) to all vectors in W, where W is as in (D5): $W = \mathrm{Cone}_{0,1}(-e_1, -\hat{v}) \cap \partial B(0,1)$.

Lemma 4.2.1. *If for some $\delta^* > 0$, Ω satisfies (D1), (D3) and (D4), then $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v}$ for any $v \in W$ sufficiently close to e_1.*

Proof. Fix $v \in W$ and let $\alpha, \beta \in [0,1]$ be such that $v = \alpha\hat{v} + \beta e_1$. If $\alpha = 0$ or $\beta = 0$ the statement follows directly from (D1) and (D4), thus we consider $\alpha, \beta > 0$. Since $v \neq \hat{v}$, $H_{\lambda^*(\hat{v}),\hat{v}} \cap H_{\lambda^*(v),v} \neq \emptyset$. Then we define Λ as

$$x \cdot e_1 = \frac{1}{\beta} x \cdot v - \frac{\alpha}{\beta} x \cdot \hat{v} = \frac{\lambda^*(v) - \alpha\lambda^*(\hat{v})}{\beta} =: \Lambda \qquad (x \in H_{\lambda^*(\hat{v}),\hat{v}} \cap H_{\lambda^*(v),v})\,. \tag{4.3}$$

First assume $\Lambda \leq \delta^*$. By (D4) we have $y \cdot e_1 > \delta^*$ and $y \cdot \hat{v} \geq \lambda^*(\hat{v})$ for any $y \in \Omega_{\delta^*,e_1}$. Then using $\Lambda \leq \delta^*$ we obtain

$$y \cdot v = \alpha(y \cdot \hat{v}) + \beta(y \cdot e_1) > \alpha\lambda^*(\hat{v}) + \beta\delta^* \geq \lambda^*(v) \qquad (y \in \Omega_{\delta^*,e_1}),$$

and therefore $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v}$, as desired.

We finish the proof, once we show that $\Lambda > \delta^*$ leads to a contradiction. Define

$$\varepsilon_0 := \beta\,(\Lambda - \delta^*) = \beta\left(\frac{\lambda^*(v) - \alpha\lambda^*(\hat{v})}{\beta} - \delta^*\right) > 0\,. \tag{4.4}$$

By (4.4)

$$\alpha(x \cdot \hat{v}) + \beta(x \cdot e_1) = x \cdot v \geq \lambda^*(v) - \varepsilon_0 \geq \alpha\lambda^*(\hat{v}) + \beta\delta^* \qquad (x \in \Omega_{\lambda^*(v)-\varepsilon_0,v})\,.$$

Thus either $x \cdot \hat{v} \geq \lambda^*(\hat{v})$ or $x \cdot e_1 \geq \delta^*$ and by (D4), any of these cases yields $x \in \Omega_{\lambda^*(\hat{v}),\hat{v}}$. Hence $\Omega_{\lambda^*(v)-\varepsilon_0,v} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$ and in particular $\Omega_{\lambda^*(v)-\varepsilon,v} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$ for any $\varepsilon \in (0, \varepsilon_0]$.

Next, the definition of $\lambda^*(v)$ implies the existence of ε with $0 < \varepsilon \leq \varepsilon_0$, and a point $Q \in \Omega_{\lambda^*(v)-\varepsilon,v}$ such that $Q^{\lambda^*(v)-\varepsilon,v} \notin \Omega$. Let d_v and $d_{\hat{v}}$ be distances of Q to the hyperplanes $H_{\lambda^*(v)-\varepsilon,v}$ and $H_{\lambda^*(\hat{v}),\hat{v}}$ respectively and observe that $d_{\hat{v}} \geq d_v$. Then

$$Q - Q^{\lambda^*(v)-\varepsilon,v} = 2d_v v\,, \quad Q - Q^{\lambda^*(\hat{v}),\hat{v}} = 2d_{\hat{v}}\hat{v}\,,$$

and consequently using $v = \alpha\hat{v} + \beta e_1$

$$Q - Q^{\lambda^*(v)-\varepsilon,v} = \alpha\frac{d_v}{d_{\hat{v}}}(Q - Q^{\lambda^*(\hat{v}),\hat{v}}) + 2\beta d_v e_1\,.$$

Since $\Omega_{\lambda^*(v)-\varepsilon,v} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$, one has $\alpha\frac{d_v}{d_{\hat{v}}} \leq 1$ and the definition of $\lambda^*(\hat{v})$ then implies $R := Q - \alpha\frac{d_v}{d_{\hat{v}}}(Q - Q^{\lambda^*(\hat{v}),\hat{v}}) \in \Omega$. We arrive to a contradiction by showing that $Q^{\lambda^*(v)-\varepsilon,v} = R - 2\beta d_v e_1 \in \Omega$ for v sufficiently close to e_1. Since $R \in \Omega$ and Ω is symmetric and convex in x_1, it is sufficient to prove that $(R + Q^{\lambda^*(v)-\varepsilon,v}) \cdot e_1 \geq 0$. Notice that $Q - \frac{d_{\hat{v}}}{e_1 \cdot \hat{v}}e_1 \in H_{\lambda^*(\hat{v}),\hat{v}} \cap \Omega \subset \bar{\Omega}_{0,e_1}$, and therefore

$$\left(Q - \frac{d_{\hat{v}}}{e_1 \cdot \hat{v}}e_1\right) \cdot e_1 \geq 0\,.$$

Then using the definitions of $R, Q^{\lambda^*(v)-\varepsilon,v}$, and $v = \alpha\hat{v} + \beta e_1$ we obtain

$$\begin{aligned} R + Q^{\lambda^*(v)-\varepsilon,v} &= Q - 2\alpha d_v\hat{v} + Q^{\lambda^*(v)-\varepsilon,v} = 2[Q - \alpha d_v\hat{v} - d_v v] \\ &= 2[(Q - \frac{d_{\hat{v}}}{e_1\cdot\hat{v}}e_1) + \frac{d_{\hat{v}}}{e_1\cdot\hat{v}}e_1 - d_v(2\alpha\hat{v} + \beta e_1)]\,. \end{aligned}$$

Since $d_{\hat{v}} \geq d_v$ and $1 > e_1\cdot\hat{v} > 0$, one has $(R + Q^{\lambda^*(v)-\varepsilon,v})\cdot e_1 \geq 0$ for any α sufficiently close to 0 (and β close to 1). $\square$

The second lemma shows that the portion of $\partial\Omega$ that is close to H_{λ,e_1} is not symmetric with respect to this hyperplane.

Lemma 4.2.2. *Given $\delta^* > 0$, consider a bounded domain Ω satisfying (D1), (D3) and (D4). Fix $\lambda \in (\delta^*, \ell(e_1))$ and let U be a connected component of Ω_{λ,e_1}. Then for any $\varepsilon > 0$ there is $z \in (\partial U)\setminus H_{\lambda,e_1}$ such that dist$(z, H_{\lambda,e_1}) < \varepsilon$ and $z^{\lambda,e_1} \in \Omega$.*

Proof. Recall, that in (D4) we fixed a unit vector $\hat{v} = (\hat{v}_1, \hat{v}')$ with $\hat{v}_1 \in (0,1)$ and $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$. Choose $x^* \in \bar{U}$ such that $x^*\cdot\hat{v} = \inf_{x\in\bar{U}} x\cdot\hat{v}$. Clearly $x^* \in \partial U \cap H_{\lambda,e_1}$. Since $\lambda > \delta^*$ and $\Omega_{\delta^*,e_1} \subset \bar{\Omega}_{\lambda^*(\hat{v}),\hat{v}}$, for any sufficiently small $\varepsilon > 0$ one has $B(x^*,\varepsilon)\cap\Omega \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$. We prove that for any such $\varepsilon > 0$ there is a point z in $B(x^*,\varepsilon)$ with the desired properties. Since $x^* \in \bar{U}$, for

$$\delta := \frac{\varepsilon}{2}\frac{\sqrt{1-\hat{v}_1^2}}{2\sqrt{1-\hat{v}_1^2}+1}\,, \tag{4.5}$$

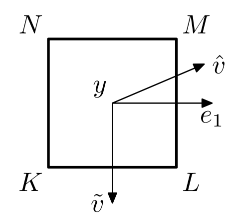
there is $y \in U \cap B(x^*,\delta)$ and $0 < \rho < \delta$ with $B(y,\rho) \subset U \cap B(x^*,\delta)$. Consider the two-dimensional plane $\mathcal{S}$ passing through y spanned by vectors e_1 and $\hat{v}$.

Then the two-dimensional closed square (see figure) $\mathcal{C}(y,\rho) \subset \mathcal{S}$ centered at y with a side of length ρ perpendicular to e_1, is a subset of $B(y,\rho)\cap\mathcal{S} \subset U$. Let $\tilde{v} \in \mathcal{S}$ be the unit vector perpendicular to e_1 with $\tilde{v}\cdot\hat{v} < 0$. An elementary calculation shows $\tilde{v}\cdot\hat{v} = -\sqrt{1-\hat{v}_1^2}$. Translate $\mathcal{C}(y,\rho)$ along $\tilde{v}$ as far as it stays inside U, that is, define

$$\kappa_0 := \sup\{\mu : \mathcal{C}(y+\kappa\tilde{v},\rho) \subset U \text{ for all } \kappa \in [0,\mu]\}.$$

Denote $\hat{y}$ and $\hat{K}, \hat{L}, \hat{M}, \hat{N}$ the center and the vertices of $\mathcal{C}(y + \kappa_0 \tilde{v}, \rho)$ ($\hat{X}$ is the image of X in translation by vector $\kappa_0 \tilde{v}$). We prove that $\hat{L}$ is a point with the desired properties.

First, since $\mathcal{C}(\hat{y}, \rho) \subset \bar{U}$, the definition of x^* yields $\hat{y} \cdot \hat{v} \geq x^* \cdot \hat{v}$ or equivalently $y \cdot \hat{v} + \kappa_0 \tilde{v} \cdot \hat{v} \geq x^* \cdot \hat{v}$ and consequently



$$\kappa_0 \leq \frac{y \cdot \hat{v} - x^* \cdot \hat{v}}{-\tilde{v} \cdot \hat{v}} \leq \frac{|y - x^*||\hat{v}|}{-\tilde{v} \cdot \hat{v}} < \frac{\delta}{-\tilde{v} \cdot \hat{v}} \,.$$

Then (4.5) implies

$$\begin{aligned} |x - x^*| &\leq |x - \hat{y}| + |\hat{y} - y| + |y - x^*| \leq \rho + \kappa_0 + \delta \\ &\leq 2\delta + \frac{\delta}{-\tilde{v} \cdot \hat{v}} < \varepsilon \qquad (x \in \mathcal{C}(\hat{y}, \rho)) \,. \end{aligned}$$

Thus $\mathcal{C}(\hat{y}, \rho) \subset B(x^*, \varepsilon)$ and in particular

$$\operatorname{dist}(\hat{L}, H_{\lambda, e_1}) \leq \operatorname{dist}(\hat{L}, x^*) < \varepsilon \,.$$

Next, by the definition of κ_0, $\operatorname{Int}(\mathcal{C}(\hat{y}, \rho)) \subset U$, where $\operatorname{Int}(\mathcal{C}(\hat{y}, \rho))$ is the interior of $\mathcal{C}(\hat{y}, \rho)$ in the topology of the plane $\mathcal{S}$. Moreover, there exists $\hat{z} \in \partial U$ that lies on the side connecting $\hat{K}$ and $\hat{L}$. Since $\mathcal{C}(\hat{y}, \rho) \subset \Omega_{\lambda^*(\hat{v}), \hat{v}}$, there is $\hat{\lambda} > \lambda^*(\hat{v})$ such that $\hat{L} \in H_{\hat{\lambda}, \hat{v}}$. Then $\mathcal{P}_{\hat{\lambda}, \hat{v}}(\operatorname{Int}\mathcal{C}(\hat{y}, \rho)) \subset \Omega$, and in particular $z \in \Omega$ for any $z \neq \hat{L}$ on the side connecting $\hat{K}$ and $\hat{L}$, sufficiently close to $\hat{L}$. Moreover, the convexity of Ω in e_1 implies $z \in \Omega$ for any $z \neq \hat{L}$ on the side connecting $\hat{K}$ and $\hat{L}$. Thus, $\hat{L} = \hat{z} \in \partial U$.

By the definition of ε, one has $\mathcal{P}_{\lambda, e_1} \hat{L} \in \bar{\Omega}_{0, e_1}$. If $\mathcal{P}_{\lambda, e_1} \hat{L} \in \partial \Omega$, then, by the convexity of Ω in e_1, the whole segment connecting $\hat{L}$ and $\mathcal{P}_{\lambda, e_1} \hat{L}$ is in $\partial \Omega$, a contradiction. Hence $\mathcal{P}_{\lambda, e_1} \hat{L} \in \Omega$.

Finally, since e_1 and $\tilde{v}$ are perpendicular,

$$\operatorname{dist}(\hat{L}, H_{\lambda, e_1}) = \operatorname{dist}(L, H_{\lambda, e_1}) > 0,$$

and therefore $\hat{L} \notin H_{\lambda, e_1}$. $\square$

4.3 Proofs of the main scalar results

In this section we assume that Ω satisfies (D1) and (D2) (not necessarily (D3)) and the nonlinearity F satisfies (N1) – (N3). At some places, where explicitly stated, we also assume (D3), (D4) or (N4). We remark that, even though (D2) is not needed in all results, we assume it throughout the section. Consider a classical solution u of (1.1) satisfying (2.1) and (2.2).

We use the notation introduced at the beginning of Chapter 2 and the following one. For any function $g : \Omega \to \mathbb{R}$, and any $\lambda \in [0, \ell)$ we set

$$V_\lambda g(x) := g(x^\lambda) - g(x), \qquad (x \in \Omega_\lambda)\,,$$

and for the solution u of (1.1) we define

$$w^\lambda(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,\infty))\,.$$

As shown in Subsection 3.1, the function w^λ solves a linear problem (3.3), (3.4) with $L \in E(\alpha_0, \beta_0, \Omega_\lambda \times (0,\infty))$. Hence the results of Chapter 3 are applicable to w^λ. We use this observation below, often without notice.

We carry out the process of moving hyperplanes in the following way. Starting from $\lambda = \ell$ we move λ to the left as long as the following property is preserved

$$\lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0. \tag{4.6}$$

We show below that the process can get started and then we examine the limit of the process given by

$$\lambda_0 := \inf\{\mu > 0 : \lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0 \text{ for each } \lambda \in [\mu, \ell)\}. \tag{4.7}$$

Remark 4.3.1. Note, that by the relative compactness of $\{u(\cdot,t) : t \geq 0\}$ in $C(\bar{\Omega})$, (4.6) is equivalent to the following property:

$$V_\lambda z(x) \geq 0 \qquad (x \in \Omega_\lambda,\ z \in \omega(u),\ \lambda \in [\lambda_0, \ell))\,. \tag{4.8}$$

Further observe that each $z \in \omega(u)$ is nonincreasing in x_1 in Ω_{λ_0}. Indeed, if $(x_1, x'), (\tilde{x}_1, x') \in \Omega_{\lambda_0}$ and $x_1 > \tilde{x}_1$, then $V_\lambda z \geq 0$ with $\lambda = (x_1 + \tilde{x}_1)/2 > \lambda_0$ gives $z(x_1, x') \geq z(\tilde{x}_1, x')$.

The following lemma shows that the process of moving hyperplanes can get started, that is, $\lambda_0 < \ell$. The proof follows arguments of [52, Lemma 4.1].

Lemma 4.3.2. *For λ_0 defined in (4.7) we have $\lambda_0 < \ell$. Moreover, if $\delta = \delta(\alpha_0, \beta_0, N) > 0$ is such that Lemma 3.2.3 holds with $k = 1$, then $|\Omega_{\lambda_0}| \geq \delta$.*

Proof. Fix $\varepsilon_0 > 0$ such that $|\Omega_{\ell-\varepsilon_0}| < \delta$, and we show that $\lambda_0 < \ell - \varepsilon_0$.

Suppose not, that is, suppose that there exist $\lambda \in (\ell - \varepsilon_0, \ell)$ and a sequence $(x_k, t_k)_{k\in\mathbb{N}} \subset \Omega_\lambda \times (0, \infty)$, with $t_k \to \infty$ as $k \to \infty$ and $(w^\lambda(x_k, t_k))^- \geq C^\lambda > 0$ for all $k \in \mathbb{N}$.

Since $|\Omega_\lambda| < \delta$ and $w^\lambda(x, t) \geq 0$ for any $(x, t) \in \partial\Omega_\lambda \times (0, \infty)$, Lemma 3.2.3 with $t_1 = t_k/2$, $t_2 = t_k$ yields

$$C^\lambda \leq \|(w^\lambda(\cdot, t_k))^-\|_{L^\infty(\Omega_\lambda)} \leq 2\|(w^\lambda(\cdot, \frac{t_k}{2}))^-\|_{L^\infty(\Omega_\lambda)} e^{-\frac{t_k}{2}} \qquad (k \in \mathbb{N}) .$$

Since w^λ is bounded, the right hand side of the previous inequality converge to 0 as $k \to \infty$, and we obtain a contradiction for sufficiently large k. $\square$

Next, we investigate the properties of functions $V_\lambda z$ and z for $\lambda \in [\lambda_0, \ell)$, where $z \in \omega(u)$.

Lemma 4.3.3. *For any $\tilde\lambda \in [\lambda_0, \ell)$, $z \in \omega(u)$ and any connected component $U_{\tilde\lambda}$ of $\Omega_{\tilde\lambda}$ the following statements hold true:*

(i) either $V_{\tilde\lambda} z \equiv 0$ or $V_{\tilde\lambda} z > 0$ in $U_{\tilde\lambda}$,

(ii) either $z \equiv 0$ or $z > 0$ in $U_{\tilde\lambda}$,

(iii) $z \equiv 0$ in $U_{\tilde\lambda}$ implies $V_{\tilde\lambda} z \equiv 0$ in $U_{\tilde\lambda}$.

Proof. The statement (i) was already proved in [52, Lemma 4.2].

To prove (ii) it is sufficient to show that $z(x^*) = 0$ for some $x^* = (x_1^*, (x^*)') \in U_{\tilde\lambda}$ implies $z \equiv 0$ in $U_{\tilde\lambda}$.

Since $z(x^*) = 0$, the monotonicity of z (see Remark 4.3.1) yields

$$z(x_1, (x^*)') = 0 \qquad (x_1 \in [x_1^*, \Gamma_{x^*}]) ,$$

where

$$\Gamma_{x^*} := \sup\{x_1 : (x_1, (x^*)') \in \Omega\} > x_1^* \,.$$

Then for any $\lambda \in (x_1^*, \Gamma_{x^*})$, (i) with $\tilde{\lambda} = \lambda$ yields $V_\lambda z \equiv 0$ in $\Omega_\lambda \cap U_{\tilde{\lambda}}$, and therefore $z(x_1, (x^*)') = 0$ for all $x_1 \in [2x_1^* - \Gamma_{x^*}, \Gamma_{x^*}]$. Since $x_1^* > 2x_1^* - \Gamma_{x^*}$, we can iterate this argument with x^* replaced by $(2x_1^* - \Gamma_{x^*}, (x^*)')$ and obtain $z(x_1, (x^*)') = 0$ for each $x_1 \in [\tilde{\lambda}, \Gamma_{x^*}]$.

Consequently $V_\lambda z \equiv 0$ in $\Omega_\lambda \cap U_{\tilde{\lambda}}$ for all $\lambda \in [\tilde{\lambda}, \Gamma_{x^*}]$. To finish the proof of (ii), it is sufficient to show $\Lambda = \ell_{U_{\tilde{\lambda}}}$, where

$$\ell_{U_{\tilde{\lambda}}} := \sup\{x_1 : (x_1, x') \in U_{\tilde{\lambda}} \text{ for some } x' \in \mathbb{R}^{N-1}\} \,,$$

and

$$\Lambda := \sup\{\mu \in (\tilde{\lambda}, \ell_{U_{\tilde{\lambda}}}) : V_\lambda z \equiv 0 \text{ in } U_{\tilde{\lambda}} \cap \Omega_\lambda \text{ for all } \lambda \in (\tilde{\lambda}, \mu)\} \geq \Gamma_{x^*} > \tilde{\lambda} \,.$$

Indeed, then, as in Remark 4.3.1, z is constant in x_1 in $U_{\tilde{\lambda}}$, and the boundary condition yields $z \equiv 0$ in $U_{\tilde{\lambda}}$ as desired.

For a contradiction assume that $\Lambda < \ell_{U_{\tilde{\lambda}}}$. Since $\tilde{\lambda} < (3\Lambda + \tilde{\lambda})/4 < \Lambda$, $V_{(3\Lambda+\tilde{\lambda})/4} z \equiv 0$ and consequently z is constant in x_1 for $x_1 \in (\tilde{\lambda}, (3\Lambda - \tilde{\lambda})/2)$. Thus by (i), $V_\lambda z \equiv 0$ for each $\lambda \in (\tilde{\lambda}, \min\{(3\Lambda - \tilde{\lambda})/2, \ell_{U_{\tilde{\lambda}}}\})$, a contradiction to the definition of Λ.

To prove (iii), observe that (using (i)) $V_\lambda z \equiv 0$ for each $\lambda \in (\tilde{\lambda}, \ell_{U_{\tilde{\lambda}}})$. Then the statement follows from the continuity. $\square$

The next proposition plays a central role in our arguments. The techniques are partly motivated by [52, Theorem 3.7], but the situation is more complicated here. Complications arise from the fact, that the solution u can be small on different connected components of Ω_{λ_0} at different times. A careful analysis of the interaction between different connected components of Ω_{λ_0} is required.

Proposition 4.3.4. *Assume $\lambda_0 > 0$. Then there is $z \in \omega(u)$ and a connected component U_{λ_0} of Ω_{λ_0} such that $V_{\lambda_0} z \equiv 0$ and $z > 0$ in U_{λ_0}.*

Proof. We proceed by a contradiction. That is (cf. Lemma 4.3.3), we assume:

$$\text{For any } z \in \omega(u) \text{ and any connected component } U_{\lambda_0} \text{ of } \Omega_{\lambda_0} \text{ either } V_{\lambda_0} z > 0 \text{ or } z \equiv 0 \text{ in } U_{\lambda_0}. \tag{4.9}$$

The definition of λ_0, yields the existence of an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to λ_0, $x_k \in \Omega_{\lambda_k}$ and $z_k \in \omega(u)$ such that $V_{\lambda_k} z_k(x_k) < 0$. Then, by (D2), we can fix a connected component U_{λ_0} of Ω_{λ_0} such that, for each $\lambda < \lambda_0$ there is $k \in \mathbb{N}$ with $x_k \in U_\lambda$, where

$$U_\mu \text{ is the connected component of } \Omega_\mu \text{ with } U_\mu \cap U_{\lambda_0} \neq \emptyset\,. \tag{4.10}$$

Next, define

$$U^*_{\lambda_0} := \bigcap_{\lambda<\lambda_0} U_\lambda\,,$$

and observe that $|U_\lambda \setminus U^*_{\lambda_0}|$ is arbitrary small if $\lambda > \lambda_0$ is sufficiently close to λ_0. (This property does not hold true, if we replace $U^*_{\lambda_0}$ by U_{λ_0}, consider for example Ω such that Ω_λ has two connected components for all $\lambda \geq \lambda_0$, but it is connected for $\lambda < \lambda_0$).

To continue we distinguish three cases

a) there is $z \in \omega(u)$ such that $V_{\lambda_0} z > 0$ on $U^*_{\lambda_0}$,

b) for all $z \in \omega(u)$, $z \equiv 0$ on $U^*_{\lambda_0}$,

c) for each $z \in \omega(u)$, $z \equiv 0$ on a connected component of $U^*_{\lambda_0}$, and for some $\tilde{z} \in \omega(u)$, $V_{\lambda_0}\tilde{z} > 0$ in a connected component $\tilde{U}_{\lambda_0}$ of $U^*_{\lambda_0}$

A contradiction with the definition of $\{x_k\}_{k\in\mathbb{N}}$ follows from the next three lemmas.

Lemma 4.3.5. *Assume that $\lambda_0 > 0$ and a) holds. Then $V_\lambda z \geq 0$ in U_λ for all $z \in \omega(u)$ and $\lambda < \lambda_0$ sufficiently close to λ_0.*

Lemma 4.3.6. *Assume that $\lambda_0 > 0$ and b) holds. Then $V_\lambda z \geq 0$ in U_λ for all $z \in \omega(u)$ and $\lambda < \lambda_0$ sufficiently close to λ_0.*

Lemma 4.3.7. *Assume that $\lambda_0 > 0$ and (4.9) holds. Then c) does not hold.*

Lemma 4.3.5 was already proved in [52, Lemma 4.3], where we replace Ω_{λ_0} by $U^*_{\lambda_0}$ and Ω_λ by U_λ. The proofs of Lemma 4.3.6 and Lemma 4.3.4 are postponed till the next section. $\square$

Now we address the question how big is the union of the connected components of Ω_{λ_0} on which the situation from Proposition 4.3.4 occurs.

Lemma 4.3.8. *Let $\delta = \delta(\alpha_0, \beta_0, N) > 0$ be such that Lemma 3.2.3 holds with $k = 1$. If $\Omega^*_{\lambda_0}$ denotes the union of connected components U of Ω_{λ_0}, for which there exists $z \in \omega(u)$ with $V_{\lambda_0} z \equiv 0$ and $z > 0$ in U, then*

$$\left|\Omega^*_{\lambda_0}\right| > \frac{\delta}{2}.$$

Proof. We proceed by contradiction, that is, we assume $\left|\Omega^*_{\lambda_0}\right| \le \frac{\delta}{2}$. Fix an open set $D \subset\subset \Omega_{\lambda_0} \setminus \Omega^*_{\lambda_0}$, convex in x_1, with $|\Omega_{\lambda_0} \setminus (\Omega^*_{\lambda_0} \cup D)| < \frac{\delta}{8}$, and fix $\varepsilon_0 > 0$ such that $|\Omega_{\lambda_0 - \varepsilon_0} \setminus \Omega_{\lambda_0}| < \frac{\delta}{8}$. Then $|\Omega_\lambda \setminus D| < \frac{3}{4}\delta$ for any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$.

By Proposition 4.3.4, the set $\Omega^*_{\lambda_0}$ is nonempty and we can choose its connected component $\tilde{U}$ and $\tilde{z} \in \omega(u)$ such that $V_{\lambda_0}\tilde{z} \equiv 0$ and $\tilde{z} > 0$ in $\tilde{U}$. Then for

$$C_\lambda := \frac{1}{6}\|(V_\lambda \tilde{z})^-\|_{L^\infty(\tilde{U})} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]) \tag{4.11}$$

we have $C_\lambda > 0$ for all $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0)$ and $C_\lambda \to 0$ as $\lambda \to \lambda_0$.
Claim. We can decrease $\varepsilon_0 > 0$ such that

$$|K_{z,\lambda}| < \frac{\delta}{4} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0), z \in \omega(u)),$$

where

$$K_{z,\lambda} := \{x \in D : V_\lambda z(x) < -C_\lambda\} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0), z \in \omega(u)).$$

We postpone the proof of this claim, and we finish the proof of the lemma first.

Fix any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0)$ and denote

$$Q := \{(x,t) \in \Omega_\lambda : w^\lambda(x,t) < -2C_\lambda\}.$$

Then (2.3) implies that for each sufficiently large t there is $z \in \omega(u)$ with $Q_t \cap D \subset K_{z,\lambda}$. Consequently for sufficiently large t

$$|Q_t| \leq |\Omega_\lambda \setminus D| + |K_t| < \frac{3}{4}\delta + \frac{1}{4}\delta = \delta\,,$$

and therefore $|Q_{[t,t+1]}| < \delta$ ($n+1$-dimensional measure). Choose $(t_k)_{k\in\mathbb{N}}$, $t_k \to \infty$ such that $u(\cdot, t_k) \to \tilde{z}$ as $k \to \infty$. Then (4.11) gives $\|(w^\lambda(\cdot, t_k))^-\|_{L^\infty(\tilde{U})} \geq 5C^\lambda$ for any sufficiently large k.

Since, by the definition of Q and (3.4), one has $(w^\lambda)^- \leq C_\lambda$ on $\partial_P Q$, Lemma 3.2.3 yields for any sufficiently large k

$$5C^\lambda \leq \|(w^\lambda)^-\|_{L^\infty(Q_{t_k})} \leq 2\max\{\|(w^\lambda)^-\|_{L^\infty(Q_{t_k-T})}e^{-T}, 2C_\lambda\},\ (0 < T < t_k)\,,$$

a contradiction for sufficiently large T.

Proof of the claim. For already fixed $\tilde{z}$ and $\tilde{U}$ denote

$$M := \sup_{\tilde{U}} \tilde{z} > 0\,. \tag{4.12}$$

Since $V_{\lambda_0} z \geq 0$ in Ω_{λ_0},

$$\begin{aligned} &V_\lambda z(x_1, x') = z(2\lambda - x_1, x') - z(x_1, x') \\ &\ = z(2\lambda - x_1, x') - z(2\lambda_0 - (2\lambda - x_1), x') + z(x_1 + 2(\lambda_0 - \lambda), x') - z(x_1, x') \\ &\qquad\qquad \geq z(x_1 + 2(\lambda_0 - \lambda), x') - z(x_1, x') \qquad ((x_1, x') \in \Omega(\lambda), z \in \omega(u))\,, \end{aligned} \tag{4.13}$$

where

$$\Omega(\lambda) := \{x = (x_1, x') \in \Omega_{\lambda_0} : (x_1 + 2(\lambda_0 - \lambda), x') \in \Omega_{\lambda_0}\} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0])\,.$$

Decrease $\varepsilon_0 > 0$ if necessary such that $D \subset \Omega(\lambda)$ for each $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$. We show that it is possible to decrease ε_0 such that $K_{z,\lambda} \neq \emptyset$ for some $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$, $z \in \omega(u)$, implies

$$\sup_D z < \frac{\delta M}{96[\mathrm{diam}(\Omega)]^N}\,. \tag{4.14}$$

Assume not, that is, assume that there is $(x_n)_{n\in\mathbb{N}} \subset D$, $(z_n)_{n\in\mathbb{N}} \subset \omega(u)$ and $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \nearrow \lambda_0$ as $n \to \infty$ such that $V_{\lambda_n} z_n(x_n) < -C_{\lambda_n}$ and (4.14) does not hold. After passing to a subsequence, we can assume $z_n \to z \in \omega(u)$ with convergence in $C(\bar{\Omega})$ and $x_n \to x_0 \in \bar{D}$, as $n \to \infty$. Then $V_{\lambda_0} z(x_0) \leq 0$ and $\|z\|_{L^\infty(\bar{D})} > 0$. Consequently by Lemma 4.3.3 (i) and (ii) with $\tilde{\lambda} = \lambda_0$, $V_{\lambda_0} z \equiv 0$ and $z > 0$ in a connected component U of $\Omega_{\lambda_0} \setminus \Omega^*_{\lambda_0}$ for which $x_0 \in U$, a contradiction to the definition of $\Omega^*_{\lambda_0}$.

Next, by (4.13), monotonicity of z on Ω_{λ_0}, convexity of D in x_1, and (4.14)

$$\begin{aligned}
|K_{z,\lambda}| &= \int_D I_{\{x \in D: V_\lambda z(x) < -C_\lambda\}} dx \leq \int_D I_{\{x \in D: z(x_1 + 2(\lambda_0 - \lambda), x') - z(x_1, x') < -C_\lambda\}} dx \\
&= \iint_D I_{\{x \in D: [z(x_1, x') - z(x_1 + 2(\lambda_0 - \lambda), x')]/C_\lambda > 1\}} dx_1 dx' \\
&\leq \frac{1}{C_\lambda} \iint_D z(x_1, x') - z(x_1 + 2(\lambda_0 - \lambda), x') dx_1 dx' \\
&\leq \frac{1}{C_\lambda} \int_D z\, dx - \frac{1}{C_\lambda} \int_{D_{2(\lambda_0 - \lambda)}} z\, dx \leq \frac{2(\lambda_0 - \lambda) \sup_D z}{C_\lambda} [\mathrm{diam}(\Omega)]^{N-1} \\
&\leq \frac{(\lambda_0 - \lambda)\delta M}{48 C_\lambda \mathrm{diam}(\Omega)} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0], z \in \omega(u)),
\end{aligned} \tag{4.15}$$

where I_A is the indicator function of a set A and $D_\mu := \{x \in D : x - \mu e_1 \in D\}$. Finally, let us estimate C_λ. Decrease $\varepsilon_0 > 0$ one more time to obtain

$$\sup_{\Omega(\lambda) \cap \tilde{U}} \tilde{z} \geq \frac{M}{2} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]). \tag{4.16}$$

Since $\tilde{z}^{\lambda_0} \equiv 0$ in $\tilde{U}$, the inequality (4.13) becomes an equality:

$$\tilde{z}(x_1 + 2(\lambda_0 - \lambda), x') - \tilde{z}(x_1, x') = V_\lambda \tilde{z}(x_1, x') \geq -6C_\lambda \qquad ((x_1, x') \in \Omega(\lambda) \cap \tilde{U}),$$

where the last inequality follows from (4.11). Hence, the function $\tilde{z}$ cannot decrease in x_1 by more than $6C_\lambda$ on an interval of length $2(\lambda_0 - \lambda)$. Moreover, $\tilde{z}(x_1 + 2(\lambda_0 - \lambda), x') = 0$ for $(x_1, x') \in \partial\Omega(\lambda) \setminus H_{\lambda_0}$. Thus using (4.16) we obtain

$$C_\lambda \geq \frac{(\lambda_0 - \lambda)M}{12 \mathrm{diam}(\Omega)} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]).$$

A substitution of this estimate into (4.15) yields the desired result. $\square$

The next lemma deals with the strict monotonicity of functions in ω-limit set. The proof can be found in [52, Lemma 4.6].

Lemma 4.3.9. *Assume that Ω_{λ_0} is connected. Then for any $z \in \omega(u)$ we have either $z \equiv 0$ on Ω_{λ_0} or else $z > 0$ and z is strictly decreasing in x_1 in Ω_{λ_0}. The latter holds in the form $z_{x_1} < 0$ if $z_{x_1} \in C(\Omega_{\lambda_0})$.*

Once we proved all auxiliary results, it is rather standard to prove our first main theorem.

Proof of theorem 2.1.2. In addition to the assumptions of this section we also assume (D3). We show that λ_0 defined in (4.7) satisfies the assertions of the theorem. First, by Remark 4.3.1, each $z \in \omega(u)$ is nonincreasing in x_1 in Ω_{λ_0}.

Assume $\lambda_0 > 0$. By Lemma 4.3.5, for each $z \in \omega(u)$ there is a connected component U of Ω_{λ_0} such that (2.4) holds true. Next, the existence of $\tilde{z} \in \omega(u)$ such that (2.5), (2.6) hold, follows from Proposition 4.3.4 and the strict monotonicity follows from Lemma 4.3.9.

Assume $\lambda_0 = 0$. Since Ω_0 is connected, Lemma 4.3.3 (ii) and (iii) with $\tilde{\lambda} = \lambda_0 = 0$, imply that for each $z \in \omega(u)$ either $z \equiv 0$ and $V_{\lambda_0} z \equiv 0$ in Ω_0 or $z > 0$ and $V_{\lambda_0} z \geq 0$ in Ω_0. It means that either $z \equiv 0$ in Ω or $z > 0$ in Ω.

If there is $\tilde{z} \in \omega(u)$ such that $\tilde{z} > 0$ in Ω, then using analogous arguments as above, moving hyperplanes starting from the left (or we replace e_1 by $-e_1$), we have

$$V_\lambda z(x) \geq 0 \qquad (x \in \Omega_\lambda^-, z \in \omega(u), \lambda \leq 0)\,, \tag{4.17}$$

where $\Omega_\lambda^- := \{x \in \Omega : x_1 < \lambda\}$. The process of the moving hyperplanes continued till 0, since $\tilde{z} > 0$ in Ω. If we set $\lambda = 0$ in (4.17), then we obtain $V_0 z(x) \leq 0$ for each $x \in \Omega_0$ and $z \in \omega(u)$. Therefore $V_0 z(x) \equiv 0$ for each $x \in \Omega_0$ and $z \in \omega(u)$.

If $\omega(u) = \{0\}$, the statement is trivial. $\square$

Let us turn our attention to the second main result.

Proof of theorem 2.1.3. In the proof we do not apply Convention 4.1.1 and we indicate explicitly the dependence of all functions and sets on a vector $v \in W$.

By Lemma 4.2.1 we can assume (if we change $\hat{v}$) that for each $v \in W$, (D4) holds with $\hat{v}$ replaced by v.

Let $\delta = \delta(\alpha_0, \beta_0, N) > 0$ be such that Lemma 3.2.3 holds with $k = 1$. We show that the theorem holds true for any $\delta^* > 0$ for which $|\Omega_{0,e_1} \setminus \Omega_{\delta^*,e_1}| < \frac{\delta}{2}$. (4.18)

In addition to the assumptions of this section we also assume (D3), (D5), (N4) and (D4) with already fixed δ^*. Analogously as before define

$$V_{\lambda,v}\zeta(x) := \zeta(x^{\lambda,v}) - \zeta(x) \qquad (x \in \Omega_{\lambda,v}, v \in W, \zeta \in \omega(u), \lambda > \lambda^*(v)) .$$

Moreover for $w^{\lambda,v} = u(x^{\lambda,v}, t) - u(x,t)$ denote

$$\lambda_0(v) = \inf\{\mu > \lambda^*(v) : \lim_{t\to\infty} \|(w^{\lambda,v}(\cdot,t))^-\|_{L^\infty(\Omega_{\lambda,v})} = 0 \text{ for each } \lambda \in [\mu, \ell(v))\}.$$

We now use moving hyperplanes in a direction $v \in W$ in a similar way as we did in the direction e_1. The hyperplanes are now $H_{\lambda,v}$, for $\lambda \in (\lambda^*(v), \ell(v))$. Then one of the following statements is true:

(i) $\lambda_0(v) = \lambda^*(v)$,

(ii) $\lambda_0(v) \in (\lambda^*(v), \ell(v))$ and there exists a connected component $U(v)$ of $\Omega_{\lambda_0(v),v}$ and $\zeta \in \omega(u)$ such that $\zeta > 0$ and $V_{\lambda_0(v),v}\zeta \equiv 0$ in $U(v)$.

To prove this, we use arguments analogous of those used in Lemmas 4.3.2, 4.3.3, 4.3.8, and in Proposition 4.3.4, where we replace the direction e_1 with v and the assumption $\lambda_0(e_1) > 0$ with $\lambda_0(v) > \lambda^*(v)$. Also the assumption (D1) is replaced by Lemma 4.2.1, (D2) by (D5), and (N3) by (N4). The assumptions (N1), (N2) remain unchanged as they are independent of a direction, and (D3) was not supposed in these results.

To prove Theorem 2.1.3 we need to show that $\lambda_0(e_1) = 0$ (the rest of the statements follow from Theorem 2.1.2). We show that the assumption $\lambda_0(e_1) > 0$ leads to a contradiction.

First assume that $\lambda_0(e_1) > \delta^*$ and the condition (i) holds true for some $v \in W\backslash\{e_1\}$. Then, by Proposition 4.3.4, there is $\tilde{z} \in \omega(u)$ and a connected component $U(e_1)$ of $\Omega_{\lambda_0(e_1),e_1}$ such that $V_{\lambda_0(e_1),e_1}\tilde{z} \equiv 0$ and $\tilde{z} > 0$ in $U(e_1)$. Also, (i) and Lemma 4.2.1 imply

$$\Omega_{\lambda_0(e_1)-(\lambda_0(e_1)-\delta^*),e_1} = \Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v} = \Omega_{\lambda_0(v),v}\,.$$

Consequently, Lemma 4.2.2 with $\varepsilon = (\lambda_0(e_1) - \delta^*)/2 > 0$ yields $\tilde{x} \in \partial\Omega \cap \partial U(e_1)$ such that $\tilde{x}^{\lambda_0(e_1),e_1} \in \Omega_{\lambda_0(e_1)-2\varepsilon,e_1} \subset \Omega_{\lambda_0(v),v}$.

Furthermore, $V_{\lambda_0(e_1),e_1}\tilde{z}(\tilde{x}) = 0$ and $\tilde{z}(\tilde{x}) = 0$, because $\tilde{x} \in \partial\Omega$. Therefore $\tilde{z}(\tilde{x}^{\lambda_0(e_1),e_1}) = 0$. Then, since $\tilde{x}^{\lambda_0(e_1),e_1} \in \Omega_{\lambda_0(v),v}$, Lemma 4.3.3 (ii) (in the direction v) with $\tilde{\lambda} = \lambda_0(v)$ yields $\tilde{z} \equiv 0$ in the connected component of $\Omega_{\lambda^*(v),v}$ that has a nonempty intersection with $U(e_1)$. Finally, Lemma 4.3.3 (ii) (in the direction e_1) with $\tilde{\lambda} = \lambda_0(e_1)$ implies $\tilde{z} \equiv 0$ in $U(e_1)$, a contradiction.

Next, we assume that $\lambda_0(e_1) > \delta^*$ and (ii) hold for all $v \in W$. Since W is uncountable and $|U(v)| > 0$ for each $v \in W$, there are $v, v' \in W$ such that $U(v) \cap U(v') \neq \emptyset$. Denote $\mathcal{A} := U(v) \cup \mathcal{P}_{\lambda_0(v),v}U(v)$, $\mathcal{B} := U(v') \cup \mathcal{P}_{\lambda_0(v'),v'}U(v')$ and without loss of generality assume $\mathcal{B} \not\subset \mathcal{A}$ (otherwise interchange v and v'). Then $\mathcal{B} \setminus \mathcal{A} \neq \emptyset$ and $\partial(\mathcal{B} \setminus \mathcal{A}) \subset \partial\mathcal{B} \cup \partial\mathcal{A}$. Next, we see that $\partial(\mathcal{B} \setminus \mathcal{A}) \not\subset \partial\mathcal{B}$, since otherwise we obtain $\mathcal{B} \setminus \mathcal{A} = \mathcal{B}$, a contradiction to $\emptyset \neq U(v) \cap U(v') \subset \mathcal{A} \cap \mathcal{B}$.

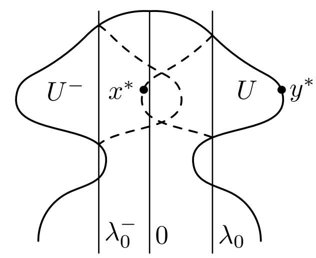
Thus, there is $\hat{x} \in \mathcal{B} \cap \partial\mathcal{A}$, or equivalently

$$\hat{x} \in \mathcal{B} \cap (\mathcal{P}_{\lambda_0(v),v}\partial U(v) \setminus \partial H_{\lambda_0(v),v})\,.$$

By ii) there is $\zeta \in \omega(u)$ such that $\zeta > 0$ in $U(v)$ and $\zeta^{\lambda_0(v),v} \equiv 0$ in $U(v)$ and in particular $\zeta(\hat{x}) = 0$. By the analogous arguments as in the previous case $\zeta \equiv 0$ in $\mathcal{B}$ and since $U(v) \cap U(v') \neq \emptyset$, also $\zeta \equiv 0$ in $U(v)$, a contradiction.

Finally, assume $0 < \lambda_0(e_1) \leq \delta^*$. If we start the process of the moving hyperplanes from the left (or we replace e_1 by $-e_1$), then using analogous arguments as above, we obtain $0 \geq \lambda_0^-(e_1) \geq -\delta^*$, where

$$\lambda_0^-(e_1) := \sup\{\mu < 0 : \lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\mathcal{P}_{0,e_1}(\Omega_{\lambda,e_1}))} = 0 \text{ for each } \lambda \in (-\ell,\mu]\}\,.$$



Without loss of generality assume $\lambda_0(e_1) \geq |\lambda_0^-(e_1)|$ (otherwise replace $u(x,t)$ by $u(x^{0,e_1},t)$). By Lemma 4.3.8, $|\Omega^*_{\lambda_0(e_1),e_1}| > \delta/2$ and since $|\Omega_{0,e_1} \setminus \Omega_{\lambda_0(e_1),e_1}| \leq \delta/2$, one has $\mathcal{P}_{\lambda_0(e_1),e_1}(\Omega^*_{\lambda_0(e_1),e_1}) \not\subset \Omega_{0,e_1} \setminus \Omega_{\lambda_0(e_1),e_1}$. Thus, there exist a connected component U of $\Omega^*_{\lambda_0(e_1),e_1}$, and $y^* \in \partial U$ such that $x_1^* < 0$, where $x^* = (x_1^*, (x^*)') := \mathcal{P}_{\lambda_0(e_1),e_1} y^*$ (see figure). Moreover, since $U \subset \Omega^*_{\lambda_0(e_1),e_1}$, there is $z \in \omega(u)$ with $z^{\lambda_0(e_1),e_1} \equiv 0$ and $z > 0$ in U. In particular

$$z(x^*) = 0 \quad \text{and} \quad z(x_1, (x^*)') > 0 \qquad (x_1 \in (x_1^*, 0]). \tag{4.19}$$

Denote by U^- the connected component of $\Omega^-_{\lambda_0^-} := \{x \in \Omega : x_1 < \lambda_0^-\}$ that contains $\mathcal{P}_{0,e_1}(U)$. Since $\lambda_0(e_1) \geq |\lambda_0^-(e_1)|$, $x^* \in U^- \cup \mathcal{P}_{\lambda_0^-(e_1),e_1}(U^-)$ and consequently Lemma 4.3.3 (ii) and (iii) with $\tilde{\lambda} = -\lambda_0(e_1)$ (with x_1 changed to $-x_1$) yields $z \equiv 0$ in $U^- \cup \mathcal{P}_{\lambda_0^-(e_1),e_1}(U^-)$. In particular $z(x_1, (x^*)') = 0$ for all $x_1 \in (x_1^*, 0]$, a contradiction to (4.19).

Hence, in all cases we found a contradiction. Therefore $\lambda_0(e_1) = 0$ and we are done. □

4.4 Proofs of Lemma 4.3.6 and Lemma 4.3.7

The assumptions in this section are the same as in Section 4.3 and Proposition 4.3.4. In particular we assume that Ω satisfies (D1) and (D2) and the nonlinearity F satisfies (N1) – (N3). We consider a classical solution u of (1.1) satisfying (2.1) and (2.2). We also return to Convention 4.1.1, that is, we do not indicate the dependence of sets or functions on e_1. About λ_0 defined in (4.7) we assume $\lambda_0 > 0$. Recall the notation U_{λ_0}, $U^*_{\lambda_0}$ and U_λ from the proof of Proposition 4.3.4 (see the paragraph containing (4.10)).

Proof of Lemma 4.3.6. Let $\delta > 0$ be such that Lemma 3.2.3 holds with $k = 1$, and fix $\varepsilon_0 > 0$ such that $\lambda_0 > \varepsilon_0$ and $|\Omega_{\lambda_0 - \varepsilon_0} \setminus \Omega_{\lambda_0}| < \delta$. We show that the conclusion of the lemma holds true for all $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$. For a contradiction assume that there is $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$, $C_\lambda > 0$ and a sequence $(x_j, t_j)_{j\in\mathbb{N}} \subset U_\lambda \times (0, \infty)$ with $t_j \to \infty$ as $j \to \infty$ and $w^\lambda(x_j, t_j) \le -C_\lambda$.

By b) and (2.3)

$$\lim_{t\to\infty} \|u(\cdot, t)\|_{L^\infty(U^*_{\lambda_0})} = 0\,, \tag{4.20}$$

and in particular there is $T > 0$ such that $\|u(\cdot, t)\|_{L^\infty(U^*_{\lambda_0})} \le \frac{C_\lambda}{4}$ for any $t \ge T$. Consequently $w^\lambda(x, t) \ge -\frac{C_\lambda}{4}$ for all $(x, t) \in U^*_{\lambda_0} \times (T, \infty)$, and therefore $x_j \in U_\lambda \setminus U^*_{\lambda_0}$, $j \in \mathbb{N}$. Now, (4.20) and an application of Lemma 3.2.3 with $k = 1$ on the set $(U_\lambda \setminus U^*_{\lambda_0}) \times (T, t_j)$ yields for all sufficiently large $j \in \mathbb{N}$

$$\begin{aligned} C_\lambda \le (w^\lambda)^-(x_j, t_j) &\le \|(w^\lambda)^-(\cdot, t_j)\|_{L^\infty(U_\lambda \setminus U^*_{\lambda_0})} \\ &\le 2 \max\left(\|(w^\lambda)^-(\cdot, T)\|_{L^\infty(U_\lambda\setminus U^*_{\lambda_0})} e^{T-t_j}, \frac{C_\lambda}{4} \right) . \end{aligned}$$

Since w^λ is bounded and T is fixed, the right hand side is less than $\frac{C_\lambda}{2}$ for sufficiently large j, a contradiction. $\square$

Proof of Lemma 4.3.7. We proceed by a contradiction, that is we assume $\lambda_0 > 0$, (4.9), and the condition c). For a domain $D \subset \Omega$, we define the inner radius of D to be

$$\operatorname{inrad}(D) := \{\rho > 0 : B(x_0, \rho) \subset D \text{ for some } x_0 \in D\}\,,$$

and if D is an open set, we let inrad(D) stand for the infimum of inner radii of all connected components of D.

Since $U^*_{\lambda_0}$ has finitely many connected components, one has $\operatorname{inrad}(U^*_{\lambda_0}) = 2r_0 = 2r_0(\lambda_0, \Omega) > 0$, and we can fix $\tilde{z} \in \omega(u)$ such that $V_{\lambda_0}\tilde{z} > 0$ holds on the largest number of connected components of $U^*_{\lambda_0}$.

Let $\gamma = \gamma(r_0, \alpha_0, \beta_0, N)$ and h_{r_0} be as in Lemma 3.2.5 corresponding to $r = r_0$ and choose $\delta = \delta(r_0, \alpha_0, \beta_0, N) > 0$ such that the conclusion of Lemma 3.2.3 holds true for $k = \gamma + 1$. Let $\varepsilon_0 > 0$ be such that $|U_{\lambda_0-\varepsilon_0} \setminus U^*_{\lambda_0}| < \delta/2$ and for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$, U_λ contains the same number of connected components of Ω_{λ_0} as $U^*_{\lambda_0}$.

Fix an open set $D \subset\subset U^*_{\lambda_0}$ satisfying $|U^*_{\lambda_0} \setminus D| < \delta/2$ such that $\operatorname{inrad}(D) \geq r_0$ and $D \cap V$ is a domain for any connected component V of $U^*_{\lambda_0}$. Then $|U_{\lambda_0-\varepsilon_0} \setminus D| < \delta$.

For already fixed $\tilde{z}$ denote by $\mathcal{U}^+$, $\mathcal{U}^0$ the set of connected components V of $U^*_{\lambda_0}$ such that $\tilde{z}^{\lambda_0} > 0$, $\tilde{z} \equiv 0$ in V, respectively. Then, by (4.9) and c), $\mathcal{U}^+$, $\mathcal{U}^0$ is a partition of the set of connected components of $U^*_{\lambda_0}$ and $\mathcal{U}^+, \mathcal{U}^0 \neq \emptyset$.

Next, fix an increasing sequence $(t_k)_{k\in\mathbb{N}}$ converging to ∞, with $u(\cdot, t_k) \to \tilde{z}$ in $C(\bar{\Omega})$ as $k \to \infty$. By the definition of $\mathcal{U}^+$, there is $q > 0$ such that for all sufficiently large n

$$w^{\lambda_0}(x, t_n) \geq 2q \qquad (x \in D \cap V, V \in \mathcal{U}^+)\,.$$

Then by the equicontinuity, with possibly decreased ε_0, there is $\vartheta > 0$ independent of n such that for all $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$

$$w^{\lambda}(x, t) \geq q \qquad ((x, t) \in (D \cap V) \times [t_n, t_n + 4\vartheta], V \in \mathcal{U}^+)\,. \tag{4.21}$$

Also, by the definition of $\mathcal{U}^0$

$$\lim_{n\to\infty} \|u(\cdot, t_n)\|_{L^\infty(V)} = 0 \qquad (V \in \mathcal{U}^0)\,. \tag{4.22}$$

Let κ_1, p be constants depending on $r_0, \vartheta, \alpha_0, \beta_0, N, \operatorname{diam}\Omega$ and $\operatorname{dist}(D, \partial U^*_{\lambda_0})$ such that Lemma 3.2.7 holds true for $(D, U, \theta) = (D \cap V, V, \vartheta)$ where V is any connected component of $U^*_{\lambda_0}$. Notice that neither κ_1 nor p depend on ε_0 or n. Let $\tilde{c}_{r_0}$ be as in Corollary 3.2.6 and set

$$\nu := \frac{1}{4} \frac{\kappa_1^2 \tilde{c}_{r_0} \sigma_{r_0}^2}{\kappa_1 \tilde{c}_{r_0} \sigma_{r_0} + e^{4\beta_0\vartheta}} \qquad \text{where} \qquad \sigma_{r_0} := \left(\frac{|B_{\frac{r_0}{2}}|}{|D|}\right)^{\frac{1}{p}} \leq 1\,. \tag{4.23}$$

A continuity argument, with possibly decreased ε_0 (see for example [52, Proof of Lemma 4.3] or the paragraph following (5.33) for an outline of the proof for systems) implies for sufficiently large n

$$\|(w^\lambda)^-(\cdot, t_n)\|_{L^\infty(U_\lambda)} \leq \nu [w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta, t_n+2\vartheta]} \qquad (V \in \mathcal{U}^+, \lambda \in (\lambda_0 - \varepsilon_0, \lambda_0))\,. \tag{4.24}$$

If necessary, decrease ε_0 again to obtain $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U^*_{\lambda_0}$ for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$. Now, fix any $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$.

Since U_λ is a domain, we can choose a domain $\tilde{D}$ with $\tilde{D} \subset\subset U_\lambda$ and $D \subset \tilde{D}$. Let $\tilde{\kappa}_1, \tilde{p}$ be constants depending on $r_0, \vartheta, \alpha_0, \beta_0, N, \operatorname{diam}\Omega$ and $\operatorname{dist}(\tilde{D}, \partial U_\lambda)$ such that Lemma 3.2.7 holds true for $(D, U, \theta) = (\tilde{D}, U_\lambda, \vartheta)$. Finally, choose T such that

$$e^T \geq 2\frac{\kappa_1}{\tilde{\kappa}_1} \text{ and } T > 4\vartheta\,. \tag{4.25}$$

Define

$$T_n := \sup\{\tau : w^\lambda(x,t) > 0, (x,t) \in \bar{V} \times [t_n, t_n + \tau), V \in \mathcal{U}^+\} \qquad (n \in \mathbb{N})\,. \tag{4.26}$$

From (4.21), we have $T_n \geq 4\vartheta$.

Since inrad $(D) \geq r_0$, then for each $V \in \mathcal{U}^+$ there is $x_0^V \in D$ with $B^V_{r_0} := B(x_0^V, r_0) \subset D \cap V$. An application of Corollary 3.2.6 with $(r, v) = (r_0, w^{\lambda_0})$ and (4.21) imply

$$w^{\lambda_0}(x,t) \geq \tilde{c}_{r_0} q e^{-\gamma(t-t_n)} \qquad ((x,t) \in B^V_{\frac{r_0}{2}} \times [t_n, t_n + T_n], V \in \mathcal{U}^+)\,. \tag{4.27}$$

Next, we show that for $\tilde{T}_n := \min\{T_n, T\}$

$$\lim_{n\to\infty} \sup_{t \in [t_n, t_n + \tilde{T}_n]} \|w^\lambda(\cdot, t)\|_{L^\infty(V\cap D)} = 0 \qquad (V \in \mathcal{U}^0)\,. \tag{4.28}$$

Otherwise, by compactness, there is $\hat{V} \in \mathcal{U}^0$, $d_0 > 0$, and a sequence $(y_m, s_m)_{m\in\mathbb{N}}$ with $(y_m, s_m) \in (D \cap \hat{V}) \times [t_{n_m}, t_{n_m} + \tilde{T}_{n_m}]$, $s_m \to \infty$ as $m \to \infty$ such that

$$|w^\lambda(y_0, s_m)| > d_0\,.$$

Passing to a subsequence we may assume $u(\cdot, s_m) \to \hat{z}$ in $C(\bar{\Omega})$ and $y_m \to y_0 \in \bar{D} \cap \hat{V}$ as $m \to \infty$ for some $\hat{z} \in \omega(u)$. Consequently

$$|V_\lambda \hat{z}(y_0)| \geq d_0\,. \tag{4.29}$$

Moreover, for each $V \in \mathcal{U}^+$, (4.27) yields $V_{\lambda_0}\hat{z} \geq \tilde{c}_{r_0} q e^{-\gamma T}$ on $B^V_{\frac{r_0}{2}}$ and therefore by Lemma 4.3.3 i), $V_{\lambda_0}\hat{z} > 0$ on V.

Hence $V_{\lambda_0}\hat{z} > 0$ on V for each $V \in \mathcal{U}^+$. But since $V_{\lambda_0}\tilde{z} > 0$ was true on the largest number of connected components of $U^*_{\lambda_0}$, $V_{\lambda_0}\hat{z} \not> 0$ in any $V \in \mathcal{U}^0$. Then (4.9) implies $\hat{z} \equiv 0$ for each $V \in \mathcal{U}^0$ and by Lemma 4.3.3 iii) also $V_{\lambda_0}\hat{z} \equiv 0$ on V for each $V \in \mathcal{U}^0$. Therefore in particular $\hat{z} \equiv 0$ on $\hat{V} \cup \mathcal{P}_{\lambda_0}\hat{V}$. Since $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U^*_{\lambda_0}$, one has $V_\lambda \hat{z} \equiv 0$ on $\bar{D} \cap \hat{V}$, a contradiction to (4.29).

Thus (4.28) holds, and in particular there exists n_0 such that

$$\|(w^\lambda)^-\|_{L^\infty((D\cap V)\times[t_n,t_n+\tilde{T}_n])} \le \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0}q}{8e^{4\beta_0\vartheta}} e^{-(\gamma+1)T} \qquad (V \in \mathcal{U}^0, n \ge n_0)\,. \quad (4.30)$$

Let us denote

$$\Gamma_0^n := \|(w^\lambda)^-(\cdot,t_n)\|_{L^\infty(U_\lambda)} \qquad (n \in \mathbb{N})\,.$$

An application of Lemma 3.2.3 on the set $(U_\lambda \setminus D) \times (t_n, t_n + \tilde{T}_n)$ and (4.30) yield

$$\begin{aligned} \|(w^\lambda)^-(\cdot,t)\|_{L^\infty(U_\lambda)} &\le 2\max\{e^{-(\gamma+1)(t-t_n)}\Gamma_0^n, e^{-(\gamma+1)T}\frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0}q}{8e^{4\beta_0\vartheta}}\} \\ &\le 2e^{-(\gamma+1)(t-t_n)}\left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0}q}{8e^{4\beta_0\vartheta}}\right) \qquad (t \in [t_n, t_n+\tilde{T}_n], n \ge n_0)\,. \end{aligned} \quad (4.31)$$

Next, since $\tilde{c}_{r_0} \le 1$, Lemma 3.2.7 with $(D, U, \theta) = (D \cap V, U_\lambda, \vartheta)$, $V \in \mathcal{U}^+$ and (4.21), (4.24), (4.31) imply

$$\begin{aligned} w^\lambda(x,t) &\ge \kappa_1[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta,t_n+2\vartheta]} - e^{4\beta_0\vartheta}\sup_{\partial_P(U_\lambda\times(t_n,t_n+4\vartheta))}(w^\lambda)^- \\ &= \frac{\kappa_1}{2}[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta,t_n+2\vartheta]} + \frac{\kappa_1}{2}[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta,t_n+2\vartheta]} \\ &\quad - e^{4\beta_0\vartheta}\sup_{\partial_P(U_\lambda\times(t_n,t_n+4\vartheta))}(w^\lambda)^- \\ &\ge \kappa_1\sigma_{r_0}\frac{1}{2}\left(\frac{\Gamma_0^n}{\nu} + q\right) - 2e^{4\beta_0\vartheta}\left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0}q}{8e^{4\beta_0\vartheta}}\right) \\ &\ge \Gamma_0^n\left(\frac{\kappa_1\sigma_{r_0}}{2\nu} - 2e^{4\beta_0\vartheta}\right) := \Gamma_1^n \\ &\qquad\qquad ((x,t) \in D\cap V \times (t_n+3\vartheta, t_n+4\vartheta), n \ge n_0)\,. \end{aligned}$$

This, (4.21) and Corollary 3.2.6 with $r = r_0$ and q replaced by $\frac{1}{2}(q + \Gamma_1^n)$ imply

$$\begin{aligned} w^\lambda(x,t) &\ge \frac{\tilde{c}_{r_0}}{2}(q+\Gamma_1^n)e^{-\gamma(t-t_n-4\vartheta)} \\ &\qquad ((x,t) \in B^V_{\frac{r_0}{2}} \times [t_n+4\vartheta, t_n+\tilde{T}_n], V \in \mathcal{U}^+, n \ge n_0)\,. \end{aligned} \quad (4.32)$$

To obtain a contradiction, and finish the proof of the lemma, we show that neither $T_n \le T$ nor $T_n > T$ is possible for infinitely many n.

Case 1. There exist an infinite subset S of positive integers such that $\tilde{T}_n = T_n \le T$ for all $n \in S$. Then for any $V \in \mathcal{U}^+$, Lemma 3.2.7 with $(D,U,\theta) = (D\cap V, U_\lambda, \vartheta)$, (4.32), (4.31) and (4.23) yield

$$\begin{aligned}
&w^\lambda(x, t_n + T_n) \\
&\qquad \ge \kappa_1 [w^\lambda]_{p,(D\cap V)\times[t_n+T_n-3\vartheta, t_n+T_n-2\vartheta]} - e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda\times(t_n+T_n-4\vartheta, t_n+T_n))} (w^\lambda)^- \\
&\qquad \ge \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0}}{2} e^{-\gamma T_n}(\Gamma_1^n + q) - 2e^{4\beta_0\vartheta} e^{-(\gamma+1)T_n}\left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \\
&\qquad \ge e^{-\gamma T_n}\Gamma_0^n\left(\frac{\kappa_1^2\tilde{c}_{r_0}\sigma_{r_0}^2}{2\nu} - 2\kappa_1\tilde{c}_{r_0}\sigma_{r_0} e^{4\beta_0\vartheta} - 2e^{4\beta_0\vartheta}\right) + \frac{\kappa_1}{4} e^{-\gamma T_n}\tilde{q}\sigma_{r_0} \\
&\qquad > 0 \qquad (x \in (\bar{D}\cap V), n \ge n_0)\,,
\end{aligned}$$

a contradiction to (4.26).

Case 2. For all sufficiently large n, $\tilde{T}_n = T < T_n$. By a similar argument as in Case 1, Lemma 3.2.7 with $(D,U,\theta) = (\tilde{D}, U_\lambda, \vartheta)$, (4.32), (4.31), (4.25) and (4.23) yield

$$\begin{aligned}
&w^\lambda(x, t_n + T) \\
&\ge \tilde{\kappa}_1 [w^\lambda]_{p,(D\cap V)\times[t_n+T-3\vartheta, t_n+T-2\vartheta]} - e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda\times(t_n+T-4\vartheta, t_n+T))} (w^\lambda)^- \\
&\ge \frac{\tilde{\kappa}_1\tilde{c}_{r_0}\sigma_{r_0}}{2} e^{-\gamma T}(\Gamma_1^n + q) - 2e^{4\beta_0\vartheta} e^{-(\gamma+1)T}\left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \\
&\ge e^{-(\gamma+1)T}\Gamma_0^n\left(\frac{\kappa_1^2\tilde{c}_{r_0}\sigma_{r_0}^2}{2\nu} - 2\kappa_1\tilde{c}_{r_0}\sigma_{r_0} e^{4\beta_0\vartheta} - 2e^{4\beta_0\vartheta}\right) + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{4} e^{-(\gamma+1)T} \\
&\ge \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{4} e^{-(\gamma+1)T} \qquad (x \in \tilde{D}, n \ge n_0)
\end{aligned} \tag{4.33}$$

Passing to a subsequence we may assume $u(\cdot, t_n + T) \to \hat{z}$ in $C(\bar{\Omega})$ as $n \to \infty$ for some $\hat{z} \in \omega(u)$. Then, by the same arguments as before ((4.27) and maximality property of $\tilde{z}$) we obtain $\hat{z}, \hat{z}^{\lambda_0} \equiv 0$ in V for all $V \in \mathcal{U}^0$. But, since $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U^*_{\lambda_0}$, one has $\hat{z}^\lambda \equiv 0$ in $D \cap V \subset \tilde{D}$, $V \in \mathcal{U}^0$, a contradiction to (4.33).

This finishes the proof. $\square$

Chapter 5

Symmetric systems

In this chapter we prove theorems for cooperative systems, that were stated in Chapter 2. To do so, we follow the basic approach and some key ideas of [52], extending the needed technical results to the present setting.

Not all these extensions are straightforward. In particular, new difficulties arise is connection with the fact that different components of the solution may be small at different times. This situation has to be handled carefully using Harnack type estimates which we develop for this purpose. As these estimates are of independent interest, we have devoted a part of the paper to linear cooperative systems, see Subsection 5.1.2. The estimates derived there extend similar results for elliptic cooperative systems as given in [3, 16, 12]. See Remark 5.1.7 for additional bibliographical comments.

Let us briefly recall the following notation, that is used throughout this chapter. For any bounded domain $\Omega \subset \mathbb{R}^N$ and for any $\lambda \in \mathbb{R}$ we denote

$$H_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\},$$
$$\Omega_\lambda = \{x \in \Omega : x_1 > \lambda\}.$$

Also we let

$$\begin{aligned} x^\lambda &= (2\lambda - x_1, x') \quad (x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}), \\ \Omega'_\lambda &= \{x^\lambda : x \in \Omega_\lambda\}, \\ \ell &= \sup\{\lambda \geq 0 : \Omega \cap H_\lambda \neq \emptyset\}, \\ S &= \{1, \dots, n\}. \end{aligned} \tag{5.1}$$

Since we use the method of the moving hyperplanes only in the direction e_1, we apply Convention 4.1.1, that is, we do not indicate the dependence of sets or functions on the direction. Also, we still employ the summation convention, that is, if an index appears twice in the same term we sum over all its possible values, usually from 1 to N. If we do not sum over all possible values, we indicate the summation explicitly.

The remainder of the chapter is organized as follows. Section 5.1 is devoted to linear systems. We prove there Harnack type estimates for positive solutions and a related result for sign-changing solutions. The proofs of our symmetry results are given in Section 5.2.

5.1 Linear equations

Similarly as in the scalar case, the proofs of our symmetry theorems use the method of moving hyperplanes and they depend on estimates of solutions of linear equations and systems. We prepare all these estimates in this section. The reader might find useful to recall notation and definitions from the beginning of Chapter 3.

In this chapter, we consider time dependent elliptic operators L of the form

$$L(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k}. \tag{5.2}$$

Unlike in the Definition 3.0.6 we assume that L does not have zero order terms. To control these terms in weakly coupled cooperative systems, we introduce the following definition.

Definition 5.1.1. Given an open set $Q \subset \mathbb{R}^{N+1}$ and a positive number β_0. We say that a matrix-valued function $(c^{ij})_{i,j\in S}$ belongs to $M^+(\beta_0, Q)$ if its entries c^{ij} are measurable functions defined on Q such that

$$|c^{ij}| \leq \beta_0 \quad (i, j \in S) \quad \text{and} \quad c^{ij} \geq 0 \quad (i, j \in S \; i \neq j).$$

5.1.1 Linearization via reflections

The arguments in this subsection are similar to those in Section 3.1. Since they differ at several points, we rather repeat them here. Especially one should notice different Hadamart's formulas, when irreducibility of the system is assumed.

Suppose that $\Omega \subset \mathbb{R}^N$ is a domain satisfying the symmetry hypotheses (D1), (D3) and let functions F_i satisfy (F1)–(F4). Let u be a nonnegative solution of (2.8), (1.9) satisfying (2.10). Using the notation introduced in (5.1), let $u^\lambda(x,t) = u(x^\lambda, t)$ and $w^\lambda(x,t) := u^\lambda(x,t) - u(x,t)$ for any $x \in \Omega_\lambda$, $t > 0$, and $\lambda \in [0, \ell)$. By (N3 cor), for each $i \in S$, $x \in \Omega_\lambda$, and $t > 0$, one has

$$\partial_t u_i^\lambda \geq F_i(t, x, u(x^\lambda, t), Du_i(x^\lambda, t), D^2 u_i(x^\lambda, t)).$$

Hence

$$\begin{aligned} \partial_t w_i^\lambda(x,t) &\geq F_i(t, x, u(x^\lambda, t), Du_i(x^\lambda, t), D^2 u_i(x^\lambda, t)) \\ &\qquad - F_i(t, x, u(x,t), Du_i(x,t), D^2 u_i(x,t)) \\ &= L_i^\lambda(x,t) w_i^\lambda + \sum_{j=1}^{n} c^{ij}(x,t) w_j^\lambda \quad (x,t) \in \Omega_\lambda \times (0, \infty), \end{aligned} \tag{5.3}$$

where

$$L_i^\lambda(x,t) = \sum_{k,m=1}^{N} a_{km}^i(x,t) \frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k=1}^{N} b_k^i(x,t) \frac{\partial}{\partial x_k}$$

and the λ-dependent coefficients a_{km}^i, b_k^i, c^{ij} are obtained from the Hadamard formula. Specifically (omitting the argument (x,t) of u and u^λ),

$$c^{ij}(x,t) = \begin{cases} \displaystyle\int_0^1 (F_i)_{u_j}(t,x,u_1^\lambda,\dots,u_{j-1}^\lambda,u_j+s(u_j^\lambda-u_j),u_{j+1},\dots, \\ \qquad\qquad u_n, Du_i, D^2u_i)\,ds, \quad \text{if } u_j^\lambda(x,t) \neq u_j(x,t), \\ \sigma, \quad \text{if } u^\lambda(x,t) = u(x,t), \end{cases}$$

$$b_k^i(x,t) = \begin{cases} \displaystyle\int_0^1 (F_i)_{p_k}(t,x,u^\lambda,\dots,u_{x_{k-1}}^\lambda,(u_i)_{x_k}+s((u_i)_{x_k}^\lambda-(u_i)_{x_k}), \\ \qquad\qquad (u_i)_{x_{k+1}},\dots,D^2u_i)\,ds, \quad \text{if } (u_i)_{x_k}^\lambda(x,t) \neq (u_i)_{x_k}(x,t), \\ 0, \quad \text{if } (u_i)_{x_k}^\lambda(x,t) = (u_i)_{x_k}(x,t), \end{cases}$$

$$a_{km}^i(x,t) = \int_0^1 (F_i)_{q_{km}}(t,x,u^\lambda,Du_i^\lambda,D^2u_i+s(D^2u_i^\lambda-D^2u_i))\,ds\,,$$

where σ is a nonnegative constant. If (F5) is not assumed, the choice of the constant σ is not relevant and we can set $\sigma = 0$; if (F5) is assumed we chose σ as in (F5).

By (F1) the coefficients are well defined on $\Omega_\lambda \times (0,\infty)$ and they are measurable functions with absolute values bounded by β_0. This and (F2) imply that $L_i^\lambda \in E(\alpha_0,\beta_0,\Omega_\lambda\times(0,\infty))$. Further, by (F4), we have $(c^{ij})_{i,j\in S} \in M^+(\beta_0,\Omega_\lambda\times(0,\infty))$ and if (F5) is satisfied, then for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$, there exist $i \in I$, $j \in J$ such that

$$c^{ij}(x,t) \geq \sigma \quad ((x,t) \in \Omega_\lambda \times (0,\infty)). \tag{5.4}$$

The Dirichlet boundary condition and nonnegativity of u_i yield

$$w_i^\lambda \geq 0 \qquad \text{on} \quad \partial\Omega_\lambda \times (0,\infty),\ i \in S. \tag{5.5}$$

5.1.2 Estimates of solutions

We now derive several estimates for solutions of a system of inequalities such as (5.3). The results here use a scalar counterparts stated in Chapter 3. As above, we formulate the results in more general setting then needed in this work.

Throughout this subsection, Q is a bounded open set in $\mathbb{R}^{N+1}$ and α_0, β_0, σ are fixed positive numbers.

We consider the following system of parabolic inequalities

$$\partial_t w_i - L_i(x,t)w_i \geq \sum_{i,j=1}^{n} c^{ij}(x,t)w_j + f_i(x,t), \quad (x,t) \in Q,\ i \in S, \tag{5.6}$$

where $L_i \in E(\alpha_0, \beta_0, Q)$ $(i \in S)$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, Q)$, and $f_i \in L^{N+1}(Q)$ $(i \in S)$. Sometimes we also assume the following condition.

(IR) There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$, there exist $i \in I$, $j \in J$ such that

$$c^{ij}(x,t) \geq \sigma \quad ((x,t) \in Q). \tag{5.7}$$

We say that w is a solution of (5.6) (or that it satisfies (5.6)) if it is an element of the space $(W^{2,1}_{N+1,loc}(Q))^n$ and (5.6) is satisfied almost everywhere. If (5.6) is complemented by a system of inequalities on $\partial_P Q$, we also require the solution to be continuous on $\bar{Q}$ and to satisfy the boundary inequalities everywhere.

The theorem following is a version of the Alexandrov-Krylov estimate for parabolic cooperative systems on general bounded sets in the space-time. The proof closely follows the ideas from [55], where Sirakov proved Alexandrov-Krylov estimate for elliptic cooperative systems. For related results in elliptic case see also [12].

Theorem 5.1.2. *Let $Q \in \mathbb{R}^{N+1}$ be an open bounded set and let $L_i \in E(\alpha_0, \beta_0, Q)$ $(i \in S)$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, Q)$. Assume*

$$\sum_{j=1}^{n} c^{ij}(x,t) \leq 0 \qquad ((x,t) \in Q, i \in S)\,. \tag{5.8}$$

If $w in (C(\bar{Q}))^n$ is a solution of (5.6)*, then*

$$\max_{i\in S} \sup_{Q} w_i^- \leq \max_{i\in S} \sup_{\partial_P Q} w_i^- + \max_{i\in S} \|f_i^-\|_{L^{N+1}(Q)} \tag{5.9}$$

If the right hand side of (5.9) *is 0 (that is, the functions w_i are nonnegative on the parabolic boundary), then the conclusion holds regardless of condition* (5.8).

Proof. First, we prove the statement under addition assumption, that Q is a smooth open set in $\mathbb{R}^{N+1}$ and the coefficients a^i_{km} of L_i are continuous functions on Q.

Fix $i \in S$. If $c^{ii} \equiv 0$ in Q, then by (5.8) and $(c^{ij})_{i,j\in S} \in M^+(\beta_0, Q)$ one has $c^{ij} \equiv 0$ in Q for each $j \in S$. For such i the statement follows from Theorem 3.2.1. Therefore assume $c^{ii} \not\equiv 0$ in Q.

Under our additional regularity assumptions on L_i and Q, there exists the solution of the problem (cf. [40, Theorem 7.17] or [38])

$$\begin{aligned} (v_i)_t - L_i(x,t)v_i - c^{ii}(x,t)v_i &= -c^{ii}(x,t), && (x,t) \in Q\,, \\ v_i &= 0, && (x,t) \in \partial_P Q\,. \end{aligned}$$

Since $c^{ii} \le -\sum_{j\neq i} c^{ij} \le 0$, one has $v_i \ge 0$ by the maximum principle. Moreover $c^{ii} \not\equiv 0$ implies $v_i \not\equiv 0$ in Q.

Lemma 5.1.3. *There exists* $\delta = \delta(N, \alpha_0, \beta_0, |\Omega|) > 0$ *such that*

$$0 \le v_i(x,t) \le 1 - \delta \qquad ((x,t) \in Q)\,.$$

We postpone the proof of the lemma and we first finish the proof of the theorem. Let z_i be the solution of

$$\begin{aligned} (z_i)_t - L_i(x,t)z_i &= -f_i^-, && (x,t) \in Q\,, \\ z_i &= 0, && (x,t) \in \partial_P Q\,. \end{aligned}$$

The solution of this problem again exists by our additional regularity assumptions. Then by Theorem 3.2.1

$$\sup_Q z_i^- \le C\|f_i^-\|_{L^{N+1}(Q)} \tag{5.10}$$

where C depends on N, α_0, β_0 and $|Q|$. Furthermore, the maximum principle yields $z_i \le 0$.

If we replace w_i by $w_i + \max_{j\in S}\sup_{\partial_P Q}(w_j)^-$ we can assume $w_i \ge 0$ on $\partial_P Q$, and by (5.8), w_i still satisfies (5.6). If we denote $K = \max_{j\in S}\sup_Q (w_j)^-$, then

$(c^{km})_{k,m\in S} \in M^+(\beta_0, Q)$ and (5.8) imply

$$\begin{aligned}(w_i)_t - L_i(x,t)w_i - c^{ii}(x,t)w_i &\geq f_i + \sum_{j\neq i} c^{ij}(x,t)w_j \\ &\geq -f_i^- - K\sum_{j\neq i} c^{ij}(x,t) \\ &\geq -f_i^- + Kc^{ii}(x,t) \qquad ((x,t)\in Q)\,.\end{aligned}$$

Let $h_i := z_i - Kv_i$. Then $h_i = 0 \leq w_i$ on $\partial_P Q$. Moreover, since $c^{ii} \leq 0$ and $z_i \leq 0$ one has

$$\begin{aligned}(h_i)_t - L_i(x,t)h_i - c^{ii}(x,t)h_i &= -f_i^- - c^{ii}(x,t)z_i + Kc^{ii} \\ &\leq -f_i^- + Kc^{ii} \leq (w_i)_t - L_i(x,t)w_i - c^{ii}(x,t)w_i \qquad ((x,t)\in Q)\,.\end{aligned}$$

Hence, by the maximum principle $w_i \geq h_i$ in Q. By Lemma 5.1.3 and (5.10) we obtain

$$\sup_Q w_i^- \leq \sup_Q (z_i^- + Kv_i) \leq C\|f_i^-\|_{L^{N+1}(Q)} + K(1-\delta)\,.$$

Taking the maximum over i in the previous inequality and an easy manipulation, yield the statement of the theorem, if L_i and Q are sufficiently regular.

Let us now remove the additional regularity assumptions imposed on Q and L_i. That is, assume that Q is an open subset of $\mathbb{R}^{N+1}$ and $L_i \in E(\alpha_0, \beta_0, Q)$. Using an approximation argument we choose a sequence of smooth domains Q^j such that $Q^j \subset Q$ and ∂Q^j approaches ∂Q in the Hausdorff metric as $j \to \infty$. Also we choose a sequence of continuous function $a_{km}^{i;j} : Q \to \mathbb{R}$ bounded by $2\beta_0$ with

$$a_{km}^{i;j}(x,t)\xi_k\xi_l \geq \frac{\alpha_0}{2}|\xi|^2 \qquad ((x,t)\in Q, \xi\in\mathbb{R}^N, i\in S, j\in\mathbb{N})\,,$$

such that $a_{km}^{i;j} \to a_{km}^i$ almost everywhere in Q as $j \to \infty$.

If we denote

$$L_i^j(x,t) = a_{kl}^{i;j}(x,t)\frac{\partial^2}{\partial_{x_k}\partial_{x_l}} + b_k^i(x,t)\frac{\partial}{\partial_{x_k}} \qquad ((x,t)\in Q)\,,$$

then w_i satisfies

$$(w_i)_t - L_i^j(x,t)w_i \geq \sum_{i,j=1}^{n} c^{ij}(x,t)w_j + f_i + (a_{kl}^i - a_{kl}^{i;j})(w_i)_{x_k x_l}, \quad (x,t) \in Q_j\,.$$

Since Q_j and L_i^j fulfill regularity assumptions required above, we obtain

$$\max_{i\in S}\sup_{Q^j}(w_i)^- \leq \max_{i\in S}\sup_{\partial_P Q^j}(w_i)^- + \|f_i^-\|_{L^{N+1}(Q)} + \|(a_{kl}^i - a_{kl}^{i;j})(w_i)_{x_k x_l}\|_{L^{N+1}(Q)}\,.$$

Since $w_i \in W_{N+1}^{1,2}(Q)$ and $a_{km}^{i;j} \to a_{k,m}^i$ almost everywhere in Q, Lebesgue dominated converge theorem implies

$$\lim_{j\to\infty}\|(a_{kl}^i - a_{kl}^{i;j})(w_i)_{x_k x_l}\|_{L^{N+1}(Q)} = 0\,.$$

Function w_i is continuous on a compact set $\bar{Q}$ and therefore it is uniformly continuous. Then

$$\lim_{j\to\infty}\max_{i\in S}\sup_{\partial_P Q^j}(w_i)^- = \max_{i\in S}\sup_{\partial_P Q}(w_i)^-$$

and

$$\lim_{j\to\infty}\max_{i\in S}\sup_{Q^j}(w_i)^- = \max_{i\in S}\sup_{Q}(w_i)^-$$

and the statement of the theorem follows.

We finish the proof of the theorem by proving Lemma 5.1.3.

Proof of Lemma 5.1.3. First observe that the function $\tilde{v}_i = 1 - v_i$ satisfies

$$\begin{aligned}(\tilde{v}_i)_t - L_i(x,t)\tilde{v}_i - c^{ii}(x,t)\tilde{v}_i &= 0, && (x,t) \in Q,\\ \tilde{v}_i &= 1, && (x,t) \in \partial_P Q\,.\end{aligned}$$

By the strong maximum principle (cf. [40]), $\tilde{v}_i > 0$ in $\bar{Q}$. Set $\hat{v}_i := g(v_i) = \tilde{v}_i^{-\zeta}$, where $\zeta > 0$ is specified below. Since g is smooth convex function

$$L_i(x,t)\hat{v}_i = g'(\tilde{v}_i)L_i(x,t)\tilde{v}_i + g''(\tilde{v}_i)a_{kl}^i(v_i)_{x_k}(v_i)_{x_l} \geq g'(\tilde{v}_i)L_i(x,t)\tilde{v}_i \quad ((x,t) \in Q)\,.$$

Thus

$$\begin{aligned}(\hat{v}_i)_t - L_i(x,t)\hat{v}_i &\leq -\zeta c^{ii}(x,t)\hat{v}_i, && (x,t) \in Q,\\ \hat{v}_i &= 1, && (x,t) \in \partial_P Q\,,\end{aligned}$$

and Theorem 3.2.1 implies

$$\|\hat{v}_i\|_{L^\infty(Q)} \leq 1 + C\zeta\beta_0\|\hat{v}_i\|_{L^\infty(Q)} \, ,$$

where C depends on N, α_0, β_0 and $|Q|$. Choosing $\zeta = 1/(2\beta_0 C)$ we obtain $\|\hat{v}_i\|_{L^\infty(Q)} \leq 2$ and therefore $v_1 = 1 - (\hat{v}_i)^{-1/\zeta} \leq 1 - 2^{-1/\zeta}$ and we are done. □

This also finishes the proof of the theorem. □

We now prove a maximum principle for small domains. It is a generalization of Lemma 3.2.3 to cooperative systems. However, for an illustration we use a different proof here. For simplicity we assume $f_i \equiv 0$ for each $i \in S$, but the proof can be carried over with nonzero f as well. One can also use the method of the proof of Lemma 3.2.3 and obtain a slightly stronger result, where the constant δ does not depend on diam Ω.

Lemma 5.1.4. *Given any $q > 0$, and $\tau, T > 0$ with $\tau < T$, there is a constant δ determined only by n, N, α_0, β_0,* diam(Ω), *and q such that for any open set $U \subset \Omega \times (\tau, T)$ with $|U_{[t,t+1]}| < \delta$ for any $t \in [\tau, T-1]$ the following holds. If $w \in C(\bar{U})$ is a solution a problem* (5.6), *with $L_i \in E(\alpha_0, \beta_0, U)$ $(i \in S)$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, U)$, and if*

$$w_i(x,t) \geq -\hat{\varepsilon} \quad ((x,t) \in \partial_P U \setminus U_\tau, \ i \in S), \tag{5.11}$$

where $\hat{\varepsilon} \geq 0$ is a constant, then

$$\max_{i\in S} \|w_i^-\|_{L^\infty(U)_t} \leq 2\max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_\tau)} e^{-q(t-\tau)}, \hat{\varepsilon}\} \quad (t \in (\tau, T)).$$

In the proof we employ the following lemma, that was stated in time independent case, in [8] and it is attributed to Varadhan. In the proof, the authors used the theory of solutions to elliptic Monge-Ampère equation based on [14] and [17]. Here, we use the theory of parabolic Monge-Ampère equation, especially results from [36] (for related results see [58]).

Lemma 5.1.5. *Given $a_0 > 0$, $b_0 \geq 1$, and $\tau \geq 0$, there exists $\delta > 0$ determined only by a_0, b_0, N, diam Ω such that for every closed set Q in $\Omega \times [\tau, \tau+1]$ with $|Q| < \delta$, there exists $h \in W^{1,2}_\infty(\Omega \times [\tau, \tau+1])$ with $1 \leq h \leq 2$ such that for every symmetric positive definite matrix $\{a_{ij}\}$ with*

$$\det(a_{ij}(x,t)) \geq a_0^{N+1} \qquad ((x,t) \in \Omega \times [\tau, \tau+1]),$$

one has

$$-h_t + a_{ij}(x,t)h_{x_i x_j} + b_0(|\nabla h| + h) < 0 \qquad ((x,t) \in Q). \tag{5.12}$$

Proof of Lemma 5.1.5. Let B_2 and B_1 be concentric balls of radius $2\,(\text{diam}\ \Omega)$ and diam Ω respectively, such that $\Omega \subset\subset B_1$.

Let $f \in C^\infty(\bar{B}_2 \times [\tau, \tau+1])$ be a nonnegative function compactly supported in $(B_2 \times [\tau, \tau+1])$ satisfying

$$f(x,t) \geq 1 \qquad ((x,t) \in Q), \quad \text{and} \quad \int_{B_2 \times (\tau, \tau+1)} f\, dxdt < 2\delta\,.$$

By [36, Example 8.2.8], there exists a function $u \in W^{1,2}_\infty(B_2 \times (\tau, \tau+1)) \cap C(\bar{B}_2 \times [\tau, \tau+1])$, concave in x and increasing in t, that satisfies parabolic Monge-Ampère equation:

$$\begin{aligned} u_t \det(-u_{x_i x_j}) &= f, \qquad && (x,t) \in B_2 \times (\tau, \tau+1), \\ u &= 0, && (x,t) \in \partial_P(B_2 \times [\tau, \tau+1])\,. \end{aligned}$$

Using Alexandrov-Krylov inequality (cf. [57, Proposition 2.1]) we obtain

$$\|u\|_{L^\infty(B_2 \times [\tau, \tau+1]} \leq C \left(\int_{B_2 \times (\tau, \tau+1)} u_t \det(-u_{x_i x_j})\, dx\, dt \right)^{\frac{1}{N+1}} < C(2\delta)^{\frac{1}{N+1}}\,.$$

where C depends on N and diam Ω. Then, since u vanishes on $\partial_P(B_2 \times [\tau, \tau+1])$ and it is nondecreasing, concave and positive function, one has

$$u, |\nabla u| \leq C(2\delta)^{\frac{1}{N+1}} \quad \text{and} \quad u_t \geq 0 \qquad ((x,t) \in B_1 \times [\tau, \tau+1])\,. \tag{5.13}$$

Define symmetric matrices $(A_{ij})_{i,j=1}^{N+1}$ and $(U_{ij})_{i,j=1}^{N+1}$ such that

$$A_{ij} = \begin{cases} a_{ij} & i,j \le N \\ 1 & i=j=N+1 \\ 0 & \text{otherwise} \end{cases} \qquad U_{ij} = \begin{cases} -u_{x_i x_j} & i,j \le N \\ u_t & i=j=N+1 \\ 0 & \text{otherwise} \end{cases} .$$

Since $(a_{ij})_{i,j=1}^{N}$ is a positive definite matrix, $(A_{ij})_{i,j=1}^{N+1}$ is positive definite as well. Since u is concave and increasing, the matrix $(U_{ij})_{i,j=1}^{N+1}$ is also positive definite.

Consequently, using the arithmetic-geometric mean theorem (see [36, Lemma 3.2.1] for details), we obtain

$$\begin{aligned} u_t - a_{ij}u_{x_i x_j} &\ge (N+1)[u_t \det(-u_{x_i x_j}) \det(a_{ij})]^{\frac{1}{N+1}} \\ &\ge a_0(N+1)[u_t \det(-u_{x_i x_j})]^{\frac{1}{N+1}} \ge a_0(N+1) \qquad ((x,t) \in Q) . \end{aligned} \tag{5.14}$$

Set $h = \frac{2b_0}{a_0(N+1)}u + 1$. Then

$$1 \le h(x,t) \le 1 + \frac{2b_0 C}{a_0(N+1)}(2\delta)^{\frac{1}{N+1}} \le 2 \quad (x,t) \in Q \quad \text{if} \quad \frac{2b_0 C}{a_0(N+1)}(2\delta)^{\frac{1}{N+1}} \le 1 .$$

If $\frac{4b_0^2 C(2\delta)^{\frac{1}{N+1}}}{a_0(N+1)} < 1$, we also have by (5.13) and (5.14)

$$\begin{aligned} -h_t + a_{ij}h_{x_i x_j} + b_0(|\nabla h| + h) &\le -2b_0 + \frac{4b_0^2 C(2\delta)^{\frac{1}{N+1}}}{a_0(N+1)} + b_0 \\ &\le -b_0 + \frac{4b_0^2 C(2\delta)^{\frac{1}{N+1}}}{a_0(N+1)} < 0 . \end{aligned}$$

Hence, the theorem is proved if, for instance, we choose δ so that

$$\frac{4b_0 C(2\delta)^{\frac{1}{N+1}}}{a_0(N+1)} = \frac{1}{2} . \qquad \square$$

Proof of Lemma 5.1.4. We claim that the assertion holds if δ is as in Lemma 5.1.5 with $b_0 := (n+\sqrt{N})\beta_0 + q + \ln 2$, $a_0 := \alpha_0$. To prove this, let $U \subset \Omega \times (\tau, T)$ and w satisfy the hypotheses of Lemma 5.1.4. Without loss of generality we may assume that $|\bar{U}_{[t,t+1]}| < \delta$ for each $t \in [\tau, T-1]$; otherwise we first prove the results for each open set $U_1 \subset\subset U$ and then use an approximation argument (or alternatively we can proceed similarly as in the proof of Lemma 3.1 in [52]). Fix $t_0 \in [\tau, T-1]$ and let g be as in the conclusion of Lemma 5.1.5 with $Q = \bar{U}_{[t_0,t_0+1]}$.

Denote by a^i_{km}, b^i_k the coefficients of L_i:

$$L_i(x,t) = \sum_{k,m=1}^{N} a^i_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k=1}^{N} b^i_k(x,t)\frac{\partial}{\partial x_k}.$$

For each $i \in S$ set $z_i := w_i/g$. Then $z_i(\cdot,t) \geq -\hat{\varepsilon}$ on $(\partial_P U_{[t_0,t_0+1]}) \setminus U_{t_0}$ and a simple computation shows

$$(z_i)_t - \tilde{L}_i(x,t)z_i \geq \sum_{\substack{j=1\\ j\neq i}}^{N} c^{ij}(x,t)z_j + \tilde{c}^{ii}(x,t)z_i \qquad (x,t) \in U_{(t_0,t_0+1)},\ i \in S, \quad (5.15)$$

where

$$\tilde{L}_i(x,t) = L_i(x,t) - \frac{2}{g(x,t)}\sum_{k,m=1}^{N} a^i_{km}(x,t)g_{x_k}(x,t)\frac{\partial}{\partial x_m}$$

and

$$\begin{aligned}\tilde{c}^{ii} &= \frac{1}{g}(-g_t + L_i g) + c^{ii} \leq \frac{1}{g}\left(-g_t + \sum_{k,m=1}^{N} a^i_{km} g_{x_k x_m} + \sqrt{N}\beta_0(|Dg| + g)\right) \\ &= \frac{1}{g}\left(-g_t + \sum_{k,m=1}^{N} a^i_{km} g_{x_k x_m} + (b_0 - q - n\beta_0 - \ln 2)(|Dg| + g)\right).\end{aligned}$$

Consequently, by (5.12)

$$\tilde{c}^{ii}(x,t) < -q - n\beta_0 - \ln 2 \qquad ((x,t) \in U_{(t_0,t_0+1)}, i \in S).$$

If we define $\hat{z}_i(x,t) := e^{(q+\ln 2)t}z_i(x,t)$, then $\tilde{z}_i$ satisfies (5.15) with $\tilde{c}^{ii}$ replaced by $\hat{c}^{ii} = \tilde{c}^{ii} + q + \ln 2 < -n\beta_0$. Hence

$$\hat{c}^{ii}(x,t) + \sum_{\substack{j=1\\ j\neq i}}^{n} c^{ij}(x,t) < 0 \qquad ((x,t) \in U_{(t_0,t_0+1)}, i \in S).$$

Also $\hat{z}_i(x,t) \geq -\hat{\varepsilon} e^{(q+\ln 2)t}$ for any $i \in S$, $(x,t) \in \partial_P U_{[t_0,t_0+1]} \setminus U_{t_0}$. Then Theorem 5.1.2 with $f_i \equiv 0$, implies

$$\max_{i\in S} \|\hat{z}_i^-\|_{L^\infty(U_t)} \leq \max\{\max_{i\in S} \|\hat{z}_i^-\|_{L^\infty(U_{t_0})}, \hat{\varepsilon} e^{(q+\ln 2)t}\} \qquad (t \in [t_0, t_0+1]),$$

or equivalently,

$$\max_{i\in S} \|z_i^-\|_{L^\infty(U_t)} \leq \max\{\max_{i\in S} \|z_i^-\|_{L^\infty(U_{t_0})} e^{(q+\ln 2)(t_0-t)}, \hat{\varepsilon}\} \qquad (t \in [t_0, t_0+1]).$$

Since $1 \leq g \leq 2$ and $z_i = w_i/g$, we have

$$\max_{i\in S} \|w_i^-\|_{L^\infty(U_t)} \leq 2\max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_{t_0})} e^{(q+\ln 2)(t_0-t)}, \hat{\varepsilon}\} \quad (t \in [t_0, t_0+1]). \tag{5.16}$$

The rest of the proof is similar to the last part of the proof of Lemma 3.2.3. Indeed, (5.16) with $t = t_0 + 1$ implies

$$\max_{i\in S} \|w_i^-\|_{L^\infty(U_{t_0+1})} \leq \max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_{t_0})} e^{-q}, 2\hat{\varepsilon}\}\,.$$

Iterating this expression for any $j \in \mathbb{N}$ with $t_0 + j \leq T$ we obtain

$$\max_{i\in S} \|w_i^-\|_{L^\infty(U_{t_0+j})} \leq \max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_{t_0})} e^{-qj}, 2\hat{\varepsilon}\}\,. \tag{5.17}$$

Since any $t \in [\tau, T]$ can be expressed in the form $t = \tau + j + s$, where $j \in \mathbb{N} \cup \{0\}$ and $s \in [0,1)$, (5.16) and (5.17) imply

$$\begin{aligned}\max_{i\in S} \|w_i^-\|_{L^\infty(U_t)} &\leq \max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_{\tau+s})} e^{-qj}, 2\hat{\varepsilon}\} \\ &\leq 2\max\{\max_{i\in S} \|w_i^-\|_{L^\infty(U_{\tau+s})} e^{-q(t-\tau)}, \hat{\varepsilon}\}\end{aligned}$$

□

Next, we formulate Harnack inequality for cooperative systems. The reader might find useful to recall the definition (3.15).

Lemma 5.1.6. *Given $d > 0$, $\varepsilon > 0$, $\theta > 0$, there are positive constants κ, κ_1, and p, determined only by n, N, $\operatorname{diam}\Omega$, α_0, β_0, d, ε and θ, such that the following statement holds. Assume that $\tau \in \mathbb{R}$; D and U are domains in Ω with $D \subset\subset U$, $dist(\bar{D}, \partial U) \geq d$, and $|D| > \varepsilon$; $L_i \in E(\alpha_0, \beta_0, U \times (\tau, \tau + 4\theta))$ for* *all $i \in S$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, U \times (\tau, \tau+4\theta))$ and $f_i \equiv 0$ for all $i \in S$. If $v = (v_1, \dots, v_n) \in (C(\bar U \times [\tau, \tau+4\theta]))^n$ is a solution of* (5.6), *then for all $i \in S$*

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \geq \kappa[v_i^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \kappa_1 \max_{j\in S} \sup_{\partial_P(U\times(\tau,\tau+4\theta))} v_j^- .$$

Remark 5.1.7. One can also state the previous lemma for nonzero $f_i \in L^{N+1}(Q)$. As a conclusion one obtains

$$\begin{aligned}\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \geq \kappa[v_i^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \kappa_1 \max_{j\in S} \sup_{\partial_P(U\times(\tau,\tau+4\theta))} v_j^- \\ - \kappa_2 \max_{j\in S} \|f_j^-\|_{L^{N+1}(Q)}\end{aligned} \tag{5.18}$$

where κ, κ_1 and κ_2 depend on n, N, $\operatorname{diam}\Omega$, α_0, β_0, d, ε and θ. The proof is analogous, with straightforward modifications as the one provided below for $f_i \equiv 0$. Since we do not need this extension for our main results, we omit it here.

If all inequalities in (5.6) are replaced by equalities, unlike in the scalar case (Lemma 3.2.7), it is not known if Lemma 5.1.6 holds true with $p = \infty$.

Lemma 5.1.6 applied to a nonnegative solution v of (5.6), gives a componentwise Harnack inequality for v:

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \geq \kappa[v_i]_{p,D\times(\tau+\theta,\tau+2\theta)}.$$

It is clear from the proof below that in this case the continuity of v up to the boundary of $U\times(\tau, T)$ is not needed. Under the irreducibility assumption (IR), the full Harnack inequality for nonnegative solutions is given in Theorem 5.1.9 below. For elliptic cooperative systems similar results have been proved in [3, 16, 12, 55]. The most general ones are those in [12], where viscosity solutions are considered. We only deal with strong solutions and the assumptions in our Harnack type results are stronger that the parabolic analogs of the assumptions in [12] (this is more than satisfactory for our applications to classical solutions of nonlinear equations). For parabolic systems in the divergence form a Harnack type result is given in [43].

Proof of Lemma 5.1.6. For $i \in S$ let

$$g_i(x,t) := -\sum_{\substack{j=1 \\ j \neq i}}^{n} c^{ij}(x,t) v_j^-.$$

Obviously, $g_i \in L^\infty(U \times (\tau, \tau + 4\theta))$. Since $c^{ij} \geq 0$ for $i \neq j$, v_i satisfies

$$(v_i)_t - L_i(x,t) v_i - c^{ii}(x,t) v_i \geq g_i(x,t), \quad (x,t) \in U \times (\tau, \tau + 4\theta).$$

Therefore by Lemma 3.2.7

$$\begin{aligned} \inf_{D \times (\tau+3\theta, \tau+4\theta)} v_i(x,t) \geq \kappa [v_i^+]_{p, D \times (\tau+\theta, \tau+2\theta)} \\ - \kappa_2 \|g_i\|_{L^{N+1}(U \times (\tau, \tau+4\theta))} - \sup_{\partial_P (U \times (\tau, \tau+4\theta))} e^{4M\theta} v_i^-, \end{aligned} \tag{5.19}$$

where κ, κ_2, and p are as in Lemma 3.2.7 and $M = \max_{i,j \in S} \sup_{U \times (\tau, \tau+4\theta)} c^{ij}(x,t)$. Now

$$\begin{aligned} \|g_i\|_{L^{N+1}(U \times (\tau, \tau+4\theta))} &\leq \tilde{\kappa}_2 \|g_i\|_{L^\infty(U \times (\tau, \tau+4\theta))} \\ &\leq \tilde{\kappa}_2 \beta_0 n \max_{j \in S} \|v_j^-\|_{L^\infty(U \times (\tau, \tau+4\theta))}, \end{aligned} \tag{5.20}$$

where $\tilde{\kappa}_2$ depends only on $\operatorname{diam}(\Omega)$ and N. Next, the function $\tilde{v} := e^{nMt} v$ satisfies the inequalities

$$(\tilde{v}_i)_t - L_i(x,t) \tilde{v}_i \geq \sum_{j=1}^{n} c^{ij}(x,t) \tilde{v}_j - nM \tilde{v}_i, \quad (x,t) \in U \times (\tau, \tau + 4\theta),\ i \in S,$$

where, by the definition of M,

$$\sum_{j=1}^{n} c^{ij}(x,t) - nM \leq 0 \quad ((x,t) \in U \times (\tau, \tau + 4\theta),\ i \in S).$$

Hence, Theorem 5.1.2 with $f_i \equiv 0$ implies

$$\max_{i \in S} \sup_{U \times (\tau, \tau+4\theta)} \tilde{v}_i^- \leq \max_{i \in S} \sup_{\partial_P (U \times (\tau, \tau+4\theta))} \tilde{v}_i^-.$$

Consequently, by (5.20),

$$\begin{aligned}\|g_i\|_{L^{N+1}(U\times(\tau,\tau+4\theta))} &\leq \tilde{\kappa}_2 n\beta_0 e^{-nM\tau}\max_{i\in S}\sup_{\partial_P(U\times(\tau,\tau+4\theta))}\tilde{v}_i^-\\ &\leq \tilde{\kappa}_2 n\beta_0 e^{4nM\theta}\max_{i\in S}\sup_{\partial_P(U\times(\tau,\tau+4\theta))} v_i^-.\end{aligned}$$

Substituting this into (5.19), we obtain the desired estimate. □

The following lemma states the local maximum principle for parabolic system. One can find analogous results for scalar elliptic problem in [32], for elliptic systems in [11, 55] and for scalar parabolic problems in [40].

Lemma 5.1.8. *For some* $(x_0, t_0) \in \mathbb{R}^{N+1}$ *and* $\delta > 0$ *assume that for all* $i \in S$ $L_i \in E(\alpha_0, \beta_0, B(x_0, \delta) \times (t_0 - \delta^2, t_0))$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, B(x_0, \delta) \times (t_0 - \delta^2, t_0))$, *and* v *is a solution of the system* (5.6) *on* $U \times (\tau, T) = B(x_0, \delta) \times (t_0 - \delta^2, t_0))$ *with the inequality sign reversed (replaced by "*$\leq$*"). Then for each* $p > 0$ *and* $\rho \in (0, 1)$ *one has*

$$\max_{j\in S}\sup_{B(x_0,\rho\delta)\times(t_0-(\rho\delta)^2,t_0)} v_j \leq C_0 \max_{j\in S}[v_j]_{p,B(x_0,\delta)\times(t_0-\delta^2,t_0)},$$

where C_0 *is a constant determined only by* δ, ρ, p, N, α_0, β_0.

The proof of this lemma can be carried out in a similar way as in the scalar case, see [40, Theorem 7.21]. Since (5.6) is coupled only in the zero-order terms, the adaptation of the proof in [40] is straightforward and is omitted. We remark that the cooperativity condition $c_{ij} \geq 0$ is not needed and can be omitted in the assumptions of this lemma.

The last result of this section is a stronger version of the Harnack inequality for irreducible systems. Observe that in the case of fully coupled systems, we are able, unlike in Theorem 5.1.6, to prove point-wise Harnack inequality, that is, under an irreducibility assumption, we can achieve $p = \infty$.

Theorem 5.1.9. *Given* $d > 0$, $\varepsilon > 0$, $\theta > 0$, *there are positive constants* $\bar{\kappa}$ *and* p, *determined only by* n, N, $\operatorname{diam}\Omega$, α_0, β_0, σ, d, ε *and* θ, *such that the following statement holds. Assume that*

(A) *D and U are domains in Ω with $D \subset\subset U$, dist$(\bar{D}, \partial U) \geq d$, and $|D| > \varepsilon$; $L_i \in E(\alpha_0, \beta_0, U \times (\tau, \tau+4\theta))$ for all $i \in S$; $(c^{ij})_{i,j \in S} \in M^+(\beta_0, U \times (\tau, \tau+4\theta))$ is such that (IR) holds; and $v = (v_1, \ldots, v_n) \in (C(\bar{U} \times [\tau, \tau+4\theta]))^n$ is a nonnegative solution of* (5.6).

Then for all $i \in S$

$$\inf_{D \times (\tau + \frac{7}{2}\theta, \tau+4\theta)} v_i(x,t) \geq \bar{\kappa} \max_{j \in S} [v_j]_{p, D \times (\tau+\theta, \tau+2\theta)}. \tag{5.21}$$

If all inequalities in (5.6) *are replaced by equations, then the conclusion holds with $p = \infty$ and with $\bar{\kappa}$ independent of ε.*

Proof. Given $d > 0$, $\varepsilon > 0$, $\theta > 0$, we first fix p and κ such that the statement of Lemma 3.2.7 is valid with d replaced by $d/(2n)$. These constants p and κ depend only on the indicated quantities.

Assume (A) is satisfied. Relabeling the components of v, we may without loss of generality assume that

$$[v_1]_{p, D \times (\tau+\theta, \tau+2\theta)} = K_0 := \max_{j \in S} [v_j]_{p, D \times (\tau+\theta, \tau+2\theta)}.$$

We may also assume that $c^{ii} \geq 0$ for all $i \in S$. Indeed, these relations are achieved by the substitution $v \to e^{-\beta_0 t} v$, which clearly does not affect the validity of the statement.

For each $k \in S$ denote $\tau_k = \tau + \frac{7}{2}\theta - \frac{\theta}{2^k}$ and fix a sequence of domains $\{U_k\}_{k=1}^n$ such that $D \subset\subset U_{k+1} \subset\subset U_k \subset\subset U$ and $\text{dist}(\bar{U}_{k+1}, \partial U_k) \geq \frac{d}{2n}$ for $k = 1, \ldots, n-1$.

We now use an induction argument. In the first step we apply Lemma 3.2.7 (see also Remark 5.1.7), with d replaced by $d/(2n)$, to the sets $U_1 \subset U$. Note that the application of Lemma 3.2.7 is legitimate by the choice of U_1. This gives

$$\inf_{U_1 \times (\tau_1, \tau+4\theta)} v_1(x,t) = \inf_{U_1 \times (\tau+3\theta, \tau+4\theta)} v_1(x,t) \geq \kappa [v_1]_{p, U_1 \times (\tau+\theta, \tau+2\theta)} \geq \kappa_1 K_0.$$

Here $\kappa_1 = \kappa \varepsilon^p / |B(0, \operatorname{diam} \Omega)|^p$ and the last inequality follows from the relations

$$\begin{aligned} [v_1]_{p, U_1 \times (\tau+\theta, \tau+2\theta)} &\geq \left(\frac{|D|}{|U_1|} \right)^p [v_1]_{p, D \times (\tau+\theta, \tau+2\theta)} \\ &\geq \left(\frac{\varepsilon}{|\Omega|} \right)^p K_0 \geq \frac{\varepsilon^p}{|B(0, \operatorname{diam} \Omega)|^p} K_0. \end{aligned}$$

Next assume that for some $k \in S$ there is a subset S_k of S with k elements such that

$$\inf_{U_k \times (\tau_k, \tau+4\theta)} v_j(x,t) \geq \kappa_k K_0 \quad (j \in S_k), \tag{5.22}$$

where κ_k is a constant depending only on the indicated quantities. If $k = n$, then the theorem is already proved: (5.22) and the relations $D \subset U_k$, $\tau_k < \tau + 7/2\theta$ give (5.21) with $\bar{\kappa} = \kappa_n$. We proceed assuming $1 \leq k < n$. By (IR), there exist $j \in S_k$ and $i \in S \setminus S_k$ such that $c^{ij} \geq \sigma$ in $U \times (\tau, \tau + 4\theta)$. Then, since $(c^{ij})_{i,j \in S} \in M^+(\beta_0, U, (\tau, \tau + 4\theta))$, $c^{ii} \geq 0$, and v is a nonnegative solution,

$$\begin{aligned} (v_i)_t - L_i(x,t)v_i &\geq \sum_{k=1}^{n} c^{ik}(x,t)v_k \geq c^{ij}(x,t)v_j \\ &\geq \sigma \kappa_k K_0 \qquad ((x,t) \in U^k \times (\tau_k, \tau + 4\theta)). \end{aligned}$$

Fix an arbitrary point $x_0 \in U_{k+1}$ and set $\rho = d/(2n)$. Since $\operatorname{dist}(\bar{U}_{k+1}, \partial U_k) \geq d/2n$, we have $B(x_0, \rho) \subset U_k$. Define a radial function $\phi : B(x_0, \rho) \to \mathbb{R}$ by $\phi(x,t) = \eta(\frac{|x-x_0|}{\rho}, t)$, where

$$\eta(r,t) = \begin{cases} (t-\tau_k)V & \text{if } r \leq \frac{1}{2},\ t \in [\tau_k, \tau+4\theta], \\ (t-\tau_k)V(1-(2r-1)^3) & \text{if } \frac{1}{2} < r \leq 1,\ t \in [\tau_k, \tau+4\theta], \end{cases}$$

and V is a positive constant to be specified below. Clearly $\phi \in C^{2,1}(\bar{B}(x_0, \rho) \times [\tau_k, \tau + 4\theta])$, $\phi(x,t) = 0$ for $|x| = \rho$ and $t \in [\tau_k, \tau + 4\theta]$, $\phi(\cdot, \tau_k) = 0$ in $B(x_0, \rho)$, and

$$\phi_t(x,t) - L_i(x,t)\phi(x,t) = V \quad (x \in B(x_0, \frac{\rho}{2}),\ t \in [\tau_k, \tau + 4\theta]).$$

Further, it is clear that there is a constant $K \geq 1$ depending only on N, β_0, θ, d, and n such that

$$\begin{aligned} \phi_t(x,t) - L_i(x,t)\phi(x,t) \leq |\phi_t(x,t) - L_i(x,t)\phi(x,t)| \leq KV \\ ((x,t) \in B(x_0, \rho) \setminus B(x_0, \frac{\rho}{2})) \times [\tau_k, \tau + 4\theta]). \end{aligned}$$

Hence, choosing $V := \sigma\kappa_k K_0/K \leq \sigma\kappa_k K_0$, we see that v_i and ϕ are, respectively, a supersolution and subsolution of the same scalar equation on $B(x_0, \rho) \times (\tau_k, \tau + 4\theta)$.

Moreover, on $\partial_P(B(x_0,\rho)\times(\tau_k,\tau+4\theta))$ we have $v_i \geq 0 = \phi$. The maximum principle therefore implies

$$v(x_0,t) \geq \phi(x_0,t) \geq \frac{\sigma\kappa_k K_0}{K}(\tau_{k+1}-\tau_k) \geq \kappa_{k+1}K_0 \quad (t \in (\tau_{k+1},\tau+4\theta)),$$

where

$$\kappa_{k+1} = \frac{\sigma\kappa_k\theta}{4K2^{k+1}}.$$

Since $x_0 \in U_{k+1}$ was arbitrary, we obtain (5.22) with k replaced by $k+1$ and with $S_{k+1} = S_k \cup \{i\}$. This completes the induction argument showing that after a finite number of steps we establish the validity of (5.22) with $k = n$, hence of (5.21), as noted above.

To prove the last statement of the theorem we combine, as usual, estimate (5.21) with a local maximum principle, as formulated in Lemma 5.1.8 below. Assume that all inequalities in (5.6) are replaced by equations and let $d > 0$ and $\theta > 0$ be given. Assume that (A) holds with condition $|D| > \epsilon$ deleted.

Let $\tilde{D}$ be a domain with the following properties:

$$D \subset\subset \tilde{D} \subset\subset U, \quad \operatorname{dist}(D,\partial\tilde{D}) > \frac{d}{4}, \quad \operatorname{dist}(\tilde{D},\partial U) > \frac{d}{4}.$$

Set $\epsilon := |B(0,d/4)|$ and note that $|\tilde{D}| > \epsilon$. With this ϵ and with d replaced by $d/4$, we apply the already proved statement of the theorem. Hence we obtain (5.21) with D replaced by $\tilde{D}$ and with some constants p and $\bar{\kappa}$ determined only by the indicated quantities, which now refer to n, N, $\operatorname{diam}\Omega$, α_0, β_0, σ, d, and θ (not ϵ). Hence

$$\inf_{D\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq \inf_{\tilde{D}\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq \bar{\kappa}\max_{j\in S}[v_j]_{p,\tilde{D}\times(\tau+\theta,\tau+2\theta)}.$$

The proof will be completed if we show that the last term can be estimated from below by CK_1, where

$$K_1 := \max_{j\in S}\sup_{D\times(\tau+\theta,\tau+2\theta)} v_j.$$

Here C is a constant depending only on the indicated quantities and such are the constants $C_1, C_2, \ldots$ in the forthcoming estimates.

Obviously, $K_1 = v_m(x_0, t_0)$ for some $m \in S$ and $(x_0, t_0) \in \bar{D} \times [\tau + \theta, \tau + 2\theta]$. Assume first that $t_0 > \tau + 5\theta/4$. Set $\delta = \min\{d/4, \sqrt{\theta}/2\}$. Lemma 5.1.8 gives

$$K_1 = v_m(x_0, t_0) \leq C_1 \max_{j\in S} [v_j]_{p,B(x_0,\delta)\times(t_0-\delta^2,t_0)} \leq C_2 \max_{j\in S} [v_j]_{p,\tilde{D}\times(\tau+\theta,\tau+2\theta)}.$$

Thus we have proved the desired estimate under the assumption $t_0 > 5\tau/4$. In other words, we have proved the estimate

$$\inf_{D\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq C_3 \max_{j\in S} \|v_j\|_{L^\infty(D\times(\tau+\frac{5\theta}{4},\tau+2\theta))} \quad (i \in S).$$

Applying this result with τ replaced by $\tau + 7\theta/12$ and θ by $\theta/3$, we next obtain

$$C_4 \max_{j\in S} \|v_j\|_{L^\infty(D\times(\tau+\theta,\tau+\frac{5\theta}{4})} \leq \inf_{D\times(\tau+\frac{7}{4}\theta,\tau+\frac{23}{12}\theta)} v_i(x,t) \leq \sup_{D\times(\tau+\frac{5}{4}\theta,\tau+2\theta)} v_i(x,t).$$

Combining this with the previous estimate, we conclude that (5.21) holds with $p = \infty$ and some constant $\bar{\kappa}$ depending only on the indicated quantities. □

5.2 Proofs of the symmetry results

In this section the assumptions are as in Section 2.2. Specifically, Ω is a bounded domain in $\mathbb{R}^N$, satisfying (D1), (D2), (D3), the nonlinearity F satisfies (F1) – (F4) and at some places, where explicitly stated, also (F5). We consider a solution u of (2.8), (1.9) satisfying (2.10) and (2.11).

We use the notation introduced at the beginning of Chapter 3 and the following one. For any function g on Ω, scalar or vector valued, and any $\lambda \in [0, \ell)$ we let

$$V_\lambda g(x) := g(x^\lambda) - g(x), \quad (x \in \Omega_\lambda).$$

We also set

$$w^\lambda(x,t) := V_\lambda u(x,t) = u(x^\lambda, t) - u(x,t) \quad (x \in \Omega_\lambda,\ t \geq 0).$$

As shown in Subsection 5.1.1, the function w^λ solves a linear problem (5.3), (5.5), with $L_i \in E(\alpha_0, \beta_0, \Omega_\lambda \times (0,\infty))$, $(c^{ij})_{i,j\in S} \in M^+(\beta_0, \Omega_\lambda \times (0,\infty))$. If (F5) is satisfied, then also the irreducibility condition (IR) holds with $U = \Omega_\lambda$, $\tau = 0$, $T = \infty$. Hence the results of Subsection 5.1.2 are applicable to w^λ. We use this observation below, often without notice.

We carry out the process of moving hyperplanes in the following way. Starting from $\lambda = \ell$ we move λ to the left as long as the following property is preserved

$$\lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0. \tag{5.23}$$

We show below that the process can get started and then examine the limit of the process given by

$$\lambda_0 := \inf\{\mu > 0 : \lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0 \text{ for each } \lambda \in [\mu,\ell)\ \}. \tag{5.24}$$

Remark 5.2.1. Note that, by compactness of $\{u(\cdot,t) : t \geq 0\}$ in E, (5.23) is equivalent to the following property: for each $z \in \omega(u)$ and $i \in S$ one has $V_\lambda z_i \geq 0$ in Ω_λ. By the definition of λ_0 and continuity of the functions in $\omega(u)$, we have

$$V_{\lambda_0} z_i(x) \geq 0 \quad (x \in \Omega_{\lambda_0},\ z \in \omega(u),\ i \in S,\ \lambda \in [\lambda_0,\ell)). \tag{5.25}$$

As in the scalar case, one can show that the function z_i is nonincreasing in x_1 in Ω_{λ_0}. Indeed, if $(x_1,x'),(\tilde{x}_1,x') \in \Omega_{\lambda_0}$ and $x_1 > \tilde{x}_1$, then $V_\lambda z_i \geq 0$ with $\lambda = (x_1+\tilde{x}_1)/2 > \lambda_0$ gives $z_i(x_1,x') \geq z_i(\tilde{x}_1,x')$.

Lemma 5.2.2. *We have $\lambda_0 < \ell$. Moreover $|\Omega_{\lambda_0}| \geq \delta$, where δ is a constant depending only on $\alpha_0, \beta_0, \Omega, n, N$.*

Proof. Let $\delta > 0$ be as in Lemma 5.1.4 with $q = 1$. If $\lambda < \ell$ is such that $|\Omega_\lambda| < \delta$, then Lemma 5.1.4 with $w = w^\lambda$, $\hat{\varepsilon} = 0$, $\tau = 0$ and $T = \infty$ gives

$$\max_{i\in S} \|(w_i^\lambda(t))^-\|_{L^\infty(\Omega_\lambda)} \leq Ce^{-t} \to 0 \text{ as } t \to \infty.$$

This and the definition of λ_0 imply that $|\Omega_{\lambda_0}| \geq \delta$. Since $|\Omega_\lambda| < \delta$ for $\lambda \approx \ell$, we have $\lambda_0 < \ell$. $\square$

Lemma 5.2.3. *For any $z \in \omega(u)$, $\lambda \in [\lambda_0,\ell)$, and any connected component U of Ω_λ the following statements hold.*

(i) *For each* $i \in S$ *either* $V_\lambda z_i \equiv 0$ *or* $V_\lambda z_i > 0$ *in* U.

(ii) *If (F5) holds and* $V_\lambda z_i \not\equiv 0$ *for some* $i \in S$, *then* $V_\lambda z_j > 0$ *in* U *for each* $j \in S$.

Proof. Fix any $\lambda \in [\lambda_0, \ell)$ and $z \in \omega(u)$, and let U be a connected component of Ω_λ. Assume that $V_\lambda z_i \not\equiv 0$ on U for some and $i \in S$. Since $V_\lambda z_i$ is continuous and nonnegative, we have

$$V_\lambda z_i > 4r_0 \quad \text{in } \bar{B}_0$$

for some open ball $B_0 \subset U$ and $r_0 > 0$. Choose an increasing sequence $t_k \to \infty$ such that $u(\cdot, t_k) \to z$ in E. Then $w^\lambda(\cdot, t_k) \to V_\lambda z$, hence for a large enough k_0 we have

$$w_i^\lambda(x, t_k) > 2r_0 \quad (x \in \bar{B}_0,\, k > k_0).$$

By the equicontinuity property (see (2.11)), there is $\vartheta > 0$ independent of k such that

$$w_i^\lambda(x, t) > r_0 \quad ((x, t) \in \bar{B}_0 \times [t_k - 4\vartheta, t_k],\, k > k_0). \tag{5.26}$$

Take now an arbitrary domain $D \subset\subset U$ with $B_0 \subset\subset D$. In view of (5.23) and (5.26), an application of Lemma 5.1.6 (with $v = w^\lambda$, $\tau = t_k - 4\vartheta$, $\theta = \vartheta$ and k sufficiently large) shows that there is $r_1 > 0$ such that for $j = i$ we have

$$w_j^\lambda(\cdot, t_k) > r_1 \quad \text{in} \quad \bar{D}. \tag{5.27}$$

Letting $k \to \infty$, we obtain

$$V_\lambda z_j \geq r_1 \qquad \text{in} \quad \bar{D}.$$

Since the domain D was arbitrary, we have $V_\lambda z_j > 0$ in U for $j = i$. This proves (i). If (F5) holds then we can also apply the full Harnack inequality, Theorem 5.1.9, to $v = w^\lambda$. In this case, (5.26) implies (5.27) for each $j \in S$ and the above arguments prove (ii). $\square$

In the next key lemma we consider the possibility $\lambda_0 > 0$. We show that it implies that each $z \in \omega(u)$ has a partial reflectional symmetry around H_{λ_0}. This will be needed in the proof of Theorem 2.2.4 and also in the proof of Theorem 2.2.1 (where the possibility $\lambda_0 > 0$ is ruled out).

Lemma 5.2.4. *If $\lambda_0 > 0$, then for each $z \in \omega(u)$ there exist $i \in S$ and a connected component U of Ω_{λ_0} such that $V_{\lambda_0} z_i \equiv 0$ in U.*

Proof. The proof is by contradiction. Assume that the statement is not true. Then, by Lemma 5.2.3, there exists $\tilde{z} \in \omega(u)$ such that

$$V_{\lambda_0} \tilde{z}_i(x) > 0 \quad (x \in \Omega_{\lambda_0},\ i \in S). \tag{5.28}$$

We show that this implies the existence of $\varepsilon_0 > 0$ such that (5.23) holds for all $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0]$, which is a contradiction to the definition of λ_0. We follow a similar scheme of arguing as in the proof proof of Lemma 4.2 in [52]. First we show that for $\lambda \approx \lambda_0$, w^λ is positive in a bounded cylinder $D \times [\bar{t}, \bar{t} + \theta]$ which has a "small" complement in $\Omega_\lambda \times [\bar{t}, \bar{t} + \theta]$ and the functions w_i^λ have small negative parts in the complement. We then show that this implies (5.23).

In what follows we assume that (5.28) holds.

Denote by $(t_k)_{k \in \mathbb{N}}$ an increasing sequence converging to ∞ for which $u(\cdot, t_k) \to \tilde{z}$ in E.

For the remainder of the proof, we fix positive constants r_0, γ, and δ as follows. First we choose $r_0 > 0$ such that each connected component of Ω_{λ_0} contains a closed ball of radius r_0. Such a choice is possible as Ω_{λ_0} has only finitely many connected components by (D2). Corresponding to $r = r_0$ (and the constants α_0, β_0 from (F1), (F2)), we take γ as in Lemma 3.2.5. Finally, let $\delta > 0$ be such that the conclusion of Lemma 5.1.4 holds for $k = \gamma + 1$.

Now let $D \subset\subset \Omega_{\lambda_0}$ be an open set such that $|\Omega_{\lambda_0} \setminus D| < \delta/2$ and such that the intersection of D with any connected component of Ω_{λ_0} is a domain containing a ball of radius r_0. In particular, D has the same number of connected components as Ω_{λ_0}. Let $\varepsilon_1 > 0$ be so small that $|\Omega_{\lambda_0 - \varepsilon_1} \setminus \Omega_{\lambda_0}| < \delta/2$, hence also

$$|\Omega_\lambda \setminus D| < \delta \quad (\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0]). \tag{5.29}$$

Using (5.28) and the equicontinuity of w^{λ_0} one shows easily (cf. the proof of (5.26) in the proof of Lemma 5.2.3) that there exist positive constants θ, d_1, and k_1 such that for each $i \in S$

$$w_i^{\lambda_0}(x,t) > 2d_1 \quad (x \in \bar{D},\ t \in [t_k, t_k + 4\theta], \quad k > k_1). \tag{5.30}$$

By the equicontinuity of the function u we have

$$\sup_{\substack{D\times[t_k,t_k+4\theta] \\ j\in S}} |w_j^{\lambda} - w_j^{\lambda_0}| \to 0 \tag{5.31}$$

as $\lambda \to \lambda_0$, uniformly with respect to k. This and (5.30) imply that, possibly with a smaller $\varepsilon_1 > 0$, for each $k > k_1$

$$w_i^{\lambda}(x,t) > d_1 \quad (i \in S,\ x \in \bar{D},\ t \in [t_k, t_k + 4\theta],\ \lambda \in (\lambda_0 - \varepsilon_1, \lambda_0]). \tag{5.32}$$

Thus we have established the positivity of w^{λ} in the bounded cylinder $\bar{D} \times [t_k, t_k + 4\theta]$.

Our next aim is to show that the functions $w_i^{\lambda}(\cdot, t_k)$ have small negative parts. Namely, we claim that given any $\varsigma > 0$ there are $k_2 \geq k_1 > 0$ and $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$

$$\|(w_i^{\lambda})^-(\cdot, t_k)\|_{L^\infty(\Omega_\lambda)} \leq \varsigma \quad (i \in S,\ k > k_2). \tag{5.33}$$

The arguments here are very similar to those used in the proof of estimate (4.10) in [52], we just recall them briefly. By (5.25), estimate (5.33) holds for $\lambda = \lambda_0$ with ς replaced by $\varsigma/2$ if k_2 is large enough. Therefore, by the equicontinuity of u, estimate (5.33) with Ω_λ replaced by Ω_{λ_0} also holds for all $\lambda \approx \lambda_0$. Next one shows, using the equicontinuity of u and Dirichlet boundary conditions, that there exists a neighborhood $\mathcal{E}$ of $\partial\Omega$, independent of λ, such that (5.33) holds with Ω_λ replaced by $\Omega_\lambda \cap \mathcal{E}$. Finally, the remaining set $\bar{\Omega}_\lambda \setminus (\mathcal{E} \cup \Omega_{\lambda_0})$ is contained in an arbitrarily small neighborhood G_0 of $H_{\lambda_0} \cap \Omega_{\lambda_0} \setminus \mathcal{E}$ if $\lambda \approx \lambda_0$. Since $w^{\lambda_0}(\cdot, t)$ vanishes on $H_{\lambda_0} \cap \Omega_{\lambda_0} \setminus \mathcal{E} \subset\subset \Omega$, using the equicontinuity we can choose G_0 so that $G_0 \subset\subset \Omega$ and (5.33) holds for $\lambda = \lambda_0$ with Ω_λ replaced by G_0 and with ς replaced by $\varsigma/2$.

Then, using the equicontinuity one more time, we obtain that (5.33) holds with Ω_λ replaced by G_0 for any $\lambda \approx \lambda_0$. Combining all these estimates we conclude that (5.33) holds for all $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$ if ε_2 is small enough.

Our final goal is to prove that (5.32) and (5.33) with a sufficiently small ς imply that if $k > k_2$, then

$$w_i^\lambda(x,t) > 0 \quad (i \in S,\ x \in \bar{D},\ t \in [t_k, \infty),\ \lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]). \tag{5.34}$$

Observe that (5.34), in conjunction with $w_i^\lambda \geq 0$ on $\partial\Omega_\lambda \times (0,\infty)$, gives $w_i^\lambda \geq 0$ on $\partial(\Omega_\lambda \setminus D) \times (t_k, \infty)$. Since $|\Omega_\lambda \setminus D| < \delta$, Lemma 5.1.4 and our choice of δ imply that for any $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$

$$\lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = \lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda\setminus D)} = 0.$$

Thus having proved (5.34), we will have the desired contradiction and the proof of Lemma 5.2.4 will be complete.

To derive (5.34) we assume that (5.33) holds, ς being a sufficiently small constant as specified below, see (5.39). Fix any $k > k_2$. Let T be the maximal element of $(t_k, \infty]$ such that

$$w_i^\lambda(x,t) > 0 \quad (i \in S,\ x \in \bar{D},\ t \in (t_k, T),\ \lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]). \tag{5.35}$$

By (5.32), $T > t_k + 4\theta$. We need to prove that $T = \infty$. Assume $T < \infty$. Then there exist $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$, $\bar{x} \in \partial D$, and $i_0 \in S$ such that $w_{i_0}^\lambda(\bar{x}, t) = 0$. We show that this leads to a contradiction by estimating $w_{i_0}^\lambda$ from below using Lemma 5.1.6. For that we first estimate the functions $(w_i^\lambda)^-$, $i \in S$, from above. We have $w_i^\lambda \geq 0$ on $\partial(\Omega_\lambda \setminus D) \times (t_k, T)$ and $|\Omega_\lambda \setminus D| < \delta$. Hence (5.35), Lemma 5.1.4, and the choice of δ give

$$\begin{aligned} \|(w_i^\lambda)^-(\cdot,t)\|_{L^\infty(\Omega_\lambda\setminus D)} &\leq 2e^{-(\gamma+1)(t-t_k)} \max_{j\in S} \|(w_j^\lambda)^-(\cdot,t_k)\|_{L^\infty(\Omega_\lambda\setminus D)} \\ &\leq 2e^{-(\gamma+1)(t-t_k)}\varsigma \quad (t \in [t_k, T],\ i \in S), \end{aligned} \tag{5.36}$$

where ς is as in (5.33).

Next we estimate $w^\lambda_{i_0}$ from below on a ball in D. Let D_0 be a connected component of D such that $\bar{x} \in \partial D_0$. By the manner in which D was chosen, D_0 contains a ball $B_0 = B(x_0, r_0)$ of radius r_0. Recall that γ was defined so that the statement of Lemma 3.2.5 holds with $r = r_0$. Let h_{r_0} be as in that statement. Then for each $i \in S$ the function $\phi(x,t) = e^{-\gamma t} h_r(x - x_0)$ satisfies

$$\begin{aligned} \partial_t \phi - L^\lambda_i(x,t)\phi &< c^{ii}(x,t)\phi, && (x,t) \in B_0 \times (t_k, \infty), \\ \phi &= 0 && (x,t) \in \partial B_0 \times (t_k, \infty), \end{aligned}$$

where $L^\lambda_i \in E(\alpha_0, \beta_0, \Omega_\lambda \times (0,\infty))$ and $(c^{ij})_{i,j\in S} \in M^+(\beta_0, \Omega_\lambda \times (0,\infty))$ are as in (5.3). Since $c^{ij} \geq 0$ for $i \neq j$, we also have

$$\partial_t \phi - L_i(x,t)\phi < \sum_{j=1}^n c^{ij}(x,t)\phi \quad (x,t) \in B_0 \times (t_k, \infty),\ i \in S.$$

We view this as a system of inequalities for the vector function $(\phi, \ldots, \phi)$; it has the opposite inequality signs than the system satisfied by w^λ. For $t = t_k$ we have by (5.32)

$$w^\lambda_i(x, t_k) > d_1 \geq d_1 \frac{\phi(x, t_k)}{\|\phi(\cdot, t_k)\|_{L^\infty(B_0)}} \quad (x \in B_0,\ i \in S).$$

Also, as $B_0 \subset D$, we have $w_j \geq 0 = \phi$ on $\partial B_0 \times (t_k, T)$. These relations justify an application of Theorem 5.1.2 to $w^\lambda - d_1(\phi, \ldots, \phi)/\|\phi(\cdot, t_k))\|_{L^\infty(B_0)}$ and we conclude that

$$w^\lambda_{i_0}(x,t) \geq d_1 \frac{\phi(x,t)}{\|\phi(\cdot, t_k)\|_{L^\infty(B_0)}} = d_1 e^{-\gamma(t - t_k)} \frac{h_{r_0}(x - x_0)}{\|h_{r_0}(\cdot - x_0)\|_{L^\infty(B_0)}} \qquad ((x,t) \in B_0 \times (t_k, T)). \tag{5.37}$$

Equipped with (5.36), (5.37), we are ready to use Lemma 5.1.6 in order to estimate $w^\lambda_{i_0}$ from below everywhere in $D_0 \times (t_k, T)$. We apply the lemma on the interval $(T - 4\theta, T)$ and the sets $D_0 \subset\subset \Omega_\lambda$, noting that $(w^\lambda_{i_0})^+ = w^\lambda_{i_0}$ in $D_0 \times (T - 4\theta, T)$ and $\operatorname{dist}(\bar{D}_0, \Omega_\lambda) \geq d := \operatorname{dist}(\bar{D}, \Omega_{\lambda_0})$. This gives

$$w^\lambda_{i_0}(x,T) \geq \kappa [w^\lambda_{i_0}]_{p, D_0 \times (T - 3\theta, T - 2\theta)} - \kappa_1 \max_{j \in S} \sup_{\partial_P(\Omega_\lambda \times (T - 4\theta, T))} (w^\lambda_j)^- \quad (x \in D_0),$$

where κ, κ_1, and p are constants depending only on d, n, N, $\operatorname{diam}\Omega$, α_0, β, θ and r_0. By (5.37), (5.36), we therefore have

$$w_{i_0}^{\lambda}(x,T) \geq \kappa e^{-\gamma(T-t_k)} G(p,r_0) - 2\kappa_1 e^{-(\gamma+1)(T-4\theta-t_k)} \quad (x \in D_0),$$

where

$$G(p,r_0) := \frac{1}{\|h_{r_0}\|_{L^\infty(B_0)}} \left(\frac{1}{|D_0|} \int_{B_0} h_r^p \right)^{\frac{1}{p}} > 0.$$

Consequently

$$\inf_{x \in D_0} w_{i_0}^{\lambda}(x,T) \geq e^{-(\gamma+1)(T-t_k)} (\kappa G(p,r_0) - \kappa_1 \varsigma). \tag{5.38}$$

We now specify the choice of ς in (5.36):

$$\varsigma := G(p,r_0)/2\kappa_1 \tag{5.39}$$

This is legitimate, as the number is independent of k and λ. Then (5.38) and the continuity of w^λ imply that $w_{i_0}^{\lambda}(\cdot,T) > 0$ on ∂D_0 contradicting $w_{i_0}^{\lambda}(\bar{x},T) = 0$. This contradiction shows that the maximal T for which (5.35) holds is equal ∞, that is, (5.34) must hold. As remarked above, this completes the proof of Lemma 5.2.4. □

The following lemma addresses the strict monotonicity of the functions in $\omega(z)$. The proof is a straightforward modification of the corresponding lemma in a scalar case, see [52, Lemma 4.6], and is omitted.

Lemma 5.2.5. *Assume that Ω_{λ_0} is connected. Then for any $z \in \omega(u)$ and any $i \in S$, either $z_i \equiv 0$ on Ω_{λ_0} or else $z_i > 0$ in Ω_{λ_0} and z_i is strictly decreasing in x_1 in Ω_{λ_0}. The latter holds in the form $(z_j)_{x_1} < 0$ if $(z_j)_{x_1} \in C(\Omega_{\lambda_0})$ for some $j \in S$.*

We need one more lemma for the proof of our symmetry results. Its assumption is identical to hypothesis (i) of Theorem 2.2.1.

Lemma 5.2.6. *Assume that there exists $\varphi = (\varphi_1, \ldots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \ldots, n\}$. Then $\lambda_0 = 0$.*

Proof. Assume $\lambda_0 > 0$. By Lemma 5.2.4, there is $i \in S$ such that $V_{\lambda_0}\varphi_i \equiv 0$ on some connected component of Ω_{λ_0}. In view of Dirichlet boundary condition, this clearly implies that φ_i vanishes somewhere in Ω a contradiction. □

Now we are ready to prove our symmetry theorems.

Proof of Theorems 2.2.1 and 2.2.4 . We show that the statements (i)–(iii) of Theorem 2.2.4 hold with λ_0 defined in (5.24). First assume $\lambda_0 > 0$. Then statement (i) follows directly from Remark 5.2.1 and Lemmas 5.2.3, 5.2.4; statement (ii) follows from (i) and Lemma 5.2.3; and statement (iii) follows from (ii) and Lemma 5.2.5.

Now we consider the case $\lambda_0 = 0$. Note that $\Omega_{\lambda_0} = \Omega_0$ is connected by (D1). By Lemma 5.2.5, for each $z \in \omega(u)$ and $i \in S$ either $z_i \equiv 0$ (and then of course $V_0 z_i \equiv 0$) or $z_i > 0$ in Ω_0. As $V_0 z_i \geq 0$ in Ω_0 by (5.25), the relation $z_i > 0$ in Ω_0 implies $z_i > 0$ in Ω. We conclude that for each $z \in \omega(u)$

$$\text{either } z_i \equiv 0 \text{ in } \Omega \text{ or } z_i > 0 \text{ in } \Omega \quad (i \in S). \tag{5.40}$$

In case (F5) holds, a stronger version of this statement follows from Lemma 5.2.3(ii):

$$\text{either } z \equiv 0 \text{ in } \Omega \text{ or } z_i > 0 \text{ in } \Omega \text{ for each } i \in S. \tag{5.41}$$

Now observe that statement (i) of Theorem 2.2.4 is trivially satisfied for any $z \in \omega(u)$ such that $z_i \equiv 0$ for some $i \in S$. Similarly, if (F5) holds, then statements (ii) and (iii) are satisfied for any such z, for it has to satisfy $z \equiv 0$ by (5.41). Thus, in view of (5.40), we only need to consider the case that there exists $\varphi \in \omega(u)$ with $\varphi_i > 0$ for all $i \in S$, that is, hypothesis (i) of Theorem 2.2.1 is satisfied. Because of this and Lemma 5.2.6, the remaining part of the proof is common to Theorems 2.2.4 and 2.2.1. We prove that hypothesis (i) implies

$$V_0 z_i \equiv 0 \quad (z \in \omega(u),\ i \in S). \tag{5.42}$$

This will complete the proof of the symmetry statements of Theorems 2.2.4 and 2.2.1 (note that hypothesis (ii) Theorem 2.2.1 implies hypothesis (i) by (5.41)). The strict monotonicity statements follow from Lemma 5.2.5 as above.

To prove (5.42), we apply the results of this section to the solution $\tilde{u}(x_1, x', t) = u(-x_1, x', t)$ in place of u ($\tilde{u}$ is indeed a solution as F is even in x_1). Denote by $\tilde{\lambda}_0$ the corresponding number defined as in (5.24) with u replaced by $\tilde{u}$. Since $\tilde{u}$ has a strictly positive element $\tilde{\varphi}$ in its ω-limit set, Lemma 5.2.6 gives $\tilde{\lambda}_0 = 0$. This clearly implies (5.42). $\square$

Proof of Theorem 2.2.3. For each $i \in S$ we have, similarly as in Subsection 5.1.1,

$$\begin{aligned}(u_i)_t &= F_i(t, x, u, Du_i, D^2u_i) - F_i(t, x, 0, 0, 0) + F_i(t, x, 0, 0, 0) \\ &= L_i(x, t)u_i + \sum_{j=1}^{n} c^{ij}(x, t)u_j + F_i(t, x, 0, 0, 0) \quad (x \in \Omega,\ t > 0)\end{aligned}$$

for suitable $L_i \in E(\alpha_0, \beta_0, \Omega\times(0, \infty))$ and $(c^{ij})_{i,j\in S} \in M^+(\beta_0, \Omega_\lambda\times(0, \infty))$. Setting $g_i(x, t) := -F_i^-(t, x, 0, 0, 0)$ and using the relations $c^{ij} \geq 0$ for $i \neq j$, we obtain the following scalar inequality for each $i \in S$

$$(u_i)_t \geq L_i(x, t)u_i + c^{ii}(x, t)u_i + g_i(x, t) \quad (x \in \Omega,\ t > 0).$$

By (2.13), $\sup_{x\in\Omega} g_i(x, t) \to 0$ as $t \to \infty$.

Such scalar inequalities are considered in the proof of Theorem 2.5 of [52]. The arguments used there can be repeated for each i to obtain the conclusion of Theorem 2.2.3.

$\square$

Proof of Theorem 2.2.5. Let λ_0 be as in (5.24). It is sufficient to prove that under the assumptions of Theorem 2.2.5 we have $\lambda_0 = 0$. Indeed, as in the proof of Theorems 2.2.1 and 2.2.4, $\lambda_0 = 0$ gives the reflectional symmetry and the strict monotonicity in $x_1 > 0$ of z_i for each $z \in \omega(u) \setminus \{0\}$ and $i \in S$. As problem (2.8), (1.9) is invariant under rotations, we can apply this to any of the solutions $\tilde{u}(x, t) = u(Rx, t)$, $R \in O(n)$. This gives the conclusion of Theorem 2.2.5.

The proof is by contradiction. Assume $\lambda_0 > 0$. Then by (5.24) (see also Remark 5.2.1) and the compactness of $\omega(z)$ in E, there exist $z \in \omega(u)$, $i \in S$, and a sequence $\lambda_k \nearrow \lambda_0$ such that $(V_{\lambda_k} z_i)^- \not\equiv 0$ in Ω_{λ_k} for $k = 1, 2, \ldots$. At the same time we have $V_{\lambda_0} z_i \equiv 0$, by Lemmas 5.2.3(ii) and 5.2.4.

We know by Lemma 5.2.5 that either $z_i \equiv 0$ in Ω_{λ_0} or $z_i > 0$ in Ω_{λ_0}.

First assume $z_i > 0$ in Ω_{λ_0}. Since $V_{\lambda_0} z_i \equiv 0$ and $z_i = 0$ on $\partial\Omega$, z_i vanishes on $\partial\Omega'_{\lambda_0} \setminus H_{\lambda_0}$ (recall that Ω'_λ is the reflection of Ω_λ in H_λ). It follows that if $x_0 \in H_{\lambda_0} \cap \partial\Omega$ and U is any neighborhood of x_0 in $\bar{\Omega}$ then z_i vanishes somewhere in $\Omega \cap U$ while at the same time $z_i \not\equiv 0$ in U. Using a rotation R which takes such a point x_0 to $(1, 0, \ldots, 0)$ and considering the solution $\tilde{u}(x,t) = u(Rx,t)$, we clearly obtain a contradiction to Lemma 5.2.5.

Next assume $z_i \equiv 0$ in Ω_{λ_0}. As $V_{\lambda_0} z_i \equiv 0$, we have $z_i \equiv 0$ in $\bar{\Omega}_{\lambda_0} \cup \Omega'_{\lambda_0}$. If there exists $x_0 \in H_{\lambda_0} \cap \partial\Omega$ such that for each neighborhood U of x_0 in $\bar{\Omega}$ one has $z_i \not\equiv 0$, then we obtain a contradiction as in the previous case. Otherwise, $z_i \equiv 0$ on some neighborhood $\mathcal{N}$ of $H_{\lambda_0} \cap \partial\Omega$. Now, for $\lambda < \lambda_0$ sufficiently close to λ_0 one has $\Omega_\lambda \subset \bar{\Omega}_{\lambda_0} \cup \Omega'_{\lambda_0} \cup \mathcal{N}$. Therefore $z_i \equiv 0$ in Ω_λ and consequently $V_\lambda z_i \geq 0$ in Ω_λ for all $\lambda \approx \lambda_0$. This is a contradiction to the existence of the sequence λ_k.

Thus in either case, $\lambda_0 > 0$ leads to a contradiction. This proves that $\lambda_0 = 0$ as needed. $\square$

References

[1] A. D. Alexandrov. A characteristic property of spheres. *Ann. Math. Pura Appl.*, 58:303–354, 1962.

[2] H. Amann. Maximum principles and principal eigenvalues. In *Ten mathematical essays on approximation in analysis and topology*, pages 1–60. Elsevier B. V., Amsterdam, 2005.

[3] A. Arapostathis, M. K. Ghosh, and S. I. Marcus. Harnack's inequality for cooperative weakly coupled elliptic systems. *Comm. Partial Differential Equations*, 24(9-10):1555–1571, 1999.

[4] A. V. Babin. Symmetrization properties of parabolic equations in symmetric domains. *J. Dynam. Differential Equations*, 6:639–658, 1994.

[5] A. V. Babin. Symmetry of instabilities for scalar equations in symmetric domains. *J. Differential Equations*, 123:122–152, 1995.

[6] A. V. Babin and G. R. Sell. Attractors of non-autonomous parabolic equations and their symmetry properties. *J. Differential Equations*, 160:1–50, 2000.

[7] H. Berestycki. Qualitative properties of positive solutions of elliptic equations. In *Partial differential equations (Praha, 1998)*, volume 406 of *Chapman & Hall/CRC Res. Notes Math.*, pages 34–44. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[8] H. Berestycki and L. Nirenberg. On the method of moving planes and the sliding method. *Bol. Soc. Brasil. Mat. (N.S.)*, 22:1–37, 1991.

[9] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.*, 47(1):47–92, 1994.

[10] I. Birindelli, È. Mitidieri, and G. Svirs. Existence of the principal eigenvalue for cooperative elliptic systems in a general domain. *Differ. Uravn.*, 35(3):325–333, 429, 1999.

[11] J. Busca and B. Sirakov. Symmetry results for semilinear elliptic systems in the whole space. *J. Differential Equations*, 163:41–56, 2000.

[12] J. Busca and B. Sirakov. Harnack type estimates for nonlinear elliptic systems and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21:543–590, 2004.

[13] X. Cabré. On the Alexandroff-Bakel′man-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 48(5):539–570, 1995.

[14] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3):369–402, 1984.

[15] A. Castro and R. Shivaji. Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric. *Comm. Partial Differential Equations*, 14(8-9):1091–1100, 1989.

[16] Z.-Q. Chen and Z. Zhao. Harnack principle for weakly coupled elliptic systems. *J. Differential Equations*, 139:261–282, 1997.

[17] S. Y. Cheng and S. T. Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x, u)$. *Comm. Pure Appl. Math.*, 30(1):41–68, 1977.

[18] Y. S. Choi and A. W. Shaker. On symmetry in semi-linear elliptic systems with nonnegative solutions. *Appl. Anal.*, 53:125–134, 1994.

[19] L. Damascelli, F. Pacella, and M. Ramaswamy. A strong maximum principle for a class of non-positone singular elliptic problems. *NoDEA Nonlinear Differential Equations Appl.*, 10(2):187–196, 2003.

[20] E. N. Dancer. Some notes on the method of moving planes. *Bull. Austral. Math. Soc.*, 46:425–434, 1992.

[21] E. N. Dancer and P. Hess. The symmetry of positive solutions of periodic-parabolic problems. *J. Comput. Appl. Math.*, 52:81–89, 1994.

[22] D. G. de Figueiredo. Monotonicity and symmetry of solutions of elliptic systems in general domains. *NoDEA Nonlinear Differential Equations Appl.*, 1:119–123, 1994.

[23] Y. Du. *Order structure and topological methods in nonlinear partial differential equations. Vol. 1*, volume 2 of *Series in Partial Differential Equations and Applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. Maximum principles and applications.

[24] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[25] D. G. De Figueiredo and J. Yang. Decay, symmetry and existence of solutions of semilinear elliptic systems. *Nonlinear Anal.*, 33(3):211–234, 1998.

[26] J. Földes. On symmetry properties of parabolic equations in bounded domains. submitted.

[27] J. Földes. Symmetry properties of asymptotically symmetric parabolic equations in $\mathbb{R}^n$. in preparation.

[28] J. Földes and P. Poláčik. On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry. *Disc. Cont. Dynamical Systems.* To appear.

[29] J. Földes and P. Poláčik. Symmetry properties of asymptotically symmetric parabolic equations in bounded domains. in preparation.

[30] B. Gidas, W.-M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68:209–243, 1979.

[31] B. Gidas, Wei Ming Ni, and L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. In *Mathematical analysis and applications, Part A*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 369–402. Academic Press, New York, 1981.

[32] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[33] M. Gruber. Harnack inequalities for solutions of general second order parabolic equations and estimates of their hölder constants. *Math. Z.*, 185(1):23–43, 1984.

[34] P. Hess and P. Poláčik. Symmetry and convergence properties for non-negative solutions of nonautonomous reaction-diffusion problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 124:573–587, 1994.

[35] B. Kawohl. Symmetrization—or how to prove symmetry of solutions to a PDE. In *Partial differential equations (Praha, 1998)*, volume 406 of *Chapman & Hall/CRC Res. Notes Math.*, pages 214–229. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[36] N. V. Krylov. *Nonlinear elliptic and parabolic equations of the second order.* D. Reidel Publishing Co., Dordrecht, Holland, 1987.

[37] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980.

[38] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasi-linear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[39] C. Li. Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains. *Comm. Partial Differential Equations*, 16:491–526, 1991.

[40] G. M. Lieberman. *Second order parabolic differential equations.* World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[41] F. Da Lio and B. Sirakov. Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations. *J. European Math. Soc.*, 9:317–330, 2007.

[42] O. Lopes and M. Mariş. Symmetry of minimizers for some nonlocal variational problems. *J. Funct. Anal.*, 254(2):535–592, 2008.

[43] L. Maddalena. Harnack's inequality for weakly coupled quasilinear parabolic systems. *Rend. Accad. Sci. Fis. Mat. Napoli (4)*, 47:5–18 (1981), 1980.

[44] H. Matano and T. Ogiwara. Monotonicity and convergence results in order-preserving systems in the presence of symmetry. *Discrete Contin. Dynam. Systems*, 5(1):1–34, 1999.

[45] H. Matano and T. Ogiwara. Stability analysis in order-preserving systems in the presence of symmetry. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(2):395–438, 1999.

[46] W.-M. Ni. Qualitative properties of solutions to elliptic problems. In *Stationary partial differential equations. Vol. I*, Handb. Differ. Equ., pages 157–233. North-Holland, Amsterdam, 2004.

[47] W.-M. Ni, P. Poláčik, and E. Yanagida. Monotonicity of stable solutions in shadow systems. *Trans. Amer. Math. Soc.*, 353(12):5057–5069 (electronic), 2001.

[48] P. Poláčik. Private communication.

[49] P. Poláčik. Symmetry properties of positive solutions of parabolic equations: a survey. preprint.

[50] P. Poláčik. Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$. I. Asymptotic symmetry for the Cauchy problem. *Comm. Partial Differential Equations*, 30(10-12):1567–1593, 2005.

[51] P. Poláčik. Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$. II. Entire solutions. *Comm. Partial Differential Equations*, 31(10-12):1615–1638, 2006.

[52] P. Poláčik. Estimates of solutions and asymptotic symmetry for parabolic equations on bounded domains. *Arch. Rational Mech. Anal.*, 183:59–91, 2007.

[53] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[54] A. W. Shaker. On symmetry in elliptic systems. *Appl. Anal.*, 41:1–9, 1991.

[55] B. Sirakov. Some estimates and maximum principles for weakly coupled systems of elliptic pde. *Nonlinear Analysis*, 70:3039–3046, 2009.

[56] W. C. Troy. Symmetry properties in systems of semilinear elliptic equations. *J. Differential Equations*, 42:400–413, 1981.

[57] K. Tso. On an Aleksandrov-Bakel′man type maximum principle for second-order parabolic equations. *Comm. Partial Differential Equations*, 10(5):543–553, 1985.

[58] R. H. Wang and G. L. Wang. On existence, uniqueness and regularity of viscosity solutions for the first initial-boundary value problems to parabolic Monge-Ampère equation. *Northeast. Math. J.*, 8(4):417–446, 1992.