Relative Methods in Symplectic Topology

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ABSTRACT

In the first part we define relative Ruan invariants that count embedded connected symplectic submanifolds which contact a fixed symplectic hypersurface V in a symplectic 4-manifold (X, ω) at prescribed points with prescribed contact orders (in addition to insertions on $X \backslash V$) for generic V. We obtain invariants of the deformation class of (X, V, ω). Two large issues must be tackled to define such invariants: (1) Curves lying in the divisor V and (2) genericity results for almost complex structures constrained to make V symplectic. Moreover, these invariants are refined to take into account rim tori decompositions. In the latter part of the paper, we extend the definition to disconnected submanifolds and construct relative Gromov-Taubes invariants.

In the second part we introduce the notion of the relative symplectic cone $\mathcal{C}_M^V$. As an application, we determine the *symplectic cone* $\mathcal{C}_M$ of certain T^2-fibrations. In particular, for some elliptic surfaces we verify the conjecture in [32]: If M underlies a minimal Kähler surface with $p_g > 0$, the symplectic cone $\mathcal{C}_M$ is equal to $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$, where $\mathcal{P}^{\alpha} = \{e \in H^2(M; \mathbb{R}) | e \cdot e > 0 \text{ and } e \cdot \alpha > 0\}$ for nonzero $\alpha \in H^2(M; \mathbb{R})$ and $\mathcal{P}^0 = \{e \in H^2(M; \mathbb{R}) | e \cdot e > 0\}$.

Contents

Chapter 1

Introduction

Symplectic Topology is the study of smooth manifolds X which admit a closed non-degenerate 2-form ω, called a symplectic form. Symplectic manifolds are interesting for many reasons, among them is their natural appearance in physics (for example in classical mechanics and string theory) and their similarity/differences to complex manifolds (All symplectic manifolds admit almost complex structures.). Many basic questions remain unanswered, such as which smooth manifolds admit symplectic structures and which classes $\alpha \in H^2(X)$ are representable by orientation compatible symplectic forms, this set being called the symplectic cone $\mathcal{C}_X$.

1.1 Relative Methods in Symplectic Topology

The search for invariants of symplectic manifolds has been very fruitful in providing partial answers to these and other questions as well as illuminating relations between symplectic topology and other fields. On the other hand, many surgery constructions on symplectic manifolds have been defined in order to construct new symplectic manifolds with certain properties. Most involve surgery along submanifolds. Therefore, understanding symplectic structures with respect to fixed submanifolds and developing relative invariants has become a central issue in researching symplectic manifolds.

One such surgery is the symplectic cut/sum constructions ([14], [43], [26]). This operation is performed on two (symplectic) manifolds X and Y, each containing a (symplectic) hypersurface V, by removing normal neighborhoods of V in each and gluing the boundaries together in an orientation compatible manner to produce a manifold $X\#_V Y$. It was shown in [14] that this can be done in the symplectic category. The behavior of symplectic forms and embedded surfaces with a view towards this surgery is the core of this work. We will define two relative invariants:

- The relative Ruan and relative Gromov-Taubes invariants which count embedded symplectic submanifolds which contact the symplectic submanifold V at prescribed points with prescribed contact orders (in addition to insertions on $X \backslash V$).
- The relative symplectic cone $\mathcal{C}_X^V$ is the set of classes $\alpha \in H^2(X, \mathbb{R})$ which are representable by an orientation compatible symplectic form making a fixed codimension 2 submanifold V symplectic.

1.2 Relative Ruan and Gromov-Taubes Invariants

Gromov's paper [17] initiated the intense study of pseudoholomorphic curves in symplectic manifolds as a means to construct symplectic invariants. This has led to a wide range of invariants for symplectic manifolds which "count" such curves. After the development of the symplectic sum and symplectic cut ([26], [14], [43]), research turned also to developing invariants relative to a fixed symplectic hypersurface V, i.e. a symplectic submanifold (or a smooth divisor) of codimension 2. A variety of relative invariants have been developed which consider curves in X which contact V in a specified manner ([22], [27], [29], [40]). In particular Gromov-Witten theory has seen great advances in the last few years: Tools such as the decomposition formulas of [27], [21] and [30] have been developed, correspondences between different invariants have been found in [42] and [18] and the invariants have been explicitly calculated for a range of manifolds, see any of the sources cited above. Gromov-Witten invariants are based on moduli spaces of maps from (nodal) Riemann surfaces to a symplectic manifold (X, ω). Much of the work involved in defining these invariants has to do with the intractable nature of these moduli spaces.

A more natural object of study in the symplectic category are embedded symplectic submanifolds. This topic was initially studied by Ruan [57], who initiated the development of invariants which count embedded submanifolds in four-manifolds. Ruan's invariant counted only connected submanifolds. In a series of fundamental papers ([60], [61]), Taubes defined Gromov-Taubes invariants, which give a delicate count of embedded, possibly disconnected submanifolds (This count refines Ruan's earlier results, in particular providing a detailed analysis of the behavior of square 0- tori and their multiple covers.), and equated them with Seiberg-Witten invariants.

In this paper, we define corresponding relative invariants, called relative Ruan invariant and relative Gromov-Taubes invariant, which count embedded symplectic submanifolds which contact the fixed symplectic hypersurface V at prescribed points with prescribed contact orders (in addition to insertions on $X \backslash V$). These invariants will be shown to be deformation invariants of the symplectic structures, note however that due to the relative setting, the symplectic structures must all make the hypersurface V symplectic. This connects these invariants to the relative symplectic cone defined [6].

We wish to emphasize the geometric nature of the relative invariants we consider. The objects that are counted are geometric sets in contrast to Gromov-Witten theory, where the underlying objects are maps. This relates these invariants to the classical curve counting methods of algebraic geometry. Notwithstanding this difference, there are relations between Ruan-type invariants and Gromov-Witten invariants, as described in [20].

The following Sections are devoted to a precise formulation of the relative spaces of submanifolds underlying the relative Ruan invariant and the properties of these spaces. Let X be a symplectic 4-manifold and V a 2-dimensional symplectic submanifold. We would like to count embedded symplectic submanifolds which intersect V. Section 2.4 defines the space of relative, connected submanifolds $\mathcal{K}_V(A, J, \mathcal{I}_A)$ in class $A \in H_2(X)$, where A is not multiply toroidal, i.e. not of the form $A = mT$ with $T^2 = 0$ and $K_\omega \cdot T = 0$, which meet the hypersurface V according to initial data $\mathcal{I}_A$. In order to understand the role of the almost complex structure J, we consider the symplectic submanifolds as embedded J-holomorphic curves. The behavior of curves in class A for generic J is rather straightforward if $A \neq \mathfrak{V}$, denoting the class of V by $\mathfrak{V}$. However, care must be taken when $A = \mathfrak{V}$: The associated differential operator is no longer surjective if $d_{\mathfrak{V}} = \frac{\mathfrak{V}^2 + c_1(\mathfrak{V})}{2} < 0$. In this case the curve V is obstructed and this influences the dimension of the spaces we consider. Generically, for a class A, we can avoid this issue; if $A \neq \mathfrak{V}$, then we simply have no curves in this class generically if $d_A < 0$. However, this does not apply to the class $\mathfrak{V}$ as we are free to choose and fix a hypersurface V at the start, so our choice may be non-generic in the sense that $d_{\mathfrak{V}} < 0$.

The following issue must also be addressed: Submanifolds which have components that lie in the hypersurface V. This is of interest in particular with a view towards a degeneration formula. In this paper, we will restrict ourselves to such classes A and hypersurfaces V which prevent this from occurring. This is addressed in Section 2.5, in particular, in 2.5.3 we state several general assumptions which we will work with in the rest of the paper: Either $d_{\mathfrak{V}} \geq 0$ or $g(V) > g(A)$, i.e. V is generic. Although these assumptions are rather annoying, in practice one can choose along which submanifold to sum or cut with some flexibility. Hence we should be able to generically avoid the problem of submanifolds lying in V with some surety.

Having understood the behavior of embedded curves under deformations of the almost complex structure (Section 3.2), we proceed to understand the properties of the space of relative *connected* submanifolds $\mathcal{K}_V(J, \mathcal{I}_A)$ for fixed initial data I_A. Under the general assumptions 2.5.3, we show in Section 4 that $\mathcal{K}$ is a smooth finite manifold which behaves well under deformations of the almost complex structure, the initial data and the symplectic structure on X. The greater part of Section 4 addresses the compactness of $\mathcal{K}_V$. Our assumptions allow us to avoid the complicated issue of curves with components lying in V when $d_{\mathfrak{V}} < 0$; in this case we cannot necessarily avoid the strata containing such curves via a genericity argument, see Lemma 4.2.2.

After finding a suitable set of almost complex structures and initial data such that the spaces $\mathcal{K}_V(A, J, \mathcal{I}_A)$ have the desirable properties, we proceed to define a number $Ru^V(A, [\mathcal{I}_A])$. After making precise this definition, we show the invariant properties of $Ru^V(A, [\mathcal{I}_A])$: It does not depend on the particular choice of J or $\mathcal{I}_A$, but rather the class $[\mathcal{I}_A]$ and the deformation class of (X, V, ω), see Theorem 5.3.1.

At this stage, we will have defined a basic invariant which counts connected non-multiply toroidal curves for generic V. From here, one can proceed in different directions. A natural inclination would be to attempt to remove the conditions placed on A and V. More general choices of V will involve different methods and will be described in a further paper. A relative Taubes invariant, giving a count of multiply toroidal A relative to V, is discussed in Section 6. Our results on generic almost complex structures in the relative setting and arguments in [61], show that the invariant defined by Taubes needs no modification in the relative setting.

On the other hand, extending the basic invariant to disconnected curves and thereby developing a relative version of the Gromov-Taubes invariant is the content of Section 7. We show that this invariant can be written as a product of the connected invariants, hence the invariant properties follow immediately.

A further direction would be to incorporate rim-tori. In [22], Ionel and Parker define a relative Gromov-Witten invariant which takes into account so called rim-tori structures. Including such structures into our count, we construct a more refined invariant than $Ru^V(A, [\mathcal{I}_A])$. In Section 5.5.1 we review their rim-tori constructions and define $Ru^V(\hat{A}, [\mathcal{I}_A])$. It is expected that the refined invariant defined in Section 5.5 can distinguish isotopy classes of hypersurfaces in the class $\mathfrak{V}$.

1.3 Relative Symplectic Cone

Given an oriented smooth manifold M known to admit symplectic structures, one would like to know which cohomology classes $\alpha \in H^2(M, \mathbb{R})$ can be represented by an orientation compatible symplectic form $\omega \in \Omega^2(M)$. We shall always restrict ourselves to symplectic forms which are compatible with the fixed orientation of the manifold M. This leads naturally to the definition of the symplectic cone:

$$\mathcal{C}_M = \{\alpha \in H^2(M) \mid [\omega] = \alpha,\ \omega \text{ is a symplectic form on } M\}. \tag{1.1}$$

In dimension 4, the symplectic cone has been determined in the following cases:

- S^2-bundles ([45]),
- symplectic T^2-bundles over T^2 ([13]),
- all $b^+ = 1$ manifolds ([34], see also [44], [2]),
- minimal manifolds underlying a Kähler surface with Kodaira dimension 0 ([32]) (A smooth 4-manifold M is said to be minimal if it contains no exceptional class, i.e. a degree 2 homology class represented by a smoothly embedded sphere of self intersection -1.), and

- Friedl and Vidussi (see [8] and [9]) determined the symplectic cone of a product S^1- bundle over any 3-manifold or a S^1-bundle over a graph manifold in terms of the Thurston norm ball of the 3-manifold.

Clearly, $\mathcal{C}_M \subset \mathcal{P}_M$, where $\mathcal{P}_M$ is the cone of classes of positive squares in $H^2(M, \mathbb{R})$. Amazingly, when M is a minimal and in the list above, the symplectic cone $\mathcal{C}_M$ is actually equal to $\mathcal{P}_M$. In particular, it holds for any M underlying a minimal Kähler surface with $p_g = 0$ or Kodaira dimension 0.

In general, $\mathcal{C}_M$ is smaller than $\mathcal{P}_M$, as there are constraints coming from the Seiberg-Witten basic classes. This is a consequence of Taubes' remarkable equivalence between Seiberg Witten invariants SW and Gromov invariants Gr ([61]). As exceptional classes and the canonical class of any symplectic structure all give rise to SW basic classes, there are corresponding constraints on $\mathcal{C}_M$.

As stated previously, there are many ways to explicitly construct new symplectic manifolds. Common among most of these methods is that some type of surgery is performed with respect to a codimension 2 symplectic submanifold V. It is natural to ask how the symplectic forms on the new manifolds relate to those on the constituent manifolds. This leads naturally to the notion of the relative symplectic cone $\mathcal{C}_M^V$ defined in Section 8. As examples, we consider T^2 fibrations over T^2 (see [13]) and manifolds with $b^+ = 1$. These are of interest, as we will consider T^2 fiber sums in the following sections.

To determine the relative cone in the $b^+ = 1$(Section 8.4), we will use the genericity results for almost complex structures J which make V pseudoholomorphic found in Section 3.2. They show that the set $\mathcal{J}_V$ of such almost complex structures J is rich enough to allow deformations of pseudoholomorphic curves, however in this example we are not restricted as much as in the GRT-setup.

The fiber sum of symplectic manifolds X and Y along symplectic embeddings of a codimension 2 symplectic manifold V, denoted $M = X\#_V Y$, as defined by Gompf ([14]) and McCarthy-Wolfson ([43]), and its inverse operation, the symplectic cut, defined by Lerman ([26]), are briefly described in Section 9.

We then proceed to show that the sum and cut operations naturally describe a cone $\mathcal{C}^{sum}$ of sum forms in terms of the relative cones of X and Y with respect to V. We also observe that in the case V having trivial normal bundle, $\mathcal{C}^{sum}$ is actually a subcone of the relative cone $\mathcal{C}_M^V$.

Furthermore, under some topological restrictions on the sum $M = X\#_V Y$ and the respective relative cones $\mathcal{C}_*^V$, we show that the *relative* symplectic cone $\mathcal{C}_M^V$ is actually equal to this subcone.

What does this imply for the symplectic cone of M? Notice that for a minimal T^2-fibration, the canonical class is proportional to the fiber class. Thus the relative symplectic cone, which is of course contained in the symplectic cone, is essentially equal to the symplectic cone. This strategy applies perfectly to fiber sums where one summand is a product T^2-fibration. This allows us to apply our construction to the following:

If the smooth manifold M underlies a minimal Kähler surface, a basic fact ([64], [11]) is that all symplectic structures on M have the same canonical class up to sign. Denote and fix one such choice $-c_1(M)$. Due to the SW constraints, for any minimal Kähler surface with $p_g > 0$, we have $\mathcal{C}_M \subset \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$. In light of this beautiful fact the following conjecture was raised:

Conjecture 1.3.1. *(Question 4.9, [32]) If M underlies a* **minimal** *Kähler surface with $p_g > 0$, the symplectic cone $\mathcal{C}_M$ is equal to $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$.*

We define $\mathcal{P}^\alpha = \{e \in \mathcal{P}_M | e \cdot \alpha > 0\}$ for nonzero $\alpha \in H^2(M;\mathbb{R})$ and $\mathcal{P}^0 = \mathcal{P}_M$. As $\mathcal{P}^0 = \mathcal{P}_M = \mathcal{C}_M$ this conjecture is known to be true when M underlies a minimal Kähler surface with Kodaira dimension 0 (Prop. 4.10, [32], see also [39]).

In Section 10 we will show that this conjecture holds for certain manifolds underlying minimal Kähler manifolds with $p_g > 0$ and Kodaira dimension 1. Many such manifolds are T^2-fibrations and can be written as a T^2-fiber sum of manifolds with $p_g = 0$ or Kodaira dimension 0, possibly with singular or multiple fibers.

Friedl and Vidussi (see [8] and [9]) determined the symplectic cone of a product S^1- bundle over any 3-manifold or a S^1-bundle over a graph manifold in terms of the Thurston norm ball of the 3-manifold. Their results overlap ours for the product T^2-fibrations $T^2 \times \Sigma_g$.

Missing from the examples in Section 8 is the K3 surface. Hopefully this could be understood in a further paper, thus rounding off the known examples of symplectic manifolds with Kodaira dimension 0. Furthermore, the notion of the relative symplectic cone is useful in the symplectic birational geometry program (see the survey paper [36]).

Moreover, it is an important ingredient in understanding the symplectic blow-down procedure in dimension 6 ([37]).

This part can also be found in [6].

1.4 Open Questions and Research Problems

Part I is only the beginning of a larger program on invariants of embedded symplectic submanifolds for symplectic 4-manifolds. As we noted, the relative invariants in this paper are only defined if no curves can limit into V. If $d_{\mathfrak{Y}} < 0$, then the dimension of the strata of curves with components mapping into V can be larger than the expected dimension. Therefore, we must develop techniques to avoid these strata when defining an invariant. In GW-theory, this is generally achieved via a virtual neighborhood construction. How to apply such techniques here is still open.

The next step, is to develop sum formulas relating absolute invariants (Gr-invariants) with the relative invariants. This will lead to rubber calculus as developed by Maulik-Pandharipande [42] and possibly show that these embedded invariants depend only on the topology of the involved pairs of manifolds, as in [41].

In the past few years, a number of relative invariants for symplectic manifolds have been defined, for example relative Seiberg-Witten, relative Oszvath-Szabo and general relative Gromov-Witten invariants. Each highlights a different aspect of the underlying symplectic manifold (or of the pair (X, V)). A detailed comparison of these invariants should provide insight into symplectic structures and make computation of these various invariants easier. Furthermore, it would be interesting to understand the relation between embedded invariants, as we have constructed, and invariants of immersed curves (general Gromov-Witten invariants, not necessarily relative). For work on relative theories and product formulas, see [59], [55], [52], [51], as well as [30], [27] and [21].

Most enticing among these is the relative Seiberg-Witten invariant. Taubes showed, that Gr-invariants can be equated with SW-invariants, hence it is expected that relative SW-invariants and relative Gromov-Taubes invariants can be equated in a similar fashion. As there are sum formulas for SW-invariants, as well as other product type formulas, this would provide new insight into the structure of our relative invariants.

Moreover, SW-invariants, and hence Gr-invariants by Taubes' work, are invariants of the smooth structure!

Explicit calculations of relative invariants for a range of examples would be useful in understanding the issues above. In order to build up a collection of such examples, we would like to determine the relative Ruan invariants of K3 surfaces, building on work in [42]. This involves proving a number of vanishing results for relative embedded surfaces and has been started in Chapter 5.4. Moreover, calculating relative Ruan invariants of $\mathbb{P}^1$-bundles would be a necessary first step towards an absolute-relative correspondence.

As noted previously, missing from Part II is the determination of any relative cone for a K3-surface. Clearly this must be remedied. The issues involved center around the structure of the moduli space of K3-surfaces and the associated (refined) period map.

The results in Part II are also expected to play a role in the determination of the Kodaira dimension of symplectic fiber sums along spheres, this is work in progress [5]. The only interesting case to be considered is the rational blow down along a sphere of self-intersection -4. In order to determine the Kodaira dimension, we need to determine the signs of K_ω^2 and $K_\omega \cdot \omega$ for the blown down manifold. This will involve understanding the structure of the symplectic forms.

Chapter 2

Relative Submanifolds and their Behavior

In this section we will describe the framework needed to define a space of submanifolds of X which meet a fixed symplectic hypersurface V. We will precisely describe the constraining data on the submanifolds as well as how to understand this from the viewpoint of sections of the bundle N as was used in the absolute case by Taubes. In contrast to Gromov-Witten theory, we shall fix certain initial data and not view our invariant as an abstract map on cohomology, though we shall allow the initial data to move in homological families. This initial data shall determine how the submanifolds meet V, they can be viewed as representatives of insertions in Gromov-Witten theory.

Let X be a compact, connected 4-dimensional smooth manifold admitting symplectic structures and fix a connected symplectic hypersurface $V \subset X$. The symplectic form ω is a nondegenerate closed 2-form on X, its class will be denoted by $[\omega] \in H^2(X, \mathbb{R})$. Denote the space of ω-compatible almost complex structures J by $\mathcal{J}_\omega$. Given any $J \in \mathcal{J}_\omega$, denote the corresponding canonical class $-K_\omega = c_1(X, \omega) = c_1(X, J)$. Note the dependence on the choice of symplectic form.

For any homology class $A \in H_2(X, \mathbb{Z})$, define

$$d_A = \frac{A^2 - K_\omega \cdot A}{2}. \tag{2.1}$$

Denoting the homology class of V by $\mathfrak{V} \in H_2(X)$, we define

$$l_A = A \cdot \mathfrak{V}. \tag{2.2}$$

Underlying the constructions in the following sections is the symplectic hypersurface V. The choice of symplectic structure ω and almost complex structure J must preserve the symplecticity of V. The symplectic structure must be chosen such that ω is an orientation compatible symplectic structure on X and restricts to an orientation compatible symplectic structure on V. Denote the set of such forms by $\mathcal{S}_X^V$. The set of classes $[\omega] \in H^2(X)$ with this property is called the relative symplectic cone $\mathcal{C}_X^V$ (see [6]).

The almost complex structure J must be chosen such that V is a J-holomorphic submanifold.

Definition 2.0.1. *Let $\omega \in \mathcal{S}_X^V$. The set $\mathcal{J}_V \subset \mathcal{J}_\omega$ is defined to be the set of almost complex structures J making V pseudoholomorphic.*

This set of almost complex structures will form the basis for our calculations throughout this paper. That it is rich enough to allow for deformations of pseudoholomorphic curves will be shown in Chapter 3.2. We will also encounter the set $\mathcal{J}_V \times [\mathcal{I}_A]$; this product we endow with the product topology.

2.1 Contact Order

For any given $J \in \mathcal{J}_V$, the contact order between a J-holomorphic submanifold C and the fixed hypersurface V is given as follows (see [46] and Lemma 3.4, [22] for details): Both submanifolds can be viewed as J-holomorphic curves for the given almost complex structure $J \in \mathcal{J}_V$. Let $f : \Sigma \to C$ be a simple J-holomorphic map from a genus $g = g([C])$ Riemann surface to X having as its image the curve C. The genus $g([C])$ is given by the adjunction formula for the curve C. Consider a point of intersection $p \in V$ of C and V. Fix local coordinates $\{v\}$ in V and let x be local coordinate in normal direction. Then either $f(\Sigma) \subset V$ or there is an integer $s > 0$ and $a_0 \in \mathbb{C}$ such that

$$f(z, \overline{z}) = (p + O(|z|), a_0 z^s + O(|z|^{s+1})). \tag{2.3}$$

Lemma 2.1.1. *Assume that $C \not\subset V$ and let $f : \Sigma \to X$ and $f' : \Sigma' \to X$ be two simple J-holomorphic embeddings of C. Then $s = s'$.*

Proof. The maps f and f' have the same image, hence by Cor. 2.5.3, [50], there exists a holomorphic map $\phi : \Sigma \to \Sigma'$ such that $f = f' \circ \phi$. Inserting this relation into Eq. 2.3, we obtain the fact that $\phi(z)$ must vanish to order 1 at $z = 0$. A series expansion of the holomorphic map ϕ in local coordinates as given above, shows that it has leading term cz, $c \in \mathbb{C}$. Moreover, comparing the second terms from Eq. 2.3, provides $a'_0\phi(z)^{s'} + O(|\phi(z)|^{s'+1}) = a_0 z^s + O(|z|^{s+1})$ which implies that the leading term in $\phi^{s'}$ must match the leading term $a_0 z^s$. This implies $s = s'$.

$\square$

Definition 2.1.2. *The contact order of C and V at $p \in V$ for $C \not\subset V$ is defined to be s.*

This definition is very intuitive and fits nicely the standard picture of contact order. However, as seen in the previous Chapter, to define an invariant, we need to consider an evaluation mapping on sections of the normal bundle to a fixed embedded curve. In particular, we need to define contact order for sections of the normal bundle N. For that reason, we will adopt the picture presented in [27]. We describe the corresponding construction briefly and then show how to obtain an "evaluation " mapping from this picture.

Consider the normal bundle N of C. Denote the boundary of the bundle by ∂N. With a view towards the embedding of the disk bundle U constructed by Taubes, we can associate to the boundary a S^1 action, such that $\partial N/S^1 = V$. Let $x(t)$ denote an orbit of the S^1 action and call $x(kt)$ a k-periodic orbit for any integer k. Remove V from X, the resulting "punctured" manifold can be viewed locally in the neighborhood of the removed hypersurface V as $\mathbb{R} \times \partial N$. Any J-holomorphic curve $u : \Sigma \to X$ which contacts V can be viewed locally in this picture, and in local coordinates can be written as $u : \Sigma_p \to \mathbb{R} \times \partial N$ with $u = (a, u_V)$ from a punctured Riemann surface Σ_p. Li-Ruan showed, that as we approach the contact point with V, $u_V \to x(kt)$ for some k-periodic orbit, whereby k is the contact order of u as described previously. Let S_k denote the space of k-periodic orbits, note that we can identify S_k with V.

Let N_V be the normal bundle of the hypersurface V and consider a single intersection point z of the curve C and V. In a neighborhood of the point z trivialize the bundle N_V. On the trivialization $U \times F_z^V$ introduce the coordinates v and x as before Eq. 2.3. Consider the fiber F_z^C of the bundle N over the point $z \in C$. A section $\mathfrak{s} \in \Gamma(N)$ intersects F_z^C at a point which we can also identify as belonging to F_z^V over a point $z' \in U$ in the trivialization. Thus, in local coordinates (v, x) at the intersection point, we can assign a contact order to the section $\mathfrak{s}$ as given by Def 2.1.2. Moreover, by removing the intersection point $\mathfrak{s} \cap F_z^C$, we can apply the orbit construction and assign to the section a k-periodic orbit with the same contact order as the curve C. Thus we define a map

$$G_k : \Gamma(N) \to S_k \tag{2.4}$$

for fixed k determined by C. This will be used to define an evaluation map needed to define the relative invariant in Chapter 5.

2.2 Initial Data

A submanifold $C \subset X$ will be constrained by two types of data: First, we fix a set of geometric objects, i.e. points, curves, etc., which the submanifold must contact. Secondly, at each of the contact points with the fixed hypersurface V, we prescribe as well the contact order $s \in \mathbb{N}$ of the two submanifolds. This contact order will be defined more precisely in 2.1. We collect them in the initial data $\mathcal{I}_A$:

Definition 2.2.1. *The initial data $\mathcal{I}$ is defined as follows: Fix integers $d, l \in \mathbb{Z}$. For $d, l \geq 0$, choose decompositions $d_1 + d_2 = d$ and $l_1 + l_2 + l_3 = l$ with $d_*, l_* \geq 0$. Then $\mathcal{I}$ consists of the following sets:*

1. *$\Omega_{d_1} \subset X \backslash V$ consisting of d_1 distinct points,*
2. *Ω_{l_1} a collection of l_1 pairs $(x, s) \in V \times \mathbb{N}$ with all x distinct,*
3. *Γ_{d_2} a collection of d_2 disjoint 1-dimensional submanifolds of $X \backslash V$,*
4. *Γ_{l_2} a collection of l_2 pairs (γ, s) with γ a 1-dimensional submanifold of V and $s \in \mathbb{N}$, all γ pairwise disjoint,*

5. $\Upsilon = \Upsilon_V^{l_3}$ *a collection of* l_3 *pairs* (V, s) *of copies of the hypersurface* V *and* $s \in \mathbb{N}$.

If $d_* = 0$ *or* $l_* = 0$, *then the respective sets are the empty set. If* $d < 0$, *then* $\mathcal{I} = \emptyset$. *If* $l < 0$, *then* $\Omega_{l_1} = \Gamma_{l_2} = \Upsilon = \emptyset$.

For a class $A \in H_2(X)$, *we denote by* $\mathcal{I}_A$ *any set of initial data* $\mathcal{I}$ *with* $d = d_A$ *and* $l = l_A$.

The collection Υ represents insertions which offer no constraint on the intersection point of the submanifold and V other than a certain contact order at the intersection point.

Definition 2.2.2. *Initial data* $\mathcal{I}_A$ *will be called proper initial data for the class* A *if the following hold:*

- *If* $d_A \geq 0$, *then*
  1. $0 \leq d_1 \leq d_A$, $0 \leq d_2 \leq 2d_A$ *and* $2d_1 + d_2 \leq 2d_A$,
  2. $2d_1 + d_2 - l_2 - 2l_3 = 2(d_A - l_A)$ *and*
  3. $A \cdot \mathfrak{V} = \sum s_i$
- *If* $d_A < 0$, *then* $\mathcal{I}_A = \emptyset$.

This definition is motivated by viewing these sets as the analogue to cohomological insertions in classical Gromov-Witten theory. A dimension count shows that we cannot expect to have any submanifolds in class A satisfying the initial data $\mathcal{I}_A$ unless $2d_1 + d_2 - l_2 - 2l_3 = 2(d_A - l_A)$. Note further, that the condition imposed on the orders s_i constrains the value of l_1: $l_1 + l_2 + l_3 \leq A \cdot \mathfrak{V} = l_A$. Moreover, this condition also ensures that if $A \cdot \mathfrak{V} > 0$, then some l_* will be nonzero. The following observation will simplify the multiply toroidal case:

Lemma 2.2.3. *Let* $d_A = 0$. *The initial data* $\mathcal{I}_A$ *is proper if and only if*

1. $d_1 = d_2 = l_1 = l_2 = 0$
2. $l_3 = A \cdot \mathfrak{V}$ *and all values of* $s_i = 1$.

Proof. If $d_1 = d_2 = l_1 = l_2 = 0$ and $l_3 = A \cdot \mathfrak{V} = l_A$, then the first two conditions for proper hold. Moreover, $s_i = 1$ and $l_3 = l_A$ imply $A \cdot \mathfrak{V} = \sum s_i$.

The definition of proper shows that $d_A = 0$ is equivalent to $d_1 = d_2 = 0$. Furthermore, we obtain that

$$-l_2 - 2l_3 = -2l_A \leq -2l_1 - 2l_2 - 2l_3 \;\Rightarrow\; -l_2 \geq 2l_1 \;\Rightarrow\; l_1 = l_2 = 0$$

and thus $l_3 = l_A$. Then $s_i = 1$ follows from $A \cdot \mathfrak{V} = \sum_{i=1}^{l_3} s_i$.

□

This result can be reformulated as follows: If $d_A = 0$, then for proper initial data we have $s_i = 1$ for all contacts, the only contacts being allowed are such, that no constraints are imposed on a curve in class A. However, a curve with $A \cdot \mathfrak{V} \neq 0$ (topological intersection number!) will generically intersect the hypersurface V in exactly l_A points each with order 1. Hence this result simply states that a curve which has no restriction d_A will not allow any restrictions in its contact with V.

Our constructions will involve operators on a universal space parametrized by almost complex structures and initial data. We will only be interested in initial data which varies in a specified manner. This is made precise in the following definition:

Definition 2.2.4. *Fix a set of initial data $\mathcal{I}_A$. Define the class $[\mathcal{I}_A]$ as the set of initial data with the following properties:*

1. $\mathcal{I}_A \in [\mathcal{I}_A]$
2. *All initial data have the same values of d_*, l_* and s_*.*
3. *Corresponding elements in the sets Ω_*, Γ_* and Υ have the same values of s_i and lie in the same homology class.*
4. *The ordering of the elements in the Γ-sets of any initial data is the same as in the given set $\mathcal{I}_A$.*

If $\mathcal{I}_A$ is proper, then the class $[\mathcal{I}_A]$ is called proper.

Note that it is not necessary to give an initial set $\mathcal{I}_A$. Simply fixing the values of d_*, l_* and s_* as well as the relevant homology classes and an ordering will define a class $[\mathcal{I}_A]$. We will consider data from this viewpoint.

2.3 Non-Degenerate Submanifolds

Consider an embedded submanifold $C \subset X$ together with an integer m and fix $J \in \mathcal{J}_V$. We can view the normal bundle N of C in X as a complex bundle with complex structure induced by the almost complex structure on X. On the other hand, we can construct a disk bundle U over C with complex structure J_0 induced from the restriction of the almost complex structure on X to C by setting $J_0|_{\text{fiber F}} = J_{\pi(F)}$ with $\pi : U \to C$. Taubes has constructed an embedding of this disk bundle U into the normal bundle N which is uniquely associated to the submanifold C and which allows for a comparison of the two complex structures. This leads to two complex valued sections $\nu \in T^{0,1}C$ and $\mu \in T^{0,1}C \otimes N^{\otimes 2}$ associated to C. These define an operator

$$Ds = \overline{\partial}s + \nu s + \mu\overline{s} \tag{2.5}$$

which is a compact perturbation of the standard $\bar{\partial}$ operator (as defined by the complex structures on the domain and target) and hence is elliptic. Moreover, this operator is canonically associated to the submanifold C. Its kernel can be viewed as the tangent space to the pseudo-holomorphic embeddings of C into X in the space of all smooth embeddings of C in X. This leads to the definition of non-degenerate:

Definition 2.3.1. *Fix an almost complex structure $J \in \mathcal{J}_V$ and a class $A \in H_2(X)$ with $d_A \geq 0$. Choose initial data $\mathcal{I}_A$. Let C be a connected, J-holomorphic submanifold in X such that $\mathcal{I}_A \subset C$. C is non-degenerate if the following hold:*

1. *If $d_A = 0$, then cokernel$(D) = \{0\}$.*
2. *If $d_A > 0$, then $D \oplus ev_{\mathcal{I}_A}$ has trivial cokernel. Here $ev_{\mathcal{I}_A}$ is the evaluation map which takes a smooth section of N to its value over the data in $\mathcal{I}_A$.*

2.3.1 Non-Degeneracy for classes with $d_A < 0$

The definition of non - degeneracy as given in Def 2.3.1 allows only for classes with $d_A \geq 0$. In the relative setting, we must allow for a further case: Assume the hypersurface V is not generic, i.e. the class $\mathfrak{V}$ has $d_{\mathfrak{V}} < 0$. Then we could still choose $A = \mathfrak{V}$ and consider curves in this class relative to V. Definition 2.3.1 can be extended to include this case as follows:

Definition 2.3.2. *Fix an almost complex structure $J \in \mathcal{J}_V$. If $d_A < 0$, then a submanifold C representing the class $A \in H_2(X)$ is called non-degenerate if it is rigid and there exist no other J-holomorphic curves C' in the class A.*

We will show, that this only applies in the setting mentioned above. If $A \neq \mathfrak{V}$, then we will not be able to find a curve C in the class A for generic almost complex structures J. Note also, that in the relative setting, we allow only almost complex structures in $\mathcal{J}_V$ for the definition of nondegeneracy.

2.4 The Space of Relative Submanifolds

We now introduce the space of relative submanifolds $\mathcal{K}_V(A, J, \mathcal{I}_A)$ for non multiply toroidal classes A, i.e. classes which are not of the form $A = mT$ for some T with $T^2 = 0$ and $K_\omega \cdot T = 0$. This definition will be rather technical, however the general idea is simple: We want to consider all connected submanifolds C, which contact V in a very controlled manner. This is determined by the initial data $\mathcal{I}_A$ and we ensure that we contact V only once for every given geometric object with the required contact order. Moreover, the curve C shall meet each geometric object in the initial data $\mathcal{I}_A$. We make this precise in the following definition:

Definition 2.4.1. *Fix $A \in H_2(X)$ and a set of proper initial data $\mathcal{I}_A$. Assume that A is not multiply toroidal. Choose an almost complex structure $J \in \mathcal{J}_V$. Denote the set $\mathcal{K}_V = \mathcal{K}_V(A, J, \mathcal{I}_A)$ of connected J-holomorphic submanifolds $C \subset X$ which satisfy*

- *If $d_A > 0$, then C*
    1. *contains Ω_{d_1} and*
    2. *intersects each member of Γ^X exactly once.*
- *If $l_A = 0$, then $C \cap V = \emptyset$.*
- *If $l_A > 0$, then C*
    1. *intersects V locally positively and transversely,*
    2. *intersects V at precisely the l_1 points of Ω_l and*

3. *intersects each member of* Γ^V *exactly once.*

4. *The remaining* l_3 *intersections with* V *are unconstrained.*

5. *Each intersection is of order* s_i *given in the initial data* $\mathcal{I}_A$ *for this component.*

2.5 Convergence of Relative Submanifolds

At this point, we must make a slight restriction in the objects we will be able to handle. We would like that no submanifold C lies in V. This is for the following two reasons: First, we would like to prove a sum formula similar to results in [27] or [21]. This will be simplified considerably by the exclusion of components in V. Secondly, philosophically speaking, we are interested only in curves which genuinely intersect V, transversely and locally positively. In this Section we will provide sufficient conditions to ensure this phenomenon is excluded either directly or due to dimension reasons.

2.5.1 Elementary Restrictions

A simple method for ensuring no component maps non-trivially into V is to consider classes such that the genus $g(A)$ given by the adjunction equality satisfies

$$g(V) > g(A).$$

2.5.2 Convergence Behavior

In order to define a relative invariant, we will need to understand the compactness properties of the relative spaces $\mathcal{K}$. This will be done in Chapter 4. In order to simplify calculations in these sections, we consider here the behavior of curves descending into V under convergence and determine their index.

Any sequence of symplectic submanifolds will converge to a limit curve by Gromov compactness. However, this limit curve may have components mapping into V. We now describe how to handle such curves, this is described in detail in [49] and [18] (see also [27] and [29], [49] contains numerous examples of this construction).

The idea is to extend the manifold X in such a manner, that the components descending into V get stretched out and become discernible. This extension is achieved by gluing X along V to the projective completion of the normal bundle N_V. This completion is denoted by $Q = \mathbb{P}(N_V \oplus \mathbb{C})$ and it comes with a natural fiberwise $\mathbb{C}^*$ action. The ruled surface Q contains two sections, the zero section V_0, which has opposite orientation to V, and the infinity section V_∞, which is a copy of V with the same orientation both of which are preserved by the $\mathbb{C}^*$ action. The manifold $X\#_{V=V_0}Q$ is symplectomorphic to X and can be viewed as a stretching of the neighborhood of V. This stretching can be done any finite number of times. Therefore consider the singular manifold $X_m = X \sqcup_{V=V_0} Q_1 \sqcup_{V_\infty=V_0} \ldots \sqcup_{V_\infty=V_0} Q_m$ which as been stretched m times. This will provide the target for the preglued submanifolds which we now describe.

Any curve in X with components lying in V can be viewed as a submanifold in X_m consisting of a number components: Each such curve has levels C_i which lie in Q_i (denoting $X = Q_0$) and which must satisfy a number of contact conditions. C_i and C_{i+1} contact along V in their respective components (Q_i, V_∞) and (Q_{i+1}, V_0) such that contact orders and contact points match up. The imposed contact conditions on V from the initial data $\mathcal{I}_A$ are imposed on the level C_m where it contacts V_∞ of Q_m whereas the absolute data is imposed on C_0.

The submanifolds C_i, viewed as maps into X_m, must satisfy certain stability conditions. In X, these are the well known standard conditions on the finiteness of the automorphism group. For those mapping into Q_i, $i > 0$, we identify any two submanifolds which can be mapped onto each other by the $\mathbb{C}^*$ action of Q.

After constructing $\{C_i\}$ in X_m we obtain a genuine curve in X meeting V as prescribed by $\mathcal{I}_A$ by gluing along V in each level. If each level was embedded, then so will the glued curve be. To show that each level remains embedded, i.e. we obtain no nodes away from the sections in Q, will be part of the task of the later sections.

The homology class of the preglued curve is defined as the sum of the homology class of C_0 and the projections of the class of C_i into $H_2(V)$.

For each such curve C, it is possible to determine the index of the associated differential operator. This is of course of central importance when determining the dimension of the spaces $\mathcal{R}$. The following Lemma sums up the result.

Lemma 2.5.1. *(Lemma 7.6, [22]; [27]) Let C be a preglued submanifold with $m+1$ levels representing the class $A \in H_2(X)$ which meets the data in $\mathcal{I}_A$ in the prolongation*

X_m as described above. Then the index of C is

$$ind(C) = d_A - m. \tag{2.6}$$

2.5.3 General Assumptions:

In the following we assume that (X, V, ω) and $A \in H_2(X)$ have been chosen so that one of the following holds:

1. $d_{\mathfrak{V}} \geq 0$ (see Lemma 4.2.2) or
2. $g(V) > g(A)$.

Remark: These statements do not change under deformations of the symplectic form ω. Hence the restriction to a specific ω will not pose a serious hurdle when we show that the relative Ruan invariant is a deformation invariant.

2.6 Main Result

The next sections are devoted to the proof of the following Proposition:

Proposition 2.6.1. *Fix a class $A \in H_2(X)$ and a proper class $[\mathcal{I}_A]$. Assume the general assumptions 2.5.3 hold and A is not multiply toroidal. Then there exists a Baire subset of $\mathcal{J}_V \times [\mathcal{I}_A]$ such that*

1. *The set $\mathcal{K}_V(A, J, \mathcal{I}_A)$ is a finite set.*
2. *If $A \neq \mathfrak{V}$, then $\mathcal{K}_V(A, J, \mathcal{I}_A)$ is empty when $d_A < 0$.*
3. *If V is an exceptional sphere, then $\mathcal{K}_V(\mathfrak{V}, J, \emptyset) = \emptyset$*
4. *Every point $h \in \mathcal{K}_V$ has the property, that each C with is non-degenerate unless possibly if C is a torus with trivial normal bundle. In this case it is m-non-degenerate for all $m > 0$.*
5. *If $(J^1, \mathcal{I}^1_A)$ are sufficiently close to $(J, \mathcal{I}_A)$, then the sets $\mathcal{K}_V$ and $\mathcal{K}^1_V$ have the same number of elements.*

Remark: Prop. 2.6.1 does not follow immediately from Taubes' results, as the relationship between the sets of generic almost complex structures in $\mathcal{J}_\omega$ and $\mathcal{J}_V$ given in Prop. 7.1, [59] and Prop. 2.6.1 is completely unclear.

Chapter 3

Generic Almost Complex Structures

In this section we will show that the set $\mathcal{J}_V$ is rich enough to allow for deformations of embedded symplectic submanifolds. To do so, we will define a suitable universal space $\mathcal{U}$ and the set of connected submanifolds $\mathcal{K}$. We show that the set $\mathcal{K}$ can be described as the zero set of a suitable section $\mathcal{F}$ of a bundle $\mathcal{B}$ over $\mathcal{U}$ and that $\mathcal{F}$ behaves as expected at its zeros.

3.1 The Universal Model

Fix $A \in H_2(X)$ and a symplectic form $\omega \in \mathcal{S}_X^V$. Let Σ be a compact, connected, oriented 2-dimensional surface of genus $g = g(A)$ as defined by the adjunction formula. Let $J \in \mathcal{J}_V$ and consider d_A and l_A as defined in 2.1 and 2.2 resp. Let $s \in \mathbb{N}^l$ be a vector of length $l \leq l_A$ such that $\sum s_i = A \cdot \mathfrak{V}$. (Clearly, if $l > l_A$, then we cannot expect under generic conditions to have any submanifolds in class A.) When d_A and l_A are nonnegative, choose initial data $\mathcal{I}_A = (\Omega, \Gamma, \Upsilon)$ using the values in s. Introduce $\mathcal{K}_V(J, \mathcal{I}_A) = \mathcal{K}$ as the set of J-holomorphic submanifolds in X which

- are abstractly diffeomorphic to Σ,
- meet the data in $\mathcal{I}_A$ as described in Def 7.1.1 and
- have fundamental class A.

The space $\mathcal{K}$ is essentially $\mathcal{K}_V$, just that we have changed the viewpoint from abstract submanifolds to those diffeomorphic to a fixed Σ. For these reasons, a good understanding of the properties of $\mathcal{K}$ is necessary to prove Prop. 2.6.1.

Let $\mathcal{J}_V \times [\mathcal{I}_A]$ be the parameter space for the universal model to be defined below and corresponding to the class of initial data in $\mathcal{K}_V(J, \mathcal{I}_A)$.

A universal space $\mathcal{U}$ for $\mathcal{K}_V(J, \mathcal{I}_A)$ consists of Diff(Σ) orbits of a 4-tuple $(i, u, J, \mathcal{I}_A)$ with

1. $u : \Sigma \to X$ an embedding off a finite set $\mathcal{E}$ of points from a Riemann surface Σ such that $u_*[\Sigma] = A$ and $u \in W^{k,p}(\Sigma, X)$ with $kp > 2$,

2. i a complex structure on Σ and $J \in \mathcal{J}_V$,

3. $\mathcal{I}_A \in [\mathcal{I}_A]$ and $\mathcal{I}_A \subset u(\Sigma)$.

Note that every map u is locally injective.

The last condition needs some explaining: The initial data $\mathcal{I}_A$ consists of two types of sets: Sets contained in $X \backslash V$ and pairs consisting of points in V and an integer s. The first set should be contained in $u(\Sigma)$, meaning the image goes through the constraints on $X \backslash V$, meeting each curve in Γ_{d_2} only once. The second set should also be contained in the image $u(\Sigma)$, however, each point should have prescribed contact order s. Furthermore, the image should meet each element in Γ_{l_2} exactly once. These are exactly the conditions imposed in the definition of the space $\mathcal{K}$.

3.2 Generic Complex Structures in $\mathcal{J}_V$

We wish to show that $\mathcal{J}_V$ has a rich enough structure to allow for genericity statements for J-holomorphic curves. A portion of these results has appeared in an Appendix in [6].

If $A \neq \mathfrak{V}$, the genericity results are proven by the standard method: We will define a map $\mathcal{F}$ from the universal model $\mathcal{U}$ to a bundle $\mathcal{B}$ with fiber $W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes u^*TX)$ over $(i, u, J, \mathcal{I}_A)$ and show that it is submersive at its zeros. Then we can apply the Sard-Smale theorem to obtain that $\mathcal{J}_V^A$ is of second category. This will involve the following technical difficulty: The spaces $\mathcal{J}_V$ and any subsets thereof which we will consider are not Banach manifolds in the C^∞-topology. However, the results we wish to obtain are for smooth almost complex structures. In order to prove our results, we need to apply Taubes trick (see [61] or [50]): This breaks up the set of smooth almost complex structures into a countable intersection of sets, each of which considers only curves satisfying certain constraints. These subsets are then shown to be open and dense by arguments restricted to C^l smooth structures, where the Sard-Smale theorem is applicable. We will not go through this technical step but implicitly assume this throughout the section, details can be found in Ch. 3 of [50].

Lemma 3.2.1. *Let $A \in H_2(X, \mathbb{Z})$, $A \neq \mathfrak{V}$, and let $\mathcal{I}_A$ be a set of initial data. Denote the set of pairs $(J, \mathcal{I}_A)$ by $\mathfrak{J}$. Let $\mathcal{J}_V^A$ be the subset of pairs $(J, \mathcal{I}_A)$ which are non-degenerate for the class A in the sense of Def. 2.3.1. Then $\mathcal{J}_V^A$ is a set of second category in $\mathfrak{J}$.*

Note that the universal model excludes multiple covers of the hypersurface V in the case that $A = a\mathfrak{V}$ for $a \geq 2$, and we can thus assume that any map $u : \Sigma \to M$ satisfies $u(\Sigma) \not\subset V$ for this proof. In particular, V could be a square 0-torus and $A = a\mathfrak{V}$ for $a \geq 2$.

Proof. Define the map $\mathcal{F} : \mathcal{U} \to \mathcal{B}$ as $(i, u, J, \mathcal{I}_A) \mapsto \overline{\partial}_{i,J} u$. Then the linearization at a zero $(i, u, J, \mathcal{I}_A)$ is given as

$$\mathcal{F}_*(\alpha, \xi, Y) = D_u \xi + \frac{1}{2}(Y \circ du \circ i + J \circ du \circ \alpha) \tag{3.1}$$

where D_u is Fredholm, Y and α are variations of the respective almost complex structures. This is a map on $H_i^{0,1}(T_{\mathbb{C}}\Sigma) \times W^{k-1,p}(u^*TX) \times T\mathcal{J}_V$. The components of (α, ξ, Y) correspond to perturbations in the complex structure i, the image $u(\sigma)$ and J.

Consider $u \in \mathcal{U}$ such that there exists a point $x_0 \in \Sigma$ with $u(x_0) \in X \backslash V$ and $du(x_0) \neq 0$ (The second condition is satisfied almost everywhere, as u is a J-holomorphic map.). Then there exists a neighborhood N of x_0 in Σ such that

1. $du(x) \neq 0$,

2. $u(x) \notin V$ for all $x \in N$.

In particular, we know that the map u is locally injective on N. Furthermore, we can find a neighborhood in N, such that there are no constraints on the almost complex structure $J \in \mathcal{J}_V$, i.e. this neighborhood does not intersect V. More precisely, Y can be chosen as from the set of ω-tame almost complex structures with no restrictions given by V. Denote this open set by N as well.

Let $\eta \in$coker$\mathcal{F}_*$. Consider any $x \in N$ with $\eta(x) \neq 0$. Then Lemma 3.2.2, [50], provides a matrix Y_0 with the properties

- $Y_0 = Y_0^T = J_0 Y_0 J_0$ with J_0 the standard almost complex structure in a local chart and
- $Y_0[du(x) \circ i(x)] = \eta(x)$.

On N choose any variation Y of J such that $Y(u(x)) = Y_0$. Then define the map $f : N \to \mathbb{R}$ by $\langle Y \circ du \circ i, \eta\rangle$. Note that $f(x) > 0$ by definition of Y. Therefore, we can find an open set N_1 in N such that $f > 0$ on that open set. Using the local injectivity of the map u and arguing as in [50], we can find a neighborhood $N_2 \subset N_1$ and a neighborhood $U \subset M$ of $u(x_0)$ such that $u^{-1}(U) \subset N_2$. Choose a cutoff function β supported in U such that $\beta(u(x)) = 1$. Hence in particular

$$\int_\Sigma \langle \mathcal{F}_*(0, 0, \beta Y), \eta\rangle > 0 \tag{3.2}$$

and therefore $\eta(x) = 0$. This result holds for any $x \in N$, therefore η vanishes on an open set.

As we have assumed $\eta \in$coker$\mathcal{F}_*$, it follows that

$$0 \;=\; \int_\Sigma \langle \mathcal{F}_*(0, \xi, 0), \eta\rangle \;=\; \int_\Sigma \langle D_u \xi, \eta\rangle$$

for any ξ. Then it follows that $D_u^* \eta = 0$ and $0 = \triangle \eta + l.o.t.$. Aronszajn's theorem allows us to conclude that $\eta = 0$ and hence $\mathcal{F}_*$ is surjective.

Thus we have the needed surjectivity for all maps admitting x_0 as described above: $u(x_0) \notin V$ and $du(x_0) \neq 0$. As stated before, this last condition is fulfilled off a finite set of points on Σ. The first holds for any map u in class A as we have assumed that $A \neq \mathfrak{V}$.

Now apply the Sard-Smale theorem to the projection onto the last two factors of $(i, u, J, \mathcal{I}_A)$ (If $\mathcal{I}_A = \emptyset$ then only onto the J-factor.). $\square$

As we have seen in the above proof, for the class $A = \mathfrak{V}$ which may have representatives which do not lie outside of V, we must be careful. In particular, it is conceivable, that the particular hypersurface V chosen may not be generic in the sense of Taubes, i.e. the set $\mathcal{J}_V$ may contain almost complex structures for which the linearization of $\overline{\partial}_J$ at the embedding of V is not surjective. The rest of this section addresses this issue. We begin by showing that the cokernel of the linearization of the operator $\overline{\partial}_J$ at a J-holomorphic embedding of V has the expected dimension:

Let j be an almost complex structure on V. Define $\mathcal{J}_V^j = \{J \in \mathcal{J}_V \mid J|_V = j\}$ and call any J-holomorphic embedding of V for $J \in \mathcal{J}_V^j$ a j-holomorphic embedding.

Lemma 3.2.2. *Fix a j-holomorphic embedding $u : (\Sigma, i) \to (X, J)$ for some $J \in \mathcal{J}_V^j$. If $d_{\mathfrak{V}} \geq 0$, then there exists a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ such that for any $J \in \mathcal{J}_V^{g,j}$ the linearization of $\overline{\partial}_{i,J}$ at the embedding u is surjective. If $d_{\mathfrak{V}} < 0$, then then there exists a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ such that the hypersurface V is rigid in X.*

Let us describe the proof before giving the exact proof. We follow ideas of Section 4, [63]. We need to show that for a fixed embedding $u : \Sigma \to X$ of V the linearization $\mathcal{F}_*$ of $\overline{\partial}_{i,J}$ at u has a cokernel of the correct dimension for generic $J \in \mathcal{J}_V^j$. To do so, we will consider the operator $\mathcal{G}(\xi, \alpha, J) := \mathcal{F}_*(\alpha, \xi, 0)$ at $(i, u, J, \mathcal{I}_A)$. We will show that the kernel of the linearization $\mathcal{F}_*$ for non-zero ξ has the expected dimension for generic J and hence the linearization of $\overline{\partial}_{i,J}$ at u also has the expected dimension. Note also, that for any $J \in \mathcal{J}_V^j$, the map u is J-holomorphic.

What is really going on in this construction? The operator $\mathcal{F}$ is a section of a bundle $\mathcal{B}$ over $\mathcal{U}$, as described above. The linearization $\mathcal{F}_*$ is a map defined on $H_i^{0,1}(T_{\mathbb{C}}\Sigma) \times W^{k-1,p}(u^*TX) \times T\mathcal{J}_V$. In our setup, we fix the complex structure along V and do not allow perturbations of this structure on V. Hence we remove the infinite dimensional component of the domain of $\mathcal{F}_*$ and are left with a finite dimensional setup.

Further, we consider a map $\mathcal{U} \to \mathcal{J}_V^j$. In this map, we fix a "constant section" u over $\mathcal{J}_V^j$, i.e. we consider the structure of the tangent spaces along a fixed j-holomorphic map u while not allowing the almost complex structure j along V to vary. Note that it is j which makes u pseudoholomorphic. Hence fixing u is akin to considering a constant section in the bundle $\mathcal{U} \to \mathcal{J}_V^j$.

We are only interested in the component of the tangent space along this section, this corresponds to the tangent space along the moduli space $\mathcal{M} = \mathcal{F}^{-1}(0)$ at the point

(u, J). However, this is precisely the component of the kernel of $\mathcal{F}_*$ with $Y = 0$, i.e. the set of pairs (ξ, α) such that $\mathcal{F}_*(\xi, \alpha, 0) = 0$, which corresponds to exactly the zeros of $\mathcal{G}$.

When considering the zeros of the map $\mathcal{G}$ viewed over $\mathcal{J}_V^j$, we know from the considerations above that this is a collection of finite dimensional vector spaces. We may remove any part of these spaces, so long as we leave an open set, which suffices to determine the dimension of the underlying space. Hence, removing $\xi = 0$, a component along which we cannot use our methods to determine the dimension of the kernel, still leaves a large enough set to be able to determine the dimension of the moduli space $\mathcal{M}$.

For this reason, we want to show that the kernel of the linearization $\mathcal{F}_*$ for non-zero ξ,or equivalently the zero set of $\mathcal{G}$ for non-zero ξ, has the expected dimension $\max\{d_{\mathfrak{V}}, 0\}$ for generic $J \in \mathcal{J}_V^j$.

Proof. The operator $\mathcal{G}$ is defined as

$$W^{k-1,p}(u^*TX) \times H_i^{0,1}(T_{\mathbb{C}}\Sigma) \times \mathcal{J}_V^j \to L^p(u^*TX \otimes T^{0,1}\Sigma)$$

$$(\xi, \alpha, J) \mapsto D_u^J \xi + \frac{1}{2} J \circ du \circ \alpha$$

where the term $D_u^J = \frac{1}{2}(\nabla \xi + J \nabla \xi \circ i)$ for some J-hermitian connection ∇ on X, say for example the Levi-Civita connection associated to J.

Let (ξ, α, J) be a zero of $\mathcal{G}$. Linearize $\mathcal{G}$ at (ξ, α, J):

$$\mathcal{G}_*(\gamma, \mu, Y) = D_u^J \gamma + \frac{1}{2} \nabla_\xi Y \circ du \circ i + \frac{1}{2} J \circ du \circ \mu.$$

As stated above, we assume nonvanishing ξ, hence we can assume that $\xi \neq 0$ on any open subset. Let $\eta \in$coker $\mathcal{G}_*$. Let $x_0 \in \Sigma$ be a point with $\eta(x_0) \neq 0 \neq \xi(x_0)$. In a neighborhood of $u(x_0) \in V$ the tangent bundle TX splits as $TX = N_V \oplus TV$ with N_V the normal bundle to V in X. With respect to this splitting, the map Y has the form

$$Y = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with all entries J-antilinear and $b|_V = 0$, thus ensuring that V is pseudoholomorphic and accounting for the fact that we have fixed the almost complex structure along V. Thus $\nabla_\xi Y$ can have a similar form, but with no restrictions on the vanishing of components along V.

Assume η projected to N_V is non-vanishing Then we can choose

$$\nabla_\xi Y = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

at x_0 such that $B(x_0)[du(x_0) \circ i(x_0)](v) = \eta^{N_V}(x_0)(v)$ and $B(x_0)[du(x_0) \circ i(x_0)](\overline{v}) = \eta^{N_V}(x_0)(\overline{v})$ for a generator $v \in T^{1,0}_{x_0}\Sigma$ and where η^{N_V} is the projection of η to N_V. Then, using the same universal model as in the previous Lemma, we can choose neighborhoods of x_0 and a cutoff function β such that

$$\int_\Sigma \langle \mathcal{G}_*(0, 0, \beta Y), \eta \rangle > 0$$

and thus any element of the cokernel of $\mathcal{G}_*$ must have $\eta^{N_V} = 0$. An argument in [63] shows that the projection of η to TV must also vanish. Therefore the map $\mathcal{G}_*$ is surjective at the embedding $u : \Sigma \to V$.

Thus the set $\{(\xi, \alpha, J)|\mathcal{G}(\xi, \alpha, J) = 0,\ J \in \mathcal{J}^j_V,\ \xi \neq 0\}$ is a smooth manifold and we may project onto the last factor. Then applying Sard-Smale, we obtain a set $\mathcal{J}^{g,j}_V$ of second category in $\mathcal{J}^j_V$, such that for any $J \in \mathcal{J}^{g,j}_V$, the kernel of the linearization of $\overline{\partial}$ at non-zero perturbations ξ of the map u is a smooth manifold of the expected dimension. In the case $d_{\mathfrak{Y}} \geq 0$, this however implies that $\mathcal{F}_*$ at (i, u, J, Ω) is surjective. Therefore, we have found a set $\mathcal{J}^{g,j}_V$ of second category in $\mathcal{J}^j_V$ such that the linearization of $\overline{\partial}_{i,J}$ at u is surjective at all elements of $\mathcal{J}^{g,j}_V$.

If however $d_{\mathfrak{Y}} < 0$, then this kernel is generically empty. This implies the rigidity of the embedding u of V.

□

In the statement of our result in Lemma 3.2.2, we fix an embedding u of the hypersurface V. This is not quite precise, as we are actually fixing the equivalence class of u in $\mathcal{U}$ under orbits of the action of Diff(Σ). However, given any two embeddings $u : (\Sigma, i) \to (X, J)$ and $v : (\Sigma, i) \to (X, \tilde{J})$ of V for $J, \tilde{J} \in \mathcal{J}^j_V$, there exists a $\phi \in \mathrm{Diff}(\Sigma)$ such that $u = v \circ \phi$. Thus, a change of embedding u will not affect the outcome of Lemma 3.2.2.

For every almost complex structure j on V the previous results provide the following:

1. A set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ with the property that the linearization of the operator $\overline{\partial}$ at a fixed j-holomorphic embedding of V is surjective ($d_{\mathfrak{V}} \geq 0$) or is injective ($d_{\mathfrak{V}} < 0$).

2. Up to a map $\phi \in \mathrm{Diff}(\Sigma)$, there is a unique j-holomorphic embedding of V for all $J \in \mathcal{J}_V^j$.

Therefore, consider the following set:

$$\mathcal{J}_V^g = \bigcup_j \mathcal{J}_V^{g,j} \subset \bigcup_j \mathcal{J}_V^j = \mathcal{J}_V.$$

Note that $\mathcal{J}_V^g$ is actually a disjoint union of sets. The following properties hold:

1. The set $\mathcal{J}_V^g$ is dense in $\mathcal{J}_V$.

2. The linearization of the operator $\overline{\partial}$ at a fixed j-holomorphic embedding of V is surjective ($d_{\mathfrak{V}} \geq 0$) or is injective ($d_{\mathfrak{V}} < 0$) for any $J \in \mathcal{J}_V^g$.

3. Up to a map $\phi \in \mathrm{Diff}(\Sigma)$, there is a unique j-holomorphic embedding of V.

We can now state the final result concerning genericity that we will need:

Lemma 3.2.3. *1. $d_{\mathfrak{V}} \geq 0$: Let $\mathcal{J}_{\mathfrak{V}}$ be the subset of pairs $(J, \mathcal{I}_{\mathfrak{V}})$ which are non-degenerate for the class $\mathfrak{V}$ in the sense of Def. 2.3.1. Then $\mathcal{J}_{\mathfrak{V}}$ is dense in $\mathfrak{I}$.*

2. $d_{\mathfrak{V}} < 0$: There exists a dense set $\mathcal{J}_{\mathfrak{V}} \subset \mathcal{J}_V$ such that V is rigid, i.e. there exist no pseudoholomorphic deformations of V and there are no other pseudoholomorphic maps in class $\mathfrak{V}$.

Proof. To begin, we will replace the set $\mathcal{J}_V$ by $\mathcal{J}_V^g$ which is a dense subset, as seen from the previous remarks. Further, for any $(J, \mathcal{I}_{\mathfrak{V}})$, $J \in \mathcal{J}_V^g$, we have surjectivity or injectivity of the linearization at the embedding of V.

Consider the case $d_{\mathfrak{V}} \geq 0$. Fix a j on V. Then consider the set $\mathcal{J}_V^{g,j}$ provided by Lemma 3.2.2. The linearization at the embedding of V is surjective for any $J \in \mathcal{J}_V^{g,j}$. For any element $(i, u, J, \mathcal{I}_{\mathfrak{V}})$ of $\mathcal{U}$ with $u(\Sigma) \not\subset V$ representing the class $\mathfrak{V}$ and $J \in \mathcal{J}_V^{g,j}$, arguments as in the proof of Lemma 3.2.1 provide the necessary surjectivity. Therefore, there exists a further set $\mathcal{J}_{\mathfrak{V}}^{g,j}$ of second category in $\mathcal{J}_V^{g,j} \times \{$ initial data$\}$ such that any pair $(J, \mathcal{I}_{\mathfrak{V}}) \in \mathcal{J}_{\mathfrak{V}}^{g,j}$ is nondegenerate, i.e. any J-holomorphic curve $u(\Sigma)$ representing $\mathfrak{V}$ is non-degenerate in the sense of Def. 2.3.1.

Define $\mathcal{J}_{\mathfrak{V}} = \bigcup_j \mathcal{J}_{\mathfrak{V}}^{g,j}$. This is a dense subset of $\mathcal{J}_V^{g,j} \times \{$ initial data$\}$ such that any pair $(J, \mathcal{I}_{\mathfrak{V}}) \in \mathcal{J}_{\mathfrak{V}}$ is nondegenerate.

If $d_{\mathfrak{V}} < 0$, then restrict to $\mathcal{J}_V^g$ as well. Thereby we have already ensured that V is rigid. Now apply the proof of Lemma 3.2.1 to the universal model $\mathcal{U}$, which we modify to allow only maps $u : (\Sigma, i) \to (X, J)$ such that $u(\Sigma) \not\subset V$. Then we can find a set $\mathcal{J}_{\mathfrak{V}}$ of second category in $\mathcal{J}_V^g$ such that there exist no maps in class $\mathfrak{V}$ other than the embedding of V. $\square$

Chapter 4

The Structure of $\mathcal{K}_V(J, \mathcal{I}_A)$

The goal of this Chapter is to show that the spaces $\mathcal{K}_V(J, \mathcal{I}_A)$ are smooth, finite, compact spaces which behave well under perturbations of J and the initial data. These results will provide the foundation for the proof of Prop. 2.6.1. We begin with the smoothness of $\mathcal{K}_V(J, \mathcal{I}_A)$, this will follow almost directly from the results of the previous Chapter.

To show compactness, we will analyse the behavior of limit curves. We will show that for generic $(J, \mathcal{I}_A)$ the limit curve is always a smooth non-multiply covered embedded symplectic submanifold, with the possible exception of the multiply toroidal case. That case will be addressed separately at the end of this Chapter.

It is important to recall the general assumptions we made in 2.5.3, they hold for all of the calculations and statements in this Chapter.

4.1 Smoothness

The results of the previous section allow us to prove the following :

Lemma 4.1.1. *Fix $A \in H_2(X)$ not multiply toroidal and a class $[\mathcal{I}_A]$. There is a set of second category $U \subset \mathcal{J}_V \times [\mathcal{I}_A]$ such that for any pair $(J, \mathcal{I}_A) \in U$ the preimage under the projection*

$$(i, u, J, \mathcal{I}_A) \to (J, \mathcal{I}_A)$$

contains only simple embeddings ($\mathcal{E} = \emptyset$).

Proof. This follows from a dimension count on the index of the associated operator. $\square$

Moreover, a similar dimension count as well as the Sard-Smale Theorem applied to the projection $(i, u, J, \mathcal{I}_A) \to (J, \mathcal{I}_A)$ proves

Lemma 4.1.2. *Fix $A \in H_2(X)$ and a class $[\mathcal{I}_A]$. There is a set of second category $U \subset \mathcal{J}_V \times [\mathcal{I}_A]$ with the following properties: When a pair $(J, \mathcal{I}_A)$ is chosen from U, then*

1. *$\mathcal{K}_V(J, \mathcal{I}_A)$ is empty if $d_A < 0$ and $A \neq \mathfrak{V}$.*
2. *If $\mathcal{I}_A$ is proper and either $d_A \geq 0$ or $A = \mathfrak{V}$, then $\mathcal{K}_V(J, \mathcal{I}_A)$ is a smooth 0-dimensional manifold and each point is non-degenerate.*
3. *Assume A is not multiply toroidal. There is an open neighborhood of pairs in $\mathcal{J}_V \times [\mathcal{I}_A]$ such that every pair therein obeys the previous assertions and the number of points in $\mathcal{K}$ is invariant in this neighborhood.*

Remark: In particular, this result also holds for the class $\mathfrak{V}$. Similar results have been proven by Jabuka, see [23].

Due to this result, we will from now on **assume that either $d_A \geq 0$ or that $A = \mathfrak{V}$.**

4.2 Compactness

We would now like to show, that every sequence of submanifolds

$$\{C_m, J_m, (\mathcal{I}_A)_m\}$$

with $(\mathcal{I}_A)_m$ a proper set of initial data in a fixed class $[\mathcal{I}_A]$ for all m, $J_m \in \mathcal{J}_V$ and C_m a point in the corresponding $\mathcal{K}$ for every m has a subsequence that converges to a J-holomorphic submanifold C in class A provided that the limit point of $(J_m, (\mathcal{I}_A)_m)$, denoted by $(J, \mathcal{I}_A)$, is chosen from a suitable Baire set in $\mathcal{J}_V \times [\mathcal{I}_A]$.

We consider first some generic results concerning the behavior of the limit curve obtained by Gromov convergence. Consider a pseudoholomorphic curve $C = \cup C_i$ composed of embedded submanifolds subject to the restrictions imposed by a fixed set of initial data $\mathcal{I}_A$. Assume for the moment, that we have a J-holomorphic map $f : \Sigma \to X$

representing the class A such that its image is C. To any multiply covered component we assign its multiplicity m_i and replace the map $f : \Sigma_i \to X$ by a simple map $\phi_i : \Sigma_i \to X$ with the same image. The same is done to any two components with the same image. The result is a collection of tuples $\{(\phi_i, \Sigma_i, m_i)\}$ with the same image as f. Moreover, we can replace the pair (ϕ_i, Σ_i) by its image C_i. Furthermore, denoting $A_i = [C_i]$, we obtain $A = \sum_i m_i A_i$. We must allow for the possibility, that one of the $A_i = \mathfrak{V}$. We therefore consider the decomposition $A = \sum_i m_i A_i + m\mathfrak{V}$, now assuming that $A_i \neq \mathfrak{V}$ for all i.

If we are in the case $\mathfrak{V}^2 < 0$ and $d_{\mathfrak{V}} \geq 0$, then $\mathfrak{V}^2 = -1$, $d_{\mathfrak{V}} = 0$ and $g(\mathfrak{V}) = 0$. This follows from standard arguments:

Lemma 4.2.1. *Let $B \in H_2(X)$ with $B^2 < 0$ and $d_B \geq 0$. Then for generic $J \in \mathcal{J}_V$ there exist no embedded irreducible curves in class B unless $B^2 = -1$ and $g = 0$.*

Proof. Assume $d_B \geq 0$. This implies $K_\omega \cdot B \leq B^2 < 0$. Using the adjunction formula to determine $K_\omega \cdot B$ provides the estimate $B^2 \geq K_\omega \cdot B \geq 2g - 2 - B^2$ which leads to $g = 0$. By assumption $0 > B^2$, hence $B^2 = K_\omega \cdot B = -1$. The result now follows from Lemma 4.1.2. $\square$

As discussed in Section 2.5, the limit curve C may have components lying in V. The results of Chapter 3.2 and Lemma 2.5.1 allow us to simplify matters somewhat:

Lemma 4.2.2. *The dimension of the stratum containing curves with $m+1$ levels and meeting the data $\mathcal{I}_A$ is*

$$\textit{dim}(m\textit{+1 levels}) = \begin{cases} -m & d_{\mathfrak{V}} \geq 0 \\ -m + |d_{\mathfrak{V}}| & d_{\mathfrak{V}} < 0 \end{cases} . \tag{4.1}$$

This finally motivates the first of the assumptions in 2.5.3 as it shows that generically we can avoid any curves with higher levels if $d_{\mathfrak{V}} \geq 0$. Therefore, we will distinguish between generic V ($d_{\mathfrak{V}} \geq 0$) and nongeneric V ($d_{\mathfrak{V}} < 0$) in this section. Moreover, if V is nongeneric, then we will need to treat carefully the case $A_i = \mathfrak{V}$.

Let us consider briefly the case $A = \mathfrak{V}$ and $d_{\mathfrak{V}} < 0$: Then the results in Chapter 3.2 show, that for generic almost complex structures, the only curve in class A is V. Therefore, $\mathcal{K}_V(J, \mathcal{I}_A = \emptyset) = \{V\}$ and hence of course compact. Due to the remark preceding Lemma 4.2.1 we consider the following two cases:

- $d_{\mathfrak{Y}} \geq 0$ and $\mathfrak{Y}^2 \geq 0$ and
- $d_{\mathfrak{Y}} < 0$ or $\mathfrak{Y}^2 = -1, d_{\mathfrak{Y}} = 0$.

4.2.1 $d_{\mathfrak{Y}} \geq 0$ and $\mathfrak{Y}^2 \geq 0$

Consider a sequence $\{C_m, J_m, (\mathcal{I}_A)_m\}$. Then Gromov compactness gives us a finite set of data $\{(\varphi_i, \Sigma_i, m_i, \mathcal{I}_i)\}$ with Σ_i a connected compact Riemann surface, φ_i a J-holomorphic map from Σ_i to X which is an embedding off of a finite set of points which may map into V or contact V in accordance with the data given by $\mathcal{I}_i$ and $m_i \in \mathbb{N}$. We choose $(J, \mathcal{I}_A)$ from a Baire set as discussed in the previous section, such that $\varphi_i(\Sigma_i) \cap \varphi_j(\Sigma_j)$ is finite for $i \neq j$ in accordance with the intersection product on homology for the classes A_j, A_i. Furthermore, denoting the push forward of the fundamental class of Σ_i by A_i, we obtain $A = \sum m_i A_i$ and the image $\cup\varphi_i(\Sigma_i)$ is connected and contacts all the data in $\mathcal{I}_A$. Moreover, $\cup \mathcal{I}_i = \mathcal{I}_A$. This implies that $\sum_i d_k^i = d_k$ $(k \in \{1, 2\})$ and $\sum_i l_k^i = l_k$ $(k \in \{1, 2, 3\})$. Further, we may assume that $2d_1^i + d_2^i - l_2^i - 2l_3^i \leq 2(d_{A_i} - l_{A_i})$ for each set $\mathcal{I}_i$. This condition simply states that the dimension of the moduli space is larger than the degrees of the insertions, thus guaranteeing that curves exist. An inequality in the opposite direction would provide too many constraints on curves in class A_i, thus effectively ruling out the existence of such a curve for generic J.

Consider any two pairs of points on V with a prescribed contact order given in the initial data. These must stay separate in the limit, as all of the initial data $\{\mathcal{I}_A\}_m$ lies in the same proper class for all m as does the limit set $\mathcal{I}_A$. Moreover, the components they lie in cannot limit to a multiple cover of the same curve, otherwise this component would need to meet more points than given by l_{A_i}. In particular, this implies that $\sum_i l_{A_i} \geq l_A$.

The properness of the initial data $\mathcal{I}_A$ as well as our estimate on the data in $\mathcal{I}_i$ allows the first estimate, the fact that $\cup \mathcal{I}_i = \mathcal{I}_A$ and $\sum_i l_{A_i} \geq l_A$ the final equality:

$$2 \sum d_{A_i} \geq \sum 2d_1^i + d_2^i - l_2^i - 2l_3^i + 2l_{A_i} = 2d_A. \tag{4.2}$$

We have thus shown, that there is a Baire set of pairs $(J, \mathcal{I}_A)$ such that

$$\sum_i d_{A_i} \geq d_A. \tag{4.3}$$

On the other hand, the following Lemma (primarily proven by Taubes [61], see also [6]) holds, we provide a proof suited to the situation at hand:

Lemma 4.2.3. *For generic pairs $(J, \mathcal{I}_A)$, either $\sum d_{A_i} < d_A$ or one of the following hold:*

1. *$\{C_m\}$ has a subsequence which converges to a J-holomorphic submanifold C with fundamental class A. Moreover, the limit curve C intersects V locally positively and transversely.*

2. *A is multiply toroidal. Furthermore, the data given by Gromov convergence consists of one triple $(\varphi_1, \Sigma_1, m_1)$ where Σ_1 is a torus, φ_i embeds Σ_1 and $A = m_1 A_1$. This includes the case $A = m\mathfrak{V}$ if V is square 0 torus.*

3. *Except possibly if $A = m\mathfrak{V}$ $(m \geq 1)$ and V is a square 0 torus, $C \not\subset V$.*

Proof. We decompose the class A as before, however distinguishing two types of classes as follows: Let B_i denote components with negative square, A_i components with non-negative square. Then write $A = m\mathfrak{V} + \sum m_i B_i + r_i A_i$. In the following we will allow the case $A = m\mathfrak{V}$.

Lemma 4.2.1 ensures, that we can find a generic set of almost complex structures such that $B_i^2 = -1$ and $g(B_i) = 0$ for all i. This in particular ensures, that $B_i \neq B_j$ for $i \neq j$ due to positivity of intersections. Moreover, if $A \cdot B_i < 0$ for some i, then we may assume that this holds for C_m with sufficiently large m in the sequence $\{C_m, J_m, (\mathcal{I}_A)_m\}$. However, each C_m is a connected J_m-holomorphic submanifold representing A, hence again by positivity of intersections, $A \cdot B_i < 0$ can only occur if $A = m_i B_i$ for some i. In particular, $A \neq \mathfrak{V}$. Again, by Lemma 4.2.1, it follows that $m_i = 1$ generically. Thus A would be represented by an embedded submanifold and the proof is done.

Assume in the following, that $A \cdot B_i \geq 0$ for all i. Lemma 4.1.2 shows, that we can find a generic set of pairs $(J, \mathcal{I}_A)$ such that $d_{A_i} \geq 0$ for each curve in class A_i, the same holds for $\mathfrak{V}$ by assumption.

Further, if $d_{A_i} \geq 0$ and $A_i^2 \geq 0$, then $d_{r_i A_i} \geq 0$ for any positive integer r_i:

$$0 \leq 2d_{A_i} \leq 2r_i d_{A_i} = -K_\omega \cdot (r_i A_i) + r_i A_i \cdot A_i \leq$$

$$\leq -K_\omega \cdot (r_i A_i) + r_i^2 A_i \cdot A_i = 2d_{r_i A_i}.$$

Note that this holds in particular for $m\mathfrak{V}$.

For such a generic choice of $(J, \mathcal{I}_A)$, let C be a connected curve representing A, which meets the initial data $\mathcal{I}_A$. Then

$$2d_A = -K_\omega \cdot (m\mathfrak{V}) + \sum_i -K_\omega \cdot (m_i B_i) + \sum_i -K_\omega \cdot (r_i A_i) +$$

$$+ m^2\mathfrak{V}^2 + \sum_i m_i^2 B_i^2 + \sum_i r_i^2 A_i^2 + 2\sum_i m\mathfrak{V} \cdot m_i B_i + 2\sum_i m\mathfrak{V} \cdot r_i A_i +$$

$$+ 2\sum_{i>j} m_i m_j B_i \cdot B_j + 2\sum_{i>j} m_i r_j B_i \cdot A_j + 2\sum_{i>j} r_i r_j A_i \cdot A_j$$

$$\geq 2md_{\mathfrak{V}} + (m^2 - m)\mathfrak{V}^2 + 2\sum_i r_i d_{A_i} + 2\sum_i m\mathfrak{V} \cdot r_i A_i + 2\sum_{i>j} r_i r_j A_i \cdot A_j +$$

$$+ \sum_i (m_i^2 - m_i) B_i^2 + 2\sum_{i>j} m_i m_j B_i \cdot B_j + 2\sum_{i>j} m_i r_j B_i \cdot A_j + 2\sum_i m\mathfrak{V} \cdot m_i B_i.$$

Consider the terms in the last line. Recalling that $B_i^2 = -1$, they can be rewritten as

$$\sum_i 2m_i A \cdot B_i - 2m_i^2 B_i^2 + (m_i^2 - m_i) B_i^2 \;=\; \sum_i 2m_i A \cdot B_i + m_i^2 + m_i \;\geq 0$$

and thus we obtain the estimate

$$2d_A \geq 2d_{\mathfrak{V}} + 2\sum_i d_{A_i}.$$

Hence either $d_A > d_{\mathfrak{V}} + \sum_i d_{A_i}$ or the following hold:

- $m_i = 0$ for all i, i.e. there are no components of negative square,
- $A_i \cdot A_j = 0 = A_i \cdot \mathfrak{V}$ for $i \neq j$,
- $m = 1$ or $d_{\mathfrak{V}} = 0$ and $\mathfrak{V}^2 = 0$ and
- $r_i = 1$ or $d_{A_i} = 0$ and $A_i^2 = 0$.

The limit curve is connected as we started with a connected curve. The second result shows that $A = rA_1$ or $A = m\mathfrak{V}$. The last two refine this to show that the curve C representing A is an embedded J-holomorphic submanifold with a single non-multiply covered connected component meeting the initial data $\mathcal{I}_A$ with $J \in \mathcal{J}_V$ or $d_A = 0$.

Thus we are done if $A \neq \mathfrak{V}$ and $d_A > 0$. Now consider the following cases:

1. $A \neq \mathfrak{V}$ and $d_A = 0$: The results above show that this implies either $A = m\mathfrak{V}$ with $m \geq 2$ and $\mathfrak{V}^2 = 0$, i.e. A is multiply toroidal or that $A \neq m\mathfrak{V}$ is (multiply) toroidal or A is represented by an embedded J-holomorphic submanifold as stated above. In the latter two cases we have $C \not\subset V$.

2. $A = \mathfrak{V}$: The limit curve is clearly an embedded curve under all circumstances, only its placement relative to V is an issue. If $C \neq V$, then we are done. If $C = V$, then either a generic choice of $\mathcal{I}_A$ will prevent this limit from occurring due to dimension reasons, see Lemma 2.5.1, or V is a square 0 torus.

If $A \cdot \mathfrak{V} > 0$, we can perturb C to be transverse to V, see [46], [47]. If $A \cdot \mathfrak{V} = 0$, then either the curves do not meet or we are in the toroidal case again.

□

4.2.2 The nongeneric Case and $\mathfrak{V}^2 = -1$, $d_{\mathfrak{V}} = 0$

Assume that $d_{\mathfrak{V}} < 0$ or that V is an exceptional sphere. Now the case $A_i = \mathfrak{V}$ must be given separate consideration. Lemma 3.2.3 resp. results on exceptional curves show, that we can find a generic set of almost complex structures, such that V is rigid and there are no other curves in class $\mathfrak{V}$. In the following, we choose only complex structures from this set.

As before, we can consider a sequence $\{C_m, J_m, (\mathcal{I}_A)_m\}$ and obtain the same result as above from Gromov compactness with one exception: The case $A_i = \mathfrak{V}$ with $m_i \geq 1$ must be considered closer: Even though we are working in the case $d_{\mathfrak{V}} < 0$, it is possible for a multiple class $m\mathfrak{V}$ to have $d_{m\mathfrak{V}} \geq 0$. For this reason, we will distinguish the following two objects:

1. Classes $A_i = \mathfrak{V}$ with $m_i > 1$ which correspond to components of the curve C in class $m\mathfrak{V}$, but which are NOT multiple covers of a submanifold in class $\mathfrak{V}$. If $\mathfrak{V}^2 < 0$, then positivity of intersections shows that any curve C can contain at most one component in class $m\mathfrak{V}$ for all m and this component must coincide with the manifold V (It could be a multiple cover of course.). This situation was studied in greater generality in [3]. Furthermore, if $\mathfrak{V}^2 \geq 0$ and a class $A_i = m\mathfrak{V}$ occurs, then the results of Lemma 3.2.1 apply. We may therefore assume , that $A_i^2 \geq 0$ in the following.

2. The specific "class" mV which corresponds to components which have as their image the hypersurface V. (Of course, the homology class associated to this "class" is $m\mathfrak{V}$. We wish to emphasize the distinction between the geometric object associated to this "class" and the previous one.) This can only occur if $d_{m\mathfrak{V}} < 0$.

Note further, that we can choose our almost complex structures such that the components corresponding to mV are rigid, while those in $m_i\mathfrak{V}$ are not. Such a decomposition is not necessary in the case $d_{\mathfrak{V}} \geq 0$, as V acts no differently than any other curve in the class $\mathfrak{V}$. In the current situation, the specific hypersurface V is singled out in the class, while all others can be excluded.

Our general assumptions in 2.5.3 rule out the existence of the specific class mV if $d_{\mathfrak{V}} < 0$. If V is an exceptional sphere, we have $d_{\mathfrak{V}} = 0$, so we need to argue differently: In this case $d_{m\mathfrak{V}} < 0$ if $m \geq 2$. Hence this rules out any classes of type (1). Any appearance of the class $\mathfrak{V}$ in a decomposition of A must therefore stem from a cover of V as there is but one representative of $\mathfrak{V}$. However, Lemma 2.5.1 shows that any such curve can be avoided through a sufficiently generic choice of J and initial data $\mathcal{I}_A$.

Hence, in all cases the calculations from the previous section apply. We again obtain an embedded curve or A is multiply toroidal.

The compactness results from this section are summarized in the following Lemma:

Lemma 4.2.4. *Fix $A \in H_2(X)$ and assume A is not (multiply) toroidal. Under the general assumptions 2.5.3, there is a set of second category $U \subset \mathcal{J}_V \times [\mathcal{I}_A]$ with the following properties: When a pair $(J, \mathcal{I}_A)$ is chosen from U, then*

1. *$\mathcal{K}_V(J, \mathcal{I}_A)$ is a finite collection of points.*
2. *Any submanifold in $\mathcal{K}_V(J, \mathcal{I}_A)$ meets V locally positively and transversely.*
3. *$C \not\subset V$*
4. *Assertion 1 is an open condition in $J_V \times [\mathcal{I}_A]$.*

Proof. Assertions (2) and (3) are proven in the previous Chapter by direct calculation. Moreover, these calculations show also that with the exception of multiply toroidal classes, the spaces $\mathcal{K}_V$ are compact. Together with Lemma 4.1.2 it follows that $\mathcal{K}$ is finite. Furthermore, a direct application of the implicit function theorem gives part 2 of the Lemma.

□

Chapter 5

The Relative Ruan Invariant

This section is devoted to the precise definition of the relative Ruan invariant. We first describe how to define a number associated to the spaces $\mathcal{K}_V(A, J, \mathcal{I}_A)$. In the following, we show that this is a deformation invariant of the symplectic structure ω on X.

Recall the assumptions in 2.5.3. The constructions in this section again assume that no curves can limit into the fixed hypersurface V. Moreover, throughout we assume that A is not multiply toroidal.

5.1 The number $Ru^V(A, [\mathcal{I}_A])$

For a fixed $A \in H_2(X)$ which is not multiply toroidal, consider the set $\mathcal{K}_V(J, \mathcal{I}_A)$ for generic pairs $(J, \mathcal{I}_A)$ and a fixed class $[\mathcal{I}_A]$. Then define the number

$$Ru^V(A, [\mathcal{I}_A]) = \sum_{C \in \mathcal{K}_V(J, \mathcal{I}_A)} r(C, \mathcal{I}_A). \tag{5.1}$$

5.2 The Definition of $r(C, \mathcal{I})$

This number is determined through an analysis of the behavior of the operator D (Eq. 2.5) under perturbation to a $\mathbb{C}$-linear operator. The methods used in this Section can be found in Kato [24] (Chaps. II and VII), McDuff-Salamon [50] (Appendix A) and the papers of Taubes in [61]. To fully understand this, consider the set $\mathcal{F}_{\mathbb{R}}$ of real linear Fredholm operators from a Banach space X to a Banach space Y of index n. This space can be decomposed into components $\mathcal{F}^k_{\mathbb{R}}$ consisting of those operators with kernel of dimension k. The minimal codimension of $\mathcal{F}^{k+1}_{\mathbb{R}}$ in $\mathcal{F}^k_{\mathbb{R}}$ is given by

$$n + 2k + 1. \tag{5.2}$$

Consider now a real analytic perturbation of D. More precisely, consider a path of operators $A_t : [0, 1] \to \mathcal{F}_{\mathbb{R}}$ with the following properties:

1. A_t depends real-analytically on the parameter t,
2. $A_0 = D$,
3. $A_t - A_0$ is a bounded 0th order deformation of D and
4. A_1 is $\mathbb{C}$-linear.

The third condition ensures that we stay in $\mathcal{F}_{\mathbb{R}}$, more precisely we even stay within the set of elliptic operators, as D is elliptic. Moreover, each of the operators has compact resolvent. The operator D is not $\mathbb{C}-$ but $\mathbb{R}-$linear, thus we view it formally as a map between the underlying real bundles. However, each of these bundles carries a holomorphic structure, hence we can consider an analytical extension of this path. This can be achieved by choosing a Sobolev completion of the domain and the range of the operator D making D a bounded operator. Then we can extend this real analytical perturbation to an analytical perturbation over a domain $U \subset \mathbb{C}$ containing $[0, 1]$ in its interior. This can be done such that the analytical perturbation preserves the third condition above. This is now a perturbation in $\mathcal{F}_{\mathbb{C}}$. Applying the results in Kato, in particular Sections II.1 and VII.1, we conclude that for the real analytic path A_t the following hold:

1. Either the kernel of A_t is nonempty for all t or it is nonempty for at most finitely many t. This result is relevant in the case $n = 0$ and $k = 0$: The path A_t will intersect the component with kernel of dimension ≥ 1 transversally. Moreover, the dimension count 5.2 shows that we can choose generic perturbations which intersect only the component with $k = 1$ but not any components with $k \geq 2$. Note that A_1 is $\mathbb{C}$-linear, hence the kernel at $t = 1$ cannot have dimension 1.

2. The dimension of the kernel of A_t can only change at a finite number of t. If $n \geq 1$ or $k \geq 1$, then we can again choose a generic perturbation which will not intersect any of the higher codimension components.

Hence, we can choose a generic real analytical path A_t such that for $(n,k) = (0,0)$ we have only a finite number of t at which the kernel has dimension 1 and for $n \geq 1$ or $k \geq 1$ we can ensure that the dimension of the kernel is preserved along the whole path over $[0,1]$.

We shall always assume that we have chosen an almost complex structure J such that D has trivial cokernel. Hence we consider only $n = k$ in the following. This implies that for a fixed value n, objects with a larger kernel will be of codimension $3k+1$ at least. This allows the following constructions:

- If $n > 0$, any continuous path connecting $D = A_0$ to a $\mathbb{C}$-linear operator and a generic real analytical path A_t as described above bound a disk such that every operator in the disk has the same size kernel and cokernel. In particular, we can use the kernel of A_1, which as a $\mathbb{C}$-linear operator carries a natural orientation induced by J, to uniformly orient all of the kernels in the disk. Moreover, as this orientation is defined by the almost complex structure J, we can choose any generic $\mathbb{C}$-linear operator and obtain the same orientation of $\ker(D)$.

- If $n = 0$, then two paths can differ by the number of crossing points of the codimension 1 stratum $\mathcal{F}^1_{\mathbb{R}}$. However, each curve connecting D and A_1 must have the same number of crossings mod 2 for a generically chosen A_1. Moreover, the almost complex structure orients the 0-dimensional kernel of any $\mathbb{C}$-linear operator, this orientation must be equivalent and is given by associating ± 1 to each point in the kernel. Hence any path connecting two $\mathbb{C}$-linear operators A_1 and A_1' must cross the stratum $\mathcal{F}^1_{\mathbb{R}}$ an even number of times, i.e. any point which has its orientation reversed must have it reversed again. Thus the number of crossings mod 2 is generically independent of the choice of A_1. This defines the spectral flow mod 2 for a real analytical path: Let N be the number of crossing points, then the spectral flow mod 2 for the path A_t is $(-1)^N$.

  Now consider any continuous path with N number of crossings. Then as in the case $n > 0$, any real analytical path A_t with N crossings of the same stratum bound a disk. Thus the spectral flow can be computed for a generic continuous path.

We have shown that this number is independent of the generic choice of continuous path and endpoint.

This is the general setup for our definition of $r(C, \mathcal{I})$. We assume in the following discussion, that $C \neq V$.

$\mathbf{d_A} = \mathbf{0}$: Choose the $\mathbb{C}$-linear operator A_1 such that it has trivial kernel and cokernel. This can be achieved generically because A_1 is $\mathbb{C}$-linear and the dimension count 5.2. Now define

$$r(C, \mathcal{I}) = (-1)^N \tag{5.3}$$

for a generic continuous path A_t connecting D and A_1. Note in particular, that we can use the path to define an orientation on the kernel of D determined by the almost complex structure.

$\mathbf{d_A} > \mathbf{0}$ To each point in the initial data $\mathcal{I}_A$ we will assign a space as follows:

- Ω_{d_1}: To each point $z \in \Omega_{d_1}$ associate the fiber $N|_z$ of the normal bundle N of the curve C. This defines a direct sum $\mathfrak{E}_{d_1} = \oplus_{z \in \Omega_{d_1}} N|_z$.

- Γ_{d_2}: Consider the intersection point of the curve C and an element $\gamma \in \Gamma_{d_2}$. Through a slight perturbation of γ, we may assume that γ intersects C in such a manner, that the quotient $N|_z/T\gamma$ is well defined. In other words, by perturbing slightly, we ensure that $T\gamma|_z$ is a line in the normal bundle fiber over z. Then associate to each γ the space $\mathfrak{g}_\gamma = N|_z/T\gamma$. Each $\mathfrak{g}_\gamma$ is oriented, hence the space $\mathfrak{E}_{d_2} = \oplus_{\gamma \in \Gamma_{d_2}} \mathfrak{g}_\gamma$ is an oriented ordered direct sum of oriented lines.

- Ω_{l_1}: This set consists of pairs, as will all of the following sets. They consist of a geometric datum and an intersection order. Recall the definition of the map G_k given by Def 2.4. To each pair (z, s) associate the space $N|_z \otimes S_s$. This space is again oriented. Hence $\mathfrak{E}_{l_1} = \oplus_{(z,s) \in \Omega_{l_1}} N|_z \otimes S_s$ is oriented.

- Γ_{l_2}: Assign the space $\mathfrak{E}_{l_2} = \oplus_{(\gamma,s) \in \Gamma_{l_2}} \mathfrak{g}_\gamma \otimes S_s$.

- Υ_{l_3}: In this set, we make no restrictions on the contact location with V. Note that S_k not only encodes contact order, but also encodes a location in the hypersurface

V, see the discussion in Section 2.1. Hence we define the space $\mathfrak{E}_{l_3} = \oplus_{(V,s)\in\Upsilon_{l_3}} S_s$. The corresponding evaluation map is similar to the one constructed in Section 2.1, we do not specify a fiber F_z^C.

Consider the linear map $H : \ker(D) \to \mathfrak{E}_{d_1} \oplus \mathfrak{E}_{d_2} \oplus \mathfrak{E}_{l_1} \oplus \mathfrak{E}_{l_2} \oplus \mathfrak{E}_{l_3}$, which is composed of evaluation maps and maps G_k defined in Def. 2.4. Choose a generic continuous path A_t and use this path to orient $\ker(D)$ as described above. Then H is a map between oriented vector spaces, moreover our calculations in the previous Sections show that this map is an isomorphism for suitably generic $(J, \mathcal{I})$. Define

$$r(C, \mathcal{I}) = \operatorname{sign}(\det(H)). \tag{5.4}$$

Lemma 5.2.1. *For generic $(J, \mathcal{I}) \in \mathcal{J}_V \times [\mathcal{I}]$ the number $r(C, \mathcal{I})$ is well-defined when $C \in \mathcal{K}_V(J, \mathcal{I})$.*

Proof. We need to show that the linear map has a determinant with a well-defined sign for generic $(J, \mathcal{I}) \in \mathcal{J}_V \times [\mathcal{I}]$. This follows from the genericity results obtained in the previous Sections and the homotopy properties discussed above.

□

Note that this definition agrees with Taubes' definition if $l_A = 0$ as well as in the case $\mathcal{I}_A = \emptyset$, albeit with a different underlying set of almost complex structures.

5.3 Invariant Properties of $Ru^V(A, [\mathcal{I}_A])$

Given the triple (X, V, ω) we denote its symplectic isotopy class by $[X, V, \omega]$. This class contains all triples $(X, \tilde{V}, \tilde{\omega})$ such that there exists a smooth one parameter family (X, V_t, ω_t) with

1. $(X, V_0, \omega_0) = (X, V, \omega)$,
2. $(X, V_1, \omega_1) = (X, \tilde{V}, \tilde{\omega})$,
3. $\omega_t \in \mathcal{S}_X^{V_t}$ and
4. $[\omega_t] = [\omega] \in H^2(X)$.

The triple (X, V, ω) is deformation equivalent to $(\tilde{X}, \tilde{V}, \tilde{\omega})$ if there exists a diffeomorphism $\phi : \tilde{X} \to X$ such that $[\tilde{X}, \phi^{-1}(V), \phi^*(\omega)] = [\tilde{X}, \tilde{V}, \tilde{\omega}]$.

Theorem 5.3.1. *The number $Ru^V(A, [\mathcal{I}_A])$ depends only on the deformation class of (X, V, ω), the class $A \in H_2(X)$, the initial class $[\mathcal{I}_A]$ and the ordering of the data in the sets Γ_*. In particular, it does not depend on a particular choice of $(J, \mathcal{I}_A)$.*

The proof of this Theorem will occupy the rest of this Section. To begin, we consider the deformation invariance. Consider a family of symplectic forms $\{\omega_t\}$ parametrized by $[0, 1]$. Let $(J_0, \mathcal{I}_A^0), (J_1, \mathcal{I}_A^1) \in \mathcal{J}_V \times [\mathcal{I}_A]$ be two pairs associated to ω_0 and ω_1 such that J_* is compatible with ω_*, $(J_*, \mathcal{I}_A^*)$ is suitably generic in the sense of the previous Sections and $[\mathcal{I}_A^1] = [\mathcal{I}_A^0]$.

Definition 5.3.2. *Define the set $\Gamma(\{\omega_t\}, [\mathcal{I}_A])$ to be the space of smooth sections $\mathfrak{s} = \{(t, J_t, \mathcal{I}_A^t)\}$ over $[0, 1]$ such that*

1. *J_t is ω_t-compatible,*
2. *$\mathcal{I}_A^t \in [\mathcal{I}_A]$ and*
3. *at $t = 0, 1$ we have the triples $(0, J_0, \mathcal{I}_A^0)$ and $(1, J_1, \mathcal{I}_A^1)$ with the associated pairs chosen above.*

The purpose of the following is to show that the number $Ru^V(A, [\mathcal{I}_A])$ is an invariant of the deformation class of the symplectic structure ω on X. To prove this, we will extend the universal space $\mathcal{U}$. Define a universal space $\mathcal{Y}$ similar to $\mathcal{U}$ consisting of Diff(Σ) orbits of a 5-tuple $(i, u, t, J, \mathcal{I}_A)$ with the following additional properties:

- For $t = 0, 1$, only the values for $(J, \mathcal{I}_A)$ chosen initially are allowed.
- For each corresponding pair of curves γ_0, γ_1 in the respective sets Γ_{d_2}, fix a smooth cobordism X_γ of γ_0 to γ_1. Repeat the same for the relative curves: For each corresponding pair of data $(\gamma_0, s), (\gamma_1, s)$ in the respective sets Γ_{l_2}, fix a smooth cobordism X_γ of γ_0 to γ_1. Require the intersection points of the image of u and the data in $\Gamma_{d_2}^t \cup \Gamma_{l_2}^t$ to lie on the submanifold $\prod_{\gamma \in \Gamma_{d_2} \cup \Gamma_{l_2}} X_\gamma \subset X^{d_2 + l_2}$.

For a fixed section $\mathfrak{s} \in \Gamma(\{\omega_t\}, [\mathcal{I}_A])$, define the space $\Xi_{\mathfrak{s}}$ via the pull-back diagram

$$\begin{tikzcd} \Xi_{\mathfrak{s}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi"'] \\ \mathfrak{s} \arrow[r, "T"'] & \mathcal{J}_V \times [\mathcal{I}_A] \end{tikzcd}$$

In other words, $\Xi_{\mathfrak{s}}$ is the collection of pairs (t, C) of $t \in [0,1]$ and submanifolds C, such that for each t, the submanifold $C \in \mathcal{K}(\mathfrak{s}(t))$ and meets the submanifold $\prod_{\gamma \in \Gamma_{d_2} \cup \Gamma_{l_2}} X_\gamma$ as described above.

Lemma 5.3.3. *Fix a smooth family $\{\omega_t\}$. Then there exists a Baire set $U \in \mathcal{J}_V \times [\mathcal{I}_A]$ such that any two points in U can be joined by a section $\mathfrak{s} : [0,1] \to \Gamma(\{\omega_t\}, [\mathcal{I}_A])$ such that $\Xi_{\mathfrak{s}}$ is a smooth 1-dimensional manifold. If A is not multiply toroidal, then $\Xi_{\mathfrak{s}}$ is compact.*

The proof of this Lemma will rely on the following result: Consider the following pull-back diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi"'] \\ \Gamma(\{\omega_t\}, [\mathcal{I}_A]) \arrow[r, "T"'] & \mathcal{J}_V \times [\mathcal{I}_A] \end{tikzcd}$$

where T is the evaluation map and π is the projection onto the last two components.

Lemma 5.3.4. *The map π is transverse to T.*

Proof. As in Chapter 3.2, we will need to distinguish the cases $A \neq \mathfrak{V}$ and $A = \mathfrak{V}$. In the former case, the results of Chapter 3.2 immediately show that the differential of π is surjective. In the latter case, we need to again be wary of $\xi = 0$, which is not in the image of $d\pi$. However, the differential of the evaluation map T can attain this value, hence again transversality is attained.

$\square$

We now turn our attention to the proof of Lemma 5.3.3:

Proof. The previous Lemma ensures that the pull-back space $\mathcal{X}$ is a smooth manifold. Consider the map $P : \mathcal{X} \to \Gamma(\{\omega_t\}, [\mathcal{I}_A])$. Applying the Sard-Smale Theorem to this map proves the smoothness of the space $\Xi_{\mathfrak{s}}$ as well as the claim on the dimension. The compactness of this space follows from the arguments on compactness in Chapter 4.

□

This shows that the spaces of connected submanifolds $\mathcal{K}_V(J, \mathcal{I})$ are invariant under deformation of the symplectic structure and do not depend upon the particular choice of J or $\mathcal{I}$. Moreover, it is clear that these spaces depend on the class $[\mathcal{I}]$ and the ordering of this class.

Hence, $Ru^V(A, [\mathcal{I}_A])$ is an invariant of the deformation class of the symplectic structure and otherwise depends on A, the hypersurface V and the ordered initial class $[\mathcal{I}_A]$.

Remark:

1. Thus far, the relative Ruan invariant is defined when V is suitably generic, either because $d_{\mathfrak{V}} \geq 0$ or $g(V)$ is large enough, and A is not multiply toroidal. The next step is to define the invariant in the remaining cases.

   Dropping the conditions on V will involve a construction with a virtual class argument and will be described elsewhere. We note here, that it is possible to apply the methods used above to define some type of an invariant even when V is not suitably generic. One such attempt could use the following lemma:

   Lemma 5.3.5. *Let (X, ω) be a symplectic 4-manifold, V a symplectic hypersurface. Let $A \in H_2(X)$ and assume that for some pair $(J, \mathcal{I}_A)$ the set $\mathcal{R}_V(A, J, \mathcal{I}_A) \neq \emptyset$. Assume further that $A \cdot \mathfrak{V} > A^2 \geq 0$. Then there exists a symplectic form $\tilde{\omega} \in \mathcal{S}_X^V$ such that $[\tilde{\omega}] \cdot \mathfrak{V} > [\tilde{\omega}] \cdot A$. Moreover, $\tilde{\omega}$ is a smooth deformation of ω through symplectic forms.*

   Proof. This is an application of Lemma 2.1 in [3]. Our assumptions imply the existence of a curve C in class A which intersects V transversally and locally positively in a finite number of points. Then [3] has shown, that there exists a symplectic form in the class $[\omega(t, s)] = t[\omega] + sA$, $(t > 0, s \geq 0)$, which makes V symplectic. In particular, for t small and s large enough we obtain

   $$[\omega(t, s)] \cdot \mathfrak{V} > [\omega(t, s)] \cdot A.$$

$\square$

This Lemma implies, that if $A \cdot \mathfrak{V} > A^2 \geq 0$ holds, then we can find a relative symplectic form on (X, V) such that the symplectic area of V is larger than the area of any curve in class A. This then precludes any components of a curve in class A lying in V and hence the spaces $\mathcal{K}_V$ are compact. However, determining the invariant properties of Ru^V will involve some further work.

If we consider multiply toroidal A, then Taubes has shown that already in the absolute case it is not possible to define a meaningful invariant allowing only connected curves. This first step towards an invariant for disconnected curves will be taken in Chapter 6.

2. Motivated by the work by Maulik and Pandharipande ([42]), it is natural to ask, whether the relative Ruan invariant depends not on the deformation class of (X, V, ω), but actually only on the class $\mathfrak{V}$. In particular, it would not depend on the precise choice of V. This would have interesting ramifications, see for example the result for K3 surfaces in Thm. 5.4.6. The result in [42] relies on the induced mapping $H^*(V) \to H^*(X)$, which in our 4-manifold setting is only interesting on the level of H^1. In particular, if we have no insertions in H^1 or if X is simply connected, then this condition would be trivially fulfilled for any representative of $\mathfrak{V}$ and hence this map would provide no way to distinguish between different deformation classes of hypersurfaces in class $\mathfrak{V}$. Furthermore, the rank of the skew-symmetric part of the restriction of the intersection pairing to the pull back of $H^1(X)$ to $H^1(V)$ was shown to depend only on the class A, see [31]. Thus, we are led to ask the following question:

   Question 5.3.6. *On a symplectic 4-manifold, can we find two hypersurfaces V_1 and V_2 representing the class $\mathfrak{V}$, such that the images of $H^1(V_1)$ and $H^1(V_2)$ in $H^1(X)$ differ?*

5.4 Examples

5.4.1 Genus 0 Curves

In this section we consider relative Ruan invariants in genus 0 relative to submanifolds V with $d_{\mathfrak{V}} \geq 0$. Unless otherwise stated, we assume that X is not rational or ruled. It was shown in [38] that all relative Gromov-Witten invariants of (X, ω) in genus 0 vanish if X is minimal. These invariants are concerned with connected curves representing the class A. Moreover, the proof in [38] shows, that for curves of genus 0 not only do the GW-invariants vanish, but for generic $(J, \mathcal{I}_A)$ the spaces underlying the invariants are empty. Hence all relative Ruan invariants are trivial as well for genus 0 classes if X is minimal. Moreover, the proof in [38] shows, that if X is non-minimal, then generically the only possible genus 0 connected curves are embedded -1-spheres. Hence, we consider only classes $A = E$ of an exceptional sphere.

For each exceptional curve E we can state the following: If $\mathfrak{V} \neq E$, then $\mathfrak{V} \cdot E \geq 0$ by positivity of intersections. Hence we have

$$\mathcal{K}_V(E, J, \mathcal{I}_E) = \begin{cases} (E, 1, \mathcal{I}_E) & \mathfrak{V} \cdot E \geq 0 \\ \emptyset & \text{otherwise} \end{cases} \tag{5.5}$$

and

$$\mathcal{I}_E = \begin{cases} \Upsilon_V^{E \cdot \mathfrak{V}} & E \cdot \mathfrak{V} > 0 \\ \emptyset & \text{otherwise} \end{cases}. \tag{5.6}$$

These results follow from the uniqueness of exceptional curves and Lemma 2.2.3. If $\mathfrak{V} = E$, then by Prop. 2.6.1, $\mathcal{K}_V(E, J, \emptyset) = \emptyset$, hence the relative Ruan invariant vanishes.

Assume that $E \neq \mathfrak{V}$. It then follows that

$$Ru^V(E, [\mathcal{I}_E]) = r(E, 1, \mathcal{I}_E). \tag{5.7}$$

The calculation of $r(E, 1, \mathcal{I}_E)$ relies on a path of operators, as described in Chapter 5, which corresponds to a change in complex structure on the normal bundles. This path can have only a finite number of points at which the kernel of the operator has dimension greater than 0. This however would imply the existence of more than one exceptional curve for certain almost complex structures, which we can rule out topologically. Hence $r(E, 1, \mathcal{I}_E) = 1$. Hence we have proven

Theorem 5.4.1. *Let X be a non-rational, non-ruled symplectic 4-manifold and V be a symplectic hypersurface with $d_{\mathfrak{V}} \geq 0$. Denote by E an exceptional curve in X with $E \cdot \mathfrak{V} \geq 0$. Then*

$$Ru^V(A, [\mathcal{I}_A]) = \begin{cases} 1 & \text{if } A = E, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

5.4.2 Algebraic K3-surfaces

Let X be a K3 surface, i.e. a surface with trivial canonical bundle and $b_1 = 0$. K3 surfaces have been extensively studied and much is known about their moduli. We review briefly some facts and introduce notation, details can be found in [1].

For a K3 surface, the Betti numbers take the values $b_1 = 0$ and $b_2 = 22$. The group $H^2(X, \mathbb{Z})$ is an even unimodular lattice with a quadratic form q given by the intersection pairing. This pairing has signature $(3, 19)$ and hence is isomorphic to the pairing $(\cdot, \cdot)$ given by $L = 3H \oplus 2(-E_8)$. Fix a lattice isomorphism $\phi : (H^2(X, \mathbb{Z}), q) \to L$, a K3 surface with a fixed choice of ϕ is called a marked K3 surface. The period of a marked K3 surface X is given by a choice of $[J] \in L \otimes \mathbb{C} \cong H^2(X, \mathbb{C})$ such that $\phi_{\mathbb{C}}^{-1}([J])$ generates $H^{2,0}(X, \mathbb{C})$, where $\phi_{\mathbb{C}}$ is the extension of ϕ to $L \otimes \mathbb{C}$. This also determines a complex structure on X, as we have the Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$. In other words, after fixing ϕ, the period is given by a point in $\mathbb{P}(L \otimes C)$. In particular, we define the period domain Ω to be

$$\Omega = \{[J] \in \mathbb{P}(L \otimes C) \mid ([J], [J]) = 0, \;\; ([J], \overline{[J]}) > 0\}.$$

The global Torelli theorem, originally proven by [56], states that every point in Ω corresponds to a marked K3 surface. We define a period map $\tau_1 : M_1 \to \Omega$ from an analytic, non-Hausdorff, smooth space M_1 parametrizing marked K3 surfaces. This map can be refined slightly as follows: Together with the period point $[J]$, we can choose a Kähler class $\kappa \in H^{1,1}(X, \mathbb{R})$. In particular, κ is characterised by the existence of a positive definite, with respect to q, plane $E([J]) \subset H^2(X, \mathbb{C})$, such that $q(\kappa, E([J])) = 0$ and $q(\kappa, \kappa) > 0$. Define the set

$$K\Omega = \{(\kappa, [J]) \in (L \otimes \mathbb{R}) \times \Omega \mid (\kappa, E([J])) = 0, \; (\kappa, \kappa) > 0\}$$

where we define $E([J])$ to be the span of $\{Re[J], \; Im[J]\}$. Then the set

$$(K\Omega)^0 = \{(\kappa, [J]) \in K\Omega \mid (\kappa, d) \neq 0 \; \text{ for any } d \text{ with } (d, d) = -2, \; ([J], d) = 0\}$$

is open in $K\Omega$ and the refined period map $\tau_2 : M_2 \to (K\Omega)^0$ from a smooth Hausdorff analytic space M_2 parametrizing marked pairs (X, κ) is an isomorphism. This map is defined by $\tau_2(X, \kappa) = (\phi_{\mathbb{C}}(\kappa), \tau_1(X))$. In particular, this isomorphism provides the following result:

Lemma 5.4.2. *$e \in \mathcal{P}_X$ is a Kähler class if and only if there is a q-positive definite $2-$plane U in $H^2(X, \mathbb{R})$ such that $e \perp U$, and $q(e, d) \neq 0$ for any integral class d in $H^2(X, \mathbb{Z})$ with $q(d, d) = -2$ and $d \perp U$.*

The following Theorem is referred to as the Lefschetz (1,1) Theorem:

Theorem 5.4.3. *(Thm IV.2.13, [1])*

$$Pic(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

In particular, for any class $A \in Pic(X)$ with $A^2 \geq -2$, either A or $-A$ is effective. Note that Pic(X) depends strongly on the complex structure, in fact, the rank of Pic(X), denoted $\rho(X, J)$, can take all values in $[1, 20]$. Therefore, consider the following Lemma:

Lemma 5.4.4. *Let X be the $K3$-surface and $A \in H^2(X, \mathbb{Z})$ with $A^2 \geq 0$. Then there exist complex structures on X such that A lies in the image of Pic(X) in $H^2(X, \mathbb{C})$.*

Proof. By Theorem 5.4.3, we need to show that $A \in H^{1,1}(X)$ for some decomposition given by a complex structure J.

If $A^2 > 0$, then A is a Kähler class and the complex structure determined by the plane U from Lemma 5.4.2 ensures $A \in Pic(X)$.

Consider now the case $A^2 = 0$. We shall make use of a nice feature for K3 surfaces, namely the existence of a hyperkähler metric g. Let X be a marked K3 surface with complex structure J determined by the marking and fix a Kähler class ω. Then there exists a unique hyperkähler metric g of class ω. Moreover, this metric induces a family of complex structures parameterized by the unit sphere of the imaginary quaternions and a corresponding family of Kähler forms. Denote this sphere by $S^2(g)$, and for each $u \in S^2(g)$, denote the corresponding Kähler form by ω_u. The span F of the ω_u is a $3-$dimensional positive-definite subspace of $H^2(X, \mathbb{R})$ (with a basis given by $\{\omega_I, \omega_J, \omega_K\}$). Let $F^\perp$ be the orthogonal complement of F, then $H^2(X, \mathbb{R}) = F \oplus F^\perp$ as $b^+ = 3$. In fact, $H^{1,1}_u = \mathbb{R}[\omega_u] \oplus F^\perp$. Moreover, $J \in S^2(g)$.

If $A^2 = 0$, then $A \cdot e > 0$ for some Kähler class e. Define the Grassmannian $Gr^+_{A,e}$ of positive definite 2-planes which are orthogonal to A and e. This Grassmannian is nonempty by Prop 3.1, [4]. Any element therein defines a complex structure such that $A \in H^{1,1}(X, \mathbb{R})$. $\square$

In the following, assume that V is a submanifold in a K3 surface X such that there exists a complex structure $J \in \mathcal{J}_V$ making (X, J) a complex surface and V a divisor. In particular, $\mathfrak{V}$ lies in the Néron-Severi lattice of (X, J). We will call such a curve V an algebraic curve. Lemma 5.4.4 shows that such submanifolds exist for any class with non negative square. In particular this implies that the Picard number $\rho(X, J) \geq 1$; K3-surfaces with this property are called algebraic K3-surfaces.

Assume V is an algebraic curve. Then there exists a point $J \in \mathcal{J}_V$ which corresponds to a marking of X. More precisely, let $\mathcal{J}^c_V \subset \mathcal{J}_V$ denote the subset of integrable almost complex structures. Define the set

$$\Omega_{\mathfrak{V}} = \{[J] \in \Omega \mid [J] \cdot \mathfrak{V} = 0\},$$

this describes those markings of X which make a submanifold in the class $\mathfrak{V}$ algebraic. Then our assumption implies $\mathcal{J}^c_V \cap \Omega_{\mathfrak{V}} \neq \emptyset$. Moreover, the following theorem shows that there exist markings such that the $\mathbb{Z}$-module $\langle \mathfrak{V} \rangle \subset L$ is the Néron-Severi lattice of X and that such points are dense in $\Omega_{\mathfrak{V}}$:

Theorem 5.4.5. *(Cor II.12.5.3, [58]; see also [53], [54]) Given a sublattice H of L of rank r such that the bilinear form restricted to H has signature $(1, r-1)$, $r \leq 20$, there exists an irreducible variety of dimension $(20 - r)$ parametrizing a family of K3-surfaces $\{X_t\}$ with markings such that H is a subset of the Néron-Severi group of any X_t. Moreover, for generic t, H is the Néron-Severi group of X_t.*

This allows us to determine the relative Ruan invariant relative to an algebraic hypersurface V:

Theorem 5.4.6. *Let X be the K3 surface and assume there exists a marking of X such that the class $\mathfrak{V}$ can be represented by an algebraic curve of genus 1 or higher. Then for generic algebraic representatives V of $\mathfrak{V}$, the invariant $Ru^V(A, [\mathcal{I}_A])$ vanishes for all $A \neq n\mathfrak{V}$. If $\mathfrak{V}$ is not toroidal, then the invariant vanishes if $n > 1$.*

Proof. We have seen, that there exists a dense $U \subset \Omega_{\mathfrak{V}}$ such that for any $[J] \in U$, we have $\rho(X, J) = 1$. This means that the only holomorphic curves are in class $\mathfrak{V}$. Hence for any embedded algebraic curve V representing $\mathfrak{V}$, we can be sure there are no relative embedded J-holomorphic submanifolds in any class other than possibly $A = n\mathfrak{V}$.

Now apply the vanishing principle in [25] to show that all the relative invariants $Ru^V(A, [\mathcal{I}_A])$ vanish.

□

Corollary 5.4.7. *The same holds true for any symplectic hypersurface V in class $\mathfrak{V}$ such that the deformation class of (X, V, ω) contains an algebraic representative of $\mathfrak{V}$.*

Remark:

1. The restriction on the genus in Theorem 5.4.6 ensures that $d_{\mathfrak{V}} = \mathfrak{V}^2 \geq 0$ is fulfilled. Hence we need not worry about higher level curves or non-embedded limits. The missing genus 0 case has been dealt with in the previous section.

2. Motivated by the results in [42], we expect there to be non-vanishing relative Ruan invariants of the form $Ru^V(\mathfrak{V}, [\mathcal{I}_{\mathfrak{V}}])$.

3. Recalling the discussion following the proof of Thm. 5.3.1, our results would show that all relative Ruan invariants with $A \neq \mathfrak{V}$ vanish, notwithstanding the particular choice of deformation class of V.

5.5 Refined relative Ruan Invariants

The purpose of this Section is to describe how two curves C and C' in $\mathcal{K}_V(A, J, \mathcal{I}_A)$ can be distinguished even though both curves lie in the same class A and meet the same set of initial data $\mathcal{I}_A$. This will be used in the next Section to define a refinement of the invariant $Ru^V(A, [\mathcal{I}_A])$. We will describe here the construction used in [22].

5.5.1 Rim Tori

The difference $C\#\overline{C'}$ of the curves C and C' lies not in X but in the open manifold X/V. More precisely, the class of the difference lies in the kernel $\mathfrak{R}$ of the map $H_2(X \backslash V) \rightarrow H_2(X)$. The key is to find an optimal space with which to describe this difference while keeping track of the data A and $\mathcal{I}_A$. To that end, let us fix notation: For a given class of initial data $[\mathcal{I}_A]$, let $V_{[\mathcal{I}_A]}$ denote the collection of all sets of pairs $((x_1, s_1), ..., (x_l, s_l))$ of intersection points in V and contact orders to be found in initial data in class $[\mathcal{I}_A]$. Define

$$V_A = \bigsqcup_{[\mathcal{I}_A]} V_{[\mathcal{I}_A]}$$

with the topology of the disjoint union. Note that this space has an induced ordering on each point $V_{[\mathcal{I}_A]}$ coming from the ordering on the class $[\mathcal{I}_A]$. Let $D(\epsilon)$ be an ϵ-disk bundle in the normal bundle to V. Then $X/\overline{D(\epsilon)}$ is diffeomorphic to X/V. Define the space

$$\hat{X} = [X/\overline{D(\epsilon)}] \cup S$$

where $S = \partial D(\epsilon)$. The manifold $\hat{X}$ is compact and, endowing S with the topology given by viewing it as a disjoint union of its fiber circles, we can consider the long exact sequence of the pair $(\hat{X}, S)$:

$$0 \to H_2(\hat{X}) \to H_2(\hat{X}, S) \to H_1(S) \to .$$

In the given topology for S, the set $H_1(S)$ can be viewed as the space of divisors on V, meaning the finite collection of points labeled with multiplicities and sign. This however is precisely the data in $\mathcal{I}_A$ relating to the intersection of the curves C and C' with the hypersurface V (the sign is always $+$). Note however, that $H_1(S)$ does not come with an ordering, it makes no distinction between data for curves meeting V in the same points with the same multiplicity but with a differing ordering on the contact data.

Combining this sequence with the map $\pi : H_2(\hat{X}, S) \to H_2(X)$ induced by the inclusion, which has as its kernel the set $\mathfrak{R}$, leads to the exact sequence

$$0 \to \mathfrak{R} \to H_2(\hat{X}, S) \to H_1(S) \times H_2(X). \tag{5.9}$$

There is a map from V_A to the set of divisors $H_1(S)$ which maps onto the set of effective divisors. This allows for the definition of the space $\mathcal{H}_V^X$ by the following pullback diagram:

$$\begin{tikzcd} \mathcal{H}_V^X \arrow[r] \arrow[d] & V_A \arrow[d] \\ H_2(\hat{X}, S) \arrow[r] & H_1(S) \end{tikzcd} \tag{5.10}$$

Combining 5.9 and 5.10 we obtain the fibration

$$\begin{tikzcd} \mathfrak{R} \arrow[r] & \mathcal{H}_V^X \arrow[d, "\mathfrak{r}"] \\ & H_2(X) \times V_A \end{tikzcd} \tag{5.11}$$

which allows us to lift a class A and its initial intersection data to a point in $\mathcal{H}_V^X$ which encodes the information on the intersection data as well as the class of the curve C in the kernel $\mathfrak{R}$. The procedure for a given curve C is as follows: Restrict the curve C to $X \backslash V$, lift to the space $\hat{X}$ and use the construction in [27] (see also the brief discussion in Section 2.1) to close the restricted curve C to a curve $\hat{C} \subset \hat{X}$. The class $[\hat{C}] \in H_2(\hat{X}, S)$ together with the intersection data from C defines a point in $\mathcal{H}_V^X$.

In order for this construction to be useful, we need a characterisation of the kernel $\mathfrak{R}$. This has been given in [22]: Let $\pi : S \to V$ be the projection map from the boundary of the ϵ-disk bundle to the hypersurface V. For every simple closed loop γ in V, $\pi^{-1}(\gamma)$ is a torus in S. Such tori are called rim tori and they generate $\mathfrak{R}$:

Lemma 5.5.1. *(Lemma 5.2, [22]) Each element in $\mathfrak{R}$ can be represented by a rim torus.*

The proof of this Lemma utilises the Gysin sequence for the oriented circle bundle $\pi : S \to V$ and the Meyer-Vietoris sequence of $(X, X \backslash V, V)$:

$$\to H_3(V) \to H_1(V) \xrightarrow{\Delta} H_2(S) \to H_2(V) \to$$

$$\to H_2(S) \xrightarrow{(\iota_*, \pi_*)} H_2(X \backslash V) \oplus H_2(V) \to H_2(X) \to$$

This leads to

$$\mathfrak{R} = \text{image } [i_* \circ \Delta : H_1(V) \to H_2(X \backslash V)]$$

which eliminates those rim tori which are homologous to zero in $X \backslash V$. We will call the set $\mathfrak{R}$ the set of rim tori in the following.

5.5.2 Refined relative Ruan Invariants

Rim tori allow us to differentiate curves lying in the same set $\mathcal{K}_V(A, J, \mathcal{I}_A)$ and thereby will allow us to refine the invariant $Ru^V(A, [\mathcal{I}_A])$. We first describe how this refinement works and then concern ourselves with the properties of this definition. Note that this refinement is not interesting for all classes $A \in H_2(X)$: If $A \cdot \mathfrak{V} \leq 0$, then our results show that either $\mathcal{K}_V(A, J, \mathcal{I}_A) = \emptyset$ or $A \cdot \mathfrak{V} = 0$.

To each point $h \in \mathcal{K}_V(A, J, \mathcal{I}_A)$ can be associated a class $\hat{A} \in \mathcal{H}_X^V$.

It is therefore possible to decompose the spaces $\mathcal{K}_V(A, J, \mathcal{I}_A)$ such that

$$\mathcal{K}_V(A, J, \mathcal{I}_A) = \bigsqcup_{\hat{A}} \mathcal{K}_V(A, \hat{A}, J, \mathcal{I}_A)$$

according to possible lifts of the point $(A, \mathcal{I}_A)$ under the map 5.11. This also serves as a definition of the spaces $\mathcal{K}_V(A, \hat{A}, J, \mathcal{I}_A)$. It is immediately clear that these spaces are finite and smooth for generic pairs $(J, \mathcal{I}_A)$. Moreover, the following Proposition, analogous to Prop. 2.6.1, holds:

Proposition 5.5.2. *Fix a class $A \in H_2(X)$ with $A \cdot \mathfrak{V} > 0$ and a proper class $[\mathcal{I}_A]$. Then there is a Baire subset of $\mathcal{J}_V \times [\mathcal{I}_A]$ such that*

1. *The set $\mathcal{K}_V(A, \hat{A}, J, \mathcal{I}_A)$ is a finite set.*
2. *Every point $h \in \mathcal{K}_V(A, \hat{A}, J, \mathcal{I}_A)$ has the property, that each C is non-degenerate.*
3. *If $(J^1, \mathcal{I}_A^1)$ is close to $(J, \mathcal{I}_A)$, then the sets $\mathcal{K}_V(A, \hat{A}, J, \mathcal{I}_A)$ and $\mathcal{K}_V(A, \hat{A}, J^1, {\mathcal{I}_A}^1)$ have the same number of elements.*

Proof. Only the last claim is not obvious from the results for $\mathcal{K}_V(A, J, \mathcal{I}_A)$. The issue is whether a small change in the data might cause rim tori to disappear or to be generated. However, this can be ruled out due to the fibration structure in 5.11. $\square$

The definition of the refined invariant now follows from the definitions in Section 5:

Definition 5.5.3. *The refined Ruan invariant for the symplectic hypersurface $V \subset X$ and the class $A \in H_2(X)$ and with initial data class $[\mathcal{I}_A]$ is denoted $Ru^V(\hat{A}, [\mathcal{I}_A])$ and is defined by*

$$Ru^V(\hat{A}, [\mathcal{I}_A]) = \sum_{C \in \mathcal{K}(A, \hat{A}, J, \mathcal{I}_A)} r(C, \mathcal{I}_A). \tag{5.12}$$

The following is trivial:

Lemma 5.5.4.

$$Ru^V(A, [\mathcal{I}_A]) = \sum_{\mathfrak{r}^{-1}A} Ru^V(\hat{A}, [\mathcal{I}_A]) \tag{5.13}$$

The invariant properties of $Ru^V(\hat{A}, [\mathcal{I}_A])$ are more subtle than in the non-refined case. Using cobordism arguments as in Section 5, it can be shown that $Ru^V(\hat{A}, [\mathcal{I}_A])$ is an invariant of the symplectic isotopy class $[X, V, \omega]$. Furthermore, deformations of only the symplectic structure ω also leave this number invariant. However, the numbers $r(C, \mathcal{I})$ depend on the orientation of the normal bundle of V as well as the almost complex structure on the normal bundle of V, hence it is unlikely that it is invariant under deformations of V. This question remains open.

Chapter 6

Relative Taubes Invariant for Tori with Trivial Normal Bundle

Taubes defined an invariant counting tori with trivial normal bundle in [60]. This invariant takes into account the bifurcation behavior of sequences of such tori. In this section, we show that this delicate count can be done in the relative settings without any modification of the Taubes invariant.

6.1 Behavior of Multiply Toroidal Classes

6.1.1 Non-Degeneracy of Multiply Toroidal Classes

Special consideration must be given to classes representing square 0 tori. This issue will occur throughout the following sections. For this reason, we make the following definition:

Definition 6.1.1. *A class $A \in H_2(X)$ is called multiply toroidal if*

- $A^2 = 0$,
- $K_\omega \cdot A = 0$ *and*
- *the class A is divisible, i.e. $A = kA'$ with $k > 1$.*

We call the class A toroidal if the first two conditions hold.

If C is a torus with trivial normal bundle, then we expand the definition of non-degeneracy:

Definition 6.1.2. *Fix an almost complex structure $J \in \mathcal{J}_V$. If C is a torus with trivial normal bundle, fix a positive integer $n \in \mathbb{Z}$ and call C n-non-degenerate if C' is non-degenerate for every holomorphic covering map $f : C' \to C$ of degree n or less.*

We can make this more precise: For any representatives of a (multiply) toroidal class, it is not possible to distinguish different values of k by marked points. Therefore, all such curves must be considered when constructing the invariant. Multiple covers of a torus with trivial normal bundle are classified by the fundamental group $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$. Consider a homomorphism $\rho : \pi_1(C) \to P_m$, P_m the permutation group on m letters. ρ defines, via a representation of P_m on $\mathbb{R}^m$, a m-plane bundle V_ρ over C. This will allow us to distinguish multiple covers of the base curve C in the normal bundle. We can naturally extend the operator D to the space of sections of $V_\rho \times N$. We can now make precise the definition of n-nondegenerate:

Definition 6.1.3. *Let C represent a (multiply) toroidal class. Fix $n \in \mathbb{Z}$. The curve C is n-nondegenerate if for all $m \in \{1, .., n\}$ and for all representations $\rho : \pi_1(C) \to P_m$ the operator D on the space $V_\rho \times N$ has trivial kernel.*

This definition ensures, that any curves which are counted and which stem from multiple covers of the curve C behave well. In particular, the following Lemma was proven by Taubes:

Lemma 6.1.4. *(Lemma 5.4, [60]) Let $A \in H_2(X)$ be toroidal and $n \in \mathbb{Z}^+$. Then there is an open and dense subset of smooth, ω-compatible almost complex structures $J \subset \mathcal{J}_\omega$ on X with the property, that every embedded, pseudoholomorphic torus in class A is n-nondegenerate.*

Further arguments, similar to those in step 7 of the proof of Prop. 7.1 in [59], together with this Lemma ensure that the space of curves in class A is finite and invariant under small deformations of the symplectic and almost complex structures.

6.1.2 Contact Order for Tori with Trivial Normal Bundle

If the curve C is a multiply covered torus with trivial normal bundle which intersects V non-trivially, then we must account for this in the contact order. Lemma 2.2.3 states, that in this case, to have proper initial data, we must have the base curve C' of C intersecting V with contact order 1 at each intersection point, where the contact order of C' and V is given by Def. 2.1.2. Thus we make the following definition

Definition 6.1.5. *Assume C is a multiply covered torus with trivial normal bundle in class $A = mA'$ such that $A' \cdot \mathfrak{V} \neq 0$. Then the contact order of the curve C at each intersection point of C' with V is given by an m-tuple $(1, ..., 1)$.*

Taubes' calculations all have initial data $\mathcal{I}_A = \emptyset$. The relative setting must deal with the presence of initial data on V:

Lemma 6.1.6. *Let $A = mT$ be multiply toroidal. Curves in this class can occur only if the only entries in $\mathcal{I}_A$ are in Υ each with an m-tuple $(1, ..., 1)$ as contact order.*

Proof. A multiply toroidal class has $d_T = 0$, hence by Lemma 2.2.3, we have $d_1 = d_2 = l_1 = l_2 = 0$ and $l_3 = T \cdot \mathfrak{V}$. Moreover, Def. 6.1.5 determines the contact structure. □

6.2 Space of Relative Tori

We define a space of relative submanifolds:

Definition 6.2.1. *Fix $A \in H_2(X)$ and a set of proper initial data $\mathcal{I}_A$. Assume that $A = mT$ is multiply toroidal. Choose an almost complex structure $J \in \mathcal{J}_V$. Denote the set $\mathcal{K}_V^T = \mathcal{K}_V^T(A, J, \mathcal{I}_A)$ of connected J-holomorphic submanifolds $C \subset X$ which satisfy*

- *If $l_A = 0$, then $C \cap V = \emptyset$ or V.*
- *If $l_A > 0$, then C*
    1. *intersects V locally positively and transversely and*
    2. *intersects V at precisely l_3 distinct points each with order given by the m-tuple $(1, ..., 1)$.*

We wish to show that Taubes' Lemmas still hold in the relative case. We consider three cases:

- $A_i \cdot \mathfrak{V} \neq 0$.
- $A_i \neq \mathfrak{V}$ and $A_i \cdot \mathfrak{V} = 0$ or
- $A_i = \mathfrak{V}$ and V is a square 0 torus.

The previous Lemma shows that we do not have to consider any restricting insertions. We begin with the first case.

Lemma 6.2.2. *Assume A is multiply toroidal and fix $n \in \mathbb{N}$. There is an open and dense subset $U \subset \mathcal{J}_V$ with the following properties: When an almost complex structure J is chosen from U, then*

1. *$\mathcal{K}_V^T(J, \mathcal{I}_A)$ is a finite collection of points and each point is n - nondegenerate. Moreover, if $A \neq m\mathfrak{V}$ and $A \cdot \mathfrak{V} = 0$, there exists a neighborhood $\mathfrak{N}_V$ of V such that no curve in class A lies therein.*
2. *There is an open neighborhood in $\mathcal{J}_V$ such that every almost complex structure therein obeys the previous assertion and the number of points of $\mathcal{K}_V^T$ is invariant in this neighborhood.*

Proof. $\mathbf{A} \cdot \mathfrak{V} \neq \mathbf{0}$ The calculations in Section 3.2 ensure the existence of a sufficiently large set of almost complex structures such that curves in class A behave as expected. Once this set has been found, the proofs of Lemma 5.3 and 5.4 in [60] can be repeated to give the result in this case.

$\mathbf{A} \cdot \mathfrak{V} = \mathbf{0}$ Assume first that $A \neq m\mathfrak{V}$. We will show, that there exists a neighborhood $\mathfrak{N}_V \subset X$ of V such that no pseudoholomorphic submanifold C of class A has $C \cap \mathfrak{N}_V \neq \emptyset$. If this holds, then all results on multiply toroidal classes from [61] hold for the class A: Outside of $\mathfrak{N}_V$ there are no restrictions on the almost complex structure $J \in \mathcal{J}_V$, hence all the results proven by Taubes for multiply toroidal classes (Lemmas 5.3,5.4) hold in this case as well. This proves the Lemma.

Assume no such neighborhood exists. Then let $\{\mathfrak{N}_i\}$ be a sequence of nested neighborhoods converging to V. Denote $A = n\mathfrak{T}$ with $\mathfrak{T}$ a toroidal class. Let $\{T_i, J_i\}$ be a sequence of tori in class $\mathfrak{T}$ such that $T_i \subset \mathfrak{N}_i$ and $J_i \in \mathcal{J}_V$. Gromov's compactness Theorem ensures that there exists a limit curve T and a limit almost complex structure $J_i \to J$ such that $g(T) = 1$ and Lemma 4.2.3 shows that for generic J the limit curve T is an embedded square 0-torus of class $q\mathfrak{T}$. The sequence of neighborhoods ensures that $T \subset V$. If $g(V) \geq 2$, then there can be no nontrivial smooth map $T^2 \to V$.

Assume that $g(V) = 1$. Then we have produced a J-holomorphic map $T^2 \to V$ in class $A \neq m\mathfrak{V}$ which is onto V. Such a nontrivial map does not exist.

Assume that $g(V) = 0$. Then we have a J-holomorphic map from a torus T^2 to the sphere. This must be a multiple cover of S^2. However, $K_\omega \cdot A = K_\omega \cdot \mathfrak{T} = 0$ and hence by the adjunction formula there exists no such map.

Consider now the second case: $A = m\mathfrak{V}$, hence any curve in this class can be decomposed into components such that we have either a multiple cover of V or a multiple cover of a curve in the class $\mathfrak{V}$ which does not meet V. For those curves not meeting V, arguments similar to the previous ones show that Taubes' results hold. We consider therefore only the case of a multiple cover of the hypersurface V. An analysis of Taubes' results shows, that the argument in the multiply toroidal case is essentially a relative argument on the fixed square 0 torus underlying the multiple cover. In our case this is the hypersurface V, hence his argument transfers completely.

$\square$

6.3 Relative Taubes Invariant

Taubes defined a number $Qu(e, n)$ for multiply toroidal classes and showed that it is an invariant of the deformation class of ω(Prop. 5.7, [60], see also Prop 7.5.3). This motivates the following slightly modified relative version:

Definition 6.3.1. *Let $n \geq 1$ be an integer and T toroidal and indivisible. Choose $(J, \mathcal{I}_A)$ from the Baire set obtained in Lemma 6.2.2. Define a relative Taubes invariant*

$$Qu^V(T, n) = \sum_{\{(C_k, m_k, \mathcal{I}_k)\}} \prod_k r(C_k, m_k, \mathcal{I}_k)$$

where we sum over all sets $\{(C_k, m_k, \mathcal{I}_k)\}$ with

1. *C_k an embedded torus in class $q_k T$ for some $q_k \leq n$,*

2. $m_k \geq 1$ *and* $n = \sum q_k m_k$ *and*

3. $\mathcal{I}_k$ *contains tuples* $(1, ..., 1)$ *of length* q_k *at each of the* $T \cdot \mathfrak{V}$ *intersection points.*

The value of $r(C, m, \mathcal{I}_k)$ is given by the value of $r(C, m)$ as defined by Taubes for multiply toroidal classes in [60].

The results in Lemma 6.2.2 show that only if $A \cdot \mathfrak{V} \neq 0$ do we need to reconsider the definition of the invariant $Qu(T, n)$ as given by Taubes. However, an analysis of the proof of Proposition 5.7, which states that $Qu(e, n)$ is an invariant of the deformation class of ω shows that this proof too relies on the existence of a sufficiently generic set of almost complex structures such that curves behave as expected for any J chosen therein. Hence the invariant defined by Taubes can be directly used in the relative setting with the modifications given above, i.e. $Qu^V(T, n)$ is also an invariant of the deformation class of (X, V, ω).

Chapter 7

Relative Gromov-Taubes Invariants

In this section we define a general invariant counting disconnected submanifolds as in the Gromov-Taubes invariants defined in [60]. These relative Gromov-Taubes invariants will make use of the invariants of the previous sections.

7.1 The Space of Relative Submanifolds

We now introduce the space of relative submanifolds $\mathcal{R}_V(A, J, \mathcal{I}_A)$. This definition will be rather technical, however the general idea is simple: We want to consider all submanifolds C, not necessarily connected, which contact V in a very controlled manner. This is determined by the initial data $\mathcal{I}_A$ and we ensure that we contact V only once for every given geometric object with the required contact order. Moreover, the curve C shall meet each geometric object in the initial data $\mathcal{I}_A$. We make this precise in the following definition:

Definition 7.1.1. *Fix $A \in H_2(X)$ and a set of proper initial data $\mathcal{I}_A$. Choose an almost complex structure $J \in \mathcal{J}_V$. Denote the set $\mathcal{R} = \mathcal{R}_V(A, J, \mathcal{I}_A)$ of unordered sets of tuples $\{(C_i, m_i, \mathcal{I}_i)\}$ of disjoint, connected J-holomorphic submanifolds $C_i \subset X$ with*

1. *positive integers m_i and*
2. *unordered subsets $\mathcal{I}_i$ of $\mathcal{I}_A$ with parameters d^i_*, l^i_**

satisfying the following constraints:

- *Let* A_i *represent the homology class of* C_i *and denote* d_{A_i} *as in 2.1 and* l_{A_i} *as in 2.2. Require* $d_{A_i} \geq 0$, $l_{A_i} \geq 0$ *and the set* $\mathcal{I}_i$ *to be a proper initial data set for the class* A_i.

- *If* $d_{A_i} > 0$, *then* C_i

  1. *contains precisely* d_1^i *members of* Ω_{d_1} *and*
  2. *intersects each member of* Γ_i^X *exactly once.*

- *If* $l_{A_i} = 0$, *then* $C_i \cap V = \emptyset$ *or* V.

- *If* $l_{A_i} > 0$, *then* C_i

  1. *intersects* V *locally positively and transversely,*
  2. *intersects* V *at precisely* l_1^i *points of* Ω_{l_1} *and*
  3. *intersects each member of* Γ_i^V *exactly once.*
  4. *The remaining* l_3^i *intersections with* V *are unconstrained.*
  5. *Each intersection is of order* s_i *given in the initial data* $\mathcal{I}_i$ *for this component (or by the tuple* $(1, .., 1)$ *if it is multiply toroidal).*

- *If* $i \neq i'$, *then* $\mathcal{I}_i \cap \mathcal{I}_{i'} = \emptyset$, *but* $\cup \mathcal{I}_i = \mathcal{I}_A$.

- *The integer* $m_i = 1$ *unless possibly if* C_i *is a torus with trivial normal bundle and* $A_i \cdot V = 0$.

- $\sum_i m_i A_i = A$.

Remark: By imposing the condition that the C_i be disjoint allows us to conclude

$$\sum_i d_1^i + d_2^i = d_A, \quad \sum_i l_1^i + l_2^i + l_3^i = l_A. \tag{7.1}$$

The points of $\mathcal{R}$ consist of submanifolds which are not necessarily connected. Therefore, a choice of almost complex structure J and initial data $\mathcal{I}_A$ must be made in such a manner, that any allowed decomposition of the class A into submanifolds respects the initial data and that all these submanifolds are pseudoholomorphic for the fixed almost complex structure. In particular, the pair $(J, \mathcal{I}_A)$ must be chosen such that it rules out any unwanted behavior of representatives of the class A and its decompositions. This motivates the following definition:

Definition 7.1.2. *A pair $(J, \mathcal{I})$ of almost complex structure $J \in \mathcal{J}_V$ and initial data $\mathcal{I}$ with $d_1 + d_2 = d$ and $l_1 + l_2 + l_3 = l$ is called r-admissible, if for each $A \in H_2(X)$ and each proper initial data $\mathcal{I}_A \subset \mathcal{I}$ the following conditions hold:*

1. *There are but finitely many connected J-holomorphic submanifolds in the class A contacting the initial data $\mathcal{I}_A$.*
2. *Each of the submanifolds above is non-degenerate.*
3. *There exist no connected J-holomorphic submanifolds in class A contacting all the data in $\mathcal{I}_A$ as well as a further insertion.*
4. *There is an open neighborhood of $(J, \mathcal{I})$ in $\mathcal{J}_V \times [\mathcal{I}]$ with the property that each point in this neighborhood obeys the previous three points while preserving the number of J-holomorphic curves throughout this neighborhood.*
5. *If $A^2 = 0 = c_1 \cdot A$, then each J-holomorphic submanifold in Point 1 is n-non-degenerate for each positive integer n*

7.2 Main Result for Disconnected Submanifolds

Proposition 7.2.1. *Fix a class $A \in H_2(X)$ and a proper class $[\mathcal{I}_A]$. Assume the general assumptions 2.5.3 hold. Then the set of r-admissible pairs $(J, \mathcal{I}_A)$ in $\mathcal{J}_V \times [\mathcal{I}_A]$ is a Baire subset. Furthermore, given an r-admissible pair, the following hold:*

1. *The set $\mathcal{R}_V(A, J, \mathcal{I}_A)$ is a finite set.*
2. *If $A \neq \mathfrak{V}$, then $\mathcal{R}_V(A, J, \mathcal{I}_A)$ is empty when $d_A < 0$.*
3. *If V is an exceptional sphere, then $\mathcal{R}_V(\mathfrak{V}, J, \emptyset) = \emptyset$*
4. *Every point $h \in \mathcal{R}_V$ has the property, that each C_i with $m_i = 1$ is non-degenerate, if $m_i > 1$ it is m_i-non-degenerate.*

5. If $(J^1, \mathcal{I}^1_A)$ are sufficiently close to $(J, \mathcal{I}_A)$, then the sets $\mathcal{R}_V$ and $\mathcal{R}^1_V$ have the same number of elements.

The general assumptions 2.5.3 either prevent maps into V or, if $d_{\mathfrak{V}} \geq 0$, then Lemma 2.5.1 ensures the existence of a Baire set with pairs $(J, \mathcal{I}_A)$ such that no component of a submanifold representing A lies in V. We may hence assume that all of our submanifolds are of this form in the following.

The set of r-admissible pairs $(J, \mathcal{I}_A)$ is Baire follows from the fact that a countable intersection of Baire sets is again Baire. To be precise: Consider a decomposition of $A = \sum m_k A_k$ with $A_k \cdot A_l = 0$ if $k \neq l$. Then for each A_k we consider the space of connected submanifolds $\mathcal{K}$. For a Baire set of pairs $(J, \mathcal{I}_{A_k})$ the set $\mathcal{K}_V$ has all the properties described in the previous sections. The space $\mathcal{J}_V \times \{\text{initial data}\}$ decomposes into a product of $\mathcal{J}_V$ and disjoint initial data sets corresponding to the classes A_k. Then the intersection of two Baire sets U_k and U_l corresponding to A_k and A_l is defined as follows: We have endowed all of the above products with the product topology. Denote p_1, p_2^* the projections onto $\mathcal{J}_V$ and the initial data set corresponding to A_*. Then let

$$U_k \cap U_l = p_1(U_k) \cap p_1(U_l) \times p_2^k(U_k) \times p_2^l(U_l) \tag{7.2}$$

define the intersection of the two Baire sets. This set is still a Baire set in $\mathcal{J}_V \times \{\text{initial data}\}_k \times \{\text{initial data}\}_l$ due to the properties of the product topology. From this the claim follows.

The properties of being r-admissible as given in Def. 7.1.2 are fulfilled by pairs obtained by a countable intersection of all Baire sets found in Sections 3.2 and 4 for any decomposition of A appearing in $\mathcal{R}_V$. Moreover, any pair $(J, \mathcal{I}_A)$ found in this intersection also satisfy assertions 2-4 of Prop. 2.6.1.

Consider now Assertion 1: $\mathcal{R}_V(A, J, \mathcal{I}_A)$ is finite. This will follow from the following result which was proven by Taubes:

Lemma 7.2.2. *(Lemma 5.5, [60]) Given $A \in H_2(X)$, there is a Baire subset of $\mathcal{J}_V \times [\mathcal{I}_A]$ such that when a pair is chosen from this set, then there are but finitely many classes in $H_2(X, \mathbb{Z})$ which can be a fundamental class of a J-holomorphic submanifold appearing (with some multiplicity) as an element in some $h \in \mathcal{R}_V$.*

7.3 The Number $GT^V(A)([\mathcal{I}_A])$

To define $GT^V(A)([\mathcal{I}_A])$ we will need to begin with a component of $h = \{(C_k, m_k, \mathcal{I}_k)\} \in \mathcal{R}_V(A, J, \mathcal{I}_A)$. To each such component, we will assign a number $r(C_k, m_k, \mathcal{I}_k)$. Once this has been defined, we will define a value $q(h)$ for the point h, this will consist of the values $r(C_k, m_k, \mathcal{I}_k)$ as well as accounting for permutations of the initial data. Finally, we define the relative invariant:

Definition 7.3.1. *The relative Gromov-Taubes invariant $GT^V(A)([\mathcal{I}_A])$ is defined by*

$$GT^V(A)([\mathcal{I}_A]) = \sum_{h \in \mathcal{R}_V(A,J,\mathcal{I}_A)} q(h). \tag{7.3}$$

If $\mathcal{R}_V(A, J, \mathcal{I}_A) = \emptyset$, then $GT^V(A)([\mathcal{I}_A]) = 0$.

For a suitably generic choice of pairs $(J, \mathcal{I}_A)$, this number is well-defined. In the following, we make this definition precise and show that this number is an invariant of the symplectic deformation class.

Remark: This number depends not only on the class $[\mathcal{I}_A]$, but actually also on the ordering given in the sets Γ_*. Ultimately, this only affects the sign of $GT^V(A)([\mathcal{I}_A])$. This will be taken into account in the definition of $q(h)$.

7.4 The Definition of $q(h)$

7.4.1 Permutations of the Initial Data

A point $h \in \mathcal{R}$ need not meet the data $\mathcal{I}_A$ in the order predicated in the class $[\mathcal{I}_A]$. This ordering determines an orientation of the corresponding moduli space of maps however and thus any permutation of it must be taken into account when defining an invariant. Our data $\mathcal{I}_A$ contains three types of geometric data: points, 1-dimensional curves and the hypersurface V. Neither the points nor the hypersurface V change the orientation under rearrangement. Recall the sets Γ_{d_2} and Γ_{l_2} and consider only the curves and the ordering. We can rearrange each set by a permutation π_{d_2} resp. π_{l_2} as follows: Consider the point $h = \{(C_k, m_k, \mathcal{I}_k)\}$. This point comes with an ordering. For each i define the sets $\Gamma_1^k \subset \Gamma_{d_2}$ and $\Gamma_2^k \subset \Gamma_{l_2}$ consisting of the data in $\mathcal{I}_k$ in the corresponding sets. Reorder the data in Γ_*^k in ascending fashion according to the ordering given in $\mathcal{I}_A$. Then $\sqcup\Gamma_*^k$ defines a permutation of the data in Γ_{d_2} resp. Γ_{l_2}.

Definition 7.4.1. *Define* $p(h) = sign(\pi_{d_2})sign(\pi_{l_2})$.

Equivalently, we could consider the set $\Gamma = \Gamma_{d_2} \sqcup \Gamma_{l_2}$ and a corresponding permutation π which consists of the two permutations π_{d_2} and π_{l_2}. Then $p(h)$ =sign(π).

The ordering on the curve h is not fixed. We must show that a relabeling of the curves C_k will leave the value of $p(h)$ unchanged. Let d_2^k and l_2^k denote the number of elements in Γ_1^k resp. Γ_2^k. Then the invariance of $p(h)$ under reordering follows from

Lemma 7.4.2. *The value* $d_2^k + l_2^k$ *is even.*

Proof. We have shown, that for generic $(J, \mathcal{I}_A)$ we can have curves in the class A only if the initial data is proper. This holds in particular for every connected component of the curve h. The condition for properness can be easily rewritten to show that $d_2^k + l_2^k$ is even.

□

7.4.2 $q(h)$

The value of $q(h)$ for the point $h = \{(C_k, m_k, \mathcal{I}_k)\}$ is given by the product

$$q(h) = p(h) \prod_k r(C_k, m_k, \mathcal{I}_k) \tag{7.4}$$

where the $r(C, m, \mathcal{I})$ are given by $r(C, \mathcal{I})$ as defined for non-multiply toroidal classes in Section 5 and by Taubes in the toroidal case.

7.5 Properties of Relative Gromov-Taubes Invariants

Theorem 7.5.1. *The number* $GT^V(A)([\mathcal{I}_A])$ *depends only on the deformation class of* (X, V, ω)*, the class* $A \in H_2(X)$*, the initial class* $[\mathcal{I}_A]$ *and the ordering of the data in the sets* Γ_**. In particular, it does not depend on a particular choice of* $(J, \mathcal{I}_A)$*.*

To prove this, we proceed to rewrite the number $GT^V(A)([\mathcal{I}_A])$ in terms of the number $Ru^V(A, [\mathcal{I}_A])$ and the toroidal contributions $Qu^V(A, m)$. To do so we introduce notation: Let $A \in H_2(X)$ be fixed. Denote by $S(A)$ the set defined by Taubes: This is the collection of unordered sets of pairs $\{(A_k, m_k)\}$ with the following properties:

1. $\{A_k\}$ is a set of distinct, non-multiply toroidal classes.
2. $m_k = 1$ unless $A_k^2 = 0$, in which case $m_k \geq 1$ can be any positive integer.
3. $A_k \cdot A_l = 0$ if $k \neq l$.
4. $A = \sum m_k A_k$.

Note that it is possible for $m_k \geq 2$ but for A_k not to be a toroidal class: The set $S(A)$ is a set of homology classes, we allow our submanifolds to be composed of multiple disjoint copies of classes with 0 self intersection. This is taken into account by this condition.

For a given tuple $y = \{(A_K, m_K)\} \in S(A)$, denote by $\tau(y)$ the set of pairs which appear in y and which satisfy one of the following conditions:

1. $A_k^2 \neq 0$ or
2. $c_1(A_k) \neq 0$.

With this notation we can prove the following Lemma, analogous to Lemma 5.6, [60]:

Lemma 7.5.2.

$$GT^V(A)(\mathcal{I}_A) = \sum_{y \in S(A)} \left[\prod_{\tau(y)} Per(y) \left[Ru^V(A_k, [\mathcal{I}_{A_k}]) \right]^{m_k} \right] \times \left[\prod_{(A_k, m_k) \notin \tau(y)} Qu^V(A_k, m_k) \right] \tag{7.5}$$

where

$$Per(y) = \frac{d_A!}{(d_1^k! d_2^k!)^{m_k}} \frac{l_A!}{(l_1^k! l_2^k! l_3^k!)^{m_k}} \frac{1}{m_k!}$$

Remark: The sum in the above Lemma may incorporate submanifolds in the three exceptional cases allowing for components V. This is the reason for defining $r(V, m) = 0$, as these submanifolds will not contribute to the above sum.

Proof. This is a resummation of the defining sum

$$GT^V(A)([\mathcal{I}_A]) = \sum_{h \in \mathcal{R}_V(A,J,\mathcal{I}_A)} q(h).$$

The key points are the following:

1. The permutation term $p(h)$ can be decomposed into a product of permutation terms $p(C_k, m_k, \mathcal{I}_k)$ stemming from the components $(C_k, m_k, \mathcal{I}_k)$ of h. These are taken into account in the number $Ru^V(A_k, [\mathcal{I}_k])$ with the ordering on $[\mathcal{I}_k]$ induced by the ordering of $[\mathcal{I}_A]$.

2. Given $h = \{(C_k.m_k, \mathcal{I}_k)\}$ we can replace any C_k by any other submanifold in $\mathcal{K}_V(J, \mathcal{I}_{A_k})$. Hence all elements of class A_k meeting the data $\mathcal{I}_k$ appear in the definition of the relative Gromov-Taubes invariant. Hence resumming along homology classes simply reorders the sum but does not add any new terms.

3. The term $Per(y)$ accounts for permutations of the initial data on the components of h. This is a purely combinatorial term, we have accounted for the topological effects of permuting the initial data in the term $Ru^V(A_k, [\mathcal{I}_k])$.

□

We have seen above, that the numbers $Ru^V(A, [\mathcal{I}_A])$ are invariants. The proof of Theorem 7.5.1 will be complete with the results for multiply toroidal classes in Section 6, in particular Lemma 6.2.2, and the following Lemma:

Proposition 7.5.3. *(Lemma 5.7, [60]) Let $T \in H_2(X)$ be an indivisible toroidal class. Let $n \geq 1$ be an integer. Then $Qu(T, n)$ depends only on the deformation class of the symplectic form.*

Example 7.5.4. *Recall the example in genus 0. In this case we determined the relative Ruan invariants for exceptional curves. Assume that $A = \sum_{i=1}^n E_i$, each E_i an exceptional sphere. Under the assumptions of the previous calculation, $E_i \cdot E_j = 0$ for $i \neq j$. It then follows that*

$$\mathcal{R}_V(A, J, \mathcal{I}_A) = \begin{cases} \{(E_1, 1, \mathcal{I}_1), ..., (E_m, 1, \mathcal{I}_m)\} & A \cdot \mathfrak{V} \geq 0 \\ \emptyset & \textit{otherwise} \end{cases} \tag{7.6}$$

which consists of one curve with m components. The invariant in the first case reduces to

$$GT^V(A)([\mathcal{I}_A]) = q(\{(E_1, 1, \mathcal{I}_1), ..., (E_m, 1, \mathcal{I}_m)\}) = \prod_{i=1}^{m} Ru^V(E_i, J, \mathcal{I}_i) = 1. \tag{7.7}$$

Part I

Relative Symplectic Cone

Chapter 8

The relative symplectic cone

This section comprises two distinct parts. The first subsection introduces notation and general concepts of use in the following subsections. The latter subsections contain the definition of the relative symplectic cone and examples.

8.1 Preliminaries

Let M be an oriented manifold and V an oriented codimension 2 submanifold, not necessarily connected. Throughout this section, it will be necessary to carefully distinguish the class of V, denoted $\mathfrak{V} \in H_2(M)$, and the specific submanifold V. Throughout this paper we will not distinguish between $\mathfrak{V}$ and its Poincaré dual. Moreover, $\mathfrak{V}$ will always be a non-zero class and g will denote the genus of V.

We introduce two sets: The set of symplectic classes which evaluate positively on $\mathfrak{V}$:

$$\mathcal{C}_M^{\mathfrak{V}} = \{\alpha \in \mathcal{C}_M \mid \alpha \cdot \mathfrak{V} > 0\}$$

and the larger set of classes with positive square which evaluate positively on $\mathfrak{V}$:

$$\mathcal{P}^{\mathfrak{V}} = \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}.$$

Clearly $\mathcal{C}_M^{\mathfrak{V}} \subset \mathcal{P}^{\mathfrak{V}}$.

The structure of the set $\mathcal{P}^{\mathfrak{V}}$ as a subset of $\mathcal{P}_M$ will be important in the following sections. To this end, recall the "light cone lemma": Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$. Then the set $\mathcal{P}_M$ consists of two connected components, separated by the hyperplane of classes with $[\omega] \cdot A = 0$. The component which contains $[\omega]$ is called the forward cone and is denoted by $\mathcal{P}^+$. Its complement will be denoted by $\mathcal{P}^- = \mathcal{P}_M \backslash P^+$.

Lemma 8.1.1. *(Light Cone Lemma)(Lemma 3.7, [48]) Suppose that (M, ω) is a symplectic 4-manifold with $b^+ = 1$. Let $a, b \in H^2(M)$ both lie in the closure $\overline{\mathcal{P}^+}$. Then $a \cdot b > 0$ unless $a = \lambda b$ and $a^2 = 0$.*

Note that if $a \in \overline{\mathcal{P}^-}$, then $-a \in \overline{\mathcal{P}^+}$. In particular, this means that any two classes in the same connected component of $\mathcal{P}_M$ have $a \cdot b > 0$. Moreover, if a, b lie in different components, then $-a, b$ lie in the same component and hence $a \cdot b < 0$.

Lemma 8.1.2. *If $b^+ = 1$, then $\mathcal{P}^{\mathfrak{V}}$ is connected if $\mathfrak{V} \cdot \mathfrak{V} \geq 0$; and $\mathcal{P}^{\mathfrak{V}}$ has two connected components if $\mathfrak{V} \cdot \mathfrak{V} < 0$.*

Proof. Assume that $\mathfrak{V} \cdot \mathfrak{V} > 0$. Then $\mathfrak{V}$ is in one of the components of $\mathcal{P}_M$ and, in particular, $\mathfrak{V} \in \mathcal{P}^{\mathfrak{V}}$. Let $a \in \mathcal{P}^{\mathfrak{V}}$, then $a \cdot \mathfrak{V} > 0$ by definition, hence by the light cone lemma a must lie in the same component of $\mathcal{P}_M$ as $\mathfrak{V}$. Hence $\mathcal{P}^{\mathfrak{V}}$ has only one component.

If $\mathfrak{V} \cdot \mathfrak{V} = 0$, then $\mathfrak{V}$ lies in the closure of a component of $\mathcal{P}_M$. As $\mathcal{P}^{\mathfrak{V}}$ contains no elements of square 0, the previous argument again shows that $\mathcal{P}^{\mathfrak{V}}$ has one component unless $\mathfrak{V} = 0$, which our assumptions exclude.

To understand the case $\mathfrak{V} \cdot \mathfrak{V} < 0$, note that the statement of the light cone lemma can be viewed as follows: If a lies in closure of the cone $\mathcal{P}_M$, then the hyperplane of classes defined by $a \cdot A = 0$ does not intersect the cone $\mathcal{P}_M$ at any point. However, in the case $\mathfrak{V} \cdot \mathfrak{V} < 0$ this can no longer be guaranteed. Hence we may have one or two connected components for $\mathcal{P}^{\mathfrak{V}}$. However, if we assume that we have only one connected component, then this implies that the hyperplane of classes with $\mathfrak{V} \cdot A = 0$ does not intersect $\mathcal{P}_M$, meaning all classes in the hyperplane have non-positive square. Moreover, $\mathfrak{V}$ is not in this hyperplane. Hence the span of the generators of the hyperplane and $\mathfrak{V}$ generate a space of the same dimension as $H^2(M)$. Furthermore, this basis is negative semi-definite, which contradicts $b^+ = 1$.

□

8.1.1 Minimality

Definition 8.1.3. *Let $\mathcal{E}_M$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1. M is said to be (smoothly) minimal if $\mathcal{E}_M$ is the empty set.*

Equivalently, M is minimal if it is not the connected sum of another manifold with $\overline{\mathbb{CP}^2}$. We say that N is a minimal model of M if N is minimal and M is the connected sum of N and a number of $\overline{\mathbb{CP}^2}$.

We also recall the notion of minimality for a symplectic manifold (M, ω): (M, ω) is said to be (symplectically) minimal if $\mathcal{E}_\omega$ is the empty set, where

$$\mathcal{E}_\omega = \{E \in \mathcal{E}_M | \ E \text{ is represented by an embedded } \omega\text{−symplectic sphere}\}.$$

A basic fact proved using SW theory ([59], [35], [33]) is: $\mathcal{E}_\omega$ is empty if and only if $\mathcal{E}_M$ is empty. In other words, (M, ω) is symplectically minimal if and only if M is smoothly minimal.

A class $K \in H^2(M, \mathbb{Z})$ is called a symplectic canonical class if there exists a symplectic form ω on M such that for any almost complex structure J tamed by ω,

$$K = K_\omega = -c_1(M, J).$$

Let $\mathcal{K}$ be the set of symplectic canonical classes of M. For any $K \in \mathcal{K}$ define

$$\mathcal{E}_K = \{E \in \mathcal{E}_M | K \cdot E = -1\}.$$

It is shown in Lemma 3.5 in [34] that for any ω,

$$\mathcal{E}_{K_\omega} = \mathcal{E}_\omega.$$

Let $S \subset \mathcal{K}$, then define the S-symplectic cone

$$\mathcal{C}_{M,S} = \{\alpha \in \mathcal{C}_M | \alpha = [\omega], \ (M, \omega) \text{ symplectic}, \ K_\omega \in S\}. \tag{8.1}$$

8.1.2 Relative inflation

The relative inflation procedure allows one to deform a symplectic form in a smooth family while keeping a fixed submanifold symplectic. Moreover, this deformation is explicitly given. The following is the precise statement:

Lemma 8.1.4. *(Lemma 2.1.A, [3]) Let $V, C \subset (M, \omega)$ be two distinct 2-dimensional symplectic submanifolds. The submanifold V may be disconnected with pairwise disjoint components. Assume that $C \cdot C \geq 0$ and that C and V intersect positively and transversally in a finite number of points. Then there exists a two form ρ, supported in an arbitrarily small neighborhood of C, with the following properties:*

- *$[\rho]$ is Poincaré dual to $[C]$,*
- *$\omega(s, t) = s\omega + t\rho$ for every $s > 0$ and $t \geq 0$ and*
- *V is a symplectic submanifold with respect to $\omega(s, t)$ for any choice of (s, t).*

Note that the second statement of the lemma concerns symplectic forms not only symplectic classes. With these preparations, we can now proceed to the relative symplectic cone.

8.2 Definition and properties

We make the following definition:

Definition 8.2.1. *A relative symplectic form on the pair (M, V) is an orientation compatible symplectic form on M such that $\omega|_V$ is an orientation compatible symplectic form on V. The relative symplectic cone of (M, V) is*

$$\mathcal{C}_M^V = \{\alpha \in H^2(M) | \, [\omega] = \alpha, \ \omega \textit{ is a relative symplectic form on } (M, V)\}. \tag{8.2}$$

The submanifold V is embedded, hence the adjunction equality $K_\omega \cdot \mathfrak{V} = 2g - 2 - \mathfrak{V} \cdot \mathfrak{V}$ must hold for the canonical class K_ω of any relative symplectic form ω on (M, V). Let $\mathcal{K}(V) = \{K \in \mathcal{K} \mid K \cdot \mathfrak{V} = 2g - 2 - \mathfrak{V} \cdot \mathfrak{V}\}$. The following is a consequence of the definition of $\mathcal{K}$:

Lemma 8.2.2. *If $\mathcal{K}(V) = \emptyset$, then there exists no symplectic form $\omega \in \mathcal{C}_M$ such that V is a ω-symplectic submanifold.*

For any $S \subset \mathcal{K}(V)$, define the S-relative symplectic cone

$$\mathcal{C}_{M,\,S}^{\mathfrak{V}} = \{\alpha \in \mathcal{C}_M^{\mathfrak{V}} \mid \alpha = [\omega], \ (M, \omega) \text{ symplectic }, \ K_\omega \in S\}. \tag{8.3}$$

Note that

$$\mathcal{C}^{\mathfrak{V}}_{M,S} = \mathcal{C}_{M,S} \cap \mathcal{P}^{\mathfrak{V}}. \tag{8.4}$$

The following lemma follows directly from the definition of the relative symplectic cone and the $\mathcal{K}(V)$-relative symplectic cone:

Lemma 8.2.3.

$$\mathcal{C}^{V}_{M} \subset \mathcal{C}^{\mathfrak{V}}_{M,\,\mathcal{K}(V)} \subset \mathcal{C}^{\mathfrak{V}}_{M} \subset \mathcal{P}^{\mathfrak{V}}$$

The inclusion $\mathcal{C}^{V}_{M} \subset \mathcal{P}^{\mathfrak{V}}$ can be strict, see Lemma 8.4.5.

Obviously there are maps

$$\mathcal{C}^{V}_{M} \hookrightarrow \mathcal{C}_{M}, \quad \mathcal{C}^{V}_{M} \to \mathcal{C}_{V}. \tag{8.5}$$

In fact, if V is the disjoint union of V_0 and V_1, then there is also a map $\mathcal{C}^{V}_{M} \to \mathcal{C}^{V_i}_{M}$. Note that the restriction mapping $\mathcal{C}^{V}_{M} \to \mathcal{C}_{V}$ is by no means generically injective. The following fact relates the relative cones to the symplectic cone:

Lemma 8.2.4. *Let $\mathcal{V}$ denote the set of oriented codimension 2 submanifolds of M. Then*

$$\bigcup_{V\in\mathcal{V}} \mathcal{C}^{V}_{M} = \mathcal{C}_{M}$$

Proof. The inclusion $\bigcup_{V\in\mathcal{V}} \mathcal{C}^{V}_{M} \subset \mathcal{C}_{M}$ follows from (8.5). Consider now a symplectic class $\alpha \in \mathcal{C}_{M}$ and denote by ω a symplectic form representing the class α . We distinguish two cases: $b^+ = 1$ and $b^+ > 1$.

Let $b^+ = 1$. Then α is trivially in the forward cone $\mathcal{P}^+$ of (M,ω). Moreover, if $\mathcal{E}_\omega \neq \emptyset$, $\alpha \cdot E > 0$ for all $E \in \mathcal{E}_\omega$. It now follows from Prop. 4.2 or Prop 4.3 in [34] that for k large enough the class $k\alpha$ is represented by a ω-symplectic surface. Therefore, if V represents the class $k\alpha$, it follows that $\alpha \in \mathcal{C}^{V}_{M}$.

If $b^+ > 1$, then the canonical class of (M,ω) for some almost complex structure J taming ω is represented by a ω-symplectic surface, see [59], Thm 0.2. Hence, if V represents the canonical class K_ω, then $\alpha \in \mathcal{C}^{V}_{M}$. □

The proof shows, that if $b^+ > 1$, we need only consider submanifolds V which are representatives of a canonical class K_ω of (M,ω) if we wish to understand $\mathcal{C}_M$ with respect to the relative cone. In particular, this shows that $\mathcal{C}_M \subset \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$ if

$b^+ > 1$ and M is minimal Kähler, which is of interest in connection with Conjecture 1.3.1. Furthermore, it seems natural to wonder, whether there exist a finite set of submanifolds $\mathcal{V}_f \subset \mathcal{V}$, such that they completely determine the symplectic cone of M. With respect to this question, a trivial but key observation connecting the relative symplectic cone and the symplectic cone is

Lemma 8.2.5. *Denote the submanifold V with opposite orientation by $\overline{V}$. Then*

$$\mathcal{C}_M^{\overline{V}} = -\mathcal{C}_M^V.$$

The following corollary will be useful in our applications in Section 10:

Corollary 8.2.6. *Suppose M underlies a minimal Kähler surface with $b^+ > 1$. If $c_1(M,\omega) = a\mathfrak{V}$ with $a \neq 0$ for some symplectic form ω and $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$, then $\mathcal{C}_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$*

We now proceed to calculate the relative cone for certain submanifolds V for two classes of symplectic manifolds: T^2 bundles over T^2 and manifolds with $b^+ = 1$.

8.3 T^2-bundles over T^2

The total spaces M of such bundles have been studied and classified by Sakamoto-Fukuhara [12], Ue [62] and Geiges [13]. In particular, with one exception, they all admit symplectic structures compatible with the bundle structure; in the case of a primary Kodaira surface this bundle structure must be specified as it is not unique. Moreover, the relative symplectic cone with respect to the fiber torus T_f^2 has been determined explicitly by Geiges.

In [12], the manifolds M are classified according to the monodromy A, B of the bundle and the Euler class (x, y). A manifold M is determined by the tuple $(A, B, (x, y))$. In [62], the total spaces are classified according to their geometric type as defined by Thurston. Furthermore, an explicit representation of each is given in terms of generators of Γ such that $M = \mathbb{R}^4 \backslash \Gamma$. For example, the four torus T^4 is given by the following data: $(Id, Id, (0, 0))$ (Id is the 2x2 identity matrix) with geometric type E^4 and $\Gamma = \mathbb{Z}^4$, i.e. $T^4 = \mathbb{R}^4 \backslash \mathbb{Z}^4$. From the explicit presentation of the generators of Γ, Geiges constructs symplectic forms, thereby determining the symplectic cones as well as the relative cones with respect to the fiber torus T^2_f. In the following we denote the class of a fiber torus T^2_f by F. We collect the data in the following table, details can be found in [62] and [13]:

type	b_1	$\mathcal{C}_M$	$\mathcal{C}_M^{T^2_f}$
T^4	4	$\mathcal{P}_M$	$\mathcal{P}_M^F$
primary Kodaira surface	3	$\mathcal{P}_M$	$\mathcal{P}_M^F$
hyperelliptic surface	2	$\mathcal{P}_M$	$\mathcal{P}_M$
(d)	2	$\mathcal{P}_M$	$\emptyset$
$(e)-(h)$	2	$\mathcal{P}_M$	$\mathcal{P}_M$

Note that the class of T^2-fibrations over T^2 provides a full range of possible relative cones, from $\emptyset$ to the maximal possible cone, see Lemma 8.2.3.

8.4 Manifolds with $b^+ = 1$

In this section, we will study the relative cone with respect to a submanifold V for manifolds with $b^+ = 1$. Particularly we will completely determine $\mathcal{C}_M^V$ when M is minimal.

The symplectic cone in this case is determined in [34], Thm. 4:

Theorem 8.4.1. *Let M be a 4-manifold with $b^+ = 1$ and $\mathcal{C}_M$ nonempty. Let $\mathcal{E}_M$ denote the set of all exceptional classes of M. Then*

$$\mathcal{C}_M = \{e \in \mathcal{P}_M \mid 0 < |e \cdot E| \ \ \textit{for all} \ \ E \in \mathcal{E}_M\}.$$

In particular, if M is minimal, then $\mathcal{C}_M = \mathcal{P}_M$.

Consider the K-symplectic cone for $K \in \mathcal{K}$, see Eq. 8.1.

Theorem 8.4.2. *(Theorem 3, [34]) Let M be a 4-manifold with $b^+ = 1$. Then $\mathcal{C}_M$ is the disjoint union of $\mathcal{C}_{M,K}$ over $K \in \mathcal{K}$. For each $K \in \mathcal{K}$, $\mathcal{C}_{M,K}$ is contained in a component of $\mathcal{P}_M$, which we call the K-forward cone $\mathcal{P}^+(K)$. Moreover, for each $K \in \mathcal{K}$,*

$$\mathcal{C}_{M,K} = \{e \in \mathcal{P}^+(K) \mid e \cdot E > 0 \ \ \textit{for all} \ \ E \in \mathcal{E}_K\}.$$

The following Theorem is the main result of this section:

Theorem 8.4.3. *Let M be a 4–manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If ω is a relative symplectic form on (M, V), then*

$$\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \subset \mathcal{C}_M^V.$$

Proof. The following Lemma will be central to the proof. We defer the proof of the Lemma to Section 8.5.

Lemma 8.4.4. *Let us fix a relative symplectic form ω on (M, V). For any $A \in H_2(M;\mathbb{Z})$ with*

$$A \cdot E > 0 \text{ for all } E \in \mathcal{E}_\omega,$$

$$A \cdot \mathfrak{V} > 0, \quad A \cdot A > 0, \quad A \cdot [\omega] > 0$$

$$(A - K_\omega) \cdot [\omega] > 0, \quad (A - K_\omega) \cdot (A - K_\omega) > 0, \quad (A - K_\omega) \cdot \mathfrak{V} > 0,$$

there exists an $\omega-$symplectic submanifold C in the class A, intersecting V transversally and positively.

Fix a relative symplectic form ω, we may assume that $[\omega] \in \mathcal{C}_M^V$ is an integral class. Let $e \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z})$. Then

$$e \cdot E > 0 \text{ for all } E \in \mathcal{E}_\omega, \quad e \cdot e > 0, \quad e \cdot \mathfrak{V} > 0.$$

Moreover, by Prop. 4.1, [34], $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is contained in one component of $\mathcal{P}_M$. Hence, noting that $[\omega] \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, it follows that $e \cdot [\omega] > 0$. Thus for large $l > 0$, the class $A = le - [\omega]$ will satisfy the assumptions of Lemma 8.4.4.

Apply Lemma 8.4.4 to the class $A = le - [\omega]$ for $l >> 0$ and Lemma 8.1.4 to the pair (V, C) with C the symplectic surface of class A obtained by Lemma 8.4.4. This proves that $s[\omega] + t(le - [\omega]) \in \mathcal{C}_M^V$, hence in particular le is in the relative cone and therefore also e, i.e.

$$\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z}) \subset \mathcal{C}_M^V.$$

It also follows that any real multiple of an integral class e in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is in the relative cone $\mathcal{C}_M^V$.

We now want to show that $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \subset \mathcal{C}_M^V$. Observe that $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is an open convex cone. Therefore, for any α in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, we can write $\alpha = \sum_{i=1}^p \alpha_i$, where the rays of α_i are in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, arbitrarily close to that of α, and each $\alpha_i = s_i \beta_i$ for some positive real number s_i and an integral class $\beta_i \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$. Note that $\beta_i \cdot \beta_j > 0$ for all i, j by Lemma 8.1.1. Inductively it can be shown that for each $q \leq p$, $\sum_{i=1}^{q} \alpha_i$ is in the relative cone $\mathcal{C}^V_M$:

First we choose a relative symplectic form ω_1 for the pair (M, V) with $[\omega_1] = \alpha_1$, which we can do by the procedure described above. For a large integer l, since $\beta_2 \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z})$, we can apply Lemma 8.4.4 to $A = l\beta_2$ to obtain a submanifold C_2. Lemma 8.1.4 applied to the pair (V, C_2) on (M, ω_1) then shows that $s[\omega_1] + tA \in \mathcal{C}^V_M$. Choosing $s = 1$ and $t = \frac{s_2}{l}$ shows that $\alpha_1 + \alpha_2 = \alpha_1 + \frac{s_2}{l}A$ is in the relative cone.

Now choose a symplectic form ω_2 for the pair (M, V) with $[\omega_2] = \alpha_1 + \alpha_2$. This completes the argument.

□

Notice that when M is minimal and $\mathfrak{V} \cdot \mathfrak{V} < 0$, $\mathcal{C}^V_M$ contains a component of $\mathcal{P}^{\mathfrak{V}}$ whenever there is a relative symplectic form whose class lies in that component.

In the rest of this subsection we assume that M is minimal. The general case is more complicated and not needed in the application in Section 10, so it will be studied elsewhere.

Lemma 8.4.5. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}^V_M \neq \emptyset$. If $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} \neq 0$, then $\mathcal{C}^{\mathfrak{V}}_{M,\, \mathcal{K}(V)}$ is contained in one component of $\mathcal{P}_M$. In particular, if $\mathfrak{V} \cdot \mathfrak{V} < 0$, the inclusion $\mathcal{C}^V_M \subset \mathcal{P}^{\mathfrak{V}}$ is strict.*

Proof. Thm 1, [34] states that under the assumptions on M there is a unique symplectic canonical class up to sign. Denote this class by K. Thm 3 and Prop. 4.1, [34] show that symplectic forms with canonical class K lie in one component of the cone $\mathcal{P}_M$ while those with canonical class $-K$ lie in the other. Hence $\mathcal{C}^{\mathfrak{V}}_{M,\, \mathcal{K}(V)}$, which contains only symplectic classes representable by symplectic forms with canonical class K, is contained in one component of $\mathcal{P}_M$. Lemmas 8.1.2 and 8.2.3 complete the proof. □

The following result follows directly from Thm 1., [34]:

Lemma 8.4.6. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}^V_M \neq \emptyset$. If $2g-2-\mathfrak{V}\cdot\mathfrak{V} \neq 0$ or $\mathfrak{V}\cdot\mathfrak{V} \geq 0$, then $\mathcal{K}(V)$ is a point.*

Denote the class in $\mathcal{K}(V)$ by K. Together with Theorem 8.4.3, this leads to the following statement in the minimal case, notice that this is equality in Thm. 8.4.3:

Corollary 8.4.7. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} \neq 0$ or $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then*

$$\mathcal{C}_M^V = \mathcal{C}_{M,K}^{\mathfrak{V}}$$

is one connected component of $\mathcal{P}$. Moreover, if $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$.

Proof. Assuming $\mathcal{C}_M^V \neq \emptyset$, we may choose a relative symplectic form ω on (M, V). Then $K_\omega = K$, hence by Thm. 8.4.3

$$\mathcal{C}_{M,\,K}^{\mathfrak{V}} = \mathcal{C}_{M,\,K_\omega}^{\mathfrak{V}} \subset \mathcal{C}_M^V.$$

Moreover, by Lemma 8.2.3 $\mathcal{C}_M^V \subset \mathcal{C}_{M,\,\mathcal{K}(V)}^{\mathfrak{V}} = \mathcal{C}_{M,\,K}^{\mathfrak{V}}$.

If $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then by Lemma 8.4.5, $\mathcal{C}_{M,\,\mathcal{K}(V)}^{\mathfrak{V}} = \mathcal{C}_{M,\,K}^{\mathfrak{V}}$ is contained in one component of $\mathcal{P}_M$. The minimal case of Thm. 8.4.2 and Eq. 8.4 imply that

$$\mathcal{C}_{M,\,K}^{\mathfrak{V}} = \mathcal{P}^+(K) \cap \mathcal{P}^{\mathfrak{V}}$$

and hence by Lemma 8.1.2, that $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$. □

Remark: If M is a minimal symplectic 4-manifold with $b^+ = 1$ and K is the unique canonical class, then it was shown in Thm. 1 and Prop. 4.1, [34], that $\mathcal{C}_M = \mathcal{C}_{M,K} \cup \mathcal{C}_{M,-K}$. This decomposition can be restated in terms of the relative cone of a submanifold V with $\mathfrak{V} \cdot \mathfrak{V} \geq 0$: $\mathcal{C}_M = \mathcal{C}_M^V \cup \mathcal{C}_M^{\overline{V}}$, see Cor. 8.2.5.

Missing from the Corollary is the case $\mathfrak{V} \cdot \mathfrak{V} < 0$ and $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} = 0$. In this case V is an embedded sphere of self-intersection -2 or -1. This follows from $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} = 0$, as then $\mathfrak{V} \cdot \mathfrak{V} < 0$ can only hold if $g = 0$. We will just deal with the case $\mathfrak{V} \cdot \mathfrak{V} = -2$ since otherwise M is not minimal. In the minimal $b^+ = 1$ case, the Corollary shows, that only if V is a sphere of self intersection -2 can $\mathcal{C}_M^V$ possibly consist of two disjoint components:

Lemma 8.4.8. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If V is an embedded sphere of self-intersection -2, then $\mathcal{C}_M^V$ has two connected components and the inclusion $\mathcal{C}_M^V \cup \mathcal{C}_M^{\overline{V}} \subset \mathcal{C}_M$ is strict.*

Proof. The proof relies on Prop. 2.4, [10]: This result provides a self-diffeomorphism of M which gives rise to a reflection on cohomology. The diffeomorphism ϕ can be chosen such that $\phi(V) = V$, though ϕ does not fix V pointwise. In fact it is an antipodal map on V, reversing the orientation of V. The effect of ϕ on cohomology is the reflection:

$$\phi : x \mapsto x + (x \cdot \mathfrak{V})\mathfrak{V}$$

which is precisely the reflection across the hyperplane of classes A with $A \cdot \mathfrak{V} = 0$.

Suppose ω is a relative symplectic form for (M, V) and K_ω is its canonical class. Notice that ϕ^* preserves the canonical class K_ω since $K_\omega \cdot \mathfrak{V} = 0$. Moreover, $\phi^*(\omega)$ is a symplectic form for $(M, \overline{V})$ since ϕ preserves the submanifold V but reverses the orientation of V and the canonical class of $\phi^*(\omega)$ is $\phi^*(K_\omega) = K_\omega$. This implies that $-\phi^*(\omega)$ is a relative symplectic form for (M, V). Notice however that the canonical class of $-\phi^*(\omega)$ is $-K_\omega$ which means that $-\phi^*(\omega)$ and ω lie in different components of $\mathcal{P}_M$. $\square$

Example: Consider the manifold $M = S^2 \times S^2$. Let V be a -2 sphere in the class $S_1 - S_2 \in H_2(M)$. Choose ω to be in the class $s_1 + 2s_2 \in \mathcal{C}_M^V$ where $s_i(S_j) = 0$ if $i = j$ and 1 if $i \neq j$. Then $\phi^*(\omega)$ is in the class

$$[\omega] + ([\omega] \cdot \mathfrak{V})\mathfrak{V} = [\omega] + \mathfrak{V} = 2s_1 + s_2.$$

Hence $-\phi^*(\omega)$ is in the class $-2s_1 - s_2$.

8.5 Proof of Lemma 8.4.4

Proof. This relies on Proposition 4.3 in [34] and the genericity results of Section 3.2.

Let us first recall the notion of A being $J-$effective and simple considered in [3]. A is $J-$effective and simple if, for a generic choice of $k(A) = \frac{-K_\omega(A)+A\cdot A}{2} \geq 0$ distinct points $\Omega_{k(A)}$ in M, there exists a connected J-holomorphic submanifold $C \subset M$ which represents A and passes through all the $k(A)$ points.

Proposition 4.3 in [34] implies that any A with

$$A \cdot E \geq 0 \text{ for all } E \in \mathcal{E},$$

$$A \cdot A > 0, \quad A \cdot [\omega] > 0,$$

$$(A - K_\omega) \cdot (A - K_\omega) > 0, \quad (A - K_\omega) \cdot [\omega] > 0,$$

is $J-$effective for any J tamed by ω.

Let $\mathcal{J}_V$ be the space of $\omega-$tamed almost complex structures making V a pseudo-holomorphic submanifold. Then for every $J \in \mathcal{J}_V$ there exists a J-holomorphic curve C in class A by the previous considerations. We consider connected nodal J-holomorphic curves C representing A with multiple components each having as their image a ω-symplectic submanifold. However, for the purposes of this Lemma, we need to exclude components which lie in V. Therefore consider a connected J-holomorphic curve C representing A: The curve C is given by a collection $\{(\phi_i, \Sigma_i)\}$ of maps ϕ_i and Riemann surfaces Σ_i. We want to reformulate this as a collection $\{(\varphi_i, C_i, m_i)\}$ of simple maps, submanifolds and multiplicities. If ϕ_i is a multiple cover, we replace it by a simple map φ_i with the same image and an integer m_i tracking the multiplicity. We also combine maps which have same image, adding the multiplicities and keeping only one copy of the map. Ultimately, we replace all the maps ϕ_i by simple embeddings φ_i with image C_i. We can therefore decompose the class A as $\sum_i m_i[C_i]$. Note that we are only interested in the submanifolds C_i, so we do allow the class $[C_i]$ to be divisible. However, we want the class to represent a submanifold, hence correspond to an embedding φ_i. We denote

1. components with $[C_i]^2 < 0$ by B_i and
2. those with $[C_i]^2 \geq 0$ which do not lie in V by A_i.

The class A_i could be a multiple of the class $\mathfrak{V}$, however due to our decomposition above we consider only maps which are not multiple covers of V. Furthermore, we could have a component $[C_0] = \mathfrak{V}$, with a multiplicity $m \geq 1$, which is a (multiple) cover of V. Thus, $A = m\mathfrak{V} + \sum_i m_i B_i + r_i A_i$.

We begin with the negative square components: Let $B \cdot B < 0$. Consider the moduli space of $\mathcal{M}(B, J, g)$ of pairs (u, j), where $j \in \mathcal{T}_g$, the Teichmüller space of a closed oriented surface Σ_g of genus g, and $u : (\Sigma_g, j) \to (M, J)$ is a somewhere injective $(j, J)-$holomorphic curve in the class B. If $B \neq \mathfrak{V}$, then arguments similar to those in the proof of Lemma 3.2.1 show that for generic $J \in \mathcal{J}_V$, the spaces $\mathcal{M}(B, J, g)$ are smooth manifolds of dimension $2(-K_\omega \cdot B + g - 1) + \dim G_g$.

By the adjunction formula the space of non-parametrized $J-$holomorphic curves has dimension

$$2(-K_\omega \cdot B + g - 1) \leq 2B \cdot B - (2g - 2).$$

Thus if $B \cdot B < 0$, $\mathcal{M}(B, J, g)$ is non-empty only if $g = 0$ and $B \cdot B = -1$. We conclude, that for a generic $J \in \mathcal{J}_V$, the only irreducible components of a cusp $A-$curve with negative self-intersection (except possibly the component C_0) have $B^2 = -1$. In particular, note that all $B_i \in \mathcal{E}_\omega$ and $k(B_i) = 0$. Hence, our assumptions imply that $A \cdot B_i > 0$ for all i.

Now let us divide the proof into two cases, in both we shall use the genericity results proven in Section 3.2:

Case 1. $k(\mathfrak{V}) \geq 0$**:** The condition $k(\mathfrak{V}) \geq 0$ can be rewritten using the adjunction formula to state that $\mathfrak{V}^2 \geq 0$ if $g(\mathfrak{V}) \geq 1$ and $\mathfrak{V}^2 \geq -1$ at worst, if $g(\mathfrak{V}) = 0$. In the following we will allow the case $A = m\mathfrak{V}$.

The results of Section 3.2, in particular Lemma 3.2.1 and Lemma 3.2.3, show, that we can find a generic set of pairs $(J, \Omega_{k(A)})$ such that $k(A_i) \geq 0$, each curve in class A_i resp. $\mathfrak{V}$ meets at most $k(A_i)$ resp. $k(\mathfrak{V})$ generic points and $\sum k(A_i) + k(\mathfrak{V}) \geq k(A)$.

Further, if $k(A_i) \geq 0$ and $A_i^2 \geq 0$, then $k(r_i A_i) \geq 0$ for any positive integer r_i:

$$0 \leq 2k(A_i) \leq 2r_i k(A_i) = -K_\omega(r_i A_i) + r_i A_i \cdot A_i \leq$$

$$\leq -K_\omega(r_i A_i) + r_i^2 A_i \cdot A_i = 2k(r_i A_i).$$

Note that this holds in particular for $m\mathfrak{V}$.

For such a generic choice of $(J, \Omega_{k(A)})$, let C be a connected curve representing A, which contains the $k(A)$ distinct points of $\Omega_{k(A)}$. Then

$$2k(A) = -K_\omega(m\mathfrak{V}) + \sum_i -K_\omega(m_i B_i) + \sum_i -K_\omega(r_i A_i) +$$

$$+ m^2\mathfrak{V}^2 + \sum_i m_i^2 B_i^2 + \sum_i r_i^2 A_i^2 + 2\sum_i m\mathfrak{V} m_i B_i + 2\sum_i m\mathfrak{V} r_i A_i +$$

$$+2\sum_{i>j} m_i m_j B_i \cdot B_j + 2\sum_{i>j} m_i r_j B_i \cdot A_j + 2\sum_{i>j} r_i r_j A_i \cdot A_j \geq$$

$$\geq 2mk(\mathfrak{V}) + 2\sum_i r_i k(A_i) + (m^2 - m)\mathfrak{V}^2 + 2\sum_i m\mathfrak{V} r_i A_i + 2\sum_{i>j} r_i r_j A_i \cdot A_j +$$

$$+\sum_i (m_i^2 - m_i) B_i^2 + 2\sum_{i>j} m_i m_j B_i \cdot B_j + 2\sum_{i>j} m_i r_j B_i \cdot A_j + 2\sum_i m\mathfrak{V} m_i B_i$$

If $\mathfrak{V}^2 = -1$, then denote $B_0 = \mathfrak{V}$ and include it in the following estimate. Fix an i and consider the terms in the last line. They can be rewritten as

$$2m_i A \cdot B_i - 2m_i^2 B_i^2 + (m_i^2 - m_i) B_i^2 \;=\; 2m_i A \cdot B_i + m_i^2 + m_i \;\geq 0$$

and thus we obtain the estimate

$$2k(A) \geq 2k(\mathfrak{V}) + 2\sum_i k(A_i).$$

Hence either $k(A) > k(\mathfrak{V}) + \sum_i k(A_i)$ or the following hold:

- $m_i = 0$ for all i, i.e. there are no components of negative square,
- $A_i \cdot A_j = 0 = A_i \cdot \mathfrak{V}$ for $i \neq j$,
- if $\mathfrak{V}^2 \geq 0$, then $m = 1$ or $k(\mathfrak{V}) = 0$ and $\mathfrak{V}^2 = 0$ and
- $r_i = 1$ or $k(A_i) = 0$.

Therefore, the curve C representing A is an embedded J-holomorphic submanifold with a single non-multiply covered component containing the set $\Omega_{k(A)}$ with $J \in \mathcal{J}_V$ or $k(A) = 0$.

In the following cases we are done:

1. $A \neq m\mathfrak{V}$
2. $A = m\mathfrak{V}$ and $k(\mathfrak{V}) > 0$: The results above imply that $m = 1$. Choose $\Omega_{k(A)}$ such that it contains a point not in V. Then C does not lie in V and intersects V locally positively.

We are left with the following case: $A = m\mathfrak{V}$ and $k(\mathfrak{V}) = 0$. However, in this case the previous results show that either $m = 1$ or $\mathfrak{V}^2 = 0$. The latter is excluded by the assumption $A^2 > 0$. The former would mean $A = \mathfrak{V}$ and thus

$$0 = 2k(\mathfrak{V}) = \mathfrak{V}^2 - K_\omega \mathfrak{V} = \mathfrak{V}(A - K_\omega)$$

which is also excluded by assumption.

In all cases, we can perturb C to be transverse to V, see [46], [47].

Case 2: $k(\mathfrak{V}) < 0$: In this case, the results of Section 3.2 show, that we can find a generic set of almost complex structures, such that V is rigid and there are no other curves in class $\mathfrak{V}$. In the following, we choose only complex structures from this set.

Even though we are working in the case $k(\mathfrak{V}) < 0$, it is possible for a multiple class $m\mathfrak{V}$ to have $k(m\mathfrak{V}) \geq 0$. For this reason, we will distinguish the following two objects:

1. Classes $A_i = m_i\mathfrak{V}$ which correspond to components of the curve C in class $m\mathfrak{V}$, but which are NOT multiple covers of a submanifold in class $\mathfrak{V}$. If $\mathfrak{V}^2 < 0$, then positivity of intersections shows that any curve C can contain at most one component in class $m\mathfrak{V}$ for all m and this component must coincide with the manifold V. This situation was studied in greater generality in [3]. Furthermore, if $\mathfrak{V}^2 \geq 0$ and a class $A_i = m\mathfrak{V}$ occurs, then the results of Lemma 3.2.1 apply. We may therefore assume , that $A_i^2 \geq 0$ in the following.

2. The specific "class" mV which corresponds to components which have as their image the submanifold V.

Note further, that we can choose our almost complex structures such that the components corresponding to mV are rigid, while those in $m_i\mathfrak{V}$ are not. Such a decomposition is not necessary in the case $k(\mathfrak{V}) \geq 0$, as we can choose a generic set of pairs such that $\mathcal{K}_V^{\mathfrak{V}}(J, \Omega)$ is smooth (see Section 3.2 for details), however we do not know if V is an element in this set, nor does this matter for the calculation. In the current situation, the specific submanifold V acts differently than other elements in the class $\mathfrak{V}$.

We now proceed as in [3]: Consider the class $\tilde{A} = A - mV = \sum r_i A_i + m_i B_i$. We assume that such a decomposition is possible, i.e. there exists a not necessarily connected pseudoholomorphic curve in class $\tilde{A}$. The case $A = mV$ will be considered afterwards. We need to show, that there exists a generic set of pairs $(J, \Omega_{k(A)})$ such that $k(A) > \sum k(A_i)$. Proceeding exactly as in [3], we obtain the following two estimates: $\sum k(A_i) \leq k'(\tilde{A})$, where k' is the modified count defined by McDuff [48]. Furthermore

$$2k(A) - 2k'(\tilde{A}) = m \underbrace{(A - K_\omega) \cdot V}_{>0 \text{ by assumption}} + \text{non-negative terms}$$

and hence, combining all these inequalities, for generic pairs $(J, \Omega_{k(A)})$ we obtain $k(A) > \sum k(A_i)$. We therefore conclude, that we can rule out such a decomposition, hence

"$A = mV$" or $A = \sum r_i A_i + m_i B_i$. In the latter case we are done by the same line of argument as in the $k(\mathfrak{V}) \geq 0$ case, albeit with the added restriction on the almost complex structures that V is rigid and no further curves in class $\mathfrak{V}$ exist. The case "$A = mV$" corresponds to the case $A = m\mathfrak{V}$ and $k(A) \geq 0$, more precisely,

$$0 < m(A - K_\omega) \cdot V = mA \cdot V - K_\omega \cdot V = A^2 - K_\omega \cdot A = k(A).$$

Thus, applying Lemma 3.2.1, we can find a generic set of pairs $(J, \Omega_{k(A)})$ such that $A = m\mathfrak{V}$ is represented by an embedded curve with deformations. Choosing $\Omega_{k(A)}$ such that it contains a point not in V ensures that a representative of A in this case does not lie in V.

As before, we can make the curve C transverse to V.

$\square$

Chapter 9

The gluing formula

We now return to Conjecture 1.3.1 which was stated in the introduction: Is the symplectic cone $\mathcal{C}_M$ equal to the $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$ for minimal Kähler manifolds? In this section we provide the theoretical framework necessary to answer this question for so called "good" sums. We first review the symplectic sum and cut operations. This leads to the definition of a good sum and the subsequent homological reformulation of these operations.

9.1 Smooth fiber sum

Let X, Y be $2n$-dimensional smooth manifolds. Suppose we are given codimension 2 embeddings $j_* : V \to *$ into X and Y of a smooth closed oriented manifold V with normal bundles N_*V. Assume that the Euler classes of the normal bundle of the embedding of V in X resp. Y satisfy $e(N_XV)+e(N_YV)=0$ and fix a fiber-orientation reversing bundle isomorphism $\Theta : N_XV \to N_YV$. By canonically identifying the normal bundles with a tubular neighborhood ν_* of $j_*(V)$, we obtain an orientation preserving diffeomorphism $\varphi : \nu_X \backslash j_X(V) \to \nu_Y \backslash j_Y(V)$ by composing Θ with the diffeomorphism that turns each punctured fiber inside out. This defines a gluing of X to Y along the embeddings of V denoted $M = X\#_{(V,\varphi)}Y$. The diffeomorphism type of this manifold is determined by the embeddings (j_X, j_Y) and the map Θ. Note also, that if V has trivial normal bundle, then this construction should actually be viewed as a sum along $V \times S^1$.

In the rest of the paper, whenever we consider a fiber sum, we fix V, the embeddings

j_* and the bundle isomorphism Θ without necessarily explicitly denoting either. This fixes the homology of the manifold $M = X\#_{(V,\varphi)}Y$.

Example 9.1.1. *Consider for example the torus $T^4 = T^2_f \times T^2$, where the first factor is the fiber direction. Let $M = T^4\#_{T^2_f}T^4$ be the sum along the fiber T^2_f. Then M is actually $T^2_f \times \Sigma_2$, as can be seen from the following:*

$$\begin{tikzcd}
T^2_f \arrow[r] & T^4 \arrow[d] & T^4 \arrow[d] & T^2_f \arrow[l] & \Longrightarrow & T^2_f \arrow[r] & T^4\#_{T^2_f}T^4 = M \arrow[d] \\
& T^2 & T^2 & & & & T^2\#T^2 = \Sigma_2
\end{tikzcd}$$

9.2 Symplectic sum and symplectic cut

We briefly describe the symplectic sum construction $M = X\#_V Y$ as defined by Gompf [14] (see also McCarthy-Wolfson [43]). Assume X and Y admit symplectic forms ω_X, ω_Y resp. If the embeddings j_* are symplectic with respect to these forms, then we obtain $M = X\#_{(V,\varphi)}Y$ together with a symplectic form ω created from ω_X and ω_Y. It was shown in [14] that this can be done without loss of symplectic volume.

Furthermore, Gompf showed that the symplectic form ω thus constructed on $M = X\#_{(V,\varphi)}Y$ from ω_X, ω_Y is unique up to isotopy. This result allows one to construct a smooth family of isotopic symplectic sums $M = X\#_{(V,\varphi_\lambda)}Y$ parametrized by $\lambda \in D^2\backslash\{0\}$ as deformations with a singular fiber $X \sqcup_V Y$ over $\lambda = 0$ (see [21], Sect. 2). Therefore, we suppress φ_λ from the notation, choosing instead to work with an isotopy class where necessary.

Thus, a symplectic sum will be denoted by $M = X\#_V Y$, a symplectic class ω on the sum will denote an isotopy class.

The symplectic cut operation of Lerman [26] functions as follows: Consider a symplectic manifold (M, ω) with a Hamiltonian circle action and a corresponding moment map $\mu : M \to \mathbb{R}$. We can assume that 0 is a regular value, if necessary by adding a constant. We can thus cut M along $\mu^{-1}(0)$ into two manifolds $M_{\mu>0}$ and $M_{\mu<0}$, both of which have boundary $\mu^{-1}(0)$. If we collapse the S^1-action on the boundary, we obtain manifolds $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ which contain a real codimension 2 submanifold

$V = \mu^{-1}(0)/S^1$. If we symplectically glue $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ along V we obtain again M.

Note that the above construction is local in nature, thus the moment map and the S^1 action need only be defined in a neighborhood of the cut.

The symplectic structure ω restricted to $M_{\mu>0}$ and $M_{\mu<0}$ reduces to a symplectic structure on $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ which have the same value on V. This motivates the sum decomposition of the symplectic cones in section 9.3.

A symplectic cut is only possible on a symplectic manifold, thus, when discussing a symplectic cut on $M = X\#_V Y$, we implicitly consider only those isotopy classes allowing moment maps μ with $V = \mu^{-1}(0)\backslash S^1$.

In the case $M = T^4\#_{T_f^2}T^4$ it is possible to understand the geometric construction underlying the cut: Consider $M = T^4\#_{T_f^2}T^4 = T^2 \times \Sigma_2$ and view Σ_2 such that we have the holes on either end and a cylindrical connecting piece S in between. Furthermore, in M this copy of Σ_2 is transverse to the fiber T^2. Choose local coordinates (λ, θ, t) on $S \times T^2$, $(\lambda, \theta) \in [-1, 1] \times [0, 2\pi]$ coordinates on S, t a coordinate on T^2. Consider an S^1 action on the second coordinate stemming from the Hamiltonian $\mu : S \to \mathbb{R}$ given by $\mu(\lambda, \theta, t) = \lambda$. Locally, any symplectic form is given by $\omega = a\ d\lambda \wedge d\theta + b\ dt + \Omega$, with $\Omega(\frac{\partial}{\partial\theta}, \cdot) = 0$ and $a \in \mathbb{R}$ nonzero.

The symplectic cut defined by μ produces $M_{\mu<0} = T_b^2 \times T^2$ with T_b^2 a punctured torus with boundary S^1. Over each point of the boundary, there is a fiber T_f^2, hence $\partial M_{\mu<0} = T^3$. Collapsing this boundary under the S^1 action produces $T^4 = T_f^2 \times T^2$. In particular, the action maps $d\lambda \wedge d\theta$ to local coordinates on a neighborhood of the collapsed boundary on T_b^2 without loss of volume.

9.3 $\mathcal{C}^V_{X\#_V Y}$ from $\mathcal{C}^V_X$ and $\mathcal{C}^V_Y$

9.3.1 The cone of sum forms

We are interested in the symplectic cone $\mathcal{C}_M$ of the 4-manifold M. Suppose this manifold can be obtained as a symplectic sum $M = X\#_V Y$. Let us fix the symplectic embeddings as well as the map Θ. In the following, we will distinguish between the manifold M and the specific viewpoint as a symplectic sum from X and Y along V by explicitly denoting $M = X\#_V Y$. Accordingly, we define the following symplectic cone associated to the symplectic sum:

Definition 9.3.1. *Suppose that $M = X\#_V Y$. Define the cone of sum forms, $\mathcal{C}^{sum}_{X\#_V Y}$, to be the set of classes of symplectic forms on M which can be obtained by summing X and Y with symplectic embeddings $\tilde{j}_*$ and bundle map $\tilde{\Theta}$ isotopic to the fixed choice j_* and Θ.*

We obtain the following result:

Theorem 9.3.2. *For a symplectic manifold $M = X\#_V Y$,*

$$\mathcal{C}^{sum}_{X\#_V Y} = \Psi(\Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y)) \tag{9.1}$$

where Φ, Ψ are the maps on cohomology corresponding to the inclusion of X, Y into $X \sqcup_V Y$ and the projection of $X\#_V Y$ onto the singular manifold $X \sqcup_V Y$ respectively (see (9.2) below).

Proof. Consider the following maps on cohomology:

$$\begin{tikzcd} H^2(X \sqcup_V Y) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & H^2(X\#_V Y) \\ H^2(X) \oplus H^2(Y) & \end{tikzcd} \tag{9.2}$$

Let $C_{X\sqcup_V Y} := \Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y) \in H^2(X \sqcup_V Y)$. We can view this set as the collection of classes of symplectic forms in X and Y which are symplectic and equal on V. More precisely, we obtain $C_{X\sqcup_V Y}$ by pulling back of $\mathcal{C}_V$ under the restriction maps r_X, r_Y from (8.5):

$$\begin{tikzcd} C_{X\sqcup_V Y} \arrow[r] \arrow[d] & \mathcal{C}^V_X \arrow[d, "r_X"] \\ \mathcal{C}^V_Y \arrow[r, "r_Y"] & \mathcal{C}_V \end{tikzcd}$$

The symplectic sum takes (X, ω_X) and (Y, ω_Y) and produces a symplectic manifold $(X\#_V Y, \omega)$. This will work for any relative symplectic forms ω_X and ω_Y identical on V (By identical we mean that the symplectomorphism used to produce the symplectic singular manifold $X \sqcup_V Y$ maps these two forms symplectically to each other along V.). Thus any symplectic class $(\alpha_X, \alpha_Y) \in C_{X\sqcup_V Y}$ can be summed to produce a symplectic class $\alpha \in \mathcal{C}^{sum}_{X\#_V Y}$. Therefore $\Psi(\Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y)) \subset \mathcal{C}^{sum}_{X\#_V Y}$.

On the other hand, given any symplectic class in $\mathcal{C}^{sum}_{X\#_V Y}$, any symplectic representative ω of such a class can be symplectically cut, such that the manifolds (X, V) and (Y, V) result with symplectic forms ω_X and ω_Y which agree on V. Hence, $\Psi^{-1}\mathcal{C}^{sum}_{X\#_V Y} \subset \Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y)$. □

Remark:

1. Theorem 9.3.2 is valid for any dimension.

2. In general, the cone of sum forms will be a strict subset of the relative cone $\mathcal{C}^V_M$. For example, consider $M = K3\#_{T^2_f}K3$, the fiber sum of two K3 surfaces along a fiber torus. This has $b_2(M) = 45$ which is also the dimension of the relative cone $\mathcal{C}^{T^2_f}_M$. On the other hand, $\mathcal{C}^{T^2_f}_{K3}$ has dimension 22. Hence the cone of sum forms must be a strict subset of the relative cone. This indicates, that a precise study of the second homology of the symplectic sum $M = X\#_V Y$ should be interesting, and we dedicate the rest of the section to this analysis.

9.3.2 The Second Homology of $M = X\#_V Y$

We assume that X, Y are 4-manifolds and that V has trivial normal bundle. The latter ensures that the class of V will exist in $H_2(X\#_V Y)$ after summing, albeit not the particular copy of V along which was summed. Denote the class of V by $f \in H_2(X\#_V Y)$. In this section, we shall describe a "natural" basis of the second homology with respect to the fiber sum operation, which will allow us to efficiently construct and deconstruct cohomology classes on $X\#_V Y$.

We begin by detailing the role of the maps involved in the symplectic sum in the structure of the second homology of $M = X\#_V Y$. The homology of M can be analyzed by the Mayer-Vietoris sequences for the triples $(X \sqcup_V Y, X, Y)$ and $(X\#_V Y, X\backslash V, Y\backslash V)$

:

$$\begin{tikzcd}
H_2(V) \arrow[r, "(j_X, j_Y)_*"] & H_2(X) \oplus H_2(Y) \arrow[r, "\phi"] & H_2(X \sqcup_V Y) \\
H_2(S_V) \arrow[u] \arrow[r, "\lambda"] & H_2(X \backslash V) \oplus H_2(Y \backslash V) \arrow[u] \arrow[r, "\rho"] & H_2(X \#_V Y) \arrow[u, "\psi"'] \\
R_V \arrow[u] & & \mathcal{R}_{X \#_V Y} \arrow[u]
\end{tikzcd} \tag{9.3}$$

The map λ, defined on classes in the homology of the normal unit circle bundle S_V, is induced by the canonical identification of the tubular neighborhoods and the normal bundles. The map ρ is identity on the classes which are supported away from V and on classes supported near V is defined by the gluing map φ, in particular by Θ. The map $\phi : H_2(X) \oplus H_2(Y) \to H_2(X \sqcup_V Y)$ produces classes with the appropriate matching conditions on V as determined by Θ in preparation for summing along V. The map $\psi : H_2(X \#_V Y) \to H_2(X \sqcup_V Y)$ is induced by the gluing map φ, in particular by the embeddings j_* and the isomorphism Θ. Then ψ correctly decomposes classes in $X \#_V Y$ in accordance with the symplectic gluing. The set $\mathcal{R}_{X \#_V Y}$ is completely determined by R_V and an understanding of how these classes bound in M. The outer columns are exact, for a detailed discussion of the kernel R_V see [22], Sect. 5.

We are in the four dimensional setting, thus when we consider the Poincaré dual diagram to (9.3), we obtain in particular the following component:

$$\begin{tikzcd}
H^2(X \sqcup_V Y) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & H^2(X \#_V Y) \arrow[r] & \mathcal{R}^D_{X \#_V Y} \arrow[r] & 0 \\
H^2(X) \oplus H^2(Y) & & &
\end{tikzcd} \tag{9.4}$$

This is precisely the diagram used in the proof of Theorem 9.3.2. This motivates the detailed discussion of the generators of the second homology which follows.

To explicitly describe the second homology of $M = X \#_V Y$, we consider the following part of the Mayer-Vietoris sequences as before:

$$
\begin{tikzcd}
H_2(X) \oplus H_2(Y) \arrow[r, "\phi"] & H_2(X \sqcup_V Y) \arrow[r, "\delta"] & H_1(V) \\
H_2(X \backslash V) \oplus H_2(Y \backslash V) \arrow[u] \arrow[r, "\rho"] & H_2(M) \arrow[u, "\psi"] \arrow[r, "(\gamma,t)"] & H_1(S_V) \simeq H_1(S^1) \oplus H_1(V) \arrow[u, "\mu"'] \\
 & \mathcal{R}_{X\#_V Y} \arrow[u] &
\end{tikzcd}
\tag{9.5}
$$

Define the subgroups $x = \rho(H_2(X\backslash V), 0)$ and $y = \rho(0, H_2(Y\backslash V))$ of $H_2(M)$. Elements in x and y are representable by submanifolds in X, Y resp. which are supported away from the submanifold V. Denote generators of x and y by x_i and y_i. Note that $x_i \cdot f = 0 = y_i \cdot f$ in the intersection form.

Define the subgroup $\Gamma = (\gamma, t)^{-1}(H_1(S^1), 0) \simeq \mathbb{Z}$. Representatives γ^M of this subgroup are submanifolds formed from submanifolds $\gamma^X \in X$ and $\gamma^Y \in Y$, these being supported in any neighborhood of V and thus affected by the sum construction. The submanifolds γ^* intersect V nontrivially and $\gamma^* \cdot f = 1$. We denote the generator of this subgroup by γ^M. Note that $\psi(\gamma^M)$ is always nontrivial: $\psi(\gamma^M) = (\gamma^X, \gamma^Y)$.

Define $\tau = (\gamma, t)^{-1}(0, H_1(V))$ and $\mathcal{R}_M = \ker\psi$. The following holds:

Lemma 9.3.3. $\psi(\tau) = coker(\phi)$

Proof. Let g_i be generators of $H_1(V)$. Then τ is generated by

$$\tau_i = (\gamma, t)^{-1}(0, g_i).$$

The commutativity of (9.5) shows

$$\delta\psi(\tau_i) = (\mu)(\gamma, t)\tau_i = \mu(0, t(\tau_i)) = t(\tau_i) = g_i. \tag{9.6}$$

Thus it follows from $g_i \neq 0$ that $\psi(\tau_i) \notin \ker\delta$. Thus

$$\psi(\tau_i) \in H_2(X \sqcup_V Y)/\ker\delta = H_2(X \sqcup_V Y)/im\phi = coker(\phi). \tag{9.7}$$

Hence $\psi(\tau) \subset coker(\phi)$.

From the definition of $coker(\phi)$ it follows that $coker(\phi) = H_2(X \sqcup_V Y)/im\phi$ and hence any nontrivial element c of the cokernel is supported in a neighborhood of V, but is not generated out of elements in $H_2(X)$ and $H_2(Y)$. In particular, $c \cdot \gamma^* = 0$. Thus any lift $\tilde{\tau}$ of an element c in the cokernel by ψ to $H_2(M)$ has $\gamma(\tilde{\tau}) = 0$. Furthermore,

$$0 \neq \delta(c) = \delta\psi(\psi^{-1}\tau) = (\mu)(\gamma, t)\tilde{\tau} = t(\tilde{\tau}), \tag{9.8}$$

so $(\gamma, t)\tilde{\tau} \in 0 \oplus H_1(V)$ is nontrivial. Therefore, $\tilde{\tau} \in \tau$ and thus $coker(\phi) \subset \psi(\tau)$. $\square$

Consider now the set $\mathcal{R}_{X\#_V Y}$. In particular, let us describe how objects in this set are generated from submanifolds of $X \backslash V$ and $Y \backslash V$. Define $\mathcal{R}_Y$ as the image of the map $i^Y_* \Delta : H_1(V) \to H_2(Y \backslash V)$ where i^Y is the inclusion of S_V into $Y \backslash V$ and $\Delta : H_1(V) \to H_2(S_V)$ stems from the Gysin sequence for the bundle $S_V \to V$. If V is 2 dimensional, then the map Δ is an injection. Furthermore, consider for each simple closed curve l in V the preimage in $\partial N_Y V$, this is a torus. Such tori are called rim tori. We restate a result in [22]:

Lemma 9.3.4. *(Lemma 5.2, [22]) Each element $R \in \mathcal{R}_Y$ can be represented by a rim torus.*

Under symplectic gluing, rim tori glue and are the elements of $\mathcal{R}_M$, in particular the elements $i^X_* \Delta l$ and $-i^Y_* \Delta l$ for some loop $l \in H_1(V)$ glue. An example of this process is the generation of non-fiber tori in $K3$ when viewed as a sum $E(1)\#_{T^2_f} E(1)$ (See [16], Sect. 3.1). These are then precisely the elements of $\mathcal{R}_{K3} = \{T^2_1, T^2_2\}$ and, in the same process, $\tau = \{S^2_1, S^2_2\}$ is produced. This accounts for the two new hyperbolic terms in the intersection form. We observe the following

Lemma 9.3.5. *Assume that $H_1(V) \to H_1(Y)$ is an injection and V has trivial normal bundle. Then Y has no rim tori and $\tau = 0 = \mathcal{R}_{X\#_V Y}$.*

Proof. To prove that Y has no rim tori, it will suffice to show, that $i_* : H_2(S_V) \to H_2(Y \backslash V)$ is trivial on elements which are trivial under the map $\pi_* : H_2(S_V) \to H_2(V)$. Therefore, consider the map $\xi : H_3(Y) \to H_2(S_V)$ where $S_V = V \times S^1$. Let $W \in H_3(Y)$, then $\xi(W) = W \cap S_V = W \cap (V \times S^1)$. In particular, $W \cap V \in H_1(V)$, thus by the injectivity assumption, if this intersection is non-trivial, it is non-trivial in $H_1(Y)$.

Therefore, the map ξ maps $H_3(Y)$ onto the space generated by $\alpha \times S^1$ and $\beta \times S^1$, where α, β are generators of $H_1(V)$. This space is the kernel of π_* and the map i_* is trivial on it.

Let us now consider $\mathcal{R}_{X\#_V Y}$. Elements of this set are constructed by symplectic gluing from elements in $X \backslash V$ and $Y \backslash V$, or equivalently, from classes in $H_2(X \backslash V)$ and $H_2(Y \backslash V)$. In particular, considering (9.3), only classes in R_V are relevant, these are precisely those classes which do not map trivially to $H_2(V)$. As we have seen above, our assumption implies that $H_2(S_V) \to H_2(Y \backslash V)$ is trivial. Hence every element in $\mathcal{R}_{X\#_V Y}$ would be trivial.

Thus $\mathcal{R}_{X\#_V Y} = 0$. Furthermore, $\tau = 0$ is clear from the assumption. □

The previous discussion allows us to explicitly state a set of generators for $H_2(M)$:

- $\{f\}$ is the fiber class present in both X and Y, in our case this is the class of V;
- $\{x_i\}$, $\{y_i\}$;
- $\{k_i\} \subset \mathcal{R}_{X\#_V Y}$, generators which are represented by submanifolds mapping to 0 in $X \sqcup_V Y$;
- $\{\gamma\}$ generated out of elements of the homology of both X and Y, e.g $[\Sigma_2]$ from copies of T^2 in Example 9.1.1. Note that this is the origin for the non-surjectivity of the map ψ: $\psi[\Sigma_2]$ will always have a fixed relative orientation of the two copies of T^2 into which Σ_2 degenerates. Thus the pairing of the tori with opposite orientation will not lie in the image of ψ.
- $\{\tau_i\} \subset \tau$; these objects will persist in $X \sqcup_V Y$ and hence contribute to its homology as well.

Given this set of generators, we can explicitly state how an element in the cone of sum forms decomposes: Given $\alpha = \sum_i a_i X_i + b_i Y_i + cF + g\Gamma + e_i \mathcal{R}_i + t_i T_i \in \mathcal{C}^{sum}$ and taking the Poincaré dual basis of the one given above, we obtain two forms $\alpha_X = \sum_i a_i X_i + c^X F^X + g\Gamma^X$ and $\alpha_Y = \sum_i b_i Y_i + c^Y F^Y + g\Gamma^Y$ on X resp. Y. Note that this is ultimately a direct result of Theorem 9.3.2.

9.3.3 Good sums

If we know the relative cones of X and Y, then, considering (9.3), we should obtain information on the structure of the relative cone on $M = X\#_V Y$ by using the Poincaré duals of the maps ϕ and ψ. For this to work nicely, one needs ϕ to be surjective and ψ to be injective. We thus make the following definition:

Definition 9.3.6. *A symplectic sum $M = X\#_V Y$ is called* <u>*good*</u> *if ϕ is surjective and ψ is injective.*

This statement is equivalent to $\tau = 0 = \mathcal{R}$, and Lemma 9.3.5 provides a simple criterion to check this.

Theorem 9.3.7. *Suppose $M = X\#_V Y$ is good and V has trivial normal bundle. If for X, Y,*

$$\mathcal{C}_*^V = \{\alpha \in \mathcal{P}_* \mid \alpha \cdot \mathfrak{V} > 0\}, \tag{9.9}$$

then

$$\mathcal{C}_{X\#_V Y}^{sum} = \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}. \tag{9.10}$$

Consequently $\mathcal{C}_M^V = \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}$.

Proof. The second result is immediate: Theorem 9.3.2 and Lemma 8.2.3 show that $\mathcal{C}_{X\#_V Y}^{sum} \subset \mathcal{C}_M^V \subset \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}$.

The first result follows, if we can show $\{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\} \subset \mathcal{C}_{X\#_V Y}^{sum}$.

We proceed as remarked above, using Theorem 9.3.2: Taking the Poincaré dual basis of the one given above, we can write each $\alpha \in \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}$ as

$$\alpha = \sum_i a_i X_i + b_i Y_i + cF + g\Gamma + e_i \mathcal{R}_i + t_i T_i; \quad g > 0. \tag{9.11}$$

As $\mathcal{R} = 0 = \tau$, the last two terms drop.

We must now show, that $\alpha \in \mathcal{C}_{X\#_V Y}^{sum}$. We thus choose a possible pair of classes α_X and α_Y in $H^2(X)$ resp. $H^2(Y)$ as determined by Theorem 9.3.2 and show that this can be done in such a way as to ensure that they are representable by a relative symplectic form. We first determine a relation which preserves the volume. In the following, we show how to choose this pair, so that they lie in their respective relative cones $\mathcal{C}_*^V$.

Hence the class α can be obtained by summing two classes representable by relative symplectic forms and thus, by Gompf's result, $\alpha \in \mathcal{C}^{sum}_{X\#_V Y}$.

Choose the candidates for classes summing to α as follows:

$$\begin{aligned} \alpha_X &= \textstyle\sum_i a_i X_i + c^X F^X + g\Gamma^X \in H^2(X) \\ \alpha_Y &= \textstyle\sum_i b_i Y_i + c^Y F^Y + g\Gamma^Y \in H^2(Y) \end{aligned} \tag{9.12}$$

where F^* and Γ^* are the Poincaré duals on X, Y. The coefficient g must be the same for both, as $g = \alpha(\mathfrak{V}) = \alpha_X(\mathfrak{V}) = \alpha_Y(\mathfrak{V})$. The class F has $F^2 = 0$ due to the triviality of the normal bundle of V, similarly $(F^*)^2 = 0$. The volume of each of these is

$$\alpha^2 = (\sum a_i X_i)^2 + (\sum b_i Y_i)^2 + (g\Gamma)^2 + \tag{9.13}$$

$$+2\sum (a_i X_i g\Gamma + b_i Y_i g\Gamma) + cFg\Gamma$$

and

$$\alpha_X^2 = (\sum a_i X_i)^2 + g^2(\Gamma^X)^2 + 2\sum \left(a_i X_i g\Gamma^X\right) + c^X F^X g\Gamma^X. \tag{9.14}$$

Thus the difference of the volumes is calculated to be

$$\alpha^2 - \alpha_X^2 - \alpha_Y^2 = (g\Gamma)^2 - \left(g\Gamma^X\right)^2 - \left(g\Gamma^Y\right)^2 \tag{9.15}$$

$$+ 2\left(\sum a_i X_i g\Gamma - \sum a_i X_i g\Gamma^X\right) \tag{9.16}$$

$$+ 2\left(\sum b_i Y_i g\Gamma - \sum b_i Y_i g\Gamma^Y\right) \tag{9.17}$$

$$+ 2\left(cFg\Gamma - c^X F^X g\Gamma^X - c^Y F^Y g\Gamma^Y\right) \tag{9.18}$$

Note the following: The morphism $\Psi : H^2(X \sqcup_V Y) \to H^2(M)$ relates the intersection forms, giving the following relations:

1. $(\Gamma^X)^2 + (\Gamma^Y)^2 = (\Gamma^X \oplus \Gamma^Y)^2 = \Psi((\Gamma^X \oplus \Gamma^Y)^2) = \Psi(\Gamma^X \oplus \Gamma^Y)^2 = \Gamma^2$
2. $\alpha_i X_i \beta\Gamma = \Psi(\alpha_i X_i(\beta\Gamma^X \oplus \beta\Gamma^Y)) = \alpha_i X_i \beta\Gamma^X$

Applying these relations, it follows immediately that 9.15 is trivial,

$$9.16 \Rightarrow a_i X_i g\Gamma - a_i X_i g\Gamma^X = a_i(g-g)X_i\Gamma^X = 0 \tag{9.19}$$

and analogously for 9.17 and Y. Furthermore, (9.18) becomes

$$cFg\Gamma - c^X F^X g\Gamma^X - c^Y F^Y g\Gamma^Y \tag{9.20}$$

$$= cFg\Gamma - \Psi(c^X F^X g\Gamma^X + c^Y F^Y g\Gamma^Y)$$
$$= cFg\Gamma - \Psi(c^X F^X + c^Y F^Y)g\Gamma.$$

The condition for this to vanish is

$$\Psi(c^X F^X + c^Y F^Y) = cF. \tag{9.21}$$

Thus, by choosing $c = c^X + c^Y$ we preserve the volume.

Now, we must show that this can be done in such a way, as to ensure $\alpha_*^2 > 0$. Choose c^X, c^Y so that volume is preserved. Then $\alpha^2 = \alpha_X^2 + \alpha_Y^2 > 0$, and we may assume $\alpha_X^2 > 0$. This holds true for any choice of c^* satisfying 9.21.

Squaring α_X and denoting $B = \sum a_i X_i + g\Gamma^X$, we obtain

$$f(c^X) = \alpha_X^2 = B^2 + 2B \cdot F^X c^X + (c^X)^2 (F^X)^2 = B^2 + 2B \cdot F^X c^X. \tag{9.22}$$

We can always solve $f(c^X) = \rho$ for any $\rho > 0$. Thus we can ensure that $\alpha^2 > \alpha_X^2 > 0$ holds. Then also $\alpha_Y^2 = \alpha^2 - \alpha_X^2 > 0$ holds, and thus each α_* must lie in $\mathcal{C}_*^V$, hence $\tilde{\alpha} = (\alpha_X, \alpha_Y) \in \mathcal{C}_{X \sqcup_V Y}$ by definition of this set. Thus $\{\alpha \in \mathcal{P}_* \mid \alpha \cdot \mathfrak{V} > 0\} \subset \mathcal{C}^{sum}_{X \#_V Y}$. $\square$

This result is of particular interest, as it shows that good sums preserve the structure of the relative cone. Thus, if X, Y have relative cones as assumed in the Theorem and the sum is good, we can apply this result repeatedly to obtain the relative cone of $nX \#_V mY$:

$$\mathcal{C}^V_{nX \#_V mY} = \{\alpha \in \mathcal{P}_{nX \#_V mY} \mid \alpha \cdot \mathfrak{V} > 0\}. \tag{9.23}$$

Chapter 10

Symplectic cone of certain T^2−fibrations

10.1 $T^2 \times \Sigma_g$

We now show that Theorem 9.3.7 can be applied to $T^2 \times \Sigma_g$. The results of the previous section assume two things: a certain form of the relative symplectic cone and that the sum be good.

Fix $Y = T^2 \times \Sigma_k$. Thus by Lemma 9.3.5 we don't need to verify the condition $\mathcal{R} = 0 = \tau$ when applying Theorem 9.3.7, i.e. all sums $X\#_V Y$ are good.

The following result follows immediately:

Theorem 10.1.1. *Let $M = T^2 \times \Sigma_k$. Then*

$$\mathcal{C}_M^{T_f^2} = \{\alpha \in \mathcal{P}_M \mid \ \alpha \cdot [T_f^2]^D > 0\}. \tag{10.1}$$

Consequently, $\mathcal{C}_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$.

Proof. We proceed by induction: Let $M = T^4$. Then the result holds due to Lemma 8.3 and Cor. 8.2.6. Summing repeatedly we obtain

$$M = T^2 \times \Sigma_k = T^4 \#_{T_f^2} (T^2 \times \Sigma_{k-1}).$$

Using the induction hypothesis, which ensures that

$$\mathcal{C}_{T^2\times\Sigma_{k-1}}^{T_f^2} = \{\alpha \in \mathcal{P}_{T^2\times\Sigma_{k-1}} \mid \ \alpha \cdot [T_f^2]^D > 0\},$$

the result now follows from Theorem 9.3.7 (see Eq. 9.23) and Cor. 8.2.6. Note that $b^+ \geq 3$ for any k, hence by Thm IV.2.7, [1], we have $p_g > 0$. $\square$

As noted in the introduction, this result also follows from results in [8] and [9].

Remark: (Fibered symplectic forms) Every class in $\mathcal{C}_M$ can be represented by a symplectic form which restricts to a symplectic form on the fibers of M. Denote the set of such forms by $\mathcal{S}$. Then this set is contractible (and nonempty) (Thm. 1.4, [15]). See also McDuff [44].

10.2 $X\#(T^2 \times \Sigma_k)$

In the following we allow the fibration to have singular or multiply covered fibers. If we sum along a generic fiber, avoiding these special fibers, we find no obstruction to applying the methods developed above.

Theorem 10.2.1. *Let X be a minimal symplectic manifold with $b^+ = 1$. Let $V \subset X$ be a torus with trivial normal bundle and $\mathcal{C}_X^V \neq \emptyset$. Consider the manifold $M = X\#_V Y$. Then*

$$\mathcal{C}_M = \mathcal{P}^{\mathfrak{V}} \cup \mathcal{P}^{-\mathfrak{V}}.$$

Proof. We begin with the trivial case: Assume that $X = S^2 \times T^2$. Then the fiber sum is a trivial sum and we obtain $M = Y$. The result was shown in Thm. 10.1.1.

Assume in the following that $X \neq S^2 \times T^2$. Using the assumptions, we obtain from Cor. 8.4.7 that $\mathcal{C}_X^V = \mathcal{P}^{\mathfrak{V}}$. Lemma 9.3.5 and Thm. 9.3.7 now show that $C_M^V = \mathcal{P}^{\mathfrak{V}} \subset \mathcal{P}_M$, hence by Lemma 8.2.4 and Lemma 8.2.5 we obtain

$$\mathcal{P}^{\mathfrak{V}} \cup \mathcal{P}^{-\mathfrak{V}} \subset \mathcal{C}_M.$$

Let ω be a relative symplectic form on (X, V). Denote the corresponding canonical class by K_ω. The adjunction equality shows that $K_\omega \cdot \mathfrak{V} = 0$. Hence, if $K_\omega \cdot K_\omega \geq 0$, it follows from Lemma 8.1.1, that $K_\omega = a\mathfrak{V}$ for some $a \in \mathbb{R}$. If $K_\omega \cdot K_\omega < 0$, then X is a S^2-bundle over a Riemann surface of genus $g \geq 2$, hence contains no torus with trivial normal bundle.

Moreover, from the symplectic sum construction, it follows for any symplectic form ω on M obtained from relative symplectic forms ω_1 and ω_2 on (X, V) resp. (Y, V) that

$$K_\omega = K_{\omega_1} + K_{\omega_2} + 2\mathfrak{V}.$$

Thus K_ω is a multiple of $\mathfrak{V}$ for such sum symplectic forms. If K_ω is a non-zero multiple of $\mathfrak{V}$, then $\mathcal{C}_M \subset \mathcal{P}^{\mathfrak{V}} \cup \mathcal{P}^{-\mathfrak{V}}$ as K_ω is a SW basic class and we have proven the Theorem.

We must show that K_ω is a non-zero multiple of $\mathfrak{V}$. The canonical class of Y is a positive multiple of $\mathfrak{V}$. Thus, if K_{ω_1} is torsion, we are done. Assume that K_{ω_1} is non-torsion. Then the classification in Prop. 5.2, [34], shows that K_{ω_1} is a negative multiple of $\mathfrak{V}$ only if $X = \mathbb{C}P^2$, $S^2 \times S^2$ or a S^2-bundle over T^2. Only the trivial S^2-bundle over T^2 admits a torus with trivial normal bundle and we have assumed that $X \neq S^2 \times T^2$. □

Corollary 10.2.2. *Let X be a minimal elliptic Kähler surface with $p_g = 0$ and $M = X \#_{T_f^2} Y$. Then Conjecture 1.3.1 holds, i.e.*

$$C_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}.$$

Proof. The condition $p_g = 0$ for the Kähler manifold X implies $b^+ = 1$, see Thm IV.2.7, [1]. The result now follows from the previous Theorem and the uniqueness of the canonical class in the Kähler class. □

Remark: The manifold X could be an Enriques surface, a hyperelliptic surface or a Dolgachev surface.

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