Shifmania
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Non-Abelian vortices

-- Results in New Directions

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Happy 60th Birthday, Misha!

Thanking him for what he has done for
our community, and at the personal level,
for our long-lasting (> 1/4 of a century)
friendship, (indirect) collaborations,
occasional fights, and mutual stimulus ...

and

Wishing him many more fruitful years of
scientific activity and of happiness ...

Ken

Results about:

I. Non-Abelian BPS vortices with arbitrary gauge groups

II. Fractional vortices

In collabration with
M. Eto, M. Nitta, S.B. Gudnason, W. Vinci,
T. Fujimori, T. Nagashima, K.Ohashi, L. Ferretti
(Pisa, Tokyo, Cambridge, Trieste)

(2008,2009)

I. Non-Abelian vortices in general gauge theories

- Model
- General construction of BPS vortices
- GNOW duality
- Vortex moduli for SO(N), USp(2N)

The models

- Gauge group $G = U(1) \times G'$
- $G' = SU(N), SO(N), USp(N), \ldots \ldots$
- Bosonic sector of "N=2" supersymmetric theories with N flavors of (s)quarks
- FI term for the U(1) factor (vortex)
- System in completely Higgs phase
- Color-flavor locked phase with exact, unbroken G'_{c+f} symmetry

$$\mathcal{L} = \mathrm{Tr}_c \left[-\frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \mathcal{D}_\mu H \left(\mathcal{D}^\mu H \right)^\dagger - \frac{e^2}{4} \left| X^0 t^0 - 2\xi t^0 \right|^2 - \frac{g^2}{4} \left| X^a t^a \right|^2 \right]$$

$$F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu \,, \qquad \hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \left[A_\mu, A_\nu \right],$$

$$\mathcal{D}_\mu = \partial_\mu + i A^0_\mu t^0 + i A^a_\mu t^a \,.$$

where

$$X = H H^\dagger = X^0 t^0 + X^a t^a + X^\alpha t^\alpha \,,$$

$$X^0 = 2\,\mathrm{Tr}_c \left(H H^\dagger t^0 \right) \,, \qquad X^a = 2\,\mathrm{Tr}_c \left(H H^\dagger t^a \right)$$

H: N Higgs fields in <u>N</u> representation written as N x N matrix form ; $\xi > 0$ is the FI (forces the system into Higgs phase)

"Color-flavor locked" vacuum

$$\langle H \rangle = \frac{v}{\sqrt{N}} \mathbf{1}_N \,, \qquad \xi = \frac{v^2}{\sqrt{2N}}$$

Totally Higgsed but with G $_{C+F}$ flavor symmetry

Bogomolny completion →

$$T = \int d^2x \, \mathrm{Tr}_c \left[\frac{1}{e^2} \left| F_{12} - \frac{e^2}{2} \left(X^0 t^0 - 2\xi t^0 \right) \right|^2 \right.$$

$$\left. + \frac{1}{g^2} \left| \hat{F}_{12} - \frac{g^2}{2} X^a t^a \right|^2 + 4 \left| \bar{\mathcal{D}} H \right|^2 - 2\xi F_{12} t^0 \right]$$

$$\geq -\xi \int d^2x \, F^0_{12}$$

Remarks

- For G=U(N), with $N_f = N$, the color-flavor locked vacuum is unique.

- $\mathcal{M}$ (vacuum moduli) is a point. The main interest in $\mathcal{V}$ (vortex moduli

- For G' = SO(N), USp(N), ... even if $N_f = N$, the color-flavor locked vacuum is just one of the possible vacua (i.e. non-trivial vacuum degeracy).

- SO(N) $\subset$ SU(N), strictly smaller. $SU(N)^C$=SL(N,C).

- $\mathcal{M}$ has a much richer structure in general.

- Study first $\mathcal{V}$ in the maximally color-flavor locked vacuum (I part of the talk)

- → Fractional vortices (II part of the talk)

BPS equations

$$\bar{\mathcal{D}}H = \bar{\partial}H + i\bar{A}H = 0 \,,$$
$$F^0_{12} = e^2 \left[\mathrm{Tr}_c \left(HH^\dagger t^0 \right) - \xi \right] \,,$$
$$F^a_{12} = g^2 \, \mathrm{Tr}_c \left(HH^\dagger t^a \right) \,.$$

z = x + i y

$\bar{A}$ = A_x - i A_y

$$T = 2\sqrt{2N}\pi\xi\nu = 2\pi v^2 \nu \,, \qquad \nu = -\frac{1}{2\pi\sqrt{2N}} \int d^2x \, F^0_{12} \,,$$

Solution of BPS equations:

$$H = S^{-1}(z,\bar{z})H_0(z) \,, \qquad \bar{A} = -iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z})$$

$H_0(z)$ moduli matrix; $S(z,\bar{z}) \in \mathbb{C}^* \times G'^{\mathbb{C}}.$

$$\Omega = SS^\dagger \qquad \Omega = \omega\,\Omega', \qquad \omega = |s|^2, \, \Omega' = S'S'^\dagger.$$

Holomorphic Invariants

$$I^i_{G'}(H) = I^i_{G'}\left(s^{-1}S'^{-1}H_0\right) = s^{-n_i}I^i_{G'}(H_0(z))$$

G' - invariants made of H

n_i: U(1) charge

I(H)= det H
for U(N)

$$I^i_{G'}(H)\Big|_{|z|\to\infty} = I^i_{\rm vev}\, e^{i\nu n_i\theta}$$

$$I^i_{G'}(H_0) = s^{n_i}I^i_{G'}(H) \stackrel{|z|\to\infty}{\longrightarrow} I^i_{\rm vev} z^{\nu n_i}\,.$$

$$\nu\, n_i \in \mathbb{Z}_+ \qquad \to \qquad \nu = \frac{k}{n_0}\,, \qquad k \in \mathbb{Z}_+$$

$\to$ $H_0(z)$

$$n_0 \equiv \mathrm{gcd}\left\{n_i\,|I^i_{\rm vev} \neq 0\right\}$$

$$G = \left[U(1) \times G'\right]/\mathbb{Z}_{n_0}$$

= 2 (SO(2N), USp(2N))
=1 SO(2N+1) ; N for SU(N)

U(1) winding #

GNOW quantization

Special (vortex) solutions

$$H_0(z) = z^{\nu 1_N + \nu_a \mathcal{H}_a} \in U(1)^{\mathbb{C}} \times G'^{\mathbb{C}} \,,$$

$$(\nu 1_N + \nu_a \mathcal{H}_a)_{ll} \in \mathbb{Z}_{\geq 0} \quad \forall\, l$$

$\mathcal{H}_a$ = Cartan subalgebra of G'

$\longrightarrow$ $\nu + \nu_a \mu_a^{(i)} \in \mathbb{Z}_{\geq 0} \quad \forall\, i$ $\qquad$ $\vec{\mu}^{(i)} = \mu_a^{(i)}$ weight vectors of G'

$\longrightarrow$ $\vec{\nu} \cdot \vec{\alpha} \in \mathbb{Z}$ $\qquad$ (❣) $\qquad$ $\vec{\alpha}$ root vectors of G'

$\longrightarrow$ Solution: $\tilde{\vec{\mu}} \equiv \vec{\nu}/2$ is a weight vector of $\tilde{G}'$

Constraints for SO, USp

SO(2M+1)

$$(I_{SO,USp})^r{}_s = (H^{\mathrm{T}} J H)^r{}_s \,, \qquad 1 \le r \le s \le N \,,$$

$$J_{\mathrm{even}} = \begin{pmatrix} 0 & 1_M \\ \epsilon\, 1_M & 0 \end{pmatrix} , \qquad J_{\mathrm{odd}} = \begin{pmatrix} 0 & 1_M & 0 \\ 1_M & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

$$\epsilon = +1 \text{ for } SO(2M) \text{ and } \epsilon = -1 \text{ for } USp(2M)$$

$\rightarrow$ $$H_0^{\mathrm{T}}(z) J H_0(z) = z^{\frac{2k}{n_0}} J + \mathcal{O}\left(z^{\frac{2k}{n_0}-1}\right)$$ k=1, 2, ...

cfr. U(N) model: det H_0 = z^k + ...

Remarks :

- (❣) formally identical to the GNOW “quantization” condition for the “non-Abelian monopoles” for YM
- (❣) formally identical to that found for “non-Abelian vortices” for YM (Spanu-Konishi)
- The latter are actually Z_N vortices (no cont. moduli)
- The former has the well-known difficulties
- Our vortices have continuous moduli connecting them (orientational moduli)
- They transform genuinly as various representations of the dual G’ group, $\tilde{G}'$

Examples of the dual pairs

G'	$\tilde{G}'$
$SU(N)$	$SU(N)/\mathbb{Z}_N$
$U(N)$	$U(N)$
$SO(2M)$	$SO(2M)$
$USp(2M)$	$SO(2M+1)$
$SO(2M+1)$	$USp(2M)$

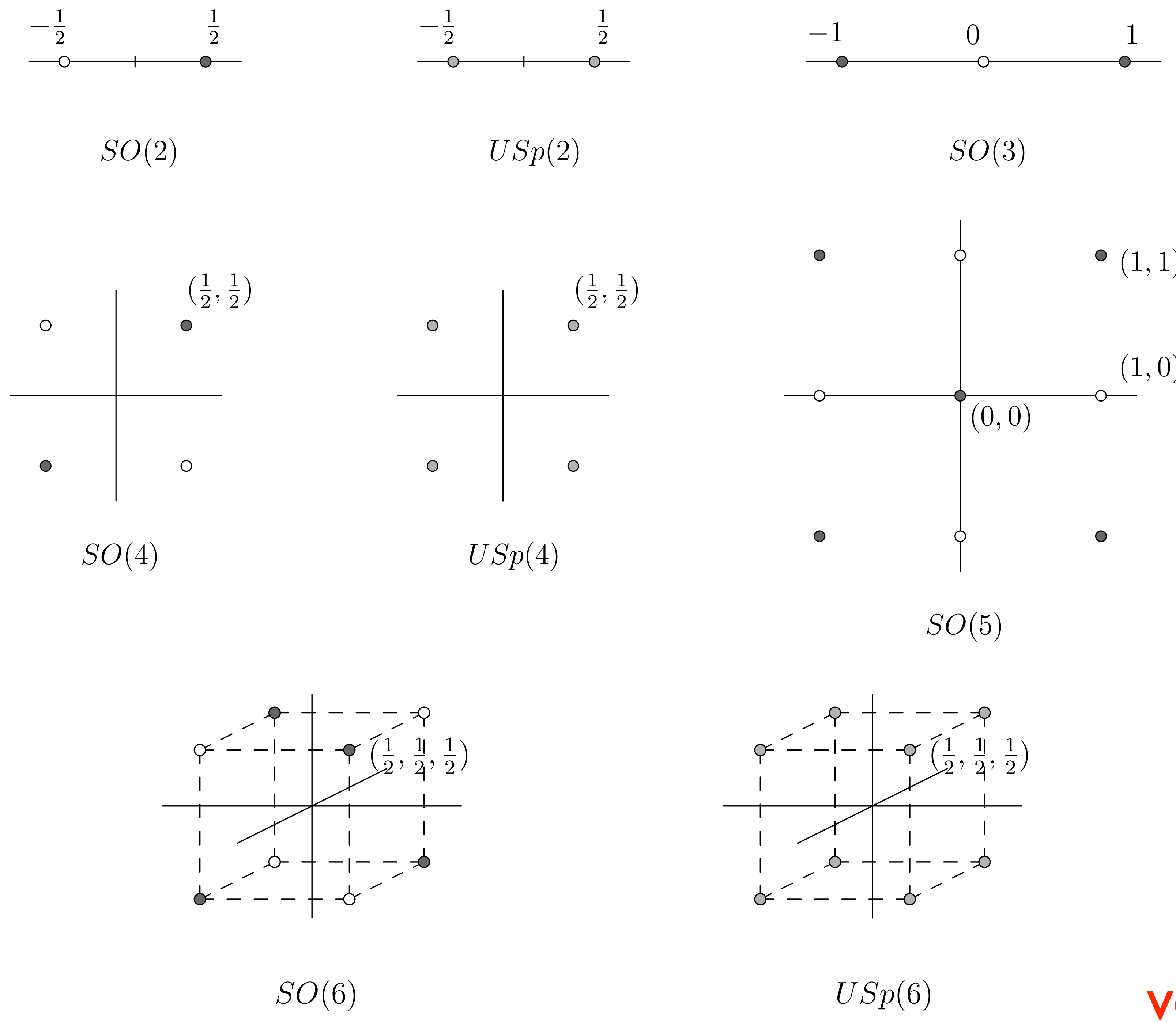
Our vortices have the orientational moduli corresponding to $\tilde{G}'$ orbits

⇨ True non-Abelian quantum monopoles via embedding of our system in G_0 such that

$$G_0 \rightarrow G \rightarrow 1$$

hep-th/0611313 Eto et. al.

arXiv:0809.1374 [hep-th] K.K. Minnesota 08



$-\frac{1}{2}$ $\frac{1}{2}$
$SO(2)$
$-\frac{1}{2}$ $\frac{1}{2}$
$USp(2)$
-1 0 1
$SO(3)$
$(\frac{1}{2}, \frac{1}{2})$
$SO(4)$
$(\frac{1}{2}, \frac{1}{2})$
$USp(4)$
$(1, 1)$
$(1, 0)$
$(0, 0)$
$SO(5)$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$SO(6)$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$USp(6)$

k=1
vortices

II. Fractional Vortices

- EAH model
- Arena of study: various Abelian and non-A generalizzations
- Vortices in degenerate vacua: origin of the vortex substructures
- Models based on CP^1
- SO(2N)

Def. (here): Vortices with minimum vorticity (the minium winding required by the topological stability) but with non-trivial tension substructures in the transverse plane

(Known examples in EAH; also torons, calorons ....)

Extended Abelian Higgs models (EAH)

Achucarro, Vachaspati, Hindmarsh

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu q\,(\mathcal{D}^\mu q)^\dagger - \frac{\lambda}{2}\left(qq^\dagger - \xi\right)^2$$

$$q = (q_1, q_2, \ldots, q_{N_f})$$

$$M \equiv \{q_i\}, \qquad \sum_{i=1}^{N_f} |q_i|^2 = \xi$$

$$\langle q\rangle = (q_0, 0, \ldots, 0), \qquad q_0 = \sqrt{\xi},$$

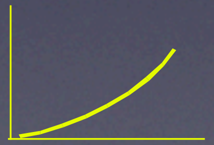
$$\mathcal{M} = \mathbb{C}P^{N_f-1} = SU(N_f)/[SU(N_f-1)\times U(1)]$$

Vacuum moduli

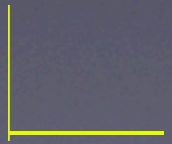
$\Pi_1(\mathcal{M}) = 1$ ➜ vortex may not be stable

ρ

① $\beta \equiv \lambda/e^2 > 1$ type II ANO vortex unstable



② $\beta < 1$ type I ANO vortex stable



③ $\beta = 1$ (BPS) Vortex has arbitrary size (sem-local vortices)

Non-trivial vortex moduli

In cases ②,③, stability guaranteed by

$$\pi_1(U(1)) = \mathbb{Z}$$

Also, in the BPS case (③)

$$\pi_2\left(\mathbb{C}P^{N_{\rm f}-1}\right) = \mathbb{Z}$$

Vortices ~ sigma model lumps:
non-trivial maps from S^2 to $\mathcal{M}$

Vortices represent U(1) fiber bundle over $\mathcal{M}$

Our models:

various Abelian and non-Abelian generalizations of EAH with $\beta=1$ (BPS): ➜

The basic ingredients for interesting $\mathcal{V}$ (vortex moduli) :

- BPS nature
- Vacuum degeneracy $\mathcal{M}$
  - Consider all vortices $\mathcal{V}$ defined at various points of $\mathcal{M}$ simultaneously
  - In general $\mathcal{M}$ turns out to be a singular manifold:

    our vortices ~ "fiber bundles over a singular manifold"

$$T = \int d^2x \left[\sum_I \frac{1}{2\, g_I^2} (F_{12}^I)^2 + \sum_\alpha \left(|(\mathcal{D}_1 H)^\alpha|^2 + |(\mathcal{D}_2 H)^\alpha|^2 \right) + \sum_I \frac{g_I^2}{2} (H^\dagger T^I H - \xi_I)^2 \right]$$

$K = L \otimes G_F$ local (L) and global (G_F) symmetries

$M = \{ q_i \mid q^\dagger T^I q = \xi^I \}.$ vacuum configuration

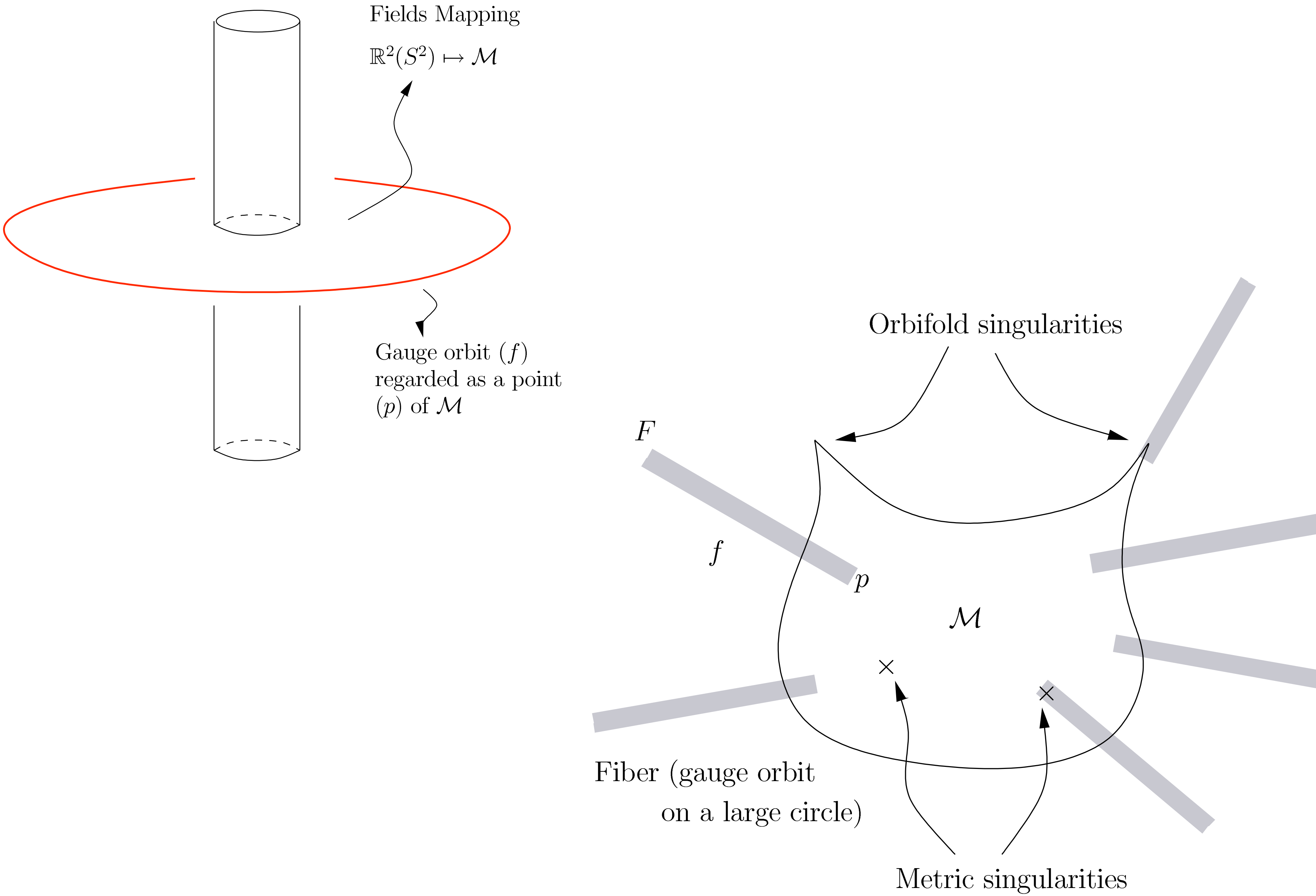
$p \in \mathcal{M} = M/F$ vacuum moduli space

$f \in F = \{ q^g \mid q^g = g q \}\ , \quad g \in L/L_0$ fiber = gauge orbits

$L_0^{\{q\}} = \{ \ell_0 \in L \mid \ell_0 q = q \}$ unbroken gauge group

$\tilde{G}^{\{q\}} = \{ g_f \in G_F \mid g_f q = \ell q \}, \quad \ell \in L$ unbroken symmetry group

Breaking of $\tilde{G}^{\{q\}}$ leads to various vortex moduli: but here we are mainly interested in vortex degeneracy arising from the vacuum degeneracy



Fields Mapping
$\mathbb{R}^2(S^2) \mapsto \mathcal{M}$
Gauge orbit (f)
regarded as a point
(p) of $\mathcal{M}$
Orbifold singularities
F
f
p
$\mathcal{M}$
Fiber (gauge orbit
on a large circle)
Metric singularities

Exact sequences of fiber bundles

$$\cdots \longrightarrow \pi_2\left(M, f\right) \longrightarrow \pi_2\left(\mathcal{M}, p\right) \longrightarrow \pi_1\left(F, f\right) \longrightarrow$$
$$\longrightarrow \pi_1\left(M, f\right) \longrightarrow \pi_1\left(\mathcal{M}, p\right) \longrightarrow \cdots$$

$p = \pi(f)$ projection

In the simple EAH model :

$$M = S^{2N-1} \, , \qquad F = S^1 \, , \qquad \mathcal{M} = S^{2N-1}/S = \mathbb{C}P^{N-1}$$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_2\left(S^{2N-1}\right) & \longrightarrow & \pi_2\left(\mathbb{C}P^{N-1}\right) & \longrightarrow & \pi_1\left(S^1\right) & \longrightarrow & \pi_1\left(S^{2N-1}\right) & \longrightarrow \cdots \\ & \| & & \| & & \| & & \| & \\ & \mathbb{1} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{1} & \end{array}$$

The minimum vortex corresponds to a minimum CP^{N-1} lump

Two main causes for (fractional) substructures

(i) When $p = p_0$ (a Z_N orbifold point) both $\pi_1(F, f)$ and $\pi_2(\mathcal{M}, p)$ make a discontinuous change.

Vortex defined near $p = p_0$ feels the presence of p_0 and look like a k=N vortex

(ii) Even when p is a regular point (not near any singularity), the fields {q} inside S^1 (a disk D^2) ~ $\mathcal{M}$: may hit either one of the singularities or simply pass the region of a large scalar curveture.

Fractional vortex substructures caused either by one of these or by a collaboration of the two ➔ examples

Models based on CP1

(1) Abelian Higgs model with (A,B), with charges (2,1).

Vacuum config. $2|A|^2 + |B|^2 = \xi \ ,$

Gauge transf: $A \sim e^{2i\alpha(x)} A, \qquad B \sim e^{i\alpha(x)} B.$

➔ $\mathcal{M} = W\mathbb{C}P^1_{2,1} \sim \dfrac{\mathbb{C}P^1}{\mathbb{Z}_2}$ (Fig.)

The coordinate of $\mathcal{M}$ is $\varphi = 2A/B^2$

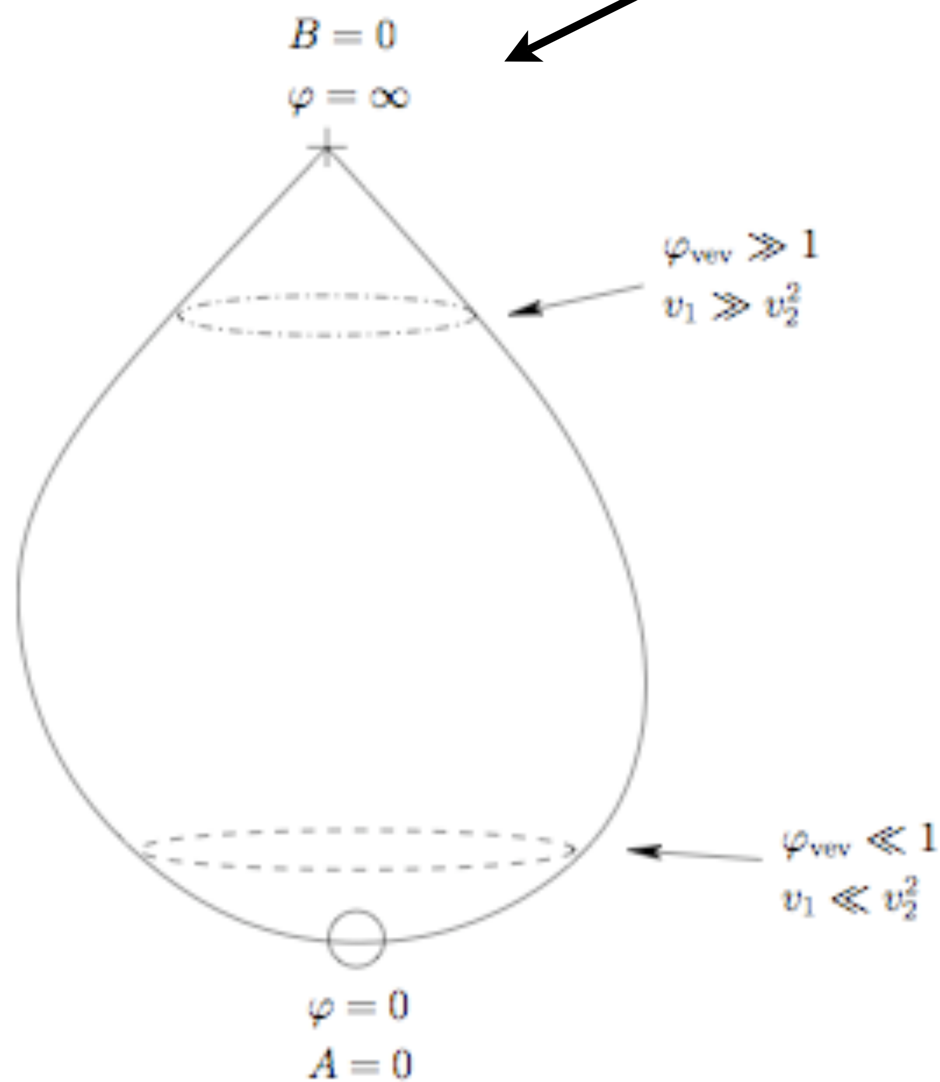
$$H = S^{-1}(z,\bar{z})H_0(z)$$

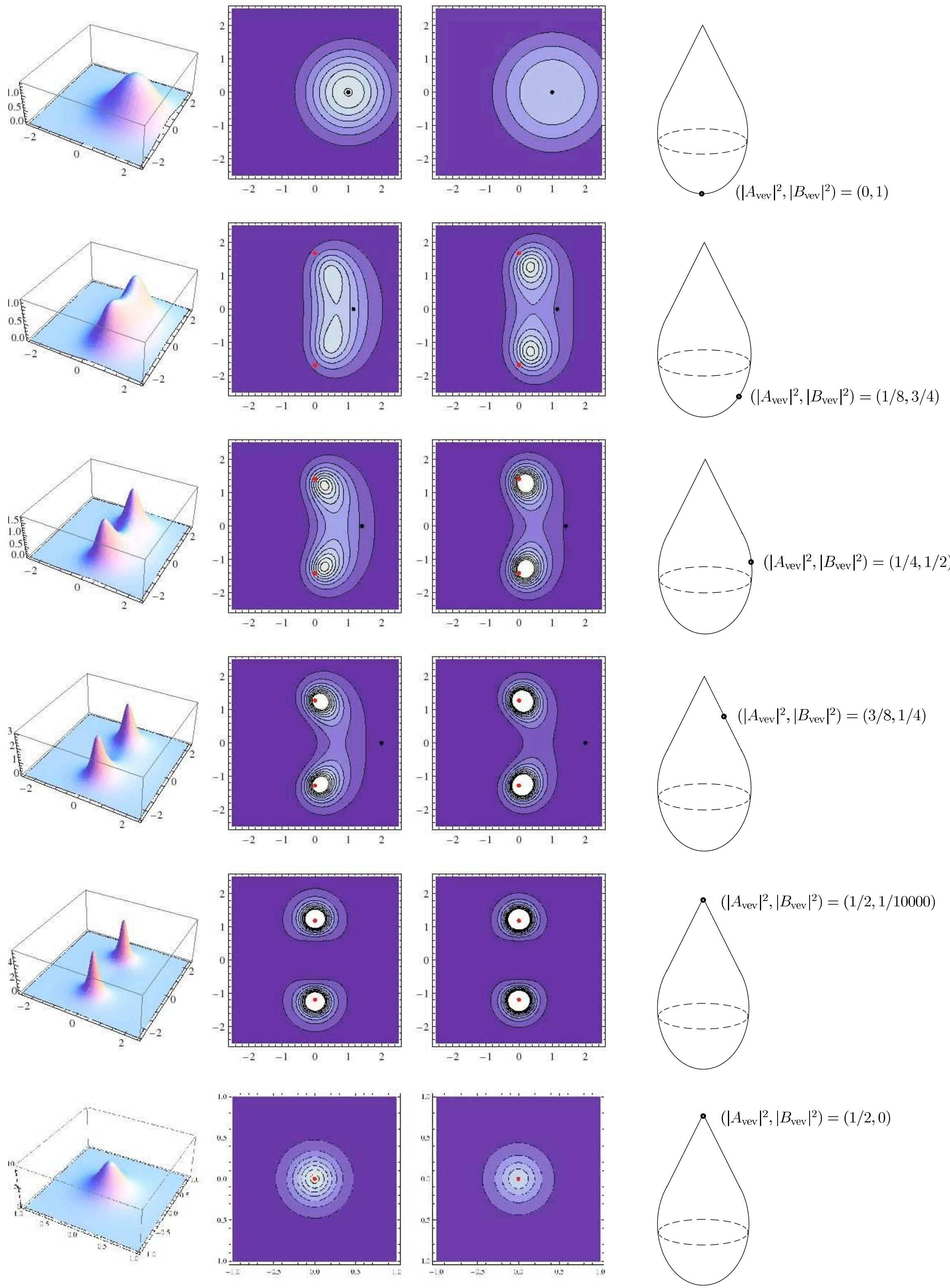
p generic $H_0^{[11]} = \begin{pmatrix} \frac{v_1}{\sqrt{2}}(z-z_1)(z-z_2) & 0 \\ 0 & v_2(z-z_3) \end{pmatrix} \ , \qquad v_1^2 + v_2^2 = \xi$

p= ∞ (B=0) $H_0^{[10]} = \begin{pmatrix} \sqrt{\xi}(z-z_1) & 0 \\ 0 & \zeta_1 \end{pmatrix}$

$$\frac{\pi_2(\mathcal{M}, p)}{\pi_2(\mathcal{M}, \infty)} = \mathbb{Z}_2 \ , \qquad \frac{\pi_1(F, f)}{\pi_1(F, f_0)} = \mathbb{Z}_2$$

Z_2 orbifold point





black dots = zeros of A;

red dots = zeros of B

Good example of

the first mechanism

Models based on CP^1

(2) U(1)xU(1) Higgs model with (A,B,C) with charges:

$$Q_1 = (2,1,1) \qquad Q_2 = (0,1,-1)$$

$$(A,B,C) \to \left(e^{i2\alpha(x)}A, e^{i\alpha(x)+i\beta(x)}B, e^{i\alpha(x)-i\beta(x)}C\right)$$

$$U(1)_1 \times U(1)_2/\mathbb{Z}_2.$$

$$\begin{aligned} M &= \{A,B,C \;\mid\; 2|A|^2+|B|^2+|C|^2=\xi_1,\; |B|^2-|C|^2=\xi_2\} \\ \mathcal{M} &= M/\left[\left(U(1)_1 \times U(1)_2\right)/\mathbb{Z}_2\right]. \end{aligned}$$

No orbifold singularity

No doubling of $\pi_1(F,f)$ or $\pi_2(\mathcal{M},p)$

An extra peak at ~ $z=z_0$, where $B(z_0)=0$

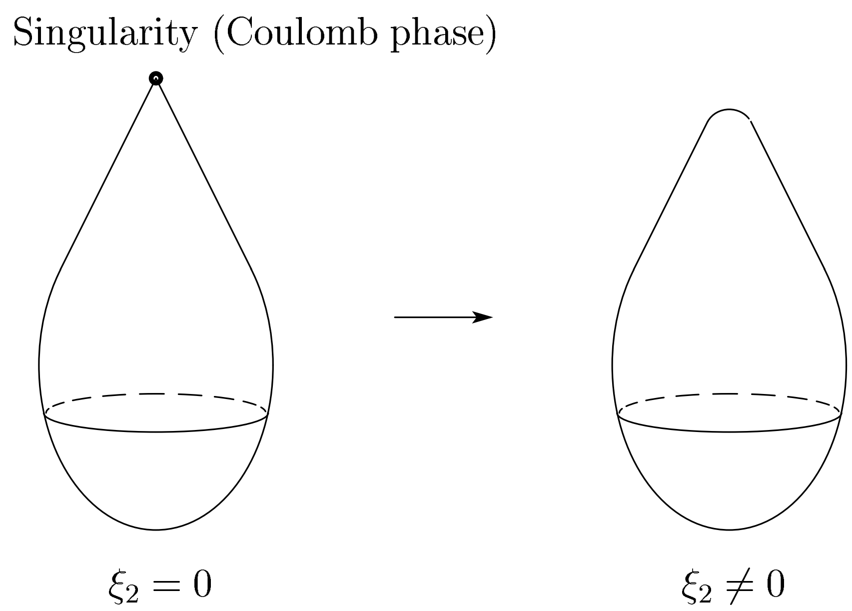


Fig. 5: A sketch of the vacuum moduli space.

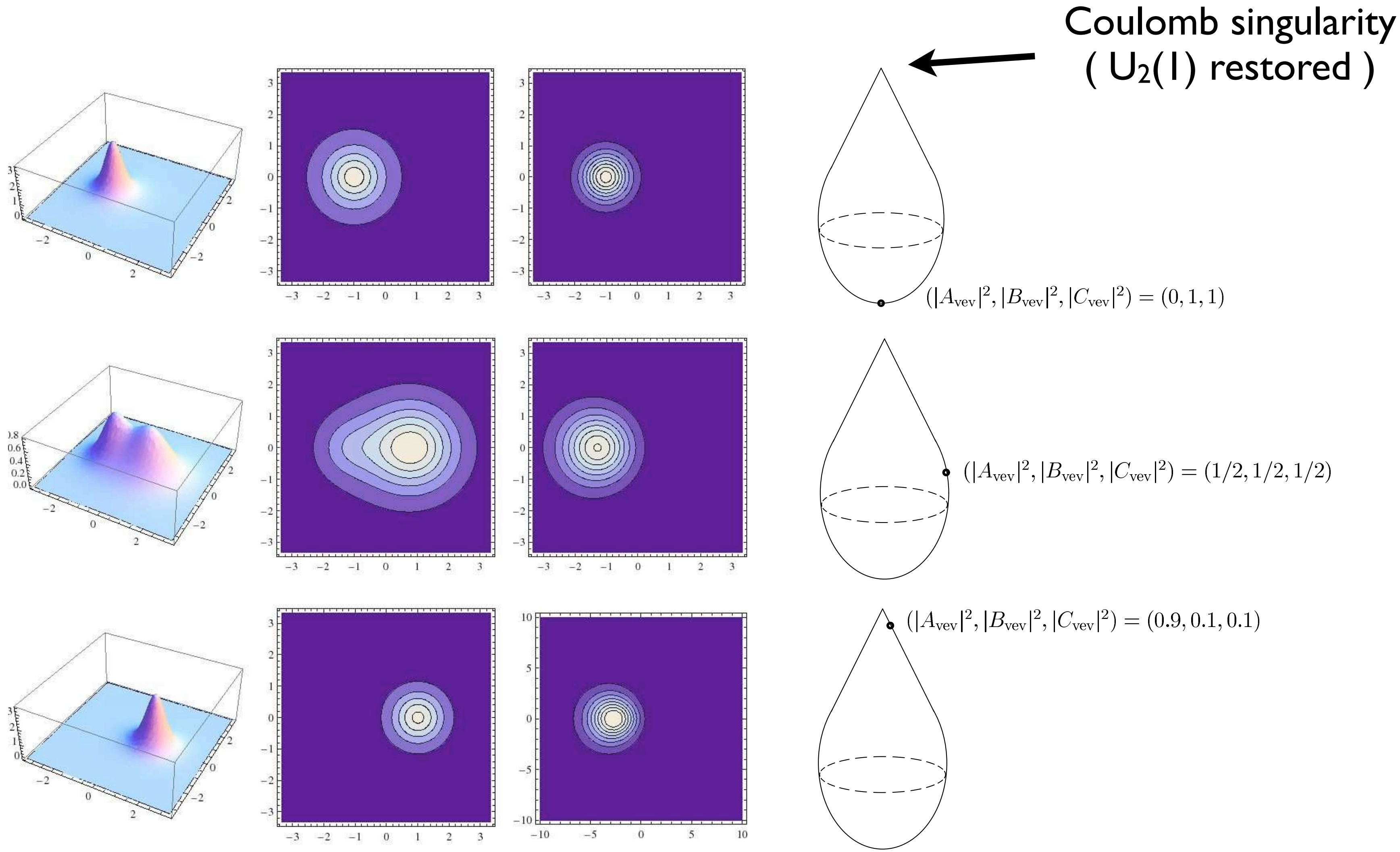


Fig. 6: The energy density (left-most) and the magnetic flux density $\boldsymbol{F}_{\mathbf{12}}^{(\mathbf{1})}$ (2nd from the left), $\boldsymbol{F}_{\mathbf{12}}^{(\mathbf{1})}$ (2nd from the right) and the boundary condition (right-most). We have chosen $\boldsymbol{\xi_1} = \mathbf{2}$, $\boldsymbol{\xi_2} = \mathbf{0}$, $\boldsymbol{e_1} = \mathbf{1}$, $\boldsymbol{e_2} = \mathbf{2}$ and $\boldsymbol{a = -1, b = 1}$ in Eq. (4.34).

Good example of the second mechanism

Models based on CP^1

(3) A model with U(1)xSU(2) with charges

	$U(1)$	$SU(2)$
A	$\mathbf{2}$	$\mathbb{1}$
B	$\mathbf{1}$	$\square$

Similar to the U(1)xU(1) model, except of course the vortex moduli includes orientational CP^1 moduli

(4) An alternative U(1)xU(1) model (A,B,C) with charges

$$Q_1 = (1,1,1) \qquad Q_2 = (0,1,-1)$$

$$\begin{aligned} M &= \{A, B, C \ \mid \ |A|^2 + |B|^2 + |C|^2 = \xi_1, \ |B|^2 - |C|^2 = \xi_2\} \ , \\ \mathcal{M} &= M/\,(U(1)_1 \times U(1)_2) \ . \end{aligned}$$

(➜ Fig.)

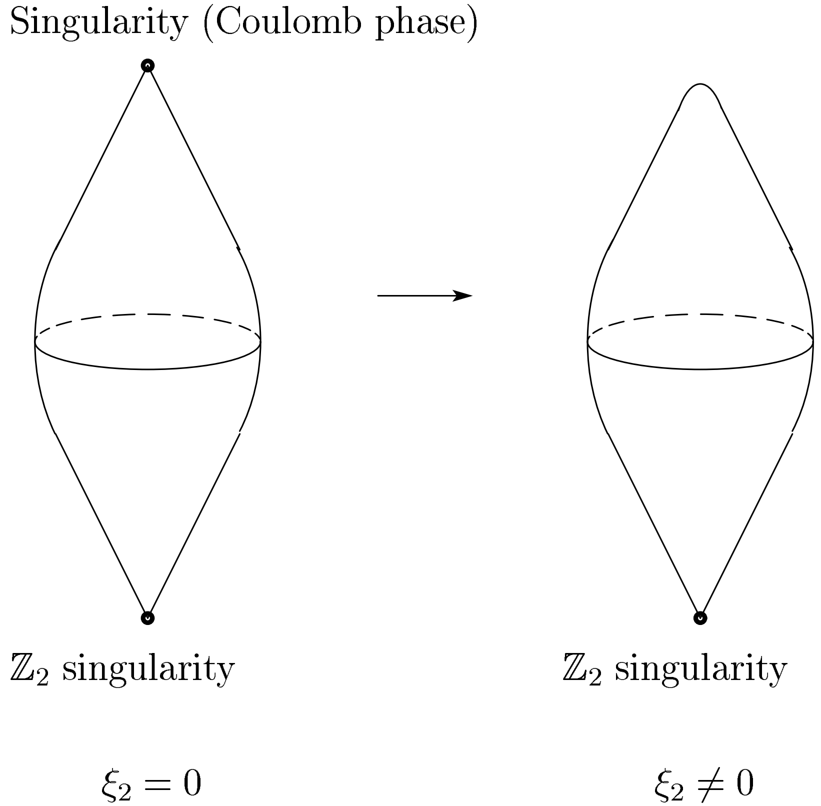
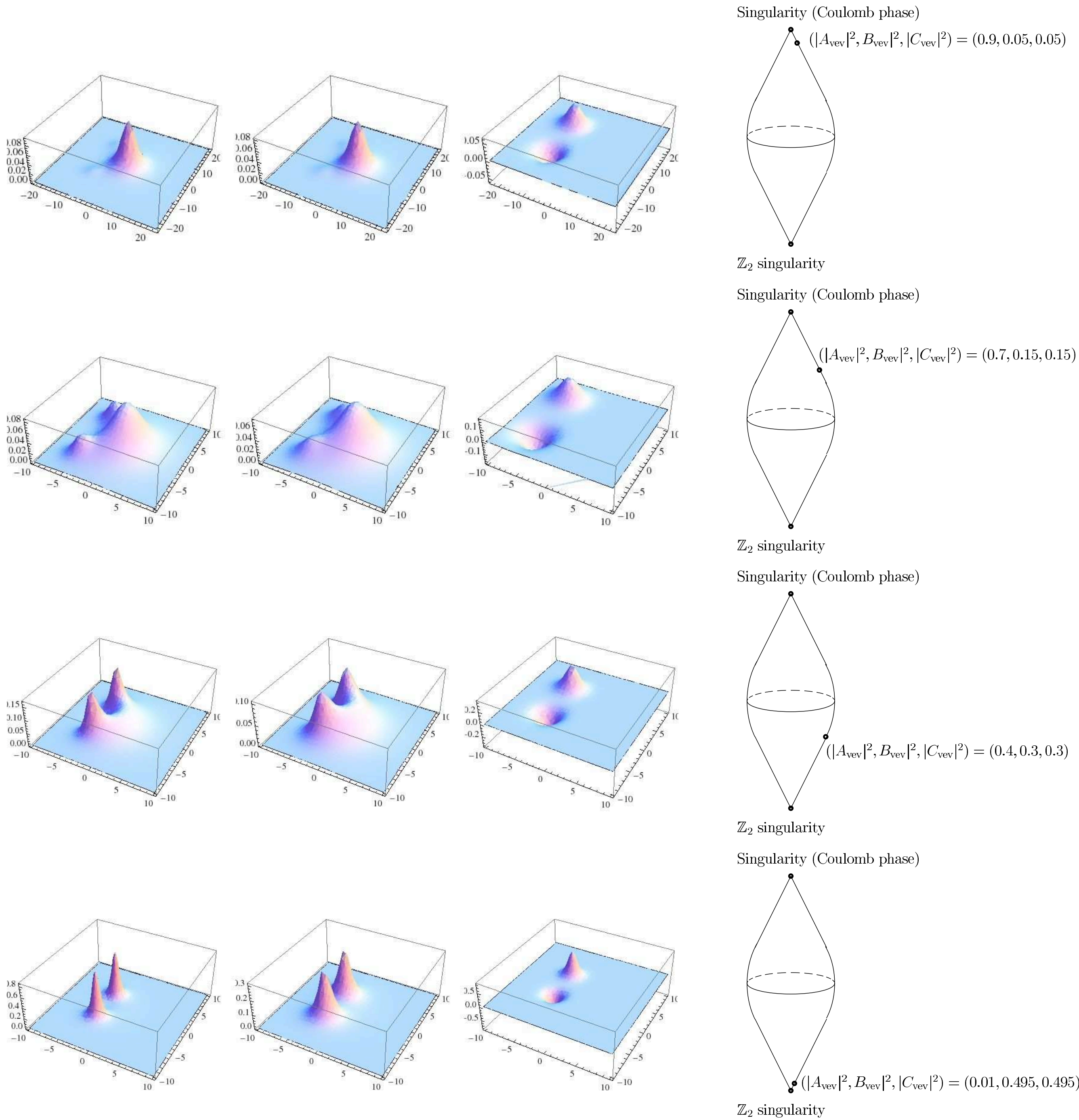


Fig. 8: The lemon space.



Both mechanisms
at work

U(1) x SO(2N) model

	$U(1)$	$SO(N)$
H	$\mathbf{1}$	$\square$

$$\langle H\rangle = \mathrm{diag}\,(v_1, v_2, \cdots, v_{2M}) \ , \qquad \sum_{i=1}^{2M} v_i^2 = \xi \ , \qquad v_i \in \mathbb{R} \ .$$

Large vacuum degeneracy

$v_1=v_2=0$ ➜ SO(2) unbroken

$v_1=v_2=v_3=0$ ➜ SO(3) unbroken, etc.

➜ stratified singular structure of $\mathcal{M}$

Below we consider the maximally color-flavor locked vacum:

$v_1=v_2= \ldots = v_{2M}= \sqrt{\xi/2M}$

$$\langle H \rangle = \sqrt{\xi} 1_{2M}$$

$$H = s^{-1}(z,\bar{z}) S'^{-1}(z,\bar{z}) H_0(z) \ ,$$

Moduli matrix for k=1 : $H_0 = \begin{pmatrix} z\mathbb{1}_M - \mathbf{Z} & \mathbf{C} \\ \mathbf{0} & \mathbb{1}_M \end{pmatrix}$

the moduli parameters

$$\mathrm{Z} = \mathrm{diag}(z_1, z_2, \ldots, z_M) \ , \qquad \mathrm{C} = \mathrm{diag}(c_1, c_2, \ldots, c_M)$$

SO(6)xU(1) model ➜ fig.

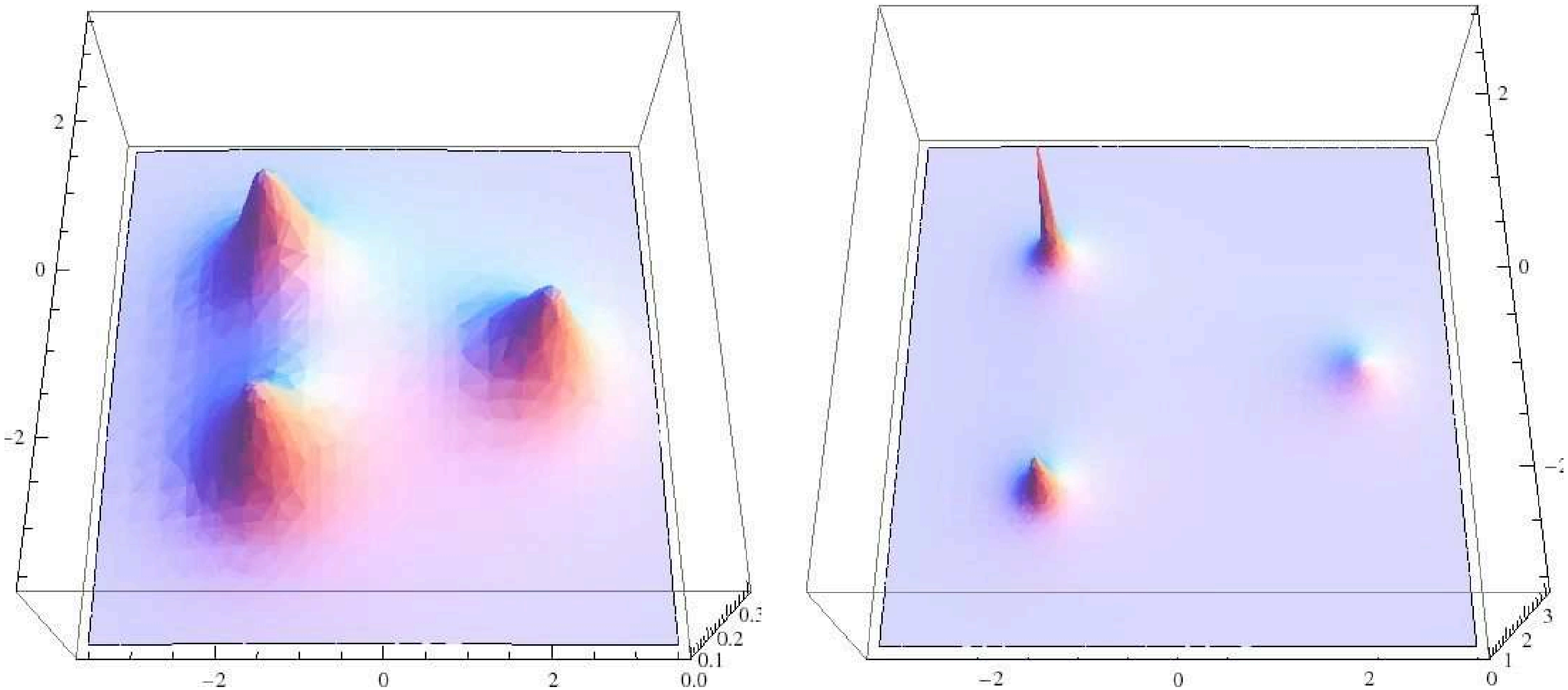


Fig. 11: The energy density of three fractional vortices (lumps) in the $\boldsymbol{U(1) \times SO(6)}$ model in the strong coupling approximation. The positions are $\boldsymbol{z_1 = -\sqrt{2} + i\sqrt{2}, z_2 = -\sqrt{2} - i\sqrt{2}, z_3 = 2}$. *Left panel*: the size parameters are chosen as $\boldsymbol{c_1 = c_2 = c_3 = 1/2}$. *Right panel*: the size parameters are chosen as $\boldsymbol{c_1 = 0, c_2 = 0.1, c_3 = 0.3}$. Notice that one peak is singular ($\boldsymbol{z_1}$) and the other two are regularized by the finite (non-zero) parameters $\boldsymbol{c_{2,3}}$.

To conclude: fractional vortices

- ➜ interesting applications
- ➜ new mathematical structures ?

(a paper about to appear)

END