Symmetries in the Gross-Neveu Phase Diagram

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crystalline phases of GN models:
gap equation and integrable hierarchies

G.Başar &GD, arxiv:0803.1501, PRL **100**, 200404 (2008)
arxiv:0806.2659, PRD **78**, 065022 (2008)
G.Başar, GD & M.Thies, arxiv: 0903.1868, PRD in press
F.Correa, GD & M.Plyushchay, arxiv: 0904.2768

energy-reflection symmetry of periodic QES systems

GD & M.Shifman, hep-th/0204224, Ann. Phys. **299**, 143 (2002)

Gross-Neveu Models

Gross/Neveu, 1974

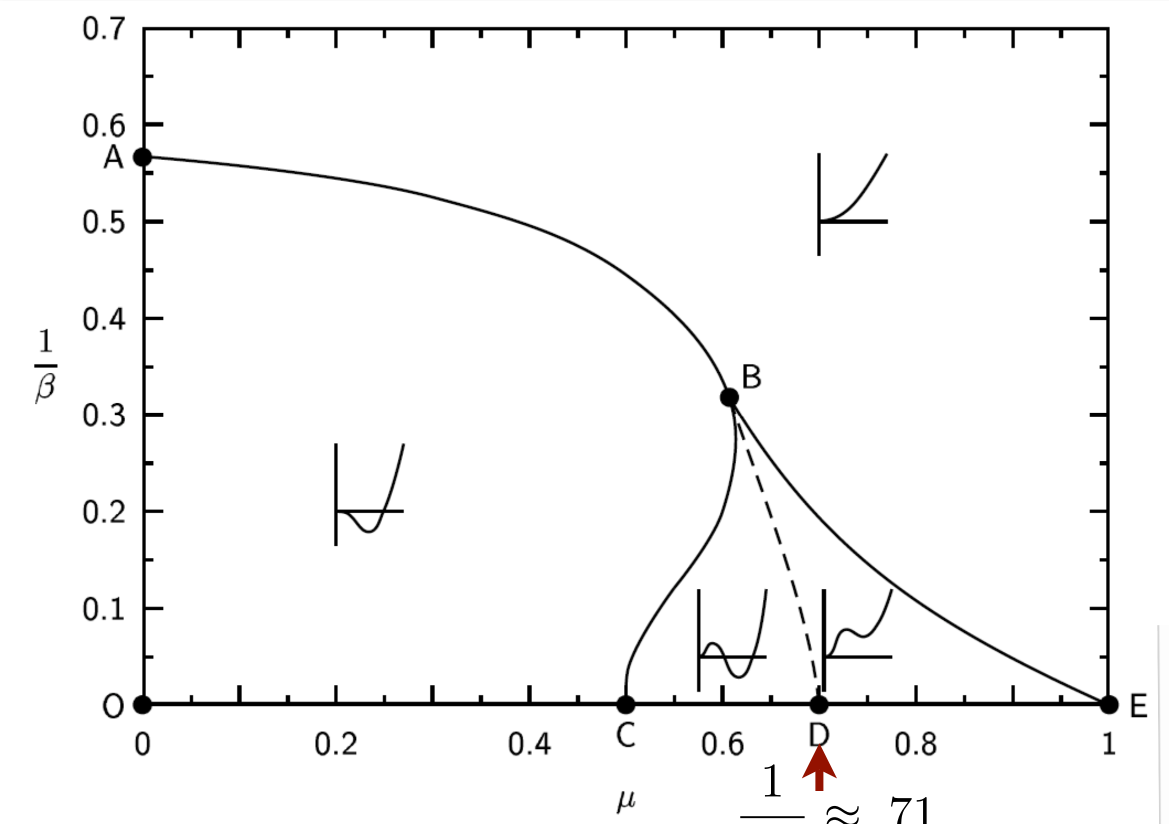
GN$_2$ $$\mathcal{L}_{\rm GN} = \bar{\psi}\, i\, \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2$$ $\psi \to \gamma^5\, \psi$

χGN$_2$
NJL$_2$ $$\mathcal{L}_{\rm NJL} = \bar{\psi}\, i\, \partial\!\!\!/ \psi + \frac{g^2}{2} \left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i \gamma^5 \psi\right)^2 \right]$$ $\psi \to e^{i\alpha\, \gamma^5}\, \psi$

- renormalizable; large N_f limit
- asymptotically free
- chiral symmetry breaking
- dynamical mass generation
- self-bound baryonic states $m_B = \frac{2}{\pi}\, m$ DHN, 1975

(T, μ) phase diagram?

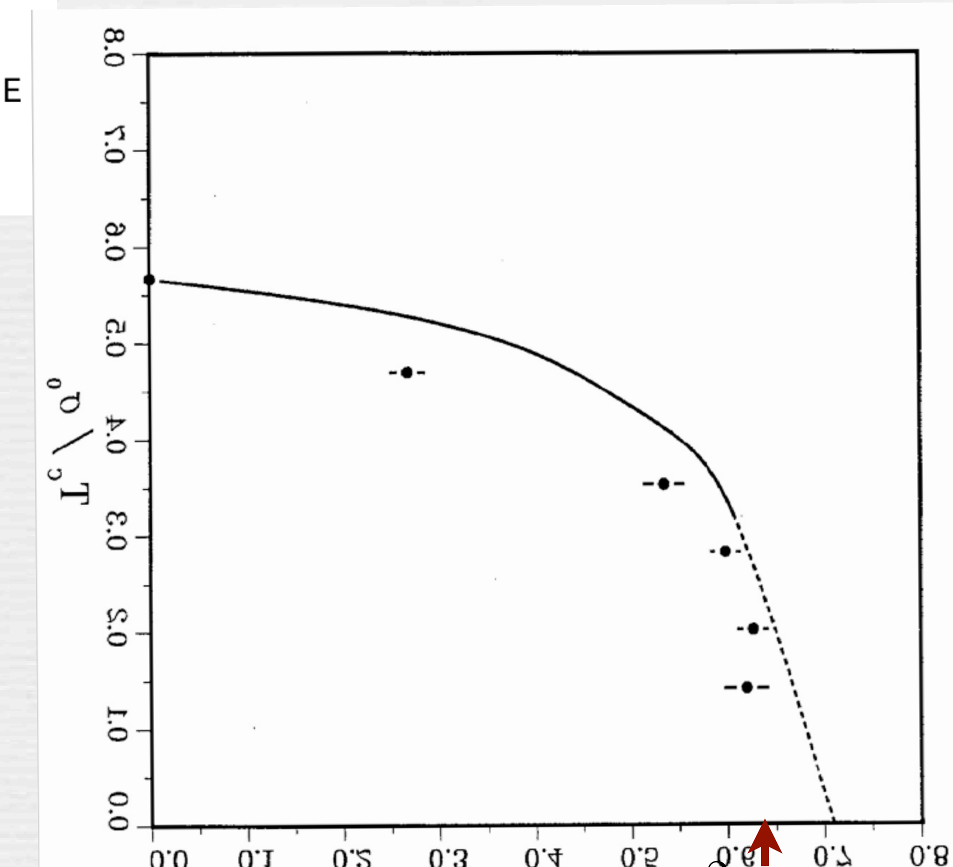
Phase diagram of Gross-Neveu model



$\frac{1}{\sqrt{2}} \approx .71$

uniform condensate

Wolff, 1985

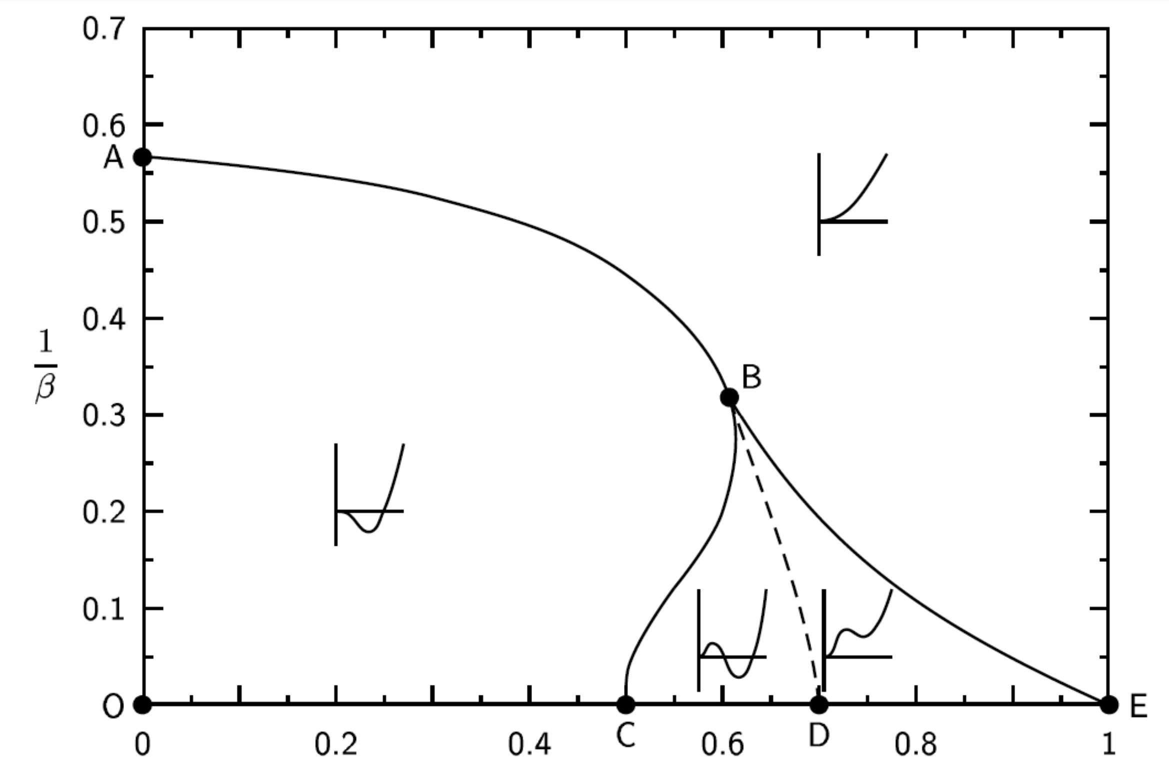


$\frac{2}{\pi} \approx .64$

lattice analysis of GN_2

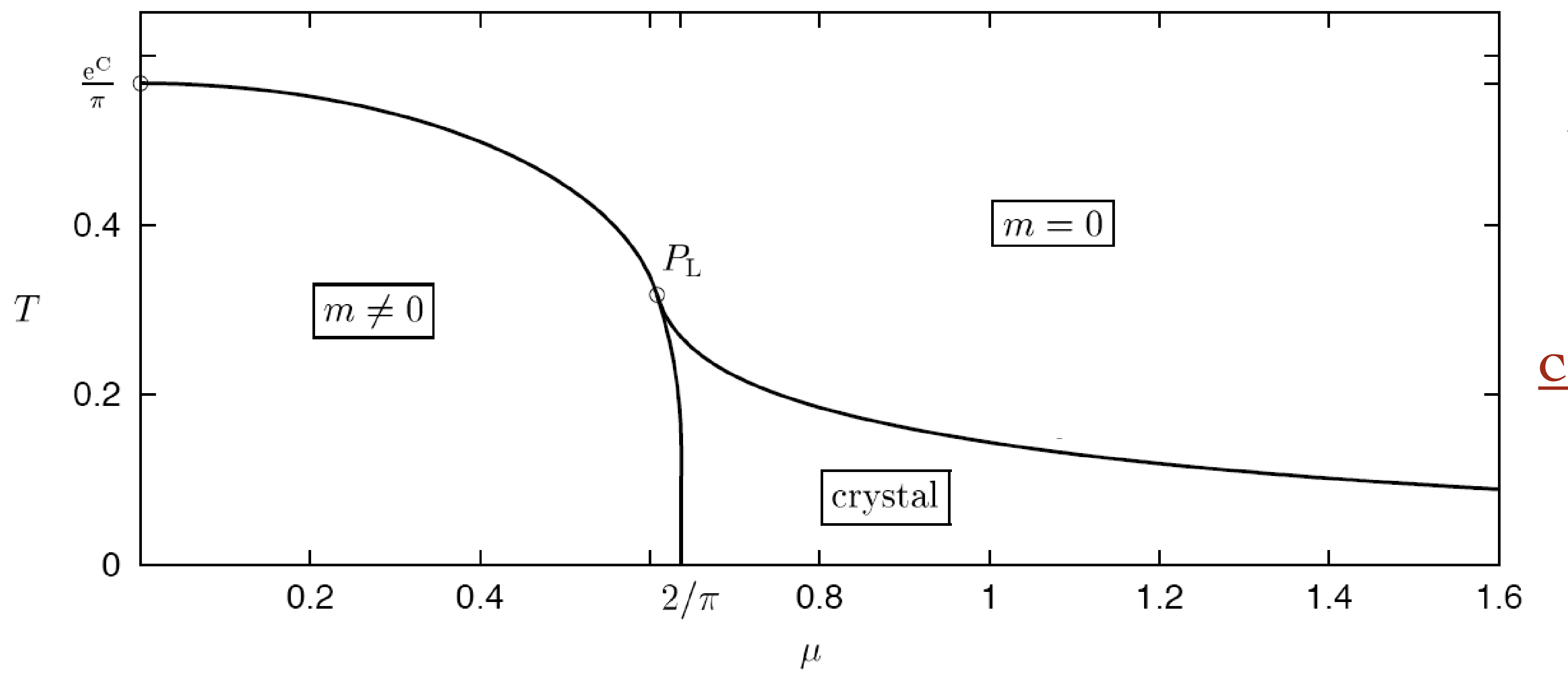
Karsch et al 1986

Phase diagram of Gross-Neveu model



uniform condensate

Wolff, 1985

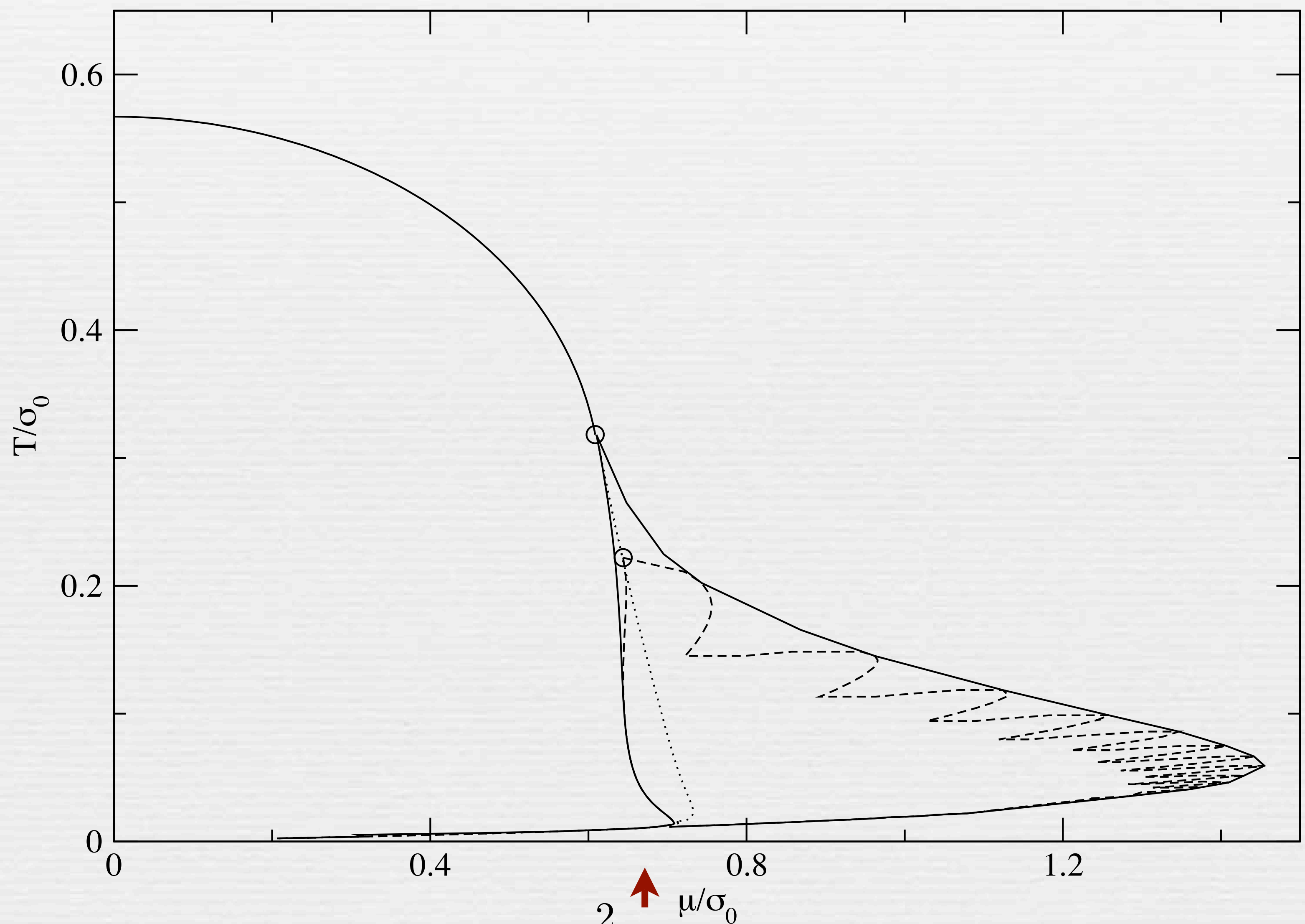


Thies & Urlichs, 2005

periodic, crystalline, phase

lattice GN_2 model

de Forcrand/Wenger 2006



$$\frac{2}{\pi} \approx .64$$

Condensed matter analogues

trans-polyacetylene = GN_2 Su, Schreiffer, Heeger, 1979

dimerization = discrete chiral symmetry of GN model

polaron crystal Brazovskii, 1980; Horovitz, 1981

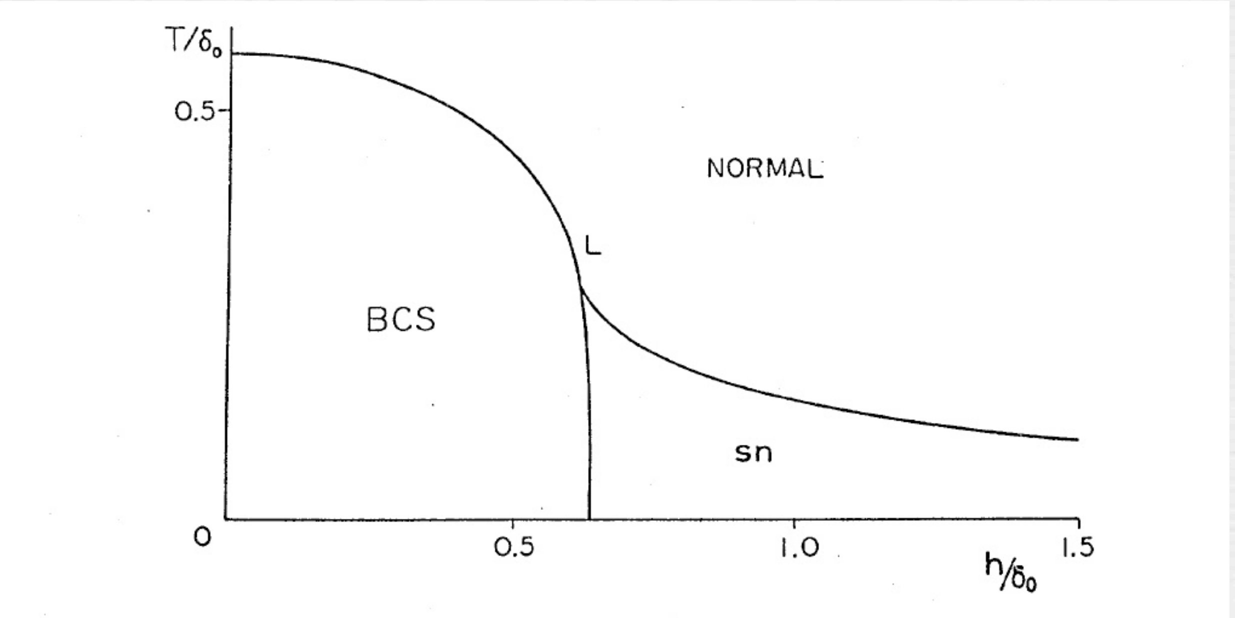
1 dim. Peierls-Fröhlich electron-phonon model

Mertsching/Fischbeck, 1981; Belokolos et al, 1981

inhomogeneous superconductors and ferromagnetism

Machida/Nakanishi, 1984

magnetic field = μ



inhomogeneous gap equation : GN_2

$$\frac{\Sigma(x)}{g^2 N} = \frac{\delta}{\delta\Sigma(x)} \ln \det \left[\partial\!\!\!/ + \Sigma(x)\right]$$

DHN(1975): inverse scattering

$\Rightarrow$ reflectionless potentials: $V_\pm = \Sigma^2 \pm \Sigma'$

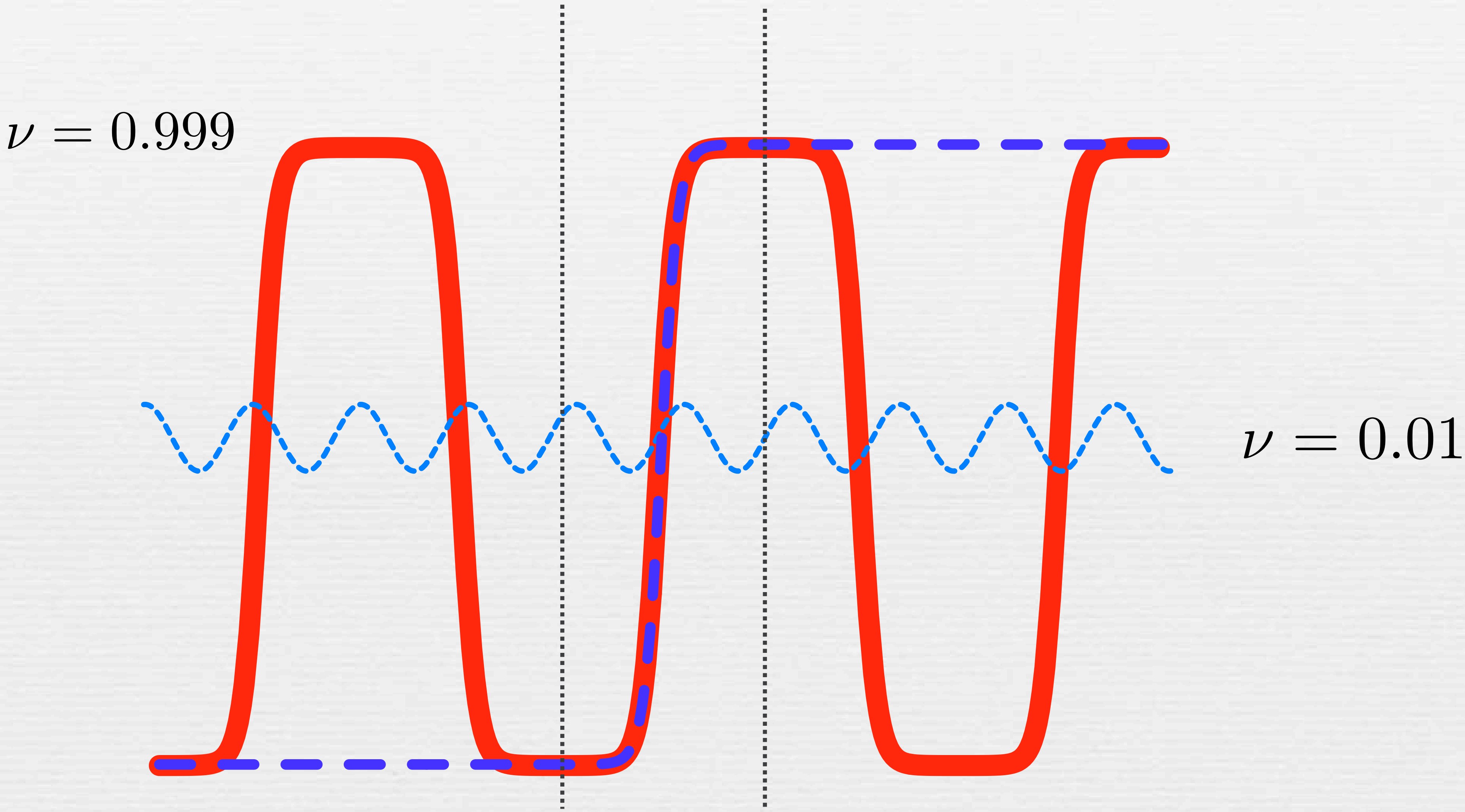
single kink: $\Sigma(x) = m \tanh(m\, x)$

Thies/Urlichs (2005): finite-gap potentials: $V_\pm = \Sigma^2 \pm \Sigma'$

kink crystal: $\Sigma(x) = m\,\nu \dfrac{\mathrm{sn}(m\,x;\nu)\,\mathrm{cn}(m\,x;\nu)}{\mathrm{dn}(m\,x;\nu)}$

kink crystal

$$\Sigma(x) = m\,\nu\,\frac{\mathrm{sn}(m\,x;\nu)\,\mathrm{cn}(m\,x;\nu)}{\mathrm{dn}(m\,x;\nu)}$$



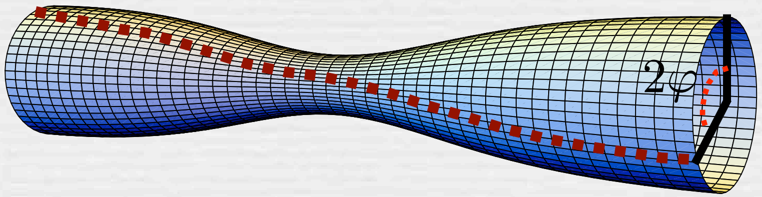
complex gap equation : NJL$_2$ $\Delta = \Sigma - i\Pi$

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta\Delta^*(x)} \ln\det\left[\partial\!\!\!/ + \left(\Sigma(x) - i\gamma^5\,\Pi(x)\right)\right]$$

Shei (1976): inv. scattering: reflectionless Dirac system

$$\Delta(x) = m\,\frac{\cosh\left(m\,\sin(\frac{\theta}{2})\,x - i\frac{\theta}{2}\right)}{\cosh\left(m\,\sin(\frac{\theta}{2})\,x\right)}$$

twisted kink

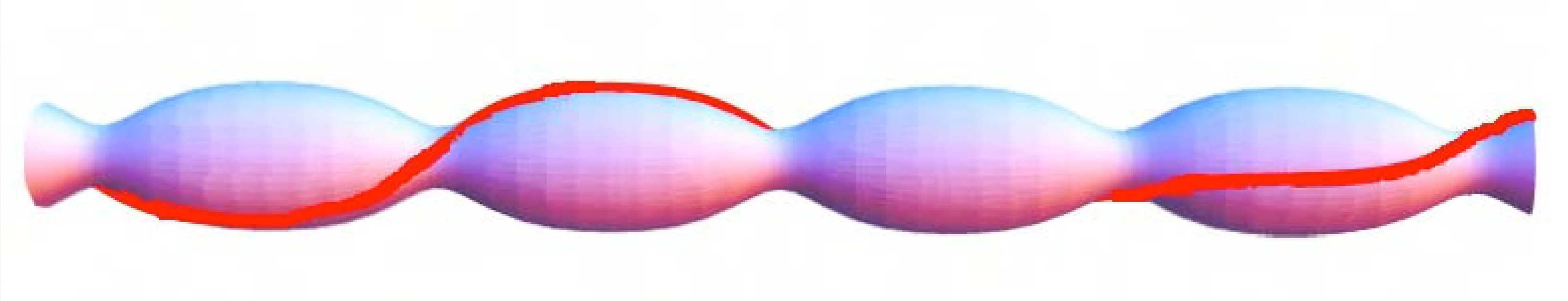


GD & Basar (2008): finite-gap Dirac system

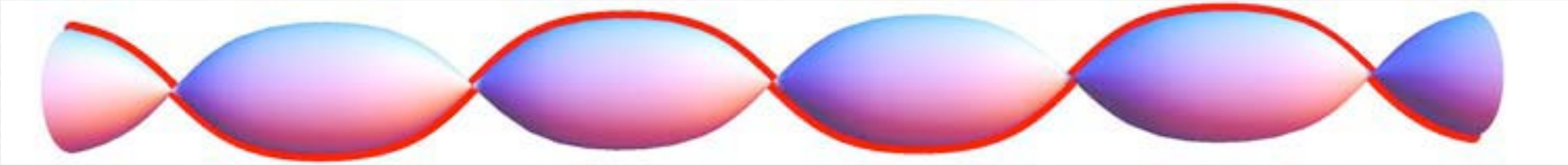
$$\Delta(x) = A\,\frac{\sigma\left(A\,x + i\mathbf{K}' - i\frac{\theta}{2}\right)}{\sigma\left(A\,x + i\mathbf{K}'\right)\sigma\left(i\frac{\theta}{2}\right)}\,e^{iQx}$$

twisted kink crystal

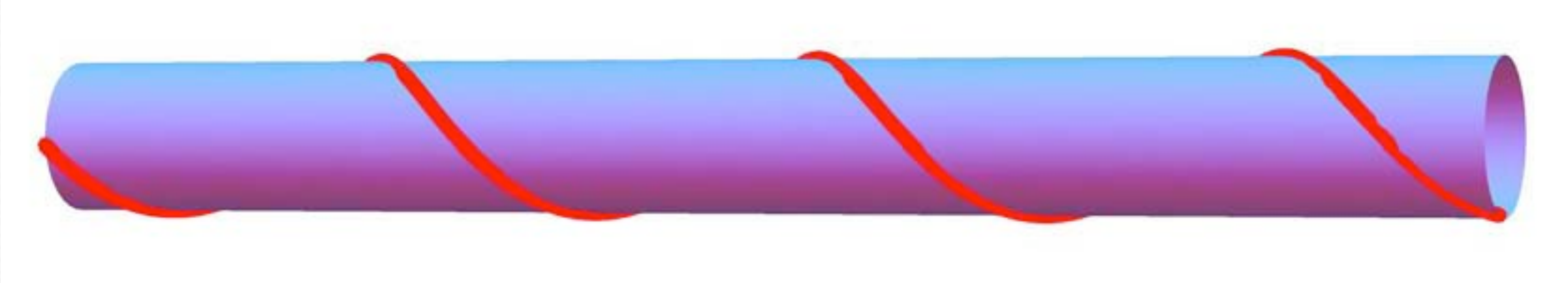
twisted kink crystal: general solution of NJL_2 gap equation



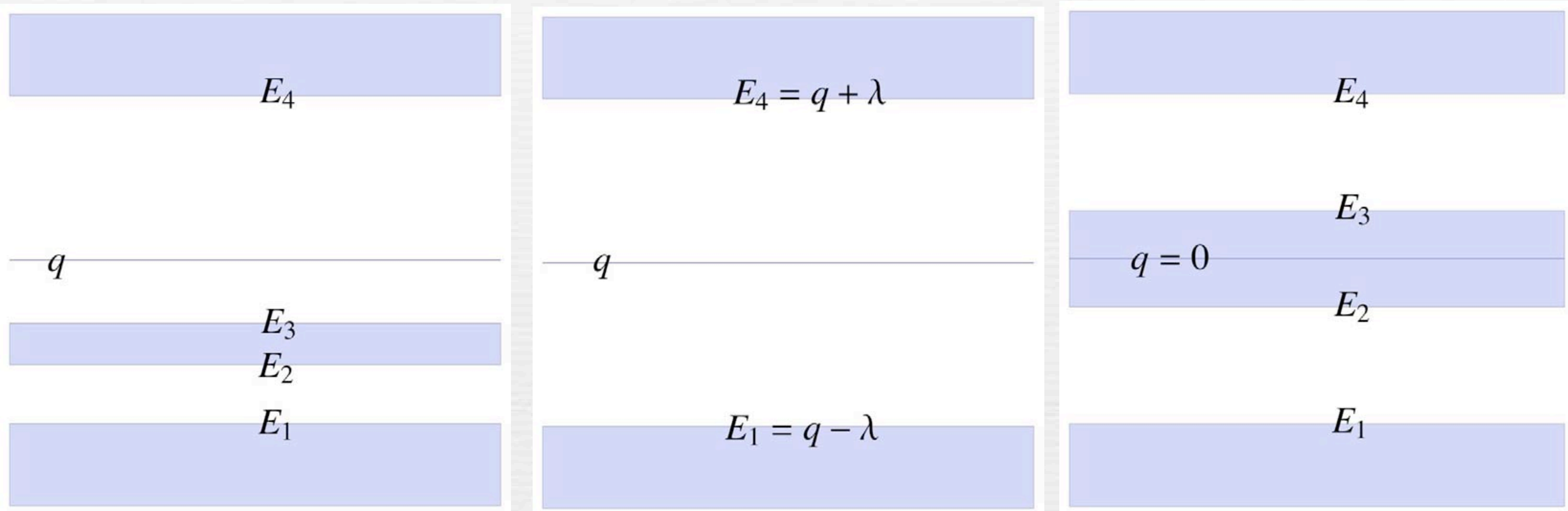
real kink crystal



spiral crystal



single-particle spectra



$$\Delta(x) = A\,\frac{\sigma\left(A\,x + i\mathbf{K}' - i\frac{\theta}{2}\right)}{\sigma\left(A\,x + i\mathbf{K}'\right)\sigma\left(i\frac{\theta}{2}\right)}\,e^{iQx}$$

solving the (complex) gap equation

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta \Delta^*(x)} \ln \det \left[\partial\!\!\!/ + \left(\Sigma(x) - i\gamma^5\, \Pi(x)\right)\right]$$

$$H = -i\gamma^5 \frac{d}{dx} + \gamma^0 \Sigma(x) + i\gamma^1 \Pi(x) \quad = \begin{pmatrix} -i\,\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\,\frac{d}{dx} \end{pmatrix}$$

Bogoliubov/de Gennes hamiltonian

resolvent : Gorkov Green's function $\quad R(x;E) \equiv \langle x|\frac{1}{H-E}|x\rangle$

spectral function $\quad \rho(E) = \frac{1}{\pi}\mathrm{Im} \int dx\, \mathrm{tr}\, R(x; E + i\epsilon)$

Eilenberger eqn: $\quad \frac{\partial}{\partial x} R(x;E)\sigma_3 = i\left[\begin{pmatrix} E & -\Delta(x) \\ \Delta^*(x) & -E \end{pmatrix}, R(x;E)\sigma_3\right]$

solving the (complex) gap equation

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta \Delta^*(x)} \ln \det \left[\partial\!\!\!/ + \left(\Sigma(x) - i\gamma^5 \, \Pi(x) \right) \right]$$

$$H = -i\gamma^5 \frac{d}{dx} + \gamma^0 \Sigma(x) + i\gamma^1 \Pi(x) \quad = \begin{pmatrix} -i\,\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\,\frac{d}{dx} \end{pmatrix}$$

spectral function $\qquad \rho(E) = \frac{1}{\pi} \mathrm{Im} \int dx \, \mathrm{tr}\, R(x; E + i\epsilon)$

two views of gap equation:

$$\ln \det[...] = -\frac{1}{\beta} \int dE \, \rho(E) \, \ln \left(1 + e^{-\beta(E-\mu)} \right)$$

$$\Delta(x) = -N \, g^2 \, \mathrm{Tr}_{D,E} \left[\gamma^0 (1 + \gamma^5) \, R(x; E) \right]$$

gap equation $\longrightarrow$ NLSE

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta \Delta^*(x)} \ln \det \left[\partial\!\!\!/ + \left(\Sigma(x) - i\gamma^5 \, \Pi(x)\right)\right]$$

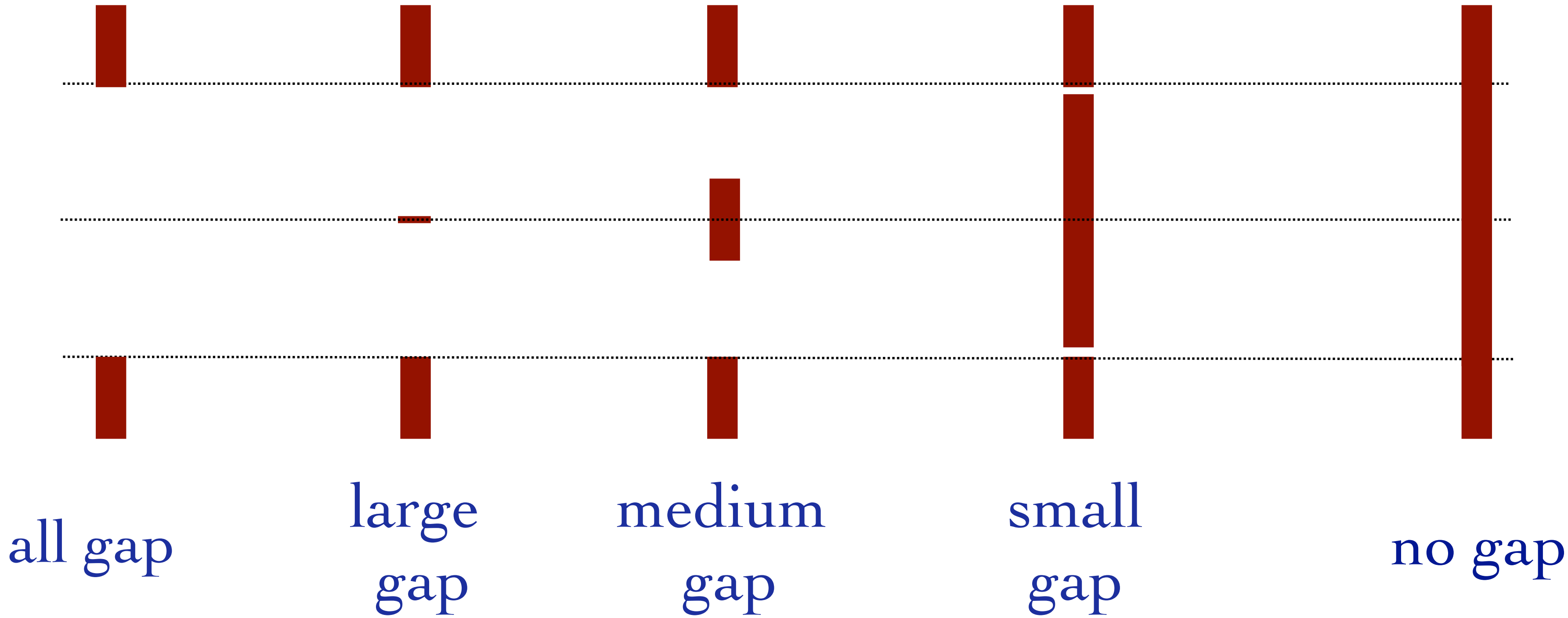
$\downarrow$ ansatz, from gap equation

$$R(x;E) = \mathcal{N}(E) \begin{pmatrix} a(E) + |\Delta(x)|^2 & b(E)\Delta(x) - i\Delta'(x) \\ b(E)\Delta^*(x) + i\Delta'^*(x) & a(E) + |\Delta(x)|^2 \end{pmatrix}$$

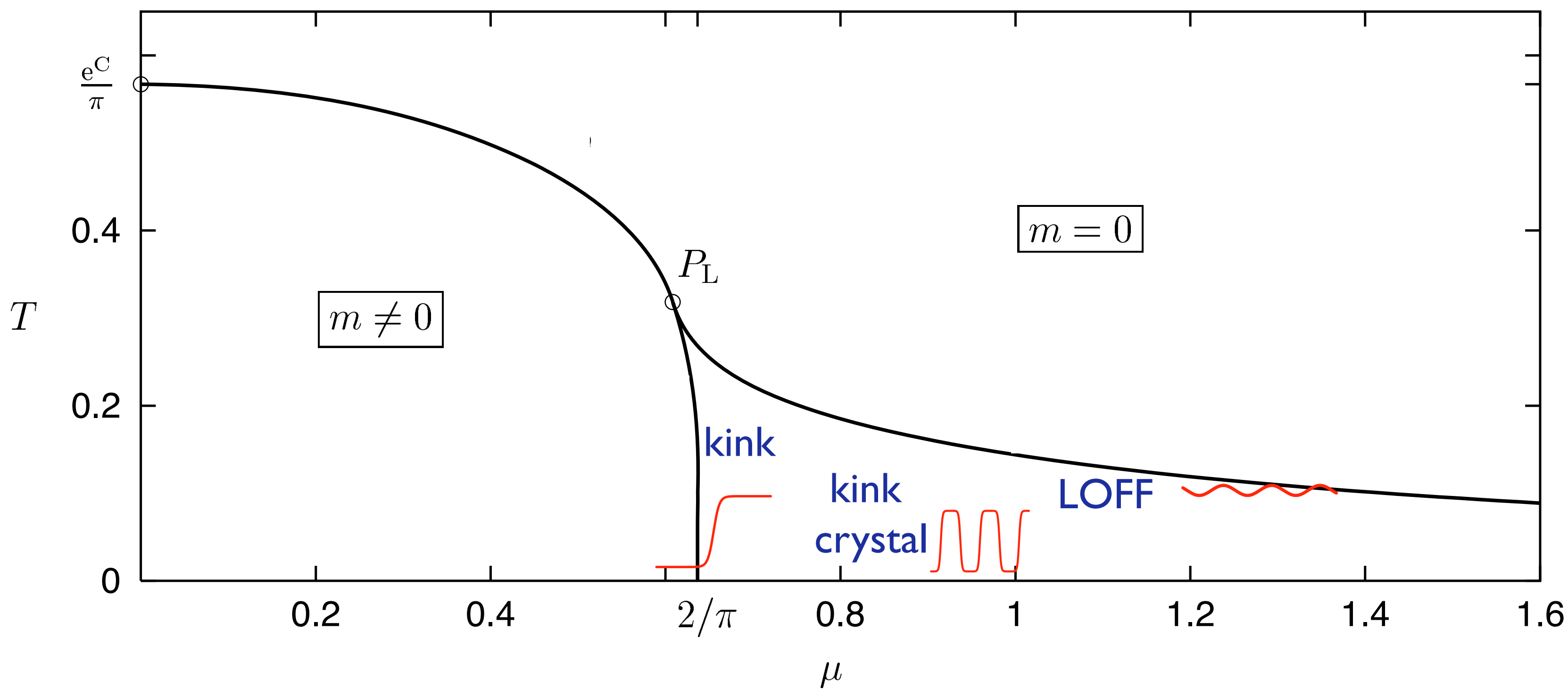
$\downarrow$ Eilenberger equation

$$\Delta'' - 2|\Delta|^2 \, \Delta + i\,(b - 2E)\,\Delta' - 2\,(a - Eb)\,\Delta = 0$$

NLSE : exactly soluble; also for exact spectral function



crossing the boundaries ...



something completely different …

Duality and energy-reflection symmetry

Quasi-exactly-soluble QM models — M. Shifman, ITEP lectures

H=polynomial in sl(2,R) generators

portion of the spectrum known algebraically

energy-reflection symmetry — M. Shifman & A. Turbiner, 1998

periodic QES systems : energy-reflection duality

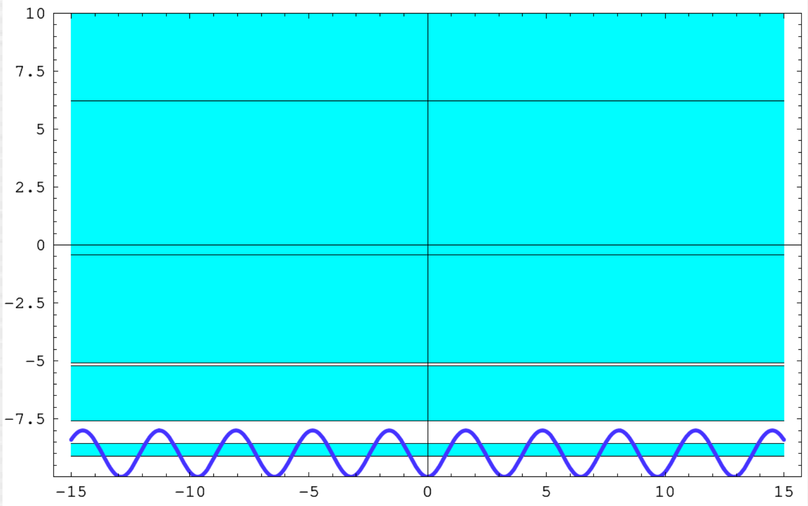
GD & M. Shifman, 2002

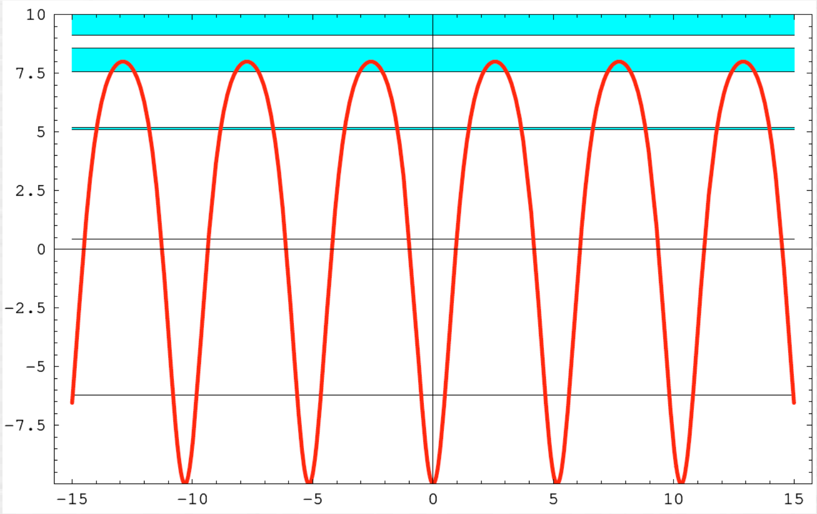
Lamé potential: $V(x) = J(J+1)\nu\,\mathrm{sn}^2(x;\nu) - \frac{1}{2}J(J+1)$

QES: J bands; edges determined algebraically

$$H = J_x^2 + \nu\, J_y^2 - \frac{1}{2}J(J+1)\mathbf{1}$$

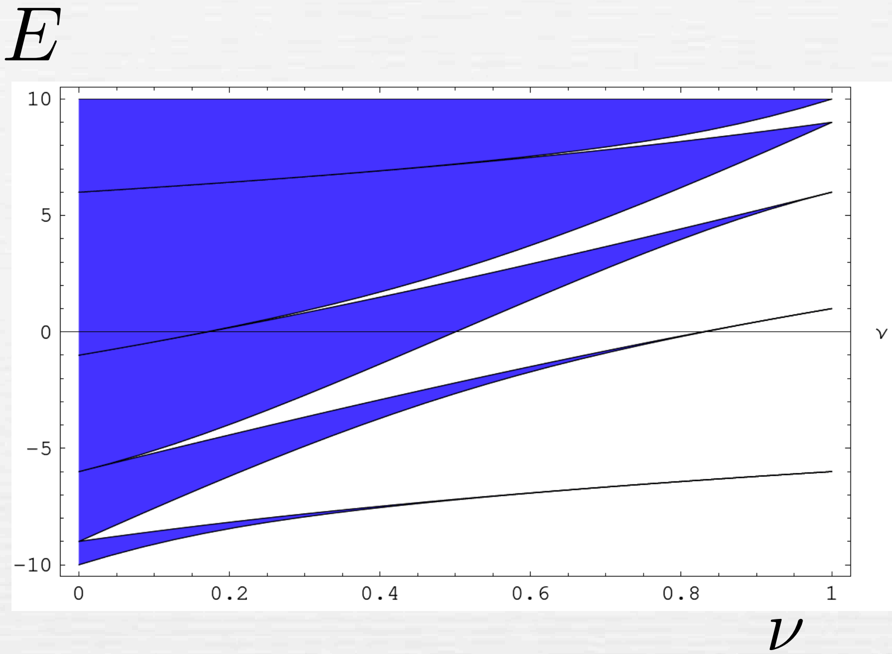
$\nu = 0.1$ $\nu = 0.9$

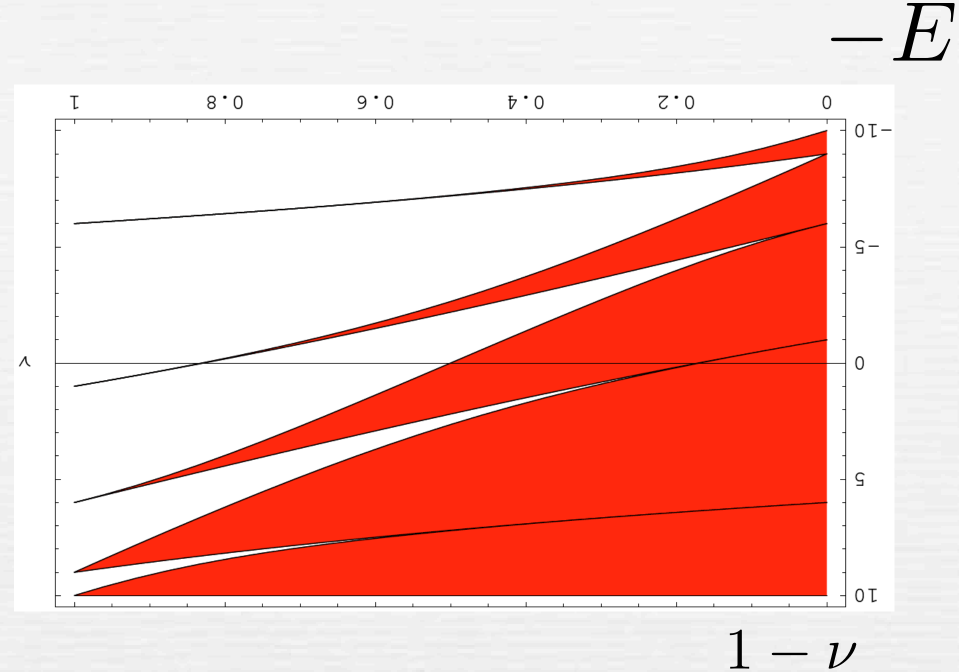


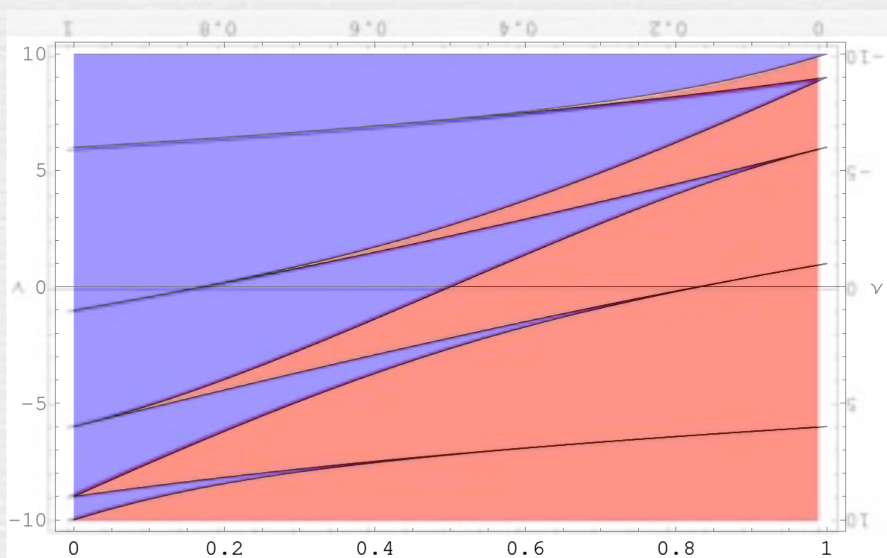


dual potentials

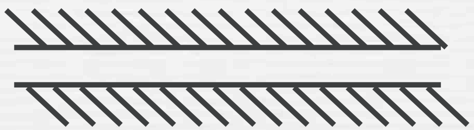
strong coupling <--> weak coupling


E
10
5
0
-5
-10
0
0.2
0.4
0.6
0.8
1
ν


−E
1 − ν


10
5
0
-5
-10
0
0.2
0.4
0.6
0.8
1

$$V(x) = J(J+1)\nu\,\mathrm{sn}^2(x;\nu) - \frac{1}{2}J(J+1)$$



WKB : $E_{\mathrm{top}} \sim \frac{J(J+1)}{2}\left(1 - \frac{2\sqrt{1-\nu}}{\sqrt{J(J+1)}} + \frac{2-\nu}{2J(J+1)} + \dots\right)$

pert. th: $\Delta E_{\mathrm{top}} \sim \frac{8J\,\Gamma(J+1/2)}{4^J\sqrt{\pi}\Gamma(J)}(\nu)^J$

$$E[\nu] \leftrightarrow -E[1-\nu]$$

perturbative/nonperturbative duality

pert. th: $E_{\mathrm{bottom}} \sim -\frac{J(J+1)}{2}\left(1 - \frac{2\sqrt{\nu}}{\sqrt{J(J+1)}} + \frac{1+\nu}{2J(J+1)} + \dots\right)$

instanton : $\Delta E_{\mathrm{bottom}} \sim \frac{8J\,\Gamma(J+1/2)}{4^J\sqrt{\pi}\Gamma(J)}(1-\nu)^J$

back to Gross-Neveu ...

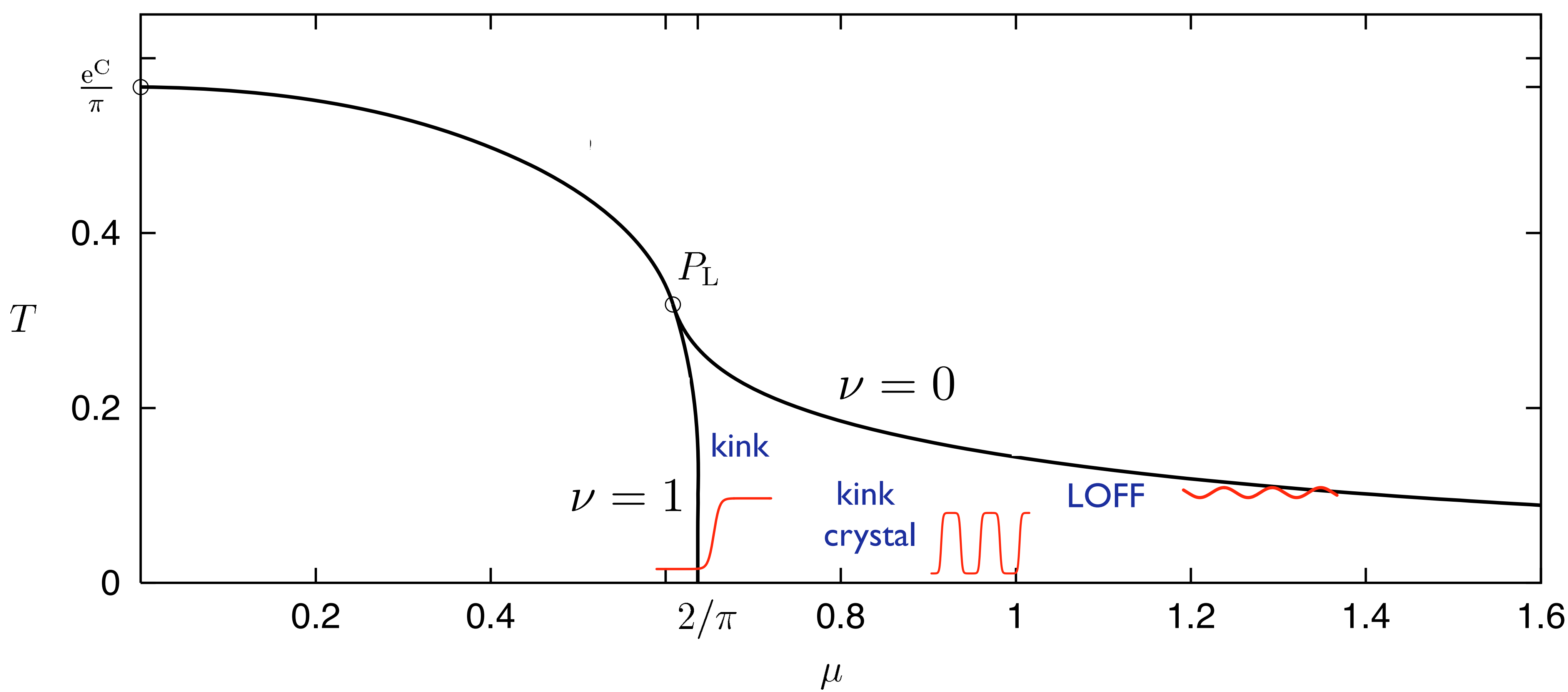
Energy-reflection symmetry in the GN_2 phase diagram

kink crystal condensate: $\Sigma(x) = m\,\nu\,\dfrac{\mathrm{sn}(m\,x;\nu)\,\mathrm{cn}(m\,x;\nu)}{\mathrm{dn}(m\,x;\nu)}$

$$\begin{aligned} V_\pm &= \Sigma^2 \pm \Sigma' \\ &= -m^2\,\nu + 2m^2\,\nu \begin{cases} \mathrm{sn}^2(m\,(x+\mathbf{K}/2);\nu) \\ \mathrm{sn}^2(m\,x;\nu) \end{cases} \end{aligned}$$

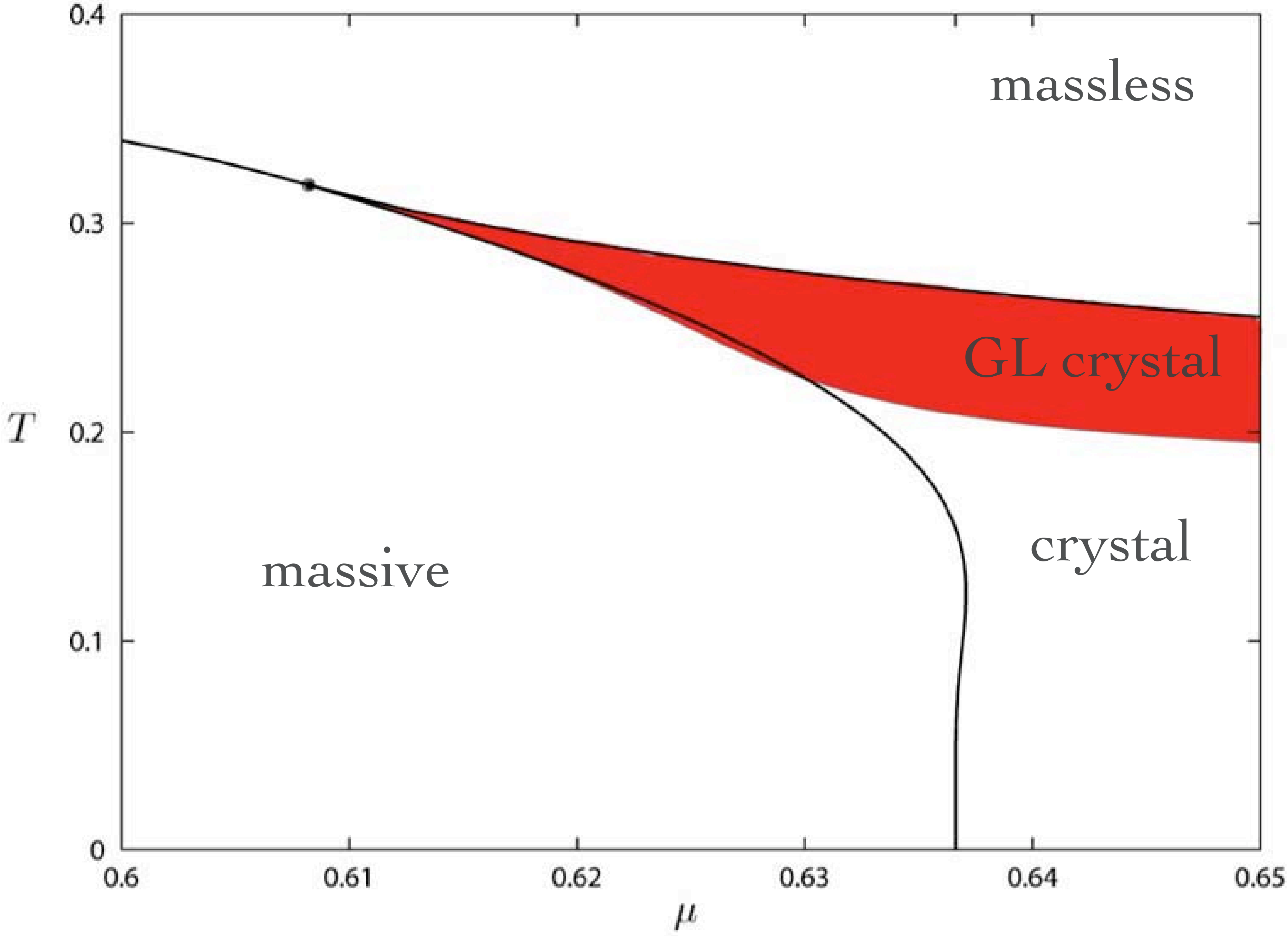
$$\nu \leftrightarrow 1-\nu$$

dense $\leftrightarrow$ dilute



GN phase diagram: dense <--> dilute duality

Ginzburg-Landau expansion



gap equation: all-orders Ginzburg-Landau expansion

$$\begin{aligned}\mathcal{L}_{\rm GL} &= c_0 + c_2|\Delta|^2 + c_3 {\rm Im}\left[\Delta(\Delta')^*\right] + c_4\left[|\Delta|^4 + |\Delta'|^2\right] \\ &\quad + c_5 {\rm Im}\left[\left(\Delta'' - 3|\Delta|^2\Delta\right)(\Delta')^*\right] \\ &\quad + c_6\left[2|\Delta|^6 + 8|\Delta|^2|\Delta'|^2 + 2{\rm Re}\left((\Delta')^2(\Delta^*)^2\right) + |\Delta''|^2\right] + \dots\end{aligned}$$

$$= \sum_n c_n(T,\mu)\, a_n(x)$$

NLSE → entire hierarchy satisfied

$$[a_n(x)]_{NLSE} = \alpha_n|\Delta(x)|^2 + \beta_n$$

GN_2: mKdV hierarchy

NJL_2: AKNS hierarchy

Conclusions

- there is a lot of symmetry in the GN phase diagram
- integrable hierachies: GN_2 = mKdV ; NJL_2 = AKNS
- thermodynamics: crystalline phases
- energy/reflection symmetry = dense/dilute duality

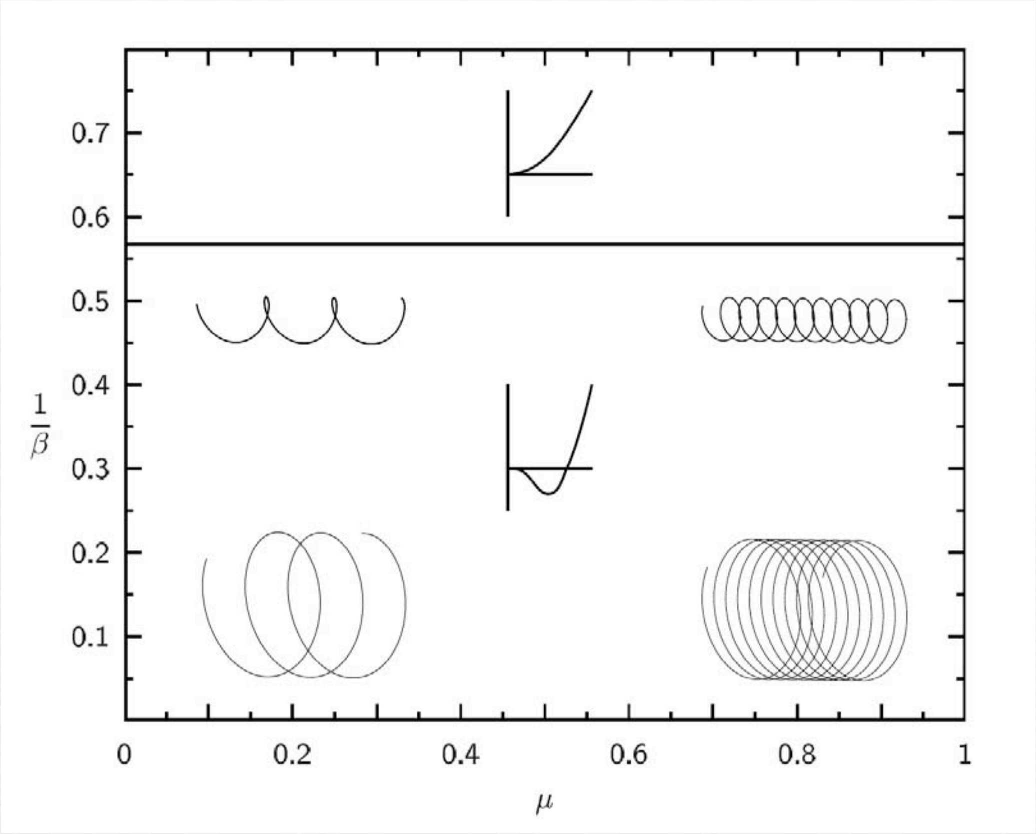
Happy Birthday Misha!

NJL$_2$ phase diagram

Schön/Thies, 2000

"chiral spiral"

$$\Delta = A\, e^{2i\mu\, x}$$



Basar,GD,Thies, 2009: twisted kink crystal --> chiral spiral