Quark Counting Rules:
Old and New Approaches

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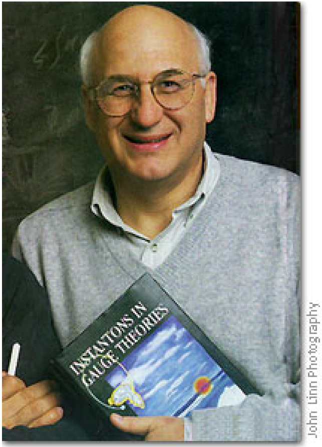
Happy Birthday Misha!



Hadronic form factors

- Hadronic form factors:
  $(1/Q^2)^{n_q - 1}$ counting rules
- Exclusive-inclusive connection:
  Parton distributions behave like $(1-x)^{2n_q - 3}$
- Expectation: some fundamental/easily visible reason

Soft mechanism

- Early idea: Feynman mechanism/Drell-Yan formula [PRL 70]

$$F(Q^2) = \int_0^1 dx \int d^2\mathbf{k}_\perp \, \Psi^*(x, \mathbf{k}_\perp + \bar{x}\mathbf{q}_\perp)\Psi(x, \mathbf{k}_\perp)$$

Take region where both $\Psi_M(x, \mathbf{k}_\perp)$ and $\Psi^*_M(x, \mathbf{k}_\perp + \bar{x}\mathbf{q}_\perp)$ are maximal:

- $|\mathbf{k}_\perp| \sim \Lambda$ is small and
- $\bar{x} \equiv 1 - x$ is close to 0, so that $|\bar{x}\mathbf{q}_\perp| \sim \Lambda$

If $|\Psi(x, \Lambda)|^2 \sim (1-x)^{2n-3}$ then

$$F(Q^2) \sim \int_0^{\Lambda/Q} \bar{x}^{2n-3} \, d\bar{x} \sim (1/Q^2)^{n-1}$$

$\Rightarrow$ Causal relation: Form of $f(x)$ determines $F(Q^2)$

Hard mechanism

- Another region in DY formula

$$F(Q^2) = \int_0^1 dx \int d^2\mathbf{k}_\perp \, \Psi^*(x, \mathbf{k}_\perp + \bar{x}\mathbf{q}_\perp)\Psi(x, \mathbf{k}_\perp)$$

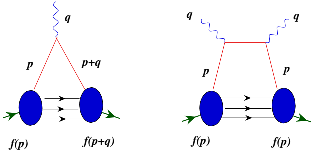
• finite x and small $|\mathbf{k}_\perp|$, e.g., region $|\mathbf{k}_\perp| \ll \bar{x}|\mathbf{q}_\perp|$, where $\Psi(x, \mathbf{k}_\perp)$ is maximal. Then

$$F_M(Q^2) \sim 2\int_0^1 dx \, |\Psi^*(x, \bar{x}\mathbf{q}_\perp)\, \varphi(x)|$$

$\Rightarrow$ form factor repeats large-$\mathbf{k}_\perp$ behavior of WF

- Mechanism was proposed by G.B. West [PRL 70] (in covariant BS-type formalism)

West's model



$$F(Q^2) \sim \int d^4p f(p) f(p+q)$$

- $f(p)$ is a function of $t \equiv p^2$ and spectator mass M^2
- If $f(t, M^2) \sim t^{-n} g(M^2)$, then $F(Q^2) \sim (1/Q^2)^n$

$$\nu W_2(x) \sim \int_{t_{\min}}^{t_{\max} \sim -2\nu} dt f^2(t, M^2) \sim (t_{\min})^{2n-1}$$

where $t_{\min} = \left(\frac{-x}{1-x}\right) \left[M^2 - (1-x)M_N^2\right]$

$$\Rightarrow \nu W_2(x) \sim (1-x)^{2n-1}$$

DY vs West model

- DY: Active parton is "on-shell" $p^2 \sim \Lambda^2$
- $F(Q^2)$ reflects the size of phase space in which $1 - x \sim \Lambda/Q$
- West model: Active parton is highly virtual
- $F(Q^2)$ reflects shape of WF for large virtualities
  $\Rightarrow$ Two mechanisms are completely different
  Surpise: $(1/Q^2)^n \Leftrightarrow (1-x)^{2n-1}$ holds in both models!
- NB: In DY model, n is not necessarily integer
- NB: In West's model, $(1/Q^2)^n$ and $(1-x)^{2n-1}$ have the same cause, but not "causing" each other

Hard mechanism & pQCD

- Integer n naturally appear in hard model: reflect number of hard propagators
- Hard exchange in a theory with dimensionless coupling constant gives $n = n_q - 1$ [BF 73]
- Consequence of scale invariance [MMT 73]
- QCD: $(\alpha_s/Q^2)^{n_q-1}$
- Suppression: $F_\pi(Q^2) \to (2\alpha_s/\pi)s_0/Q^2$
  $\left[s_0 = 4\pi^2 f_\pi^2 \approx 0.7\,\mathrm{GeV}^2\right]$
- Known: $\alpha_s/\pi \sim 0.1$ is penalty for an extra loop
- AdS/QCD model: $F_\pi(Q^2) \to s_0/Q^2$ [Grigoryan, AR]

AdS/QCD

AdS/QCD claims nonperturbative explanation of quark counting rules

Reason: conformal invariance & short-distance behavior of normalizable modes $\Phi(\zeta)$

Form factor in AdS/CFT [Polchinsky,Strassler]

$$F(Q^2) = \int_0^{1/\Lambda} \frac{d\zeta}{\zeta^3} \, \Phi_{P'}(\zeta) J(Q,\zeta) \Phi_P(\zeta)$$

Nonnormalizable mode: $J(Q,\zeta) = \zeta Q K_1(\zeta Q) \equiv \mathcal{K}_1(\zeta Q)$

Normalizable modes for mesons: $\Phi(\zeta) = C\zeta^2 J_{L+1}(\beta_{L,k}\zeta\Lambda)$

For large Q: $\mathcal{K}_1(\zeta Q) \sim e^{-\zeta Q} \Rightarrow$ only small $\zeta \lesssim 1/Q$ work

$$\Rightarrow \; F_{L=0}(Q^2) \to 1/Q^4$$

Wrong power?

Hard-Wall AdS/QCD

- 5-dimensional space: $\{x^\mu, z\} \equiv X^M$
- AdS$_5$ metric with hard wall

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right), \qquad 0 \leq z \leq z_0 = 1/\Lambda\ ,$$

- 5-dimensional vector gauge field $A_M(X)$ with $M = \mu, z$
- AdS/QCD correspondence with 4D field $A_\mu(x)$

$$A_\mu(x, z=0) = A_\mu(x)$$

- 5D gauge action for vector field

$$S_{\rm AdS} = -\frac{1}{4g_5^2}\int d^4x\, dz\, \sqrt{g}\, {\rm Tr}\left(F_{MN}F^{MN}\right)$$

- Field-strength tensor $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$
- Coupling constant $g_5^2 = 6\pi^2/N_c$ is small in large-N_c limit

Bulk-to-boundary Propagator

- Free-field satisfies $\Box_5 A(X) = 0$ or

$$\Box_4 A(x,z) + z\partial_z \left(\frac{1}{z} \partial_z A(x,z) \right) = 0$$

- In momentum 4D representation

$$z\partial_z \left(\frac{1}{z} \partial_z \tilde{A}(p,z) \right) + p^2 \tilde{A}(p,z) = 0 \qquad (*)$$

- AdS/QCD correspondence

$$\tilde{A}_\mu(p,z) = \tilde{A}_\mu(p) \frac{V(p,z)}{V(p,0)}$$

- Bulk-to-boundary propagator $V(p,z)$ satisfies $(*)$
- Gauge invariant boundary condition $F_{\mu z}(x,z_0) = 0$ on IR wall $\Rightarrow$ Neumann b.c. $\partial_z V(p,z_0) = 0$

Bound state expansion

- Solution for $V(p,z)$ with Neumann b.c. ($P=\sqrt{p^2}$)

$$V(p,z) = Pz\left[Y_0(Pz_0)J_1(Pz) - J_0(Pz_0)Y_1(Pz)\right]$$

- Bound state expansion (uses Kneser-Sommerfeld formula)

$$\frac{V(p,z)}{V(p,0)} \equiv \mathcal{V}(p,z) = -\sum_{n=1}^{\infty} \frac{g_5 f_n}{p^2 - M_n^2}\,\psi_n(z)$$

- Masses: $M_n = \gamma_{0,n}/z_0$ (Bessel zeros: $J_0(\gamma_{0,n}) = 0$))
- "Decay constants"

$$f_n = \frac{\sqrt{2}M_n}{g_5 z_0 J_1(\gamma_{0,n})}$$

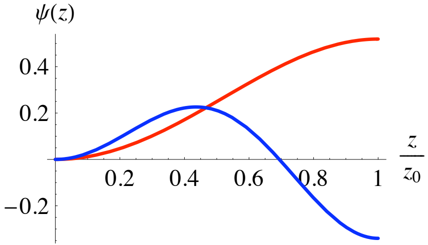
- "ψ" wave functions

$$\psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})}\, zJ_1(M_n z)$$

Wave functions of ψ type

- Obey equation of motion with $p^2 = M_n^2$
- Satisfy $\psi_n(0) = 0$ at UV and $\partial_z \psi_n(z_0) = 0$ at IR boundary
- Normalized according to

$$\int_0^{z_0} \frac{dz}{z} \, |\psi_n(z)|^2 = 1$$



- Do not look like bound state w.f. in quantum mechanics

Wave functions of ϕ type

- Introducing ϕ wave functions

$$\phi_n(z) \equiv \frac{1}{M_n z}\, \partial_z \psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_{0,n})}\, J_0(M_n z)$$

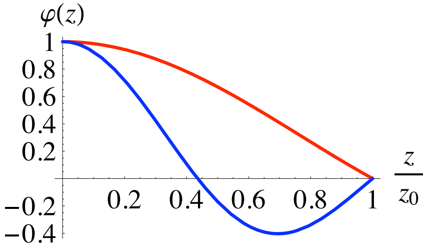
- Reciprocity:

$$\psi_n(z) = -\frac{z}{M_n}\, \partial_z \phi_n(z)$$

- Give couplings $g_5 f_n / M_n$ as their values at the origin
- Satisfy Dirichlet b. c. $\phi_n(z_0) = 0$ at confinement radius
- Are normalized by

$$\int_0^{z_0} dz\, z\, |\phi_n(z)|^2 = 1$$

Wave functions of ϕ type

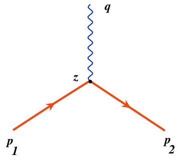


- Are analogous to bound state wave functions in quantum mechanics
- ψ w.f. correspond to vector-potential
- ϕ w.f. correspond to field-strength

Three-Point Function

- "Mercedes-Benz" form

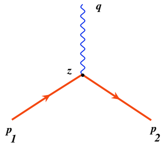
$$W(p_1, p_2, q) = \int_0^{z_0} \frac{dz}{z}\, \mathcal{V}(p_1, z)\, \mathcal{V}(p_2, z)\, \mathcal{V}(q, z)$$



- For spacelike q (with $q^2 = -Q^2$)

$$\mathcal{V}(iQ, z) \equiv \mathcal{J}(Q, z) = Qz \left[K_1(Qz) + I_1(Qz) \frac{K_0(Qz_0)}{I_0(Qz_0)} \right]$$

Form Factors



- Bound-state expansion

$$\mathcal{J}(Q,z) = \sum_{m=1}^{\infty} \frac{g_5 f_m}{Q^2 + M_m^2}\, \psi_m(z)$$

- Infinite tower of vector mesons [Son,Stephanov,Strassler]
- Transition form factors

$$F_{nk}(Q^2) = \int_0^{z_0} \frac{dz}{z}\, \mathcal{J}(Q,z)\, \psi_n(z)\, \psi_k(z)$$

Diagonal form factors

- In terms of ψ functions

$$F_{nn}(Q^2) = \int_0^{z_0} \frac{dz}{z} \, \mathcal{J}(Q,z) \, |\psi_n(z)|^2$$

- In terms of ϕ functions

$$F_{nn}(Q^2) = \frac{1}{1 + Q^2/2M_n^2} \int_0^{z_0} dz \, z \, \mathcal{J}(Q,z) \, |\phi_n(z)|^2$$

- Define

$$\mathcal{F}_{nn}(Q^2) = \int_0^{z_0} dz \, z \, \mathcal{J}(Q,z) \, |\phi_n(z)|^2$$

- Direct analogue of diagonal bound state form factors in quantum mechanics

Form Factors

- Three form factors for vector mesons

$$
\begin{aligned}
&\langle \rho^+(p_2,\epsilon')|J^\mu_{\rm EM}(0)|\rho^+(p_1,\epsilon)\rangle \\
&= -\epsilon'_\beta \epsilon_\alpha \Big[\, \eta^{\alpha\beta}(p_1^\mu + p_2^\mu)\, G_1(Q^2) \\
&\qquad +(\eta^{\mu\alpha} q^\beta - \eta^{\mu\beta} q^\alpha)(G_1(Q^2) + G_2(Q^2)) \\
&\qquad -\frac{1}{M^2} q^\alpha q^\beta (p_1^\mu + p_2^\mu)\, G_3(Q^2)\, \Big]
\end{aligned}
$$

- Hard-wall model gives

$$
-\epsilon'_\beta \epsilon_\alpha \left[\eta_{\alpha\beta}(p_1+p_2)_\mu + 2(\eta_{\alpha\mu} q_\beta - \eta_{\beta\mu} q_\alpha)\right] F_{nn}(Q^2)
$$

- Prediction: $G_1(Q^2) = G_2(Q^2) = F_{nn}(Q^2); G_3(Q^2) = 0$ [SS]
- Moments: magnetic $\mu = 2$, quadrupole $D = -1/M^2$, same result as for pointlike meson (Brodsky & Hiller)

+++ Form Factor

- +++ component of 3-point correlator gives combination

$$\mathcal{F}(Q^2) = G_1(Q^2) + \frac{Q^2}{2M^2} G_2(Q^2) - \left(\frac{Q^2}{2M^2}\right)^2 G_3(Q^2)$$

- For ρ-meson, $\mathcal{F}(Q^2)$ coincides with IMF LL transition that has leading $\sim 1/Q^2$ behavior in pQCD

Large-Q^2 behavior of $\mathcal{F}(Q^2)$

- Hard-wall model prediction

$$\mathcal{F}(Q^2) = \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z)\, |\phi(z)|^2$$

- For large Q:

$$\mathcal{J}(Q,z) \to zQK_1(Qz) \sim e^{-Qz}$$

- Only $z \sim 1/Q$ contribute $\Rightarrow \phi(z)$ may be substituted by $\phi(0)$
- Asymptotic normalization of $\mathcal{F}(Q^2)$ is given by

$$\frac{|\phi(0)|^2}{Q^2} \int_0^{\infty} d\chi\, \chi^2\, K_1(\chi) = 2\, \frac{|\phi(0)|^2}{Q^2}$$

- Same power of $1/Q^2$ as in pQCD, but no α_s/π factor

Soft-Wall model

- Take model with z^2 barrier (Karch et al.)
- Equation for bulk-to-boundary propagator $V(p,z)$

$$z\partial_z \left[\frac{1}{z}\, e^{-\kappa^2 z^2} \partial_z V \right] + p^2\, e^{-\kappa^2 z^2} V = 0$$

- Solution normalized to 1 for $z = 0$ ($a = -p^2/4\kappa^2$)

$$\mathcal{V}(p,z) = a \int_0^1 dx\, x^{a-1} \, \exp\left[-\frac{x}{1-x}\, \kappa^2 z^2 \right] \; ,$$

- Propagator has poles at locations $p^2 = 4(n+1)\kappa^2 \equiv M_n^2$

$$\mathcal{V}(p,z) = \kappa^2 z^2 \sum_{n=0}^{\infty} \frac{L_n^1(\kappa^2 z^2)}{a+n+1} = \sum_{n=0}^{\infty} \frac{g_5 f_n}{M_n^2 - p^2}\, \psi_n(z)$$

Wave Functions

- ψ wave functions

$$\psi_n(z) = z^2 \sqrt{\frac{2}{n+1}}\, L_n^1(\kappa^2 z^2)$$

- Coupling constants

$$g_5 f_n = \frac{1}{z}\, e^{-\kappa^2 z^2}\, \partial_z \psi_n(z)\Big|_{z=\epsilon\to 0} = \sqrt{8(n+1)}\kappa^2$$

- ϕ wave functions

$$\phi_n(z) = \frac{1}{M_n z} e^{-\kappa^2 z^2} \partial_z \psi_n(z) = \frac{2}{M_n} e^{-\kappa^2 z^2} L_n^0(\kappa^2 z^2)$$

$$\phi_0(z) = \sqrt{2} e^{-\kappa^2 z^2} \quad , \quad \phi_1(z) = \sqrt{2} e^{-\kappa^2 z^2}(1-\kappa^2 z^2)$$

Form Factors & ρ-Meson Dominance

- Form factor of the lowest state

$$\mathcal{F}_{00}(Q^2) = 2\int_0^\infty dz\, z\, e^{-\kappa^2 z^2}\, \mathcal{J}(Q,z)$$

- Using representation for $\mathcal{J}(Q,z)$ gives

$$\mathcal{F}_{00}(Q^2) = \frac{1}{1+Q^2/M_0^2}$$

- Exact vector dominance is due to overlap integral

$$\mathcal{F}_{m,00} \equiv 2\int_0^\infty dz\, z^3\, e^{-z^2}\, L_m^1(z^2) = \delta_{m0}$$

Large-Q^2 behavior

- Large-Q^2 behavior of $\mathcal{F}$ form factor

$$\mathcal{F}_{nn}(Q^2) \to \frac{\Phi_n^2(0)}{Q^2} \int_0^\infty d\chi\, \chi^2\, K_1(\chi) = \frac{2\,\Phi_n^2(0)}{Q^2}$$

- In hard-wall model:

$$\Phi_0^{\mathrm{H}}(0) = \frac{\sqrt{2} m_\rho}{\gamma_{0,1} J_1(\gamma_{0,1})} \Rightarrow \mathcal{F}_\rho^{\mathrm{H}}(Q^2) \to \frac{2.56 m_\rho^2}{Q^2}$$

- In soft-wall model:

$$\Phi_0^{\mathrm{S}}(0) = \frac{m_\rho}{\sqrt{2}} \Rightarrow \mathcal{F}_\rho^{\mathrm{S}}(Q^2) \to \frac{m_\rho^2}{Q^2}$$

Action including χSB

- Full action of hard-wall model

$$S^B_{\rm AdS} = {\rm Tr}\int d^4x \int_0^{z_0} dz \left[\frac{1}{z^3}(D^M X)^\dagger (D_M X) + \frac{3}{z^5} X^\dagger X - \frac{1}{8g_5^2 z}\left(B^{MN}_{(L)} B_{(L)MN} + B^{MN}_{(R)} B_{(R)MN}\right)\right]$$

- $DX = \partial X - iB_{(L)}X + iXB_{(R)}$, $B_{(L,R)} = V \pm A$,
  $X(x,z) = v(z)U(x,z)/2$,
  Chiral field: $U(x,z) = \exp\left[2it^a\pi^a(x,z)\right]$, $t^a = \sigma^a/2$
  Pion field: $\pi^a(x,z)$
  $v(z) = (m_q z + \sigma z^3)$ with $m_q \sim$ quark mass, $\sigma \sim$ condensate
- Longitudinal component of axial field

$$A^a_{\parallel\, M}(x,z) = \partial_M \psi^a(x,z)$$

gives another pion field $\psi^a(x,z)$

Pion wave function Ψ

- Model satisfies Gell-Mann–Oakes–Renner relation $m_\pi^2 \sim m_q$
- Chiral limit $m_q = 0$: analytic result for $\Psi(z) \equiv \psi(z) - \pi(z)$

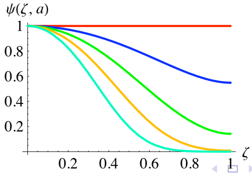
$$\Psi(z) = z\,\Gamma(2/3)\left(\frac{\alpha}{2}\right)^{1/3}\left[I_{-1/3}\left(\alpha z^3\right) - I_{1/3}\left(\alpha z^3\right)\frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)}\right]$$

where $\alpha = g_5\sigma/3$

- $\Psi(z)$ satisfies $\Psi(0) = 1$, Neumann b.c. $\Psi'(z_0) = 0$ and

$$f_\pi^2 = -\frac{1}{g_5^2}\left(\frac{1}{z}\partial_z\Psi(z)\right)_{z=\epsilon\to 0}$$

$\Psi(z) \to \psi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

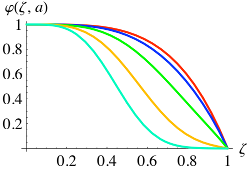
Pion wave function Φ

- Conjugate wave function

$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2}\left(\frac{1}{z}\,\partial_z \Psi(z)\right) = -\frac{2}{s_0}\left(\frac{1}{z}\,\partial_z \Psi(z)\right)$$

- Characteristic scale $s_0 = 4\pi^2 f_\pi^2 \approx 0.67\,\mathrm{GeV}^2$
- $\Phi(z)$ satisfies $\Phi(0) = 1$ and Dirichlet b.c. $\Phi(z_0) = 0$

$\Phi(z) \to \phi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

Parameters of model

- z_0 is fixed through ρ-meson mass: $z_0 = z_0^\rho = (323\ \mathrm{MeV})^{-1}$
- From $\Phi(0) = 1$, it follows that

$$g_5^2 f_\pi^2 = 3 \cdot 2^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} \frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)} \alpha^{2/3}$$

- Experimental f_π is obtained for $\alpha = (424\,\mathrm{MeV})^3$
- Then $a \equiv \alpha z_0^3$ equals $2.26 \equiv a_0$
- Note: $I_{2/3}(a)/I_{-2/3}(a) \approx 1$ for $a \gtrsim 1$
  $\Rightarrow$ value of f_π is basically determined by α alone

Pion Form Factor

- In terms of $\Psi(z)$:

$$F_\pi(Q^2) = \frac{1}{g_5^2 f_\pi^2} \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[\left(\frac{\partial_z \Psi}{z} \right)^2 + \frac{g_5^2 v^2}{z^4} \Psi^2(z) \right]$$

- Normalization can be checked from

$$F_\pi(Q^2) = -\int_0^{z_0} dz\, \mathcal{J}(Q,z)\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big)$$

that gives

$$F_\pi(0) = -\int_0^{z_0} dz\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big) = \Psi(0)\, \Phi(0) = 1$$

Pion Charge Radius

- In terms of f_π:

$$\langle r_\pi^2 \rangle \Big|_{a \gtrsim 2} = \frac{3}{4\pi^2 f_\pi^2} + \frac{1}{2\pi^2 f_\pi^2} \ln\left(\frac{\alpha z_0^3}{0.566}\right) \approx 0.34 \mathrm{fm}^2$$

- Compare to Nambu-Jona-Lasinio model

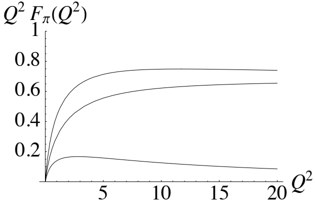
$$\langle r_\pi^2 \rangle_{\mathrm{NJL}} = \underbrace{\frac{3}{2\pi^2 f_\pi^2}}_{0.34 \mathrm{fm}^2} + \underbrace{\frac{1}{8\pi^2 f_\pi^2} \ln\left(\frac{m_\sigma^2}{m_\pi^2}\right)}_{0.11 \mathrm{fm}^2}$$

- Pion of hard-wall AdS/QCD model is too small

Pion Form Factor at Large Q^2

- Form factor in terms of $\Psi(z)$ and $\Phi(z)$:

$$F_\pi(Q^2) = \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[g_5^2 f_\pi^2 \Phi^2(z) + \frac{9\alpha^2}{g_5^2 f_\pi^2}\, z^2\, \Psi^2(z) \right]$$



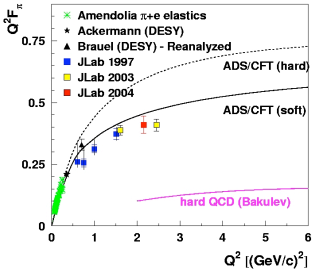
- Total (in GeV^2)
- Φ^2 term
- Ψ^2 term

- For large Q, only $z \sim 1/Q$ work:

$$F_\pi(Q^2) \to \frac{2\, g_5^2 f_\pi^2 \Phi^2(0)}{Q^2} = \frac{4\pi^2 f_\pi^2}{Q^2} \equiv \frac{s_0}{Q^2}$$

Pion Form Factor

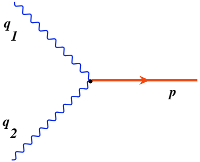
- Comparison with experiment



- Pion is too small
- pQCD has $2\alpha_s/\pi$ factor due to one-gluon exchange:

$$F_\pi^{\rm pQCD}(Q^2) \to \frac{2\alpha_s}{\pi} \cdot \frac{s_0}{Q^2} \sim 0.2\, F_\pi^{\rm AdS/QCD}(Q^2)$$

Anomalous Amplitude



- $\pi^0\gamma^*\gamma^*$ form factor

$$\int \langle \pi, p | T\left\{ J^{\mu}_{\rm EM}(x)\, J^{\nu}_{\rm EM}(0)\right\} |0\rangle e^{-iq_1 x} d^4 x$$

$$= \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, \frac{N_c}{12\pi^2 f_\pi}\, K_{\gamma^*\gamma^*\pi^0}\left(Q_1^2, Q_2^2\right)$$

$p = q_1 + q_2$ and $q^2_{1,2} = -Q^2_{1,2}$

- For real photons in QCD is fixed by axial anomaly

$$K_{\gamma^*\gamma^*\pi^0}(0,0) = 1$$

Extending AdS/QCD Model

- Need to have isoscalar fields $\Rightarrow$ gauging $U(2)_L \otimes U(2)_R$

$$\mathcal{B}_\mu = t^a B_\mu^a + \mathbf{1}\,\frac{\hat{B}_\mu}{2}$$

- Need Chern-Simons term

$$S_{\rm CS}^{(3)}[\mathcal{B}] = \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}\int d^4x\,dz\,(\partial_z\mathcal{B}_\mu)\left[\mathcal{F}_{\nu\rho}\mathcal{B}_\sigma + \mathcal{B}_\nu\mathcal{F}_{\rho\sigma}\right]$$

- Anomalous form factor conforming to QCD anomaly

$$\begin{aligned} K(Q_1^2,Q_2^2) &= \Psi(z_0)\mathcal{J}(Q_1,z_0)\mathcal{J}(Q_2,z_0) \\ &- \int_0^{z_0}\mathcal{J}(Q_1,z)\mathcal{J}(Q_2,z)\,\partial_z\Psi(z)\,dz \end{aligned}$$

- Check:

$$K(0,0) = \Psi(z_0) - \int_0^{z_0}\partial_z\Psi(z)\,dz = \Psi(0) = 1$$

$\gamma^*\gamma^*\pi^0$ Form Factors

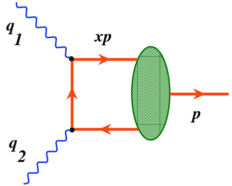
- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\,\Phi(z)\,z\,dz$$

- One real photon:

$$K(0, Q^2) \to \frac{\Phi(0)s_0}{2Q^2} \int_0^{\infty} d\chi\,\chi^2\,K_1(\chi) = \frac{s_0}{Q^2}$$

$\gamma^*\gamma\pi^0$ Form Factor in pQCD



- In pQCD:

$$K^{\rm pQCD}(0,Q^2) = \frac{s_0}{3Q^2}\int_0^1 \frac{\varphi_\pi(x)}{x}\,dx \equiv \frac{s_0}{3Q^2}\,I^\varphi$$

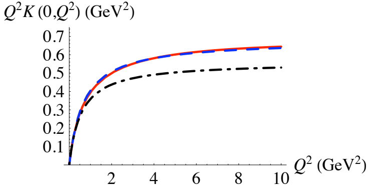
- Coincides with AdS/QCD model if $I^\varphi = 3$, e.g., for $\varphi_\pi(x) = 6x(1-x)$ (asymptotic DA)

Comparison with data

- Brodsky-Lepage interpolation

$$K^{\mathrm{BL}}(0,Q^2) = \frac{1}{1+Q^2/s_0}$$

- Our model (red) is very close to BL interpolation (blue)



- CLEO data represented by black dash-dotted line
- NLO pQCD fits data. Fits give DA's with $I^{\varphi} \approx 3$

Equal large photon virtualities

- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \Phi(z)\, z\, dz$$

- Equal photon virtualities:

$$K(Q^2, Q^2) \to \frac{\Phi(0) s_0}{Q^2} \int_0^{\infty} d\chi\, \chi^3 \,[K_1(\chi)]^2 = \frac{s_0}{3Q^2}$$

- pQCD result does not depend on pion DA

$$K^{\mathrm{pQCD}}(Q^2, Q^2) = \frac{s_0}{3} \int_0^1 \frac{\varphi_\pi(x)\, dx}{xQ^2 + (1-x)Q^2} = \frac{s_0}{3Q^2}$$

- and coincides with AdS/QCD model!

Non-equal large photon virtualities

- Take $Q_1^2 = (1+\omega)Q^2$ and $Q_2^2 = (1-\omega)Q^2$
- Leading-order pQCD gives in this case

$$K^{\rm pQCD}(Q_1^2, Q_2^2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)\, dx}{1+\omega(2x-1)} \equiv \frac{s_0}{3Q^2} I^\varphi(\omega)$$

- AdS/QCD model gives

$$\frac{\Phi(0)s_0}{2Q^2} \sqrt{1-\omega^2} \int_0^\infty d\chi\, \chi^3\, K_1(\chi\sqrt{1+\omega})K_1(\chi\sqrt{1-\omega})$$
$$= \left(\frac{s_0}{3Q^2}\right) \left\{ \frac{3}{4\omega^3} \left[2\omega - (1-\omega^2)\, \ln\left(\frac{1-\omega}{1+\omega}\right)\right]\right\}$$

- $\{\ldots\}$ coincides with pQCD $I^\varphi(\omega)$ for $\varphi(x) = 6x(1-x)$

AdS/pQCD duality

- Use representation

$$\chi K_1(\chi) = \int_0^\infty e^{-\chi^2/4u-u}\, du \ ,$$

- And integrate over χ to get

$$K(Q_1^2, Q_2^2) \rightarrow \frac{s_0}{Q^2} \int_0^\infty \int_0^\infty \frac{u_1 u_2\, e^{-u_1-u_2} du_1 du_2}{u_2(1+\omega) + u_1(1-\omega)} \ .$$

- Change $u_2 = x\lambda$, $u_1 = (1-x)\lambda$ and integrate over λ:

$$K(Q_1^2, Q_2^2) \rightarrow \frac{s_0}{3Q^2} \int_0^1 \frac{6\, x(1-x)\, dx}{1+\omega(2x-1)}$$

- Coincides with the pQCD formula if $\varphi_\pi(x) = 6\, x(1-x)$

Bound-state decomposition

- GVMD for bulk-to-boundary propagator:

$$\mathcal{J}(Q,z) = \sum_{n=1}^{\infty} \frac{g_5 f_n \psi_n^V(z)}{Q^2 + M_n^2}$$

- Form factor $K(Q_1^2, Q_2^2)$ has double GVMD representation

$$K(Q_1^2, Q_2^2) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{A_{n,k}}{(1 + Q_1^2/M_n^2)(1 + Q_2^2/M_k^2)}$$

- But we know that $K(Q^2, Q^2) \sim 1/Q^2$!

How double GVMD gives $1/Q^2$

- Soft-wall model integral

$$K^{s}(Q_1^2, Q_2^2) = 2\kappa^2 \int_0^\infty \mathcal{J}^{s}(Q_1, z)\, \mathcal{J}^{s}(Q_2, z)\, e^{-\kappa^2 z^2}\, z dz$$

- Gives ($a_i = Q_i^2/M^2$ and $M = 2\kappa$ is mass scale)

$$K^{s}(Q_1^2, Q_2^2) = \sum_{n=0}^{\infty} \frac{a_1}{(a_1+n)(a_1+n+1)} \frac{a_2}{(a_2+n)(a_2+n+1)}$$

- Each term behaves like $1/Q_1^2 Q_2^2$, but

$$K^{s}(Q^2, Q^2) \to a^2 \int_0^\infty \frac{dn}{(n+a)^4} = \frac{1}{3a} = \frac{M^2}{3Q^2}$$

- Higher resonances are important!

Summary

- Form Factors in AdS/QCD given by QM-like formulas
- Only one mechanism $z \sim 1/Q$ for large Q
- IMF (LL) form factor of vector meson indeed behaves like $1/Q^2$ for large Q^2
- Exact ρ-dominance for $\mathcal{F}(Q^2)$ in soft-wall model
- Large-Q^2 asymptotics is s_0/Q^2 vs. pQCD $(2\alpha_s/\pi)s_0/Q^2$
- Overshoots data: AdS/QCD pion is too small
- Anomalous amplitude:
  1. Extension to $U(2)_L \otimes U(2)_R$ and Chern-Simons term
  2. Fixing normalization by conforming to QCD anomaly
  3. Large-Q^2 behavior coincides with pQCD calculations for asymptotic pion DA
  4. Double GVMD does not contradict to $1/Q^2$ asymptotics

Conclusion

- AdS/QCD provides instructive model
  for what may happen
  with form factors in QCD

Happy Birthday Misha!