# Poset and Polytope Perspectives On Alternating Sign Matrices

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Finally, I would like to thank Jesus Christ "in whom are hidden all the treasures of wisdom and knowledge" for illumining this little bit of his creation for me to discover.

## Dedication

*This thesis is dedicated to Ryan my constant encourager and Hazel my sweet little joy.*

JESSICA PALENCIA STRIKER

Number of words: 200

## Abstract

Alternating sign matrices (ASMs) are square matrices with entries 0, 1, or $-1$ whose rows and columns sum to 1 and whose nonzero entries alternate in sign. We put ASMs into a larger context by studying a certain tetrahedral poset and its subposets. We prove the order ideals of these subposets are in bijection with a variety of interesting combinatorial objects, including ASMs, totally symmetric self–complementary plane partitions (TSSCPPs), Catalan objects, tournaments, and totally symmetric plane partitions. We prove product formulas counting these order ideals and give the rank generating function of some of the corresponding lattices of order ideals. We also reformulate a known expansion of the tournament generating function as a sum over ASMs and prove a new expansion as a sum over TSSCPPs.

We define the alternating sign matrix polytope as the convex hull of $n \times n$ alternating sign matrices and prove its equivalent description in terms of inequalities. We count its facets and vertices and describe its projection to the permutohedron as well as give a complete characterization of its face lattice in terms of modified square ice configurations. Furthermore we prove that the dimension of any face can be easily determined from this characterization.

Advisor: Professor Dennis Stanton

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# Chapter 1

# Introduction

Alternating sign matrices (ASMs) are both mysterious and compelling. They are simply defined as square matrices with entries 0, 1, or $-1$ whose rows and columns sum to 1 and alternate in sign, but have proved quite difficult to understand (and even count). In this work we present two new perspectives which will help to shed light on these objects and put them in a larger context.

In Chapter 2 we seek to put ASMs into a larger context by viewing them from the poset perspective. Toward this end we study a tetrahedral poset $T_n$ and various subposets consisting of all the elements of $T_n$ and only certain ordering relations. We prove that the order ideals of these various subposets are in bijection with a variety of interesting combinatorial objects, including ASMs, tournaments, totally symmetric self–complementary plane partitions, and totally symmetric plane partitions. We prove product formulas counting the order ideals of these posets and give the rank generating function of some of the corresponding lattices of order ideals as well as discuss the connections between these different combinatorial objects.

In Chapter 3 we explore ASMs from the polytope perspective. We first define the alternating sign matrix polytope as the convex hull of $n \times n$ alternating sign matrices and prove its equivalent description in terms of inequalities. This is analogous to the well known result of Birkhoff and von Neumann that the convex hull of the permutation matrices equals the set of all nonnegative doubly stochastic matrices. We count the facets and vertices of the alternating sign matrix polytope and describe its projection to the permutohedron as well as give a complete characterization of its face lattice in terms of modified square ice configurations. Furthermore we prove that the dimension of any face can be easily determined from this characterization.

## 1.1 Background

In 1983 William Mills, David Robbins, and Howard Rumsey made an elegant conjecture about the enumeration of alternating sign matrices.

**Definition 1.1.1.** Alternating sign matrices (ASMs) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- the entries in each row and column sum to 1
- nonzero entries in each row and column alternate in sign

See Figure 1.1 for the seven ASMs with three rows and three columns.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Figure 1.1: The $3 \times 3$ ASMs

ASMs have an interesting connection to physics in that they are in bijection with the statistical physics model of square ice with domain wall boundary conditions [23].

The horizontal molecules correspond to $+1$, the vertical molecules correspond to $-1$, and all other molecules correspond to 0. An ASM and the corresponding square ice configuration are shown in Figure 1.2.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

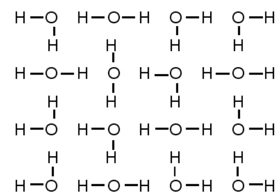


Figure 1.2: An ASM and the corresponding square ice configuration

Mills, Robbins, and Rumsey conjectured that the total number of $n \times n$ alternating sign matrices is given by the expression

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1.1.1}$$

They were unable to prove this for all $n$, so for the next 13 years it remained an intriguing mystery for the mathematical community until Doron Zeilberger proved it [43]. For such a simple statement, though, Zeilberger's proof was quite complicated. Shortly thereafter, Greg Kuperberg noticed the connection between ASMs and square ice and gave a shorter proof using these insights [23]. This connection with physics has strengthened since then with the conjecture of Razumov and Stroganov which says that the enumeration of subclasses of ASMs gives the ground state probabilities of the dense O(1) loop model in statistical physics [33]. This conjecture has been further refined and various special cases have been proved (see for example [13]), but no general proof is known.

Another curious fact is that (1.1.1) counts two other types of mathematical objects: descending plane partitions with largest part less than or equal to $n$ and totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box. See Figures 1.3 and 1.4 for examples of these objects when $n = 3$.

We begin by defining plane partitions.

**Definition 1.1.2.** A *plane partition* is a two dimensional array of positive integers which weakly decrease across rows from left to right and weakly decrease down columns.

We can visualize a plane partition as a stack of unit cubes pushed up against the corner of a room. If we identify the corner of the room with the origin and the room with the positive orthant, then denote each unit cube by its coordinates in $\mathbb{N}^3$, we obtain the following equivalent definition of a plane partition. A plane partition $\pi$ is a finite set of positive integer lattice points $(i, j, k)$ such that if $(i, j, k) \in \pi$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, and $1 \leq k' \leq k$ then $(i', j', k') \in \pi$.

Using this alternate definition, we can define the following symmetry properties. A plane partition $\pi$ is *symmetric* if whenever $(i, j, k) \in \pi$ then $(j, i, k) \in \pi$ as well. $\pi$ is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i)$ and $(k, i, j)$ are in $\pi$ as well. A plane partitions is *totally symmetric* if it is both symmetric and cyclically symmetric, so that if $(i, j, k) \in \pi$ then all six permutations of $(i, j, k)$ are also in $\pi$.

**Definition 1.1.3.** A totally symmetric self–complementary plane partition (TSSCPP) inside a $2n \times 2n \times 2n$ box is a totally symmetric plane partition which is equal to its complement. This means that the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself.

We now define descending plane partitions.

**Definition 1.1.4.** Descending plane partitions (DPPs) are arrays of positive integers with each row $i$ indented by $i-1$ units, weak decrease across rows, and strict decrease down columns, along with the condition that the number of parts in each row is strictly less than the largest part in that row and is greater than or equal to the largest part in the next row.

See Figures 1.3 and 1.4 for the seven TSSCPPs inside a $6 \times 6 \times 6$ box and the seven DPPs with part size less than or equal to 3. In [1] George Andrews showed that

$$
\begin{array}{cccccc}
6&6&6&3&3&3\\
6&6&6&3&3&3\\
6&6&6&3&3&3\\
3&3&3&&&\\
3&3&3&&&\\
3&3&3&&&
\end{array}
\qquad
\begin{array}{cccccc}
6&6&6&4&3&3\\
6&6&5&3&3&2\\
6&6&6&3&3&3\\
4&3&3&1&&\\
3&3&3&&&\\
3&3&2&&&
\end{array}
\qquad
\begin{array}{cccccc}
6&6&6&4&3&3\\
6&6&5&4&3&3\\
6&6&4&3&2&2\\
4&4&3&2&&\\
3&3&2&&&\\
3&3&2&&&
\end{array}
\qquad
\begin{array}{cccccc}
6&6&6&5&5&3\\
6&5&5&4&3&1\\
6&5&4&3&2&1\\
5&4&3&2&1&\\
5&3&2&1&1&\\
3&1&1&&&
\end{array}
$$

$$
\begin{array}{cccccc}
6&6&6&5&5&3\\
6&5&5&3&3&1\\
6&5&5&3&3&1\\
5&3&3&1&1&\\
5&3&3&1&1&\\
3&1&1&&&
\end{array}
\qquad
\begin{array}{cccccc}
6&6&6&5&4&3\\
6&6&5&3&3&2\\
6&5&5&3&3&1\\
5&3&3&1&1&\\
4&3&3&1&&\\
3&2&1&&&
\end{array}
\qquad
\begin{array}{cccccc}
6&6&6&5&4&3\\
6&6&5&4&3&2\\
6&5&4&3&2&1\\
5&4&3&2&1&\\
4&3&2&1&&\\
3&2&1&&&
\end{array}
$$

Figure 1.3: TSSCPPs inside a $6 \times 6 \times 6$ box

$$
\emptyset
\qquad
2
\qquad
3
\qquad
\begin{array}{cc} 3&1 \end{array}
\qquad
\begin{array}{cc} 3&2 \end{array}
\qquad
\begin{array}{cc} 3&3 \end{array}
\qquad
\begin{array}{cc} 3&3\\ &2 \end{array}
$$

Figure 1.4: DPPs with largest part less than or equal to 3

TSSCPPs inside a $2n \times 2n \times 2n$ box are counted by (1.1.1). In [28] Mills, Robbins and Rumsey proved that the generating function for DPPs with largest part less than or equal to $n$ weighted by $q$ to the sum of the entries is equal to the $q$-ification of (1.1.1),

$$\prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}, \tag{1.1.2}$$

where $n!_q = (1+q)(1+q+q^2)\ldots(1+q+q^2+\ldots+q^{n-1})$. No weight has yet been found on ASMs or TSSCPPs that yields (1.1.2) as the generating function, though much of this work was inspired by the search for such a weight. Also, no one has found a bijection between any two of these three classes of objects even though they are each counted by (1.1.1). In Section 2.5 we will discuss the possibility of a bijection between ASMs and TSSCPPs from the poset perspective and in Section 2.4 we will discuss bijections between objects which contain ASMs and TSSCPPs as subsets.

# Chapter 2

# The poset perspective

In order to establish a larger context within which to examine alternating sign matrices and totally symmetric self-complementary plane partitions we construct a tetrahedral poset containing subposets corresponding to each of these objects. We begin with a brief review of poset terminology following Chapter 3 of [36]. We also make a note whenever our notation differs from the notation of [36].

## 2.1 Poset terminology

Poset is a shortened form of the phrase *partially ordered set*. A partially ordered set is an extension of the familiar idea of a totally ordered set such as the integers. In the integers any two elements are comparable; it is easy to tell which of two integers is the largest. This is not always the case in a poset since elements of a poset are allowed to be incomparable.

**Definition 2.1.1.** A poset is a set of elements $P$ along with an ordering relation "$\leq$" satisfying the following axioms:

- For all $x \in P$, $x \leq x$. (reflexivity)
- If $x \leq y$ and $y \leq x$ then $y = x$. (antisymmetry)

- If $x \leq y$ and $y \leq z$ then $x \leq z$. (transitivity)

We will use the symbol "$<$" to denote strict inequality, that is, $x < y$ if and only if $x \leq y$ and $x \neq y$. Given a poset $P$, an element $y \in P$ is said to *cover* $x \in P$ if $x < y$ and there exists no $z \in P$ such that $x < z < y$. A poset can be represented by its *Hasse diagram*. To draw the Hasse diagram of a poset we represent poset elements as vertices and then draw directed edges between these vertices such that there is a directed edge from $x$ to $y$ if and only if $y$ covers $x$. In this chapter we will find it necessary to extend this definition slightly at times by drawing edges in the Hasse diagram from certain $x \in P$ to $y \in P$ when $x < y$ but $y$ does not cover $x$.

A *chain* in a poset is a set of elements in which all the pairs of elements are comparable. All the posets in this chapter will be *ranked* which means we can define a function $f$ on the elements of the poset such that $f(x) = 0$ if $x$ is a minimal poset element, and $f(y) = f(x) + 1$ if $y$ covers $x$. The *rank* $n$ of $P$ is defined as the rank of the largest maximal element of $P$. The *rank generating function* of a ranked poset is $F(P, q) = \sum_{i=0}^{n} a_i q^i$ where $a_i$ equals the number of poset elements of rank $i$. Thus $F(P, 1)$ equals the cardinality of $P$.

We say $Q$ is a *subposet* of $P$ if the elements of $Q$ are a subset of the elements of $P$ and the partial ordering on $Q$ is such that if $x \leq y$ in $Q$ then $x \leq y$ in $P$. This corresponds to the notion of *weak subposet* in [36]. To obtain the *dual* of a poset $P$, denoted $P^*$, simply define a new ordering relation $\leq^*$ which is the reverse of the old ordering relation $\leq$. That is, the dual $P^*$ of a poset $P$ consists of all the poset elements of $P$ and an ordering relation $\leq^*$ such that if $x \leq y$ in $P$ then $y \leq^* x$ in $P^*$.

Given any poset $P$ we can find the lattice of order ideals $J(P)$ of $P$. An *order ideal* of a poset is a subset $S$ of the poset elements such that if $y \in S$ and $x \leq y$ then $x \in S$. The order ideals of a poset can be made into a poset themselves by taking set containment as the partial order. Denote the poset of order ideals of $P$ as $J(P)$. $J(P)$ is a *lattice* which means that every pair of elements of $J(P)$ has a least upper bound and a greatest lower bound. Finally, $J(P)$ is also *distributive* because the least upper bound and greatest lower bound as operators satisfy the distributive laws. So we call $J(P)$ a *distributive lattice*. Every finite distributive lattice can be constructed in the above manner. The fundamental theorem of finite distributive lattices says that given a finite distributive lattice $L$ there exists a unique finite poset $P$ for which $L = J(P)$ [36].

## 2.2 Introduction to the tetrahedral poset

In this section we define posets $P_n$ and $T_n$ using certain unit vectors in $\mathbb{R}^3$. We also state the theorems which will be proved and explained throughout the rest of this chapter concerning the order ideals of subposets of $P_n$ and $T_n$ and their connections to well–known combinatorial objects such as ASMs, TSSCPPs, Catalan objects, and tournaments. We also discuss bijections between the order ideals of subposets of $T_n$ and certain arrays of integers with inequality conditions on the entries. Finally, we state and prove isomorphisms between the order ideals of certain subposets of $T_n$.

We begin by constructing the pyramidal poset $P_n$ by use of certain unit vectors. Define the vectors $\overrightarrow{r} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$, $\overrightarrow{g} = (0, 1, 0)$, and $\overrightarrow{b} = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$. We use these vectors to define $P_n$ by drawing its Hasse diagram. First let the elements of $P_n$ be defined as the coordinates of all the points reached by linear combinations of $\overrightarrow{r}$ and $\overrightarrow{g}$. That is, as a set $P_n = \{c_1 \overrightarrow{r} + c_2 \overrightarrow{g},\ c_1, c_2 \in \mathbb{Z}_{\geq 0},\ c_1 + c_2 \leq n - 2\}$. To obtain the partial order on $P_n$ let all the vectors $\overrightarrow{r}$ and $\overrightarrow{g}$ used to define the elements of $P_n$ be directed edges in the Hasse diagram, and additionally draw into the Hasse diagram as directed edges the vectors $\overrightarrow{b}$ between poset elements wherever possible. See Figure 2.1 for an example of this construction. Thus it is easy to check that the Hasse diagram of $P_n$ has $\binom{n}{2}$ vertices and $3\binom{n-1}{2}$ edges.

It will be useful for what follows to count the number of order ideals of $P_n$.

**Lemma 2.2.1.** *The number of order ideals of $P_n$ is $2^{n-1}$. The rank generating function of $J(P_n)$ is $\displaystyle\prod_{j=1}^{n-1}(1 + q^j)$.*

*Proof.* The proof is by induction on $n$. Suppose $P_{n-1}$ has $2^{n-2}$ order ideals and rank generating function $\prod_{j=1}^{n-2}(1+q^j)$. Let $a$ be the element of $P_n$ with the largest $x$ coordinate. Thus $a = (n-2)\overrightarrow{r}$. Let $I$ be an order ideal of $P_n$. If $a \in I$ then the $n-2$ poset elements $c\overrightarrow{r}$ with $c < n-2$ are also in $I$, leaving a copy of $P_{n-1}$ from which to choose more elements to be in $I$ (and yielding a weight of $q^{n-1}$ for the principle order ideal of $a$). So there are $2^{n-1}$ order ideals of $P_n$ which include $a$. Now suppose $a \notin I$. Then any element larger than $a$ in the partial order is also not in $I$ (that is any element of the form $c_1\overrightarrow{r} + c_2\overrightarrow{g}$ with $c_1 + c_2 = n-2$), so $I$ must be an order ideal of the subposet consisting of all elements of $P_n$ not greater than or equal to $a$ (that is, elements of the form $c_1\overrightarrow{r} + c_2\overrightarrow{g}$ with $c_1 + c_2 < n-2$). This subposet is $P_{n-1}$. So there are $2^{n-2}$ order ideals of $P_n$ not containing $a$. Thus there are $2 \cdot 2^{n-2} = 2^{n-1}$ order ideals of $P_n$ and the rank generating function is

$$F(P_n, q) = (1+q^{n-1})\prod_{j=1}^{n-2}(1+q^j) = \prod_{j=1}^{n-1}(1+q^j).$$

□

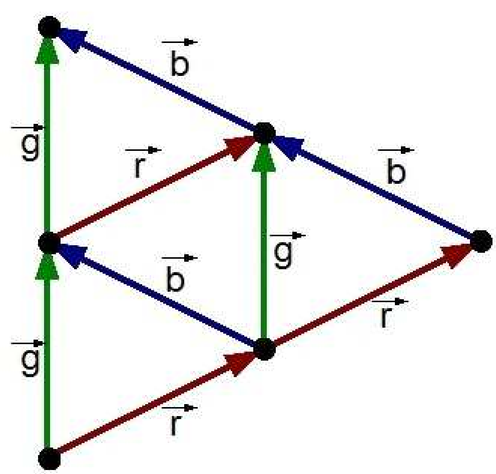



Figure 2.1: The Hasse diagram of $P_4$ with each vector labeled

Let $P_n(S)$ where $S \subseteq \{r, b, g\}$ denote the poset $P_n$ with only the edges whose colors are in $S$. Note that $P_n(\{r, b\})$ has the same number of order ideals as $P_n$ since whenever two vertices are connected by a green edge there is also a path from one to the other consisting of a red edge and a blue edge. From our construction we know that the other two posets $P_n(S)$ with $|S| = 2$, $P_n(\{b, g\})$ and $P_n(\{r, g\})$, are dual posets (see Figure 2.1). We now count the order ideals of the isomorphic posets $P_n(\{b, g\})$ and $P_n^*(\{r, g\})$, obtaining a famous number as our answer.

**Lemma 2.2.2.** *$|J(P_n(\{b, g\}))| = |J(P_n(\{r, g\}))| = C_n = \frac{1}{n+1}\binom{2n}{n}$. The rank generating function of $J(P_n(\{b, g\})) = J(P_n^*(\{r, g\}))$ equals $C_n(q)$ where $C_n(q)$ is the Carlitz–Riordan $q$–Catalan number defined by the recurrence*

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q) \tag{2.2.1}$$

*with initial conditions $C_0(q) = C_1(q) = 1$.*

*Proof.* We prove this theorem by constructing a bijection between order ideals of $P_n(\{b, g\})$ and Dyck paths of $2n$ steps. A Dyck path is a lattice path in the plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ which never goes below the $x$–axis. Dyck paths from $(0, 0)$ to $(2n, 0)$ are counted by the Catalan number $C_n$ [36]. To construct our bijection we will rotate the axes of our Dyck path slightly as in Figure 2.2. The Carlitz–Riordan $q$–Catalan numbers $C_n(q)$ weight Dyck paths by $q$ to the power of the number of complete unit squares under the path (or rhombi in our rotated picture) [12]. Our claim is that Dyck paths with this weight are in bijection with order ideals of the posets $P_n(\{b, g\})$ weighted by number of elements in the order ideal. The bijection proceeds by overlaying the Dyck path on $P_n(\{b, g\})$ as in Figure 2.2 and circling every poset element which is strictly below the path. Each circled element is the southeast corner of a unit rhombus under the Dyck path. Thus the order ideal consisting of all the circled elements corresponds to the Dyck path and the weight is preserved. Therefore $|J(P_n(\{b, g\}))| = C_n$ and the rank generating function $F(J(P_n(\{b, g\})), q)$ equals $C_n(q)$. Then since $P_n(\{b, g\}) \simeq P_n^*(\{r, g\})$ the result follows by poset isomorphism. ☐

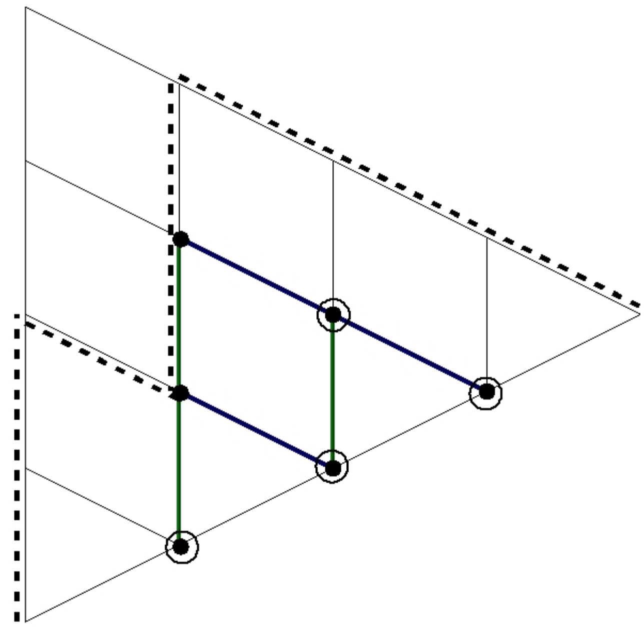


Figure 2.2: A Dyck path (dashed line) overlaid on $P_4(\{b, g\})$. The order ideal corresponding to this path consists of all the circled poset elements.

We now construct the tetrahedral poset $T_n$ as a three–dimensional analogue of the poset $P_n$. Define the unit vectors $\overrightarrow{y} = (\frac{\sqrt{3}}{6}, \frac{1}{2}, \frac{\sqrt{6}}{3})$, $\overrightarrow{o} = (\frac{-\sqrt{3}}{3}, 0, \frac{\sqrt{6}}{3})$, $\overrightarrow{s} = (\frac{-\sqrt{3}}{6}, \frac{1}{2}, -\frac{\sqrt{6}}{3})$. We use these vectors along with the previously defined vectors $\overrightarrow{r}$, $\overrightarrow{g}$, and $\overrightarrow{b}$ to define $T_n$ by drawing its Hasse diagram. First let the elements of $T_n$ be defined as the coordinates of all the points reached by linear combinations of $\overrightarrow{r}$, $\overrightarrow{g}$, and $\overrightarrow{y}$. Thus as a set $T_n = \{c_1\overrightarrow{r} + c_2\overrightarrow{g} + c_3\overrightarrow{y},\ c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0},\ c_1 + c_2 + c_3 \leq n - 2\}$. To obtain the partial order on $T_n$ let all the vectors $\overrightarrow{r}$, $\overrightarrow{g}$, and $\overrightarrow{y}$ used to define the elements of $T_n$ be directed edges in the Hasse diagram, and additionally draw into the Hasse diagram as directed edges the vectors $\overrightarrow{b}$, $\overrightarrow{o}$, and $\overrightarrow{s}$ between poset elements wherever possible. See Figure 2.3 for an example of this construction.

This construction yields a poset whose Hasse diagram is in the shape of a tetrahedron. We now think of each edge in the Hasse diagram as having a color corresponding to the first letter of the unit vector as follows: $\overrightarrow{r}$ equals red, $\overrightarrow{o}$ equals orange, $\overrightarrow{y}$

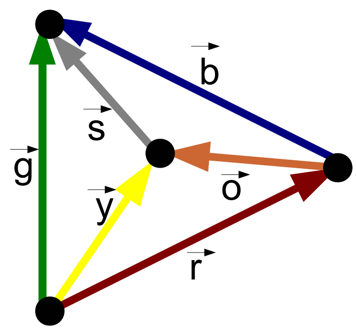



Figure 2.3: The Hasse diagram of $T_3$ with each vector labeled

equals yellow, $\overrightarrow{g}$ equals green, $\overrightarrow{b}$ equals blue, and $\overrightarrow{s}$ equals silver (see Figure 2.4). The partial order of $T_n$ is defined so that the corner vertex with edges colored red, green, and yellow is the smallest element, the corner vertex with edges colored silver, green, and blue is the largest element, and the other two corner vertices are ordered such that the one with silver, yellow, and orange edges is above the one with orange, red, and blue edges. $T_n$ restricted to only the red, green, and blue edges is isomorphic to the disjoint sum of $P_j$ for $2 \leq j \leq n$. Thus $T_n$ can be thought of as the poset which results from beginning with the poset $P_n$, overlaying the posets $P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ successively, and connecting each $P_i$ to $P_{i-1}$ in a certain way by the orange, yellow, and silver edges. The order ideals of $T_n$ are in bijection with totally symmetric plane partitions inside an $(n-1) \times (n-1) \times (n-1)$ box (see Section 2.8).

We now examine the number of order ideals of the elements of $T_n$ when we drop the ordering relations corresponding to edges of certain colors in the Hasse diagram. There are $2^6 = 64$ possible combinations of the six colors, but some of the colors are induced by the combination of others, so the number of distinct possibilities to consider is reduced. We wish to consider only subsets of the colors which include all the induced edges. Thus whenever we include red and blue we must also include green. Similarly, orange and silver induce blue, silver and yellow induce green, and red and orange induce yellow. We summarize this in the following definition.

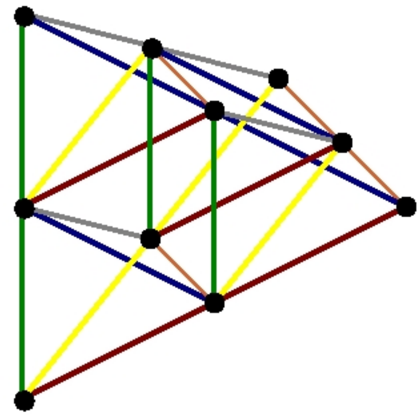


Figure 2.4: The Hasse diagram of $T_4$

**Definition 2.2.3.** Let a subset $S$ of the six colors {red, blue, green, orange, yellow, silver} (abbreviated $\{r, b, g, o, y, s\}$) be called *admissible* if all of the following are true:

- If $\{r, b\} \subseteq S$ then $g \in S$
- If $\{o, s\} \subseteq S$ then $b \in S$
- If $\{s, y\} \subseteq S$ then $g \in S$
- If $\{r, o\} \subseteq S$ then $y \in S$

This reduces the 64 possibilities to 40 admissible sets of colors to investigate. Surprisingly, for almost all of these admissible sets of colored edges, there exists a nice product formula for the number of order ideals of $T_n$ restricted to these edges and a bijection between these order ideals and an interesting set of combinatorial objects.

Our notational convention will be such that given an admissible subset $S$ of the colors $\{r, b, g, o, y, s\}$, $T_n(S)$ denotes the poset formed by the vertices of $T_n$ together with all the edges whose colors are in $S$. The induced colors will be in parentheses. We summarize below (with more explanation to come in this and the following sections) the posets $T_n(S)$ and lattices of order ideals $J(T_n(S))$ associated to each of the 40 sets of colors $S$, grouping them according to the number of colors in $S$. See Figure 2.5 for a graphical representation of the following list.

- Zero colors: Boolean algebra (one (empty) set of colors)
- One color: Disjoint sum of chains - number of order ideals given by products of factorials (six singleton sets of colors)
- Two colors: Two non-isomorphic classes of posets with different numbers of order ideals
  - A disjoint sum of posets whose order ideals are counted by binomial coefficients (three sets of colors)
  - A disjoint sum of posets whose order ideals are counted by Catalan numbers (eight sets of colors)
- Three colors:
  - Two non-isomorphic classes of posets both with $2^{\binom{n}{2}}$ order ideals
    * A connected poset whose order ideals are in bijection with semistandard Young tableaux of staircase shape $\delta_n = (n-1)\ (n-2)\ \ldots\ 3\ 2\ 1$ (five sets of colors)
    * A disjoint sum of the $P_j$ for $2 \leq j \leq n$ whose order ideals are in bijection with tournaments on $n$ vertices (four sets of colors)
  - One poset class with no product formula known for the number of order ideals (two sets of colors)
- Four colors: Two non-isomorphic classes of posets with the same number of order ideals but no known bijection
  - ASM (one set of colors)
  - TSSCPP (six sets of colors)

- Five colors: Two non-isomorphic posets with different numbers of order ideals and no nice product formula known
  - ASM $\cap$ TSSCPP (two sets of colors)
  - The intersection of two different realizations of TSSCPPs (one set of colors)
- All six colors: Totally symmetric plane partitions (one set of colors)

We now state the product formulas for the number of order ideals of $T_n(S)$ for $S$ an admissible set of colors, along with the rank generating functions wherever we have them. (Recall that $F(P, q)$ denotes the rank generating function for the poset $P$.) For the sake of comparison we have also written each formula as a product over the same indices $1 \leq i \leq j \leq k \leq n-1$ in a way which is reminiscent of the MacMahon box formula. See Figures 2.6 through 2.11 for the Hasse diagram of a poset from each class for $n = 4$. For the proofs and discussion related to the more difficult theorems see the remaining sections in this chapter.

**Theorem 2.2.4.**

$$F(J(T_n(\emptyset)), q) = (1+q)^{\binom{n+1}{3}} = \prod_{1 \leq i \leq j \leq k \leq n-1} \frac{[2]_q}{[1]_q}.$$

*Proof.* $J(T_n(\emptyset))$ is simply the Boolean algebra of rank $\binom{n+1}{3}$ whose generating function is as above. □

**Theorem 2.2.5.** *For any color $x \in \{r, b, y, g, o, s\}$*

$$F(J(T_n(\{x\})), q) = \prod_{j=1}^{n} j!_q = \prod_{1 \leq i \leq j \leq k \leq n-1} \frac{[i+1]_q}{[i]_q}.$$

*Proof.* $T_n(\{x\})$ is the disjoint sum of $n-j$ chains of length $j-1$ as $j$ goes from 1 to $n-1$. So the number of order ideals is the product of the number of order ideals of each chain, which can be expressed using factorials. The $q$–case is also clear. □

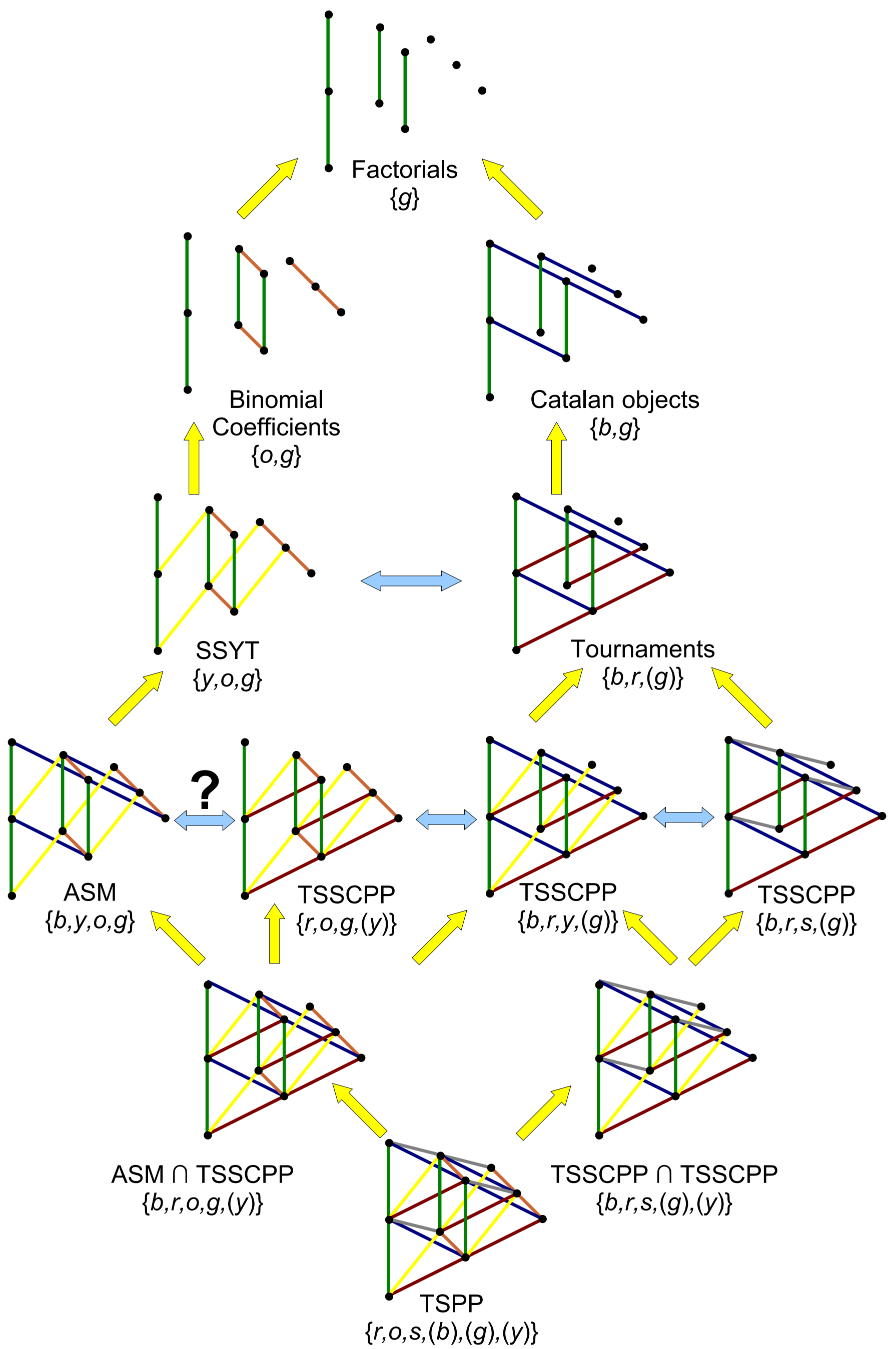



Figure 2.5: The big picture of inclusions and bijections between order ideals $J(T_n(S))$. The bijection between three color posets is in Section 2.4 and the bijections between TSSCPP posets is by poset isomorphism (which will be discussed later in this section). The only missing bijection between sets of the same size is between ASM and TSSCPP.

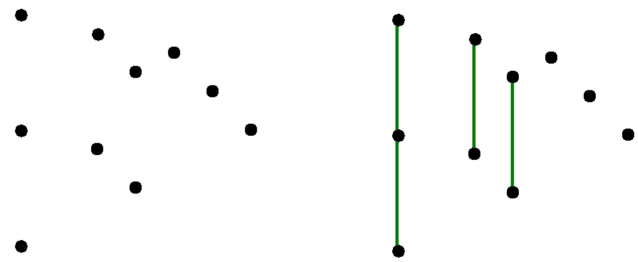


Figure 2.6: Left: $T_4(\emptyset)$ Right: $T_4(\{g\})$

The proof of the following two theorems is in Section 2.3.

**Theorem 2.2.6.** *If $S \in \{\{g,o\},\{r,s\},\{b,y\}\}$ then*

$$F(J(T_n(S)),q) = \prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q = \prod_{1 \le i \le j \le k \le n-1} \frac{[j+1]_q}{[j]_q}.$$

**Theorem 2.2.7.** *If $S_1 \in \{\{b,g\},\{b,s\},\{y,o\},\{g,s\}\}$ and $S_2 \in \{\{r,y\},\{r,g\},\{y,g\},$ $\{b,o\}\}$ then*

$$|J(T_n(S_1))| = |J(T_n(S_2)| = \prod_{j=1}^{n} C_j = \prod_{j=1}^{n} \frac{1}{j+1}\binom{2j}{j} = \prod_{1 \le i \le j \le k \le n-1} \frac{i+j+2}{i+j},$$

*where $C_j$ is the $j$th Catalan number.*

$$F(J(T_n(S_1)),q) = F(J(T_n^*(S_2)),q) = \prod_{j=1}^{n} C_j(q),$$

*where the $C_j(q)$ are the Carlitz–Riordan $q$–Catalan numbers.*

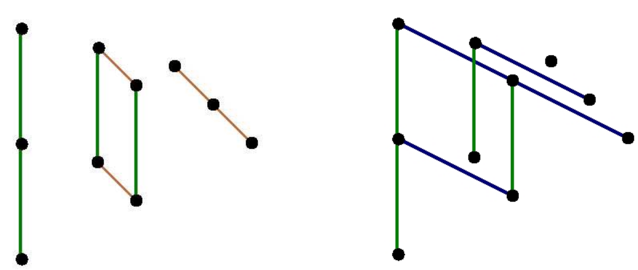


Figure 2.7: Left: Binomial poset $T_4(\{o,g\})$ Right: Catalan poset $T_4(\{b,g\})$

The proof of the next theorem is one of the main subjects of Section 2.4.

**Theorem 2.2.8.** *If $S$ is an admissible subset of $\{r, b, g, o, y, s\}$, $|S| = 3$, and $S \notin \{\{r, g, y\}, \{s, b, r\}\}$ then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n-1}(1 + q^j)^{n-j} = \prod_{1 \le i \le j \le k \le n-1} \frac{[i + j]_q}{[i + j - 1]_q}.$$

*Thus if we set $q = 1$ we have $|J(T_n(S))| = 2^{\binom{n}{2}}$.*

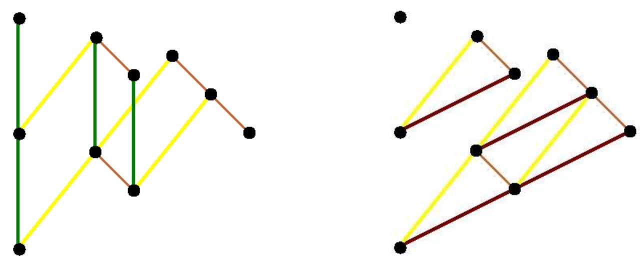


Figure 2.8: Left: $T_4(\{o, g, y\})$  Right: $T_4(\{r, (y), o\})$

There seems to be no nice product formula for the number of order ideals of the dual posets $T_n(\{r, g, y\})$ and $T_n(\{s, b, r\})$ which are the only two ways to pick three adjacent colors while not inducing any other colors. We have calculated the number of order ideals for $n = 1$ to 6 to be: 1, 2, 9, 96, 2498, 161422.

The proof of the following theorem and the relationship between these order ideals and ASMs and TSSCPPs is the main subject of Section 2.5.

**Theorem 2.2.9.** *If $S$ is an admissible subset of $\{r, b, g, o, y, s\}$ and $|S| = 4$ then*

$$|J(T_n(S))| = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \prod_{1 \le i \le j \le k \le n-1} \frac{i + j + k + 1}{i + j + k - 1}.$$

There are two different cases for five colors: one case consists of the dual posets $T_n(\{(g), (b), o, y, s\})$ and $T_n(\{r, b, (g), o, (y)\})$ and the other case is $T_n(\{r, b, s, (y), g\})$. A nice product formula has not yet been found for either case. See further discussion of these posets in Section 2.7.

Finally, the order ideals of the full tetrahedron $T_n$ are counted by the following theorem, which is proved in Section 2.8.

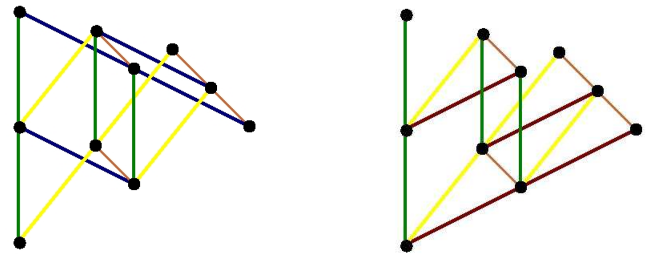


Figure 2.9: Left: ASM poset $T_4(\{o, g, y, b\})$ Right: TSSCPP poset $T_4(\{r, (y), o, g\})$

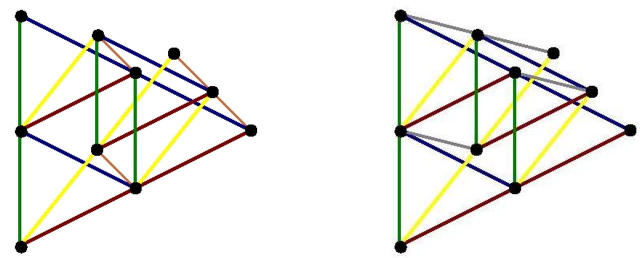


Figure 2.10: Left: $T_4(\{o, (g), (y), b, r\})$ Right: $T_4(\{g, (y), b, r, s\})$

**Theorem 2.2.10.**

$$|J(T_n)| = \prod_{1 \le i \le j \le n-1} \frac{i+j+n-2}{i+2j-2} = \prod_{1 \le i \le j \le k \le n-1} \frac{i+j+k-1}{i+j+k-2}.$$

Next we give two bijections between order ideals of $T_n(S)$, $S$ an admissible subset of $\{r, b, g, y, o, s\}$, and arrays of integers with certain inequality conditions. These bijections will be used frequently throughout the rest of the chapter to prove the theorems stated above and other facts about $T_n(S)$ and the related combinatorial objects.

**Definition 2.2.11.** Let $S$ be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Define $X_n(S)$ to be the set of all integer arrays $x$ of staircase shape $\delta_n = (n-1)\ (n-2) \ldots 3\ 2\ 1$ whose entries $x_{i,j}$ satisfy both $0 \le x_{i,j} \le j$ and the following inequality conditions corresponding to the additional colors in $S$:

- orange: $x_{i,j} \le x_{i+1,j}$
- red: $x_{i,j} \le x_{i-1,j+1}$

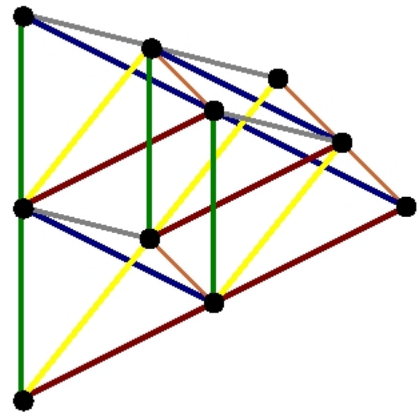


Figure 2.11: TSPP poset $T_4$

- yellow: $x_{i,j} \leq x_{i,j+1}$
- blue: $x_{i,j} \leq x_{i+1,j-1} + 1$
- silver: $x_{i,j} \leq x_{i,j-1} + 1$

**Proposition 2.2.12.** *If $S$ is an admissible subset of $\{r, b, g, y, o, s\}$ and $g \in S$ then $X_n(S)$ is in weight–preserving bijection with the set of order ideals $J(T_n(S))$ where the weight of $x \in X_n(S)$ equals the sum of the entries of $x$ and the weight of $I \in J(T_n(S))$ equals $|I|$.*

*Proof.* Let $S$ be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Recall that $T_n$ is made up of the layers $P_k$ where $2 \leq k \leq n$ and that $P_k$ contains $k-1$ green–edged chains of length $k-1, \ldots, 2, 1$. For each $P_k \subseteq T_n$ let the $k-1$ green chains inside $P_k$ determine the entries $x_{i,j}$ of an integer array on the diagonal where $i+j=k$. In particular, given an order ideal $I$ of $T_n(S)$ form an array $x$ by setting $x_{i,j}$ equal to the number of elements in the induced order ideal of the green chain of length $j$ inside $P_{i+j}$. This defines $x$ as an integer array of staircase shape $\delta_n$ whose entries satisfy $0 \leq x_{i,j} \leq j$. Also since each entry $x_{i,j}$ is given by an induced order ideal and since each element of $T_n$ is in exactly one green chain we know that $|I| = \sum_{i,j} x_{i,j}$. Thus the weight is preserved.

Now it is left to determine what the other colors mean in terms of the array entries. Since the colors red and blue connect green chains from the same $P_k$ we see that inequalities corresponding to red and blue should relate entries of $x$ on the same northeast to southwest diagonal of $x$. So if $r \in S$ then $x_{i,j} \leq x_{i-1,j+1}$ and if $b \in S$ then $x_{i,j} \leq x_{i+1,j-1} + 1$. The colors yellow, orange, and silver connect $P_k$ to $P_{k+1}$ for $2 \leq k \leq n-1$. So from our construction we see that if $o \in S$ then $x_{i,j} \leq x_{i+1,j}$, if $y \in S$ then $x_{i,j} \leq x_{i,j+1}$, and if $s \in S$ then $x_{i,j} \leq x_{i,j-1} + 1$. □

A useful transformation of the elements in $X_n(S)$ is given in the following definition.

**Definition 2.2.13.** Let $S$ be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Define $Y_n(S)$ to be the set of all integer arrays $y$ of staircase shape $\delta_n = (n-1)\ (n-2) \ldots 3\ 2\ 1$ whose entries $y_{i,j}$ satisfy both $i \leq y_{i,j} \leq j+i$ and the following inequality conditions corresponding to the additional colors in $S$:

- orange: $y_{i,j} < y_{i+1,j}$
- red: $y_{i,j} \leq y_{i-1,j+1} + 1$
- yellow: $y_{i,j} \leq y_{i,j+1}$
- blue: $y_{i,j} \leq y_{i+1,j-1}$
- silver: $y_{i,j} \leq y_{i,j-1} + 1$

**Proposition 2.2.14.** *If $S$ is an admissible subset of $\{r, b, g, y, o, s\}$ and $g \in S$ then $Y_n(S)$ is in weight–preserving bijection with $J(T_n(S))$ where the weight of $y \in Y_n(S)$ is given by $\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} (y_{i,j} - i)$ and the weight of $I \in J(T_n(S))$ equals $|I|$.*

*Proof.* Let $S$ be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. For $x \in X_n(S)$ consider the array $y$ with entries $y_{i,j} = x_{i,j} + i$. The transformation $y_{i,j} = x_{i,j} + i$ affects the meaning of the colors in terms of inequalities. Any inequality which compares elements in two different rows is altered. Thus orange now corresponds to

$y_{i,j} < y_{i+1,j}$, red now corresponds to $y_{i,j} \leq y_{i-1,j+1} + 1$, blue now corresponds to $y_{i,j} \leq y_{i+1,j-1}$, and the other inequalities are unaffected. Also since $y_{i,j} = x_{i,j} + i$ it follows that the weight of $y$ is given by $\sum_{i,j}(y_{i,j} - i)$. Thus $Y_n(S)$ is in weight–preserving bijection with $X_n(S)$ so that $Y_n(S)$ is in weight–preserving bijection with $J(T_n(S))$. $\square$

We make one final transformation of these arrays which will be needed later. It will sometimes be helpful to think of the arrays $Y_n(S)$ as having a 0th column with entries $1\ 2\ 3 \dots n$.

**Definition 2.2.15.** Let $S$ be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Define $Y_n^+(S)$ to be the set of all arrays $y \in Y_n(S)$ with an additional column (call it column 0) with entries $y_{i,0} = i$ for $1 \leq i \leq n$.

Thus $y \in Y_n^+(S)$ is an array of shape $n\ (n-1) \dots 3\ 2\ 1$ with fixed 0th column $1\ 2\ 3 \dots n$.

**Proposition 2.2.16.** *If $S$ is an admissible subset of $\{r, b, g, y, o, s\}$ and $g \in S$ then $Y_n^+(S)$ is in weight–preserving bijection with $J(T_n(S))$ where the weight of $y \in Y_n^+(S)$ is given by* $\displaystyle\sum_{i=1}^{n-1}\sum_{j=0}^{n-i}(y_{i,j} - i)$ *and the weight of $I \in J(T_n(S))$ equals $|I|$.*

*Proof.* $Y_n^+(S)$ is trivially in bijection with $Y_n(S)$ by the removal of the 0th column. The weight is preserved since $y_{i,0} = i$ so that $\sum_{i=1}^{n-1}(y_{i,0} - i) = 0$. Therefore the added column entries contribute nothing to the weight. $\square$

Finally we should discuss the isomorphisms between the posets $T_n(S)$ for admissible $S$, since these isomorphisms will be used frequently throughout the rest of this chapter to prove the theorems listed in this section.

**Proposition 2.2.17.** *If $|S_1| = |S_2| = 1$ then $T_n(S_1) \simeq T_n(S_2)$.*

*Proof.* Each poset $T_n(S)$ with $|S| = 1$ is isomorphic to the same disjoint union of chains. $\square$

**Proposition 2.2.18.** $T_n(\{r, s\}) \simeq T_n(\{b, y\}) \simeq T_n(\{g, o\})$.

*Proof.* The following vector relation holds: $\overrightarrow{r} \cdot \overrightarrow{s} = \overrightarrow{b} \cdot \overrightarrow{y} = \overrightarrow{g} \cdot \overrightarrow{o} = 0$. Thus each pair of vectors is perpendicular, thus by the construction of $T_n$ these posets are isomorphic. □

**Proposition 2.2.19.** $T_n(\{b, g\}) \simeq T_n(\{s, b\}) \simeq T_n(\{y, o\}) \simeq T_n(\{g, s\})$.

*Proof.* In $T_n$ each $\overrightarrow{b}$ vector shares its terminal point with a $\overrightarrow{g}$ vector. This is also true of $\overrightarrow{s}$ and $\overrightarrow{b}$, $\overrightarrow{y}$ and $\overrightarrow{o}$, and $\overrightarrow{g}$ and $\overrightarrow{o}$. Thus the posets are isomorphic. □

**Proposition 2.2.20.** $T_n(\{g, y\}) \simeq T_n(\{y, r\}) \simeq T_n(\{r, g\}) \simeq T_n(\{b, o\})$.

*Proof.* In $T_n$ each $\overrightarrow{g}$ vector shares its initial point with a $\overrightarrow{y}$ vector. This is also true of $\overrightarrow{r}$ and $\overrightarrow{y}$, $\overrightarrow{r}$ and $\overrightarrow{g}$, and $\overrightarrow{b}$ and $\overrightarrow{o}$. Therefore the posets are isomorphic. □

**Proposition 2.2.21.** $T_n(\{b, g\}) \simeq T_n^*(\{g, y\})$.

*Proof.* Each $\overrightarrow{b}$ vector shares its terminal point with a $\overrightarrow{g}$ vector and each $\overrightarrow{g}$ vector shares its initial point with a $\overrightarrow{y}$ vector. Thus $T_n(\{b, g\}) \simeq T_n^*(\{g, y\})$ by the definition of a dual poset. □

**Proposition 2.2.22.**

$$T_n(\{b, r, (g)\}) \simeq T_n(\{o, s, (b)\}) \simeq T_n(\{r, o, (y)\}) \simeq T_n(\{y, s, (g)\}).$$

*Proof.* The following vector relation holds: $\overrightarrow{r} + \overrightarrow{b} = \overrightarrow{g}$. Similarly, $\overrightarrow{o} + \overrightarrow{s} = \overrightarrow{b}$, $\overrightarrow{r} + \overrightarrow{o} = \overrightarrow{y}$, and $\overrightarrow{y} + \overrightarrow{s} = \overrightarrow{g}$. Thus the posets are isomorphic. □

**Proposition 2.2.23.**

$$T_n(\{g, y, o\}) \simeq T_n(\{r, g, s\}) \simeq T_n(\{o, b, y\}) \simeq T_n(\{b, g, o\}) \simeq T_n(\{b, g, y\}).$$

*Proof.* In $T_n$ each $\overrightarrow{y}$ vector shares its initial point with a $\overrightarrow{g}$ vector and its terminal point with an $\overrightarrow{o}$ vector. Also, each $\overrightarrow{g}$ vector shares its initial point with an $\overrightarrow{r}$

vector and its terminal point with an $\overrightarrow{s}$ vector, each $\overrightarrow{o}$ vector shares its initial point with a $\overrightarrow{b}$ vector and its terminal point with a $\overrightarrow{y}$ vector, each $\overrightarrow{b}$ vector shares its initial point with an $\overrightarrow{o}$ vector and its terminal point with a $\overrightarrow{g}$ vector, and each $\overrightarrow{g}$ vector shares its initial point with a $\overrightarrow{y}$ vector and its terminal point with a $\overrightarrow{b}$ vector. Therefore the posets are isomorphic. □

**Proposition 2.2.24.** $T_n(\{b, r, y, (g)\}) \simeq T_n(\{r, o, g, (y)\}) \simeq T_n(\{r, s, y, (g)\})$.

*Proof.* $T_n(\{b, r, (g)\})$, $T_n(\{r, o, (y)\})$, and $T_n(\{s, y, (g)\})$ are all isomorphic. We get from $T_n(\{b, r, (g)\})$ to $T_n(\{b, r, y, (g)\})$, $T_n(\{r, o, (y)\})$ to $T_n(\{r, o, g, (y)\})$, and $T_n(\{s, y, (g)\})$ to $T_n(\{r, s, y, (g)\})$ by adding the additional color of unit vector whose initial vertex is the same as two out of three of the old colors. Thus the posets are isomorphic. □

**Proposition 2.2.25.** $T_n(\{b, r, s, (g)\}) \simeq T_n(\{o, s, g, (b)\}) \simeq T_n(\{b, s, y, (g)\})$.

*Proof.* $T_n(\{b, r, (g)\})$, $T_n(\{o, s, (b)\})$, and $T_n(\{s, y, (g)\})$ are all isomorphic. We get from $T_n(\{b, r, (g)\})$ to $T_n(\{b, r, s, (g)\})$, $T_n(\{o, s, (b)\})$ to $T_n(\{o, s, g, (b)\})$, and $T_n(\{s, y, (g)\})$ to $T_n(\{b, s, y, (g)\})$ by adding the additional color of unit vector whose terminal vertex is the same as two out of three of the old colors. Thus the posets are isomorphic. □

**Proposition 2.2.26.** $T_n(\{b, r, y, (g)\}) \simeq T_n^*(\{b, r, s, (g)\})$.

*Proof.* $T_n(\{b, r, y, (g)\})$ and $T_n(\{b, r, s, (g)\})$ are each equal to $T_n(\{b, r, (g)\})$ with an additional color of unit vectors added. In the first poset, each $\overrightarrow{y}$ vector shares its initial point with a $\overrightarrow{g}$ vector and an $\overrightarrow{r}$ vector. In the second poset, each $\overrightarrow{s}$ vector shares its terminal point with a $\overrightarrow{g}$ vector and an $\overrightarrow{r}$ vector. Thus $T_n(\{b, r, y, (g)\}) \simeq T_n^*(\{b, r, s, (g)\})$ by the definition of a dual poset. □

**Proposition 2.2.27.** $T_n(\{(g), (b), o, y, s\}) \simeq T_n^*(\{r, b, (g), o, (y)\})$.

*Proof.* The only difference between $T_n(\{(g), (b), o, y, s\})$ and $T_n(\{r, b, (g), o, (y)\})$ is that the first lacks $\overrightarrow{r}$ vectors and the second lacks $\overrightarrow{s}$ vectors. But $\overrightarrow{s}$ and $\overrightarrow{r}$ are perpendicular, so $T_n(\{(g), (b), o, y, s\})$ and $T_n(\{r, b, (g), o, (y)\})$ are dual. □

## 2.3 The two color posets—Catalan numbers and binomial coefficients

In Section 2.2 we stated Theorems 2.2.6 and 2.2.7 about the weighted enumeration of $J(T_n(S))$ when $|S| = 2$. One count was a product of Catalan numbers and the other of binomial coefficients. In this section we restate, discuss, and prove these theorems.

We begin by proving the enumeration of the two–color posets $J(T_n(S))$ when $S \in \{\{g, o\}, \{r, s\}, \{b, y\}\}$.

**Theorem 2.2.6.** *For* $S \in \{\{g, o\}, \{r, s\}, \{b, y\}\}$

$$F(J(T_n(S)), q) = \prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q = \prod_{1 \le i \le j \le k \le n-1} \frac{[j+1]_q}{[j]_q}. \tag{2.3.1}$$

*Proof.* Consider the staircase shape arrays $Y_n(\{g, o\})$ which strictly decrease down columns and have no conditions on the rows. Thus in a column of length $j$ there must be $j$ distinct integers between 1 and $n$; this is counted by $\binom{n}{j}$. If we give a weight to each of these integers of $q$ to the power of that integer minus its row, we have a set $q$–enumerated by the $q$–binomial coefficient $\begin{bmatrix} n \\ j \end{bmatrix}_q$. Thus $\prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q$ is the generating function of the arrays $Y_n(\{g, o\})$. Recall from Proposition 2.2.14 that $Y_n(\{g, o\})$ is in weight–preserving bijection with $J(T_n(\{g, o\}))$ where the weight of $y \in Y_n(\{g, o\})$ is given by $\sum_{i,j}(y_{i,j} - i)$ and the weight of $I \in J(T_n(\{g, o\}))$ equals $|I|$. Thus the rank generating function of the order ideals $F(J(T_n(\{g, o\})), q)$ equals $\prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q$. By Proposition 2.2.18 the posets $T_n(\{r, s\})$ and $T_n(\{b, y\})$ are both isomorphic to $T_n(\{g, o\})$. Therefore the result follows by poset isomorphism. $\square$

Now suppose $S$ is admissible and $|S| = 2$ but $S \notin \{\{g, o\}, \{r, s\}, \{b, y\}\}$. Then Theorem 2.2.7 applies.

**Theorem 2.2.7.** *If* $S_1 \in \{\{b, g\},\ \{b, s\},\ \{y, o\},\ \{g, s\}\}$ *and* $S_2 \in \{\{r, y\},\ \{r, g\},\ \{y, g\},\ \{b, o\}\}$ *then*

$$|J(T_n(S_1))| = |J(T_n(S_2)| = \prod_{j=1}^{n} C_j = \prod_{j=1}^{n} \frac{1}{j+1}\binom{2j}{j} = \prod_{1 \le i \le j \le k \le n-1} \frac{i+j+2}{i+j}, \tag{2.3.2}$$

*where $C_j$ is the $j$th Catalan number.*

$$F(J(T_n(S_1)),q) = F(J(T_n^*(S_2)),q) = \prod_{j=1}^{n} C_j(q), \tag{2.3.3}$$

*where the $C_j(q)$ are the Carlitz–Riordan $q$–Catalan numbers.*

*Proof.* $T_n(\{b,g\})$ is isomorphic to the disjoint sum of $P_j(\{b,g\})$ for $2 \le j \le n$. Recall from Lemma 2.2.2 that $|J(P_j(\{b,g\}))| = C_j = \frac{1}{j+1}\binom{2j}{j}$ and the rank generating function $F(J(P_j(\{b,g\})))$ equals $C_j(q)$ where $C_j(q)$ is the Carlitz-Riordan $q$-Catalan number defined by the recurrence $C_j(q) = \sum_{k=1}^{j} q^{k-1}C_{k-1}(q)C_{j-k}(q)$ with initial conditions $C_0(q) = C_1(q) = 1$. Thus the number of order ideals $|J(T_n(\{b,g\}))|$ equals the product $\prod_{j=2}^{n} C_j$ and the rank generating function $F(J(T_n(\{b,g\})),q)$ equals the product $\prod_{j=2}^{n} C_j(q)$. Then since $C_1 = C_1(q) = 1$ we can write the product as $j$ goes from 1 to $n$. Finally, the posets $T_n(S_1)$ for any choice of $S_1 \in \{\{b,g\}, \{b,s\}, \{y,o\}, \{g,s\}\}$ and the posets $T_n^*(S_2)$ for any $S_2 \in \{\{r,y\}, \{r,g\}, \{y,g\}, \{b,o\}\}$ are all isomorphic (see Propositions 2.2.19, 2.2.20, and 2.2.21), thus the result follows by poset isomorphism. $\square$

Note that the rank generating function of this theorem is not the direct $q$–ification of the counting formula (2.3.2). This is because the rank generating function does not factor nicely so it cannot be written as a nice product formula, but instead as a recurrence. The direct $q$–ification of (2.3.2) is still the generating function of $J(T_n(\{b,g\}))$, but with a different weight. $\prod_{j=1}^{n} \frac{1}{[j+1]_q}\begin{bmatrix}2j\\j\end{bmatrix}_q$ is the product of the first $n$ MacMahon $q$–Catalan numbers. The MacMahon $q$–Catalan numbers $\frac{1}{[j+1]_q}\begin{bmatrix}2j\\j\end{bmatrix}_q$ count Dyck paths with $2j$ steps (or Catalan words of length $2j$) weighted by major index. The major index of a Dyck path is the sum of all $i$ such that step $i$ of the Dyck path is an increasing step and step $i+1$ is a decreasing step. Therefore $\prod_{j=1}^{n} \frac{1}{[j+1]_q}\begin{bmatrix}2j\\j\end{bmatrix}_q$ is the generating function for order ideals $I \in J(T_n(\{b,g\}))$ with weight not corresponding to $|I|$ but rather to the product of the major index of each Dyck path in bijection with the order ideal of $P_j$ induced by the elements of $I$ contained in $P_j$. See [17] for a nice exposition of both the MacMahon and Carlitz–Riordan $q$–Catalan numbers.

## 2.4 Three color posets—tournaments and SSYT

In this section we prove Theorem 2.2.8 which states that the order ideals of the three color posets $J(T_n(S))$ where $S$ is admissible, $|S| = 3$, and $S \notin \{\{r,g,y\},\{s,b,r\}\}$ are counted by the very simple expression $2^{\binom{n}{2}}$. This is also the number of graphs on $n$ labeled vertices and equivalently the number of tournaments on $n$ vertices. A tournament is a complete directed graph with labeled vertices. We will first prove the weighted counting formula of Theorem 2.2.8 and then discuss bijections between the order ideals of the three color posets and tournaments. In the next section we will discuss the possibility of using the ideas from these bijections to find a bijection between ASMs and TSSCPPs.

We will prove Theorem 2.2.8 using two lemmas since there are two nonisomorphic classes of posets $T_n(S)$ where $S$ is admissible, $|S| = 3$, and $S \notin \{\{r,g,y\},\{s,b,r\}\}$. The first lemma is the case where $T_n(S)$ is a disjoint sum of posets and the second lemma is the case where $T_n(S)$ is a connected poset.

**Lemma 2.4.1.** *Suppose* $S \in \{\{o,s,(b)\},\{s,y,(g)\},\{o,r,(y)\},\{b,r,(g)\}\}$. *Then*

$$F(J(T_n(S)),q) = \prod_{j=1}^{n-1}(1+q^j)^{n-j} = \prod_{1\le i\le j\le k\le n-1} \frac{[i+j]_q}{[i+j-1]_q}.$$

*Proof.* $T_n(\{b,r,(g)\})$ is a disjoint sum of the $P_j$ posets for $2 \le j \le n$ whose order ideals are counted by $2^{j-1}$. Furthermore, the rank generating function of $J(P_j)$ is given by $\prod_{i=1}^{j-1}(1+q^i)$ thus $F(T_n(\{b,r,(g)\}),q)$ is the product of $\prod_{i=1}^{j-1}(1+q^i)$ for $2 \le j \le n$. Rewriting the product we obtain $\prod_{j=2}^{n}\prod_{i=1}^{j-1}(1+q^i) = \prod_{j=1}^{n-1}(1+q^j)^{n-j}$. From Proposition 2.2.22 we know that the posets $T_n(S)$ where $S \in \{\{o,s,(b)\},\{s,y,(g)\},\{o,r,(y)\}\}$ are isomorphic to $T_n(\{b,r,(g)\})$ so the result follows by poset isomorphism. □

Next we state the second lemma which proves Theorem 2.2.8 for the second case in which $T_n(S)$ is a connected poset.

**Lemma 2.4.2.** *Suppose* $S \in \{\{r,g,s\},\{o,b,y\},\{y,g,o\},\{b,g,o\},\{y,g,b\}\}$*. Then*

$$F(J(T_n(S)),q) = \prod_{j=1}^{n-1}(1+q^j)^{n-j} = \prod_{1\le i\le j\le k\le n-1} \frac{[i+j]_q}{[i+j-1]_q}.$$

We give two proofs of this lemma. In the first proof we view each order ideal as a nest of non–intersecting lattice paths and express the total number of such paths as a determinant using the theorem of Lindstrom, Gessel, and Viennot [16]. We evaluate this determinant using a formula of Krattenthaler [22]. In the second proof we note that the order ideals in question are in bijection with semistandard Young tableaux (SSYT) of staircase shape whose generating function is the staircase shape Schur function. The principle specialization of this Schur function yields the desired rank generating function.

In the first proof of this lemma we will need the following theorem of Gessel, Viennot, and Lindstrom as stated by Stembridge in [37]. We will use the following notation of Stembridge. Let $D$ be an acyclic digraph. For any pair of vertices $u, v$ in $D$, let $\mathcal{P}(u,v)$ denote the set of (directed) $D$–paths from $u$ to $v$. Given any pair of $n$–tuples $\mathbf{u} = (u_1,\ldots,u_n)$ and $\mathbf{v} = (v_1,\ldots,v_n)$ of vertices, let $\mathcal{P}(\mathbf{u},\mathbf{v})$ denote the set of $n$–tuples of paths $(R_1,\ldots,R_n)$ with $R_i \in \mathcal{P}(u_i,v_i)$ and let $\mathcal{P}_0(\mathbf{u},\mathbf{v})$ denote the subset of $\mathcal{P}(\mathbf{u},\mathbf{v})$ consisting of nonintersecting $n$–tuples of paths.

**Theorem 2.4.3** (Gessel–Viennot, Lindstrom)**.** *Let* $\mathbf{u} = (u_1,u_2,\ldots,u_n)$ *and* $\mathbf{v} = (v_1,v_2,\ldots,v_n)$ *be two $n$–tuples of vertices in an acyclic digraph $D$. If for all $i,j,k,\ell$ with $i<j, k<\ell$ every path in $\mathcal{P}(u_i,v_\ell)$ intersects every path in $\mathcal{P}(u_j,v_k)$, then*

$$GF\left[\mathcal{P}_0(\mathbf{u},\mathbf{v})\right] = \det_{1\le i,j\le n}\left(GF[\mathcal{P}(u_i,v_j)]\right)$$

*where $GF[\mathcal{P}]$, for $\mathcal{P}$ a collection of paths, is the sum over each path $R\in\mathcal{P}$ of the product of the weights associated to the edges of $R$.*

*First proof of Lemma 2.4.2.* The arrays $X_n(\{g,y,o\})$ are in weight–preserving bijection with nests of non–intersecting lattice paths $\mathcal{P}_0(\mathbf{u},\mathbf{v})$ where $\mathbf{u} = (u_1,\ldots,u_{n-1})$,

$\mathbf{v} = (v_1, \ldots, v_{n-1})$, $u_i = (-i, i)$, and $v_i = (n-2i, 2i)$. Column $i$ of $\alpha \in X_n(\{g, y, o\})$ determines the path $R_i$ by letting the height above the starting height $y = i$ of each successive horizontal step of $R_i$ be given by each successive entry of column $i$. Then the nonintersecting condition on the paths corresponds to the weak increase across rows of $\alpha$. Let the weight of a horizontal edge in $R_i$ be given by $q$ to the power of the height of that edge above the starting point of $R_i$. See Figure 2.12 for a nest of non–intersecting lattice paths corresponding to an array in $X_4(\{g, y, o\})$.

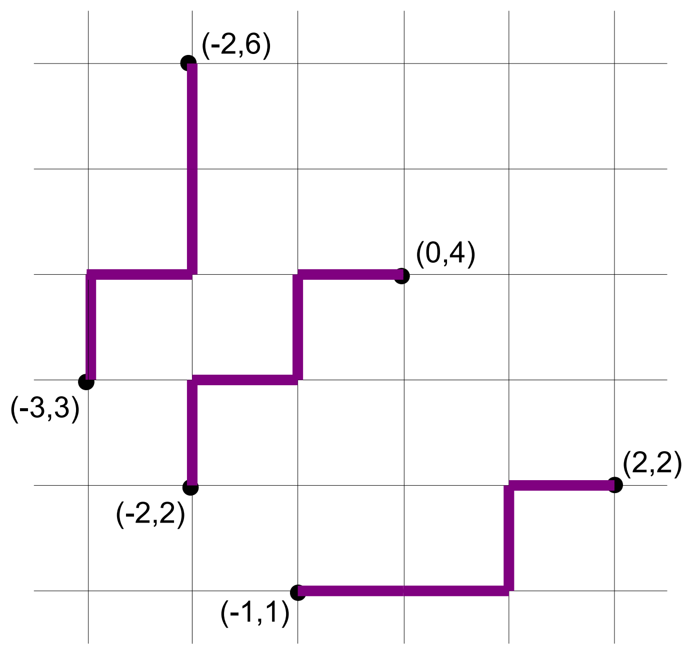



Figure 2.12: A nest of non–intersecting lattice paths corresponding to an array in $X_4(\{g, y, o\})$ of weight $q^5$

Then by Theorem 2.4.3 where we take $D$ to be the lattice $\mathbb{Z}^2$, the generating function for non–intersecting lattice paths with these starting and ending points equals the following determinant:

$$\begin{aligned} GF\left[\mathcal{P}_0(\mathbf{u}, \mathbf{v})\right] &= \det_{1\le i,j\le n-1}\left(GF[\mathcal{P}(u_j, v_i)]\right) = \det_{1\le i,j\le n-1}\left(q^{2i(i-j)}\begin{bmatrix} n \\ 2i-j\end{bmatrix}_q\right) \\ &= \prod_{i=1}^{n-1} q^{i(2i-n)} \det_{1\le i,j\le n-1}\left(q^{(n-2i)j}\begin{bmatrix} n \\ n-2i+j\end{bmatrix}_q\right). \end{aligned}$$

We then use the following equation (Theorem 26 equation (3.12) from [22]):

$$\det_{1\le i,j\le m}\left(q^{jL_i}\begin{bmatrix}A\\L_i+j\end{bmatrix}_q\right)=q^{\sum_{i=1}^m iL_i}\frac{\prod_{1\le i<j\le m}[L_i-L_j]_q}{\prod_{i=1}^m (L_i+m)!_q}\frac{\prod_{i=1}^m (A+i-1)!_q}{\prod_{i=1}^m (A-L_i-1)!_q}.$$

If we set $L_i = n-2i, A = n$, and $m = n-1$, then the power of $q$ outside the determinant exactly cancels and we are left with:

$$\begin{aligned}
&\prod_{i=1}^{n-1} q^{i(2i-n)} \det_{1\le i,j\le n-1}\left(q^{(n-2i)j}\begin{bmatrix}n\\n-2i+j\end{bmatrix}_q\right)\\
&\quad=\frac{\prod_{1\le i<j\le n-1}[2j-2i]_q\prod_{i=1}^{n-1}(n+i-1)!_q}{\prod_{i=1}^{n-1}(2n-2i-1)!_q(2i-1)!_q}\\
&\quad=\prod_{i=1}^{n-1}\frac{(2i)!_q^{n-i-1}(n+i-1)!_q}{(2i-1)!_q^{n-i+1}}\\
&\quad=\prod_{i=1}^{n-1}\frac{[2i]_q^{n-i-1}(n+i-1)!_q}{(2i-1)!_q^{2}}=\prod_{i=1}^{n-1}\frac{[2i]_q^{n-i}}{i!_q}=\prod_{i=1}^{n-1}[2]_{q^i}^{n-i}=\prod_{i=1}^{n-1}(1+q^i)^{n-i}.
\end{aligned}$$

This is the generating function of the arrays $X_n(\{g,y,o\})$. By Proposition 2.2.12 the arrays $X_n(\{g,y,o\})$ are in weight preserving bijection with the order ideals $J(T_n(\{g,y,o\}))$. Therefore $F(J(T_n(\{g,y,o\})),q) = \prod_{j=1}^{n-1}(1+q^j)^{n-j}$. From Proposition 2.2.23 the posets $T_n(S)$ where $S \in \{\{r,g,s\},\{o,b,y\},\{b,g,o\},\{y,g,b\}\}$ are isomorphic to $T_n(\{g,y,o\})$, so the result follows by poset isomorphism. □

*Second proof of Lemma 2.4.2.* The arrays $Y_n(\{g,y,o\})$ are by Definition 2.2.13 equivalent to SSYT of staircase shape $\delta_n$, thus their generating function is given by the Schur function $s_{\delta_n}(x_1,x_2,\ldots,x_n)$. Now

$$s_{\delta_n}(x_1,x_2,\ldots,x_n)=\frac{\det(x_i^{2(n-j)})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n}=\prod_{1\le i<j\le n}\frac{x_i^2-x_j^2}{x_i-x_j}=\prod_{1\le i<j\le n}(x_i+x_j)$$

using the algebraic Schur function definition and the Vandermonde determinant (see for example [25]). The principle specialization of this generating function yields the $q$–generating function $F(J(T_n(\{g,y,o\})),q) = \prod_{j=1}^{n-1}(1+q^j)^{n-j}$. The posets $T_n(S)$ where $S \in \{\{r,g,s\},\{o,b,y\},\{b,g,o\},\{y,g,b\}\}$ are isomorphic to $T_n(\{g,y,o\})$ so the result follows by poset isomorphism. □

Equipped with Lemmas 2.4.1 and 2.4.2 we now prove Theorem 2.2.8.

**Theorem 2.2.8.** *If $S$ is an admissible subset of $\{r, b, g, o, y, s\}$, $|S| = 3$, and $S \notin \{\{r, g, y\}, \{s, b, r\}\}$ then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n-1}(1 + q^j)^{n-j} = \prod_{1 \le i \le j \le k \le n-1} \frac{[i + j]_q}{[i + j - 1]_q}. \tag{2.4.1}$$

*Thus if we set $q = 1$ we have $|J(T_n(S))| = 2^{\binom{n}{2}}$.*

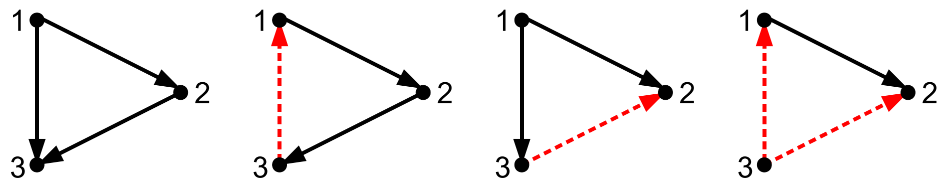
*Proof.* By Lemma 2.4.1, if $S \in \{\{o, s, (b)\}, \{s, y, (g)\}, \{o, r, (y)\}, \{b, r, (g)\}\}$ then the generating function $F(J(T_n(S)), q)$ is as above. By Lemma 2.4.2, if $S \in \{\{r, g, s\}, \{o, b, y\}, \{y, g, o\}, \{b, g, o\}, \{y, g, b\}\}$ then generating function $F(J(T_n(S)), q)$ is as above. These are the only admissible subsets $S$ of $\{r, b, g, o, y, s\}$ with $|S| = 3$ and $S \notin \{\{r, g, y\}, \{s, b, r\}\}$. □

We now discuss the second topic of this section—bijections between the order ideals of the three color posets and tournaments. First we give the bijection for the order ideals of the disjoint three–color posets. See Figure 2.13 for this bijection for $n = 3$.

**Theorem 2.4.4.** *There exists an explicit bijection between the order ideals of the poset $T_n(\{b, r, (g)\})$ and tournaments on $n$ vertices (and therefore by poset isomorphism between the order ideals of any of the disjoint three color posets of Lemma 2.4.1 and tournaments).*

*Proof.* For this proof we will use the modified arrays $Y_n^+(S)$ from Definition 2.2.15. Recall that $Y_n^+(S)$ are exactly the arrays $Y_n(S)$ with an additional column 0 added with fixed entries $1\ 2\ 3 \ldots\ n$. The colors blue and red correspond to inequalities on $Y_n^+(\{b, r, (g)\})$ such that as one goes up the southwest to northeast diagonals at each step the next entry has the choice between staying the same and decreasing by one. That is, for $\alpha \in Y_n^+(\{b, r, (g)\})$, $\alpha_{i,j} = \alpha_{i+1,j-1}$ or $\alpha_{i+1,j-1} - 1$ for all $j \ge 1$. Therefore since each of the $\binom{n}{2}$ entries of the array not in the 0th column has exactly two choices of values given the value of the entry to the southwest, we may consider each array entry $\alpha_{i,j}$ with $j \geq 1$ to symbolize the outcome of the game between $i$ and $i+j$ in a tournament. If $\alpha_{i,j} = \alpha_{i+1,j-1}$ say the outcome of the game between $i$ and $i+j$ is an upset and otherwise not. Thus tournaments on $n$ vertices are in bijection with the arrays $Y_n^+(\{b, r, (g)\})$. Therefore by Proposition 2.2.16 tournaments on $n$ vertices are in bijection with the order ideals of $T_n(\{b, r, (g)\})$. $\square$

$$\begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 1 & 3 \\ 2 & 3 & \\ 3 & & \end{array}$$




$$\begin{array}{lll} 1 & 2 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 2 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 2 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{array}$$

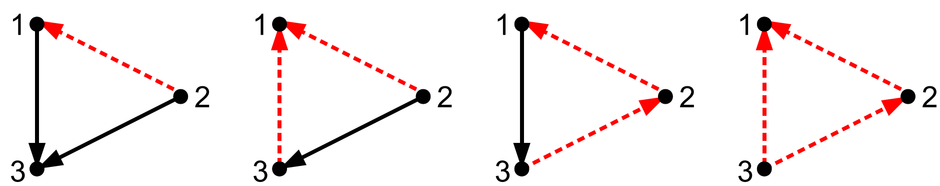



Figure 2.13: The bijection between the arrays $Y_3^+(\{b, r, (g)\})$ and tournaments on three vertices. The tournament upsets (red dashed directed edges) correspond to the array entries equal to their southwest diagonal neighbor (red entries).

The bijection between the order ideals of the connected three color poset $T_n(\{o, y, g\})$ and tournaments on $n$ vertices is due to Sundquist in [39] and involves repeated use of jeu de taquin and column deletion to go from $Y_n(\{o, y, g\})$ (which are SSYT of shape $\delta_n$ and largest entry $n$) to objects which he calls tournament tableaux.

**Theorem 2.4.5** (Sundquist)**.** *There exists an explicit bijection between staircase shape SSYT* $\delta_n = (n-1)\ (n-2)\cdots 2\ 1$ *with entries at most* $n$ *and tournaments on* $n$ *vertices (and therefore by poset isomorphism between the order ideals of any of the connected three color posets of Lemma 2.4.2 and tournaments).*

We now describe Sundquist's bijection. Begin with a SSYT $\alpha$ of shape $\delta_n$ and entries at most $n$ (that is, an element of $Y_n(\{o, y, g\})$). If there are $k$ 1s in $\alpha$ then apply jeu de taquin $k$ times to $\alpha$. If the resulting shape is not $\delta_{n-1}$ apply column deletion to the elements of $\delta_n \backslash \delta_{n-1}$ beginning at the bottom of the tableau. Put the elements removed from $\alpha$ in either step in increasing order in the first row of a new tableau $\beta$.

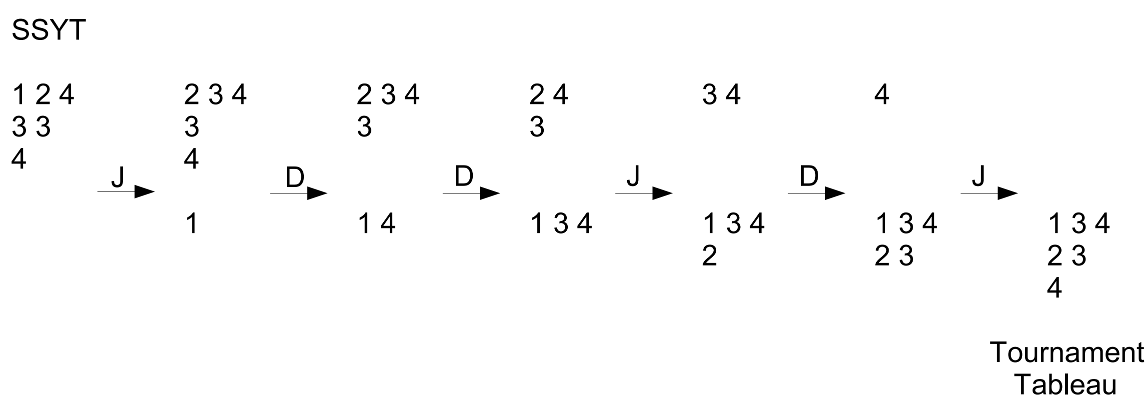



Figure 2.14: An example of Sundquist's bijection between a SSYT of shape $\delta_4$ and a tournament tableau. The resulting tournament tableau represents the tournament on 4 vertices with upsets between vertices 1 & 3, 1 & 4, 2 & 3, and 3 & 4.

Continue this process in like manner. At the $i$th iteration remove the entries equal to $i$ from $\alpha$ by repeated use of jeu de taquin, then bump by column deletion the elements of $\delta_{n-i+1} \backslash \delta_{n-i}$. Then make all the elements removed or bumped out of $\alpha$ during the $i$th iteration into the $i$th row of $\beta$. $\beta$ is called a tournament tableau and has the property that in the $i$th row the elements are greater than or equal to $i$ and all elements strictly greater than $i$ in row $i$ must not be repeated in that row. Thus there is an upset between $i$ and $j$ in the corresponding tournament ($i < j$) if and only if $j$ appears in row $i$ of the tournament tableau. See Figure 2.14 for an example of this bijection, where the arrows labeled J are where jeu de taquin was applied and the arrows labeled D are where column deletion was applied.

## 2.5 The four color posets—ASMs and TSSCPPs

In this section we prove Theorem 2.2.9 about the number of the order ideals of the four color posets by way of two propositions. The first proposition shows that $n \times n$ ASMs are in bijection with the order ideals $J(T_n(\{b, y, o, g\}))$, and the second proposition shows that TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with $J(T_n(\{r, g, o, (y)\}))$. Recall that there is only one ASM poset inside $T_n$ but we have claimed that there are six different sets of colors $S$ such that $J(T_n(S))$ corresponds to TSSCPPs. Thus once we have the correspondence between $J(T_n(\{r, g, o, (y)\}))$ and TSSCPPs inside a $2n \times 2n \times 2n$ box, we obtain the bijection for the other five sets of colors through poset isomorphism. Then in this section we also discuss a generalization of the $q$–ification (1.1.2) of the ASM counting formula which makes connections between ASMs and Catalan numbers.

Given an $n \times n$ ASM $A$ it is customary to consider the following bijection to objects called monotone triangles of order $n$ [8]. For each row of $A$ note which columns have a partial sum (from the top) of 1 in that row. Record the numbers of the columns in which this occurs in increasing order. This gives a triangular array of numbers 1 to $n$. This process can be easily reversed, and is thus a bijection. Monotone triangles can be defined as objects in their own right as follows [8].

**Definition 2.5.1.** Monotone triangles of order $n$ are all triangular arrays of integers with bottom row $1\ 2\ 3\ \ldots\ n$ and integer entries $a_{ij}$ such that $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$ and $a_{ij} \le a_{i,j+1}$.

We construct a bijection between monotone triangles of order $n$ and the arrays $Y_n(\{b, y, o, g\})$ in the following manner (see Figure 2.15). The bottom row of a monotone triangle of order $n$ is always $1\ 2\ 3\ \ldots\ n$, so we can omit writing it. If we then rotate the monotone triangle clockwise by $\frac{\pi}{4}$ we obtain a semistandard Young tableau of staircase shape $\delta_n$ whose northeast to southwest diagonals are weakly increasing. These are the arrays $Y_n(\{b, y, o, g\})$ (see Definition 2.2.13). Thus we have the following proposition.

$$
\begin{array}{ccccc}
4 \times 4 \text{ ASM} & & \text{Monotone triangle of order } 4 & & Y_4(\{b,y,o,g\}) \\
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
& \Longleftrightarrow &
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 1 & & 4 & & \\
 & 1 & & 3 & & 4 & \\
1 & & 2 & & 3 & & 4
\end{array}
& \Longleftrightarrow &
\begin{array}{lll}
1 & 1 & 2 \\
3 & 4 & \\
4 & &
\end{array}
\end{array}
$$

Figure 2.15: An example of the bijection between $n \times n$ ASMs and $Y_n(\{b, y, o, g\})$

**Proposition 2.5.2.** *The order ideals $J(T_n(\{b, y, o, g\}))$ are in bijection with $n \times n$ alternating sign matrices.*

*Proof.* From the above discussion we see that $n \times n$ ASMs are in bijection with the arrays $Y_n(\{b, y, o, g\})$. Then by Proposition 2.2.14, the arrays $Y_n(\{b, y, o, g\})$ are in bijection with the order ideals $J(T_n(\{b, y, o, g\}))$. $\square$

It is well known that the lattice of monotone triangles of order $n$ (which, as we have seen, is equivalent to $J(T_n(\{b, y, o, g\}))$) is the smallest lattice containing the strong Bruhat order on the symmetric group as a subposet, i.e. it is the MacNeille

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 1 | 1 1 | 1 2 | 1 2 | 2 2 | 1 3 | 2 3 |
| 2 | 3 | 2 | 3 | 3 | 3 | 3 |

Figure 2.16: The arrays $Y_3(\{b, y, o, g\})$ which are in bijection with $3 \times 3$ ASMs

completion of the Bruhat order [24]. This relationship was used in [34] to calculate the order dimension of the strong Bruhat order in type $A_n$. The order dimension of a poset $P$ is the smallest $d$ such that $P$ can be embedded as a subposet of $\mathbb{N}^d$ with componentwise order [40]. The proof of the order dimension of $A_n$ proceeds by looking at the poset $T_n(\{g, b\})$, noting that each component is the product of chains, and using these chains to form a symmetric chain decomposition. Then (by another theorem in [34]) the order dimension equals the number of chains in this symmetric chain decomposition, thus the order dimension of $A_n$ under the strong Bruhat order equals $\lfloor \frac{(n+1)^2}{4} \rfloor$.

The rank generating function $F(J(T_n(\{b, y, o, g\})), q)$ is not equal to the $q$–product formula (1.1.2) and does not factor nicely. The previously discussed class of tetrahedral posets whose number of order ideals factors but whose rank generating function does not factor is the two color class in which $T_n(\{b, g\})$ belongs. The number of order ideals of posets in this class is given by a product of Catalan numbers (see Section 2.3). The rank generating function for the lattice of order ideals of these posets is not given by a product formula but rather by a product of the Carlitz–Riordan $q$–Catalan numbers which are defined by a recurrence rather than an explicit formula. This gives hope that even though $F(J(T_n(\{b, y, o, g\})), q)$ does not factor nicely, perhaps it may be found to satisfy some sort of recurrence.

The other six posets $J(T_n(S))$ with $S$ admissible and $|S| = 4$ are each in bijection with totally symmetric self–complementary plane partitions (see Definition 1.1.3). We give this bijection for $S = \{g, r, o, (y)\}$ and infer the other bijections through poset isomorphism using Propositions 2.2.24, 2.2.25, and 2.2.26. See Figure 2.18 for the six different sets of arrays each corresponding to TSSCPPs inside a $6 \times 6 \times 6$ box.

Because of the symmetry conditions, there are relatively few boxes in any TSSCPP which determine it. So we can restrict to a fundamental domain in the following manner (see Figure 2.17). Given a TSSCPP $t = \{t_{i,j}\}_{1 \le i,j \le 2n}$ take a fundamental domain consisting of the triangular array of integers $\{t_{i,j}\}_{n+1 \le i \le j \le 2n}$. In this triangular array $t_{i,j} \ge t_{i+1,j} \ge t_{i+1,j+1}$ since $t$ is a plane partition. Also for these values of $i$ and $j$ the entries $t_{i,j}$ satisfy $0 \le t_{i,j} \le 2n + 1 - i$. Now if we reflect this array about a vertical line then rotate clockwise by $\frac{\pi}{4}$ we obtain a staircase shape array $x$ whose entries $x_{i,j}$ satisfy the conditions $x_{i,j} \le x_{i,j+1} \le x_{i+1,j}$ and $0 \le x_{i,j} \le j$. The set of all such arrays is equivalent to $X_n(\{g, r, o, (y)\})$ (see Definition 2.2.11). Now add $i$ to each entry in row $i$ to obtain the arrays $Y_n(\{g, r, o, (y)\})$. This gives us the following proposition.

$$
\begin{array}{cccccccc}
8 & 8 & 8 & 8 & 6 & 6 & 4 & 4 \\
8 & 8 & 8 & 8 & 6 & 5 & 4 & 4 \\
8 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
8 & 8 & 6 & 5 & 4 & 3 & 2 & 2 \\
6 & 6 & 5 & 4 & 3 & 2 & \cdot & \cdot \\
6 & 5 & 4 & 3 & 2 & 1 & \cdot & \cdot \\
4 & 4 & 3 & 2 & \cdot & \cdot & \cdot & \cdot \\
4 & 4 & 2 & 2 & \cdot & \cdot & \cdot & \cdot
\end{array}
\iff
\begin{array}{cccc}
3 & & & \\
2 & 1 & & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot
\end{array}
\iff
\begin{array}{cccc}
\cdot & \cdot & 1 & 3 \\
\cdot & \cdot & 2 & \\
\cdot & \cdot & & \\
\cdot & & &
\end{array}
\iff
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 2 & 4 & \\
3 & 3 & & \\
4 & & &
\end{array}
$$

Figure 2.17: An example of the bijection between TSSCPPs inside a $2n \times 2n \times 2n$ box and $Y_n(\{r, g, o, (y)\})$

**Proposition 2.5.3.** *The order ideals* $J(T_n(\{r, g, o, (y)\}))$ *are in bijection with totally symmetric self–complementary plane partitions inside a* $2n \times 2n \times 2n$ *box.*

*Proof.* From the above discussion we see that TSSCPPs are in bijection with the arrays $Y_n(\{r, g, o, (y)\})$. Then by Proposition 2.2.14, the arrays $Y_n(\{r, g, o, (y)\})$ are in bijection with the order ideals $J(T_n(\{r, g, o, (y)\}))$. □

$$
\begin{aligned}
Y_3(\{r,g,o,(y)\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\} \\
Y_3(\{r,b,(g),y\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\} \\
Y_3(\{y,s,(g),r\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 2 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\} \\
Y_3(\{y,s,(g),b\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\} \\
Y_3(\{o,s,(b),g\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 1 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\} \\
Y_3(\{r,b,(g),s\}) &= \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\}
\end{aligned}
$$

Figure 2.18: The six sets of arrays which are in bijection with TSSCPPs inside a $6 \times 6 \times 6$ box

Equipped with Propositions 2.5.2 and 2.5.3 we are now ready to prove Theorem 2.2.9.

**Theorem 2.2.9.** *If $S$ is an admissible subset of $\{r,b,g,o,y,s\}$ and $|S| = 4$ then*

$$|J(T_n(S))| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \prod_{1 \le i \le j \le k \le n-1} \frac{i+j+k+1}{i+j+k-1}. \tag{2.5.1}$$

*Proof.* The posets $T_n(S)$ for $S$ admissible and $|S| = 4$ are $T_n(\{g,y,b,o\})$, the three isomorphic posets $T_n(\{r,o,(y),g\})$, $T_n(\{r,b,(g),y\})$, and $T_n(\{y,s,(g),r\})$ (see Proposition 2.2.24), and the three posets dual to these, $T_n(\{y,s,(g),b\})$, $T_n(\{o,s,(b),g\})$, $T_n(\{r,b,(g),s\})$ (see Propositions 2.2.25 and 2.2.26). In Proposition 2.5.2 we showed that the order ideals of $T_n(\{g, y, b, o\})$ are in bijection with $n \times n$ ASMs and in Proposition 2.5.3 we showed that the order ideals of $T_n(\{r, o, (y), g\})$ are in bijection with TSSCPPs inside a $2n \times 2n \times 2n$ box. Therefore by poset isomorphism TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with the order ideals of any of $T_n(\{r, o, (y), g\})$, $T_n(\{r, b, (g), y\})$, $T_n(\{y, s, (g), r\})$, $T_n^*(\{y, s, (g), b\})$, $T_n^*(\{o, s, (b), g\})$, or $T_n^*(\{r, b, (g), s\})$. Thus by the enumeration of ASMs in [43] and [23] and the enumeration of TSSCPPs in [1] we have the above formula for the number of order ideals. □

As we have already noted, ASMs and TSSCPPs are both counted by (2.5.1), but there has not been found an explicit bijection between them. Through the poset perspective we have found multiple sets of objects containing both ASMs and TSSCPPs as subsets. We have seen that $Y_n(g, y, o, b)$ (corresponding to ASMs) and $Y_n(g, (y), o, r)$ (corresponding to TSSCPPs) are both contained in $Y_n(g, y, o)$ (corresponding to SSYT of shape $\delta_n$ with entries at most $n$), thus these SSYT are one such set. The existence of supersets containing both ASMs and TSSCPPs gives hope that one may be able to find a bijection between ASMs and TSSCPPs using the involution principle. We have not found a superset for which the involution principle might work for all $n$, since in each superset we have considered, we have found a value of $n$ such that the larger set of objects containing ASMs and TSSCPPs is of the wrong parity. It may be worthwhile to try and use not only the inclusions of ASM and TSSCPP into SSYT but also the inclusion of TSSCPP into tournaments along with the bijection of Sundquist between tournaments and SSYT (see Theorem 2.4.5) to construct a bijection between ASMs and TSSCPPs. See Figure 2.19 for the big picture of the known bijections and inclusions among ASMs, SSYT, tournaments, and TSSCPPs.

As in the two–color Catalan case of Theorem 2.2.7, the direct $q$–ification of the counting formula (2.5.1), does not equal the rank generating function of either ASMs or TSSCPPs, but there may be a way to find a weight on ASMs or TSSCPPs which would yield the $q$–product formula as generating function. An interesting generaliza-

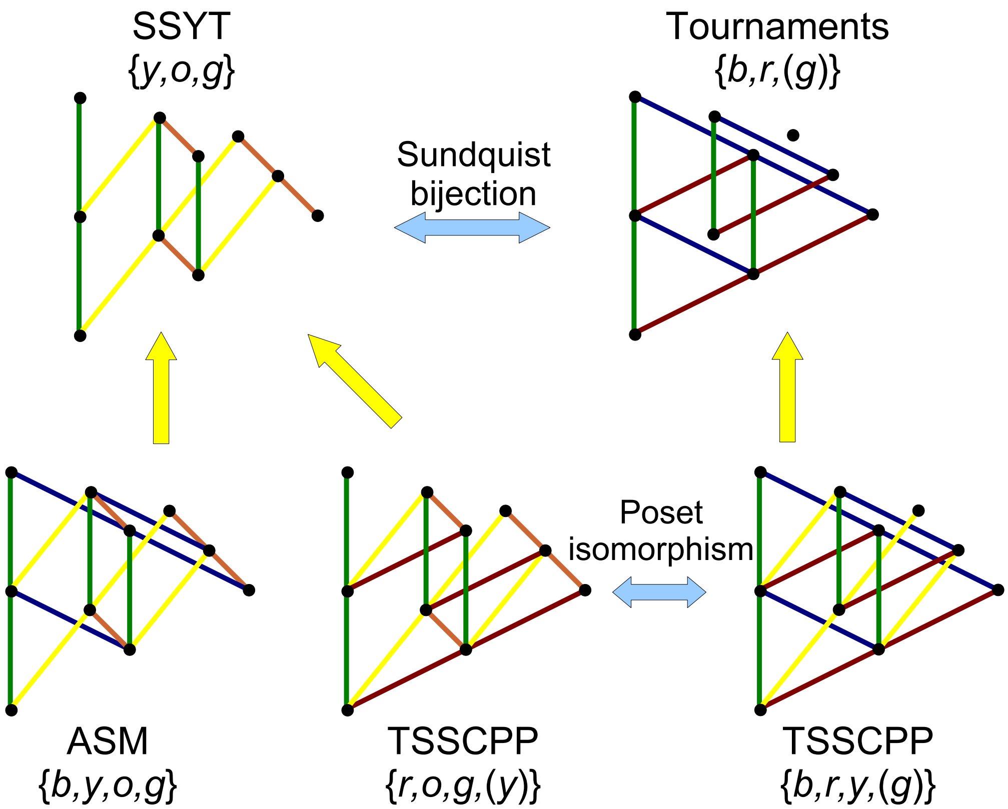



Figure 2.19: The big picture of bijections and inclusions among the three and four color tetrahedral subposets. The two sided arrows represent bijections between sets of order ideals. The one sided arrows represent inclusions of one set of order ideals into another.

tion of the $q$–product formula which has connections with the MacMahon $q$–Catalan numbers is found in the thesis of Thomas Sundquist [39]. Sundquist studies the following product formula:

$$A(n,p;q) = \frac{s_{(p\delta_n)'}(np)}{s_{p\delta_n}(n)} q^{-\binom{p}{2}\sum_{i=1}^{n-1} i^2} = \prod_{k=0}^{n-1} \frac{(np+k)!_q k!_q}{(kp+k+p)!_q (pk+k)!_q} \tag{2.5.2}$$

where $s_\lambda$ is the Schur function and $\delta_n$ is the staircase shape partition.

$A(n,2;q)$ reduces to the $q$–ified ASM product formula.

$$A(n,2;q) = \prod_{k=0}^{n-1} \frac{(3k+1)!_q}{(n+k)!_q}$$

$A(2,p;q)$ reduces to the MacMahon $q$–Catalan numbers.

$$A(2,p;q) = \frac{1}{[p+1]_q} \begin{bmatrix} 2p \\ p \end{bmatrix}_q$$

Sundquist was able to prove that $A(n,p;q)$ is a polynomial in $q$, but he was unable to find a combinatorial interpretation (even for $A(n,p;1)$) for general $n$ and $p$. It is well-known that $\frac{1}{[p+1]_q}\begin{bmatrix} 2p \\ p \end{bmatrix}_q$ for fixed $p$ is the generating function for Catalan sequences (or Dyck paths) weighted by major index. $A(n,p;q)$ suggests a kind of generalization of major index to a larger class of objects including those counted by the $q$–ASM numbers, but that larger class of objects is still unknown as is an interpretation for this $q$–weight on ASMs or TSSCPPs.

$A(n,p;1)$ is known to have connections to physics, however. An expression equal to $A(n,p;1)$ appears in [14] (Equation (5.5)) and comes from a weighted sum of components of the ground state of the $A_{k-1}$ IRF model in statistical physics. In this paper the authors comment that a combinatorial interpretation for $A(n,p;1)$ as a generalization of ASMs would be very interesting.

## 2.6 Connections between ASMs, TSSCPPs, and tournaments

In this section we discuss the expansion of the tournament generating function as a sum over ASMs and derive a new expansion of the tournament generating function as a sum over TSSCPPs. We discuss the implications of the combination of these two expansions and also describe which subsets of tournaments correspond to TSSCPPs and ASMs. This description will be straightforward for TSSCPPs and indirect for ASMs.

The alternating sign matrix conjecture first arose when David Robbins and Howard Rumsey expanded a generalization of the determinant of a square matrix called the $\lambda$-determinant as a sum over all $n \times n$ alternating sign matrices ([35]) and wondered how many elements would be in the sum. When one combines this $\lambda$-determinant formula with the Vandermonde determinant, one obtains the following theorem. (This theorem is implicit in [35], but appears explicitly in [8].) In this theorem we will need the notion of inversion number for ASMs. The inversion number of an ASM $A$ is defined as $I(A) = \sum A_{ij}A_{k\ell}$ where the sum is over all $i, j, k, \ell$ such that $i > k$ and $j < l$.

**Theorem 2.6.1** (Robbins–Rumsey)**.** *Let $A_n$ be the set of $n \times n$ alternating sign matrices, and for $A \in A_n$ let $I(A)$ denote the inversion number of $A$ and $N(A)$ the number of $-1$ entries in $A$, then*

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{A \in A_n} \lambda^{I(A)} \left(1 + \lambda^{-1}\right)^{N(A)} \prod_{i,j=1}^{n} x_j^{(n-i)A_{ij}}. \tag{2.6.1}$$

Note that there are $\binom{n}{2}$ factors in the product on the left–hand side, thus $\prod_{1 \le i < j \le n}(x_i + \lambda x_j)$ is the generating function for tournaments on $n$ vertices where each factor of $(x_i + \lambda x_j)$ represents the outcome of the game between $i$ and $j$ in the tournament. If $x_i$ is chosen then the expected winner, $i$, is the actual winner, and if $\lambda x_j$ is chosen then $j$ is the unexpected winner and the game is an upset. Thus in each monomial in the expansion of $\prod_{1\le i<j\le n}(x_i+\lambda x_j)$ the power of $\lambda$ equals the number of upsets and the power of $x_k$ equals the number of wins of $k$ in the tournament corresponding to that monomial.

We rewrite Theorem 2.6.1 in different notation which will also be needed later. For any staircase shape integer array $\alpha \in Y_n^+(S)$ (recall Definition 2.2.15) let $E_{i,k}(\alpha)$ be the number of entries of value $k$ in row $i$ equal to their southwest diagonal neighbor, $E^i(\alpha)$ be the number of entries in (southwest to northeast) diagonal $i$ equal to their southwest diagonal neighbor, and $E_i(\alpha)$ be the number of entries in row $i$ equal to their southwest diagonal neighbor, that is, $E_i(\alpha) = \sum_k E_{i,k}(\alpha)$. Also let $E(\alpha)$ be the total number of entries of $\alpha$ equal to their southwest diagonal neighbor, that is, $E(\alpha) = \sum_i E_i(\alpha) = \sum_i E^i(\alpha)$. We now define variables for the content of $\alpha$. Let $C_{i,k}(\alpha)$ be the number of entries in row $i$ with value $k$ and let $C_k(\alpha)$ be the total number of entries of $\alpha$ equal to $k$, that is, $C_k(\alpha) = \sum_i C_{i,k}(\alpha)$. Let $N(\alpha)$ be the number of entries of $\alpha$ strictly greater than their neighbor to the west and strictly less than their neighbor to the southwest. When $\alpha \in Y_n^+(\{b,y,o,g\})$ then $N(\alpha)$ equals the number of $-1$ entries in the corresponding ASM. For example, let $\alpha$ be the following array in $Y_5^+(\{b,y,o,g\})$ which is in bijection with the ASM $A$ shown below.

$$\begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & \\ 3 & 4 & 4 & & \\ 4 & 5 & & & \\ 5 & & & & \end{array} \iff \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Then the parameters on the array $\alpha$ are as follows. $E_{1,2} = E_{1,3} = E_{3,4} = E_{4,5} = 1$ and $E_{i,j} = 0$ for all other choices of $i$ and $j$. The number of diagonal equalities in each row of $\alpha$ is given by $E_1 = 2, E_2 = 0, E_3 = 1, E_4 = 1, E_5 = 0$. The number of diagonal equalities in each column of $\alpha$ is given by $E^1 = 0, E^2 = 0, E^3 = 1, E^4 = 1, E^5 = 2, E = 4$. The number of $-1$ entries in $A$ is $N = 1$. The content parameters on $\alpha$ are given by $C_{1,1} = 2, C_{1,2} = 2, C_{1,3} = 1, C_{2,2} = 2, C_{2,3} = 2, C_{3,3} = 1,$

$C_{3,4} = 2, C_{4,4} = 1, C_{4,5} = 1, C_{5,5} = 1, C_1 = 2, C_2 = 4, C_3 = 4, C_4 = 3, C_5 = 2.$

Using this notation we reformulate Theorem 2.6.1 in the following way.

**Theorem 2.6.2.** *The generating function for tournaments on $n$ vertices can be expanded as a sum over the ASM arrays $Y_n^+(\{b, y, o, g\})$ in the following way.*

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{\alpha \in Y_n^+(\{b,y,o,g\})} \lambda^{E(\alpha)} (1+\lambda)^{N(\alpha)} \prod_{k=1}^{n} x_k^{C_k(\alpha)-1}. \tag{2.6.2}$$

*Proof.* First we rewrite Equation 2.6.1 by factoring out $\lambda^{-1}$ from each $(1 + \lambda^{-1})$.

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{A \in A_n} \lambda^{I(A)-N(A)} (1+\lambda)^{N(A)} \prod_{i,j=1}^{n} x_j^{(n-i)A_{ij}}.$$

Let $\alpha \in Y_n^+(\{b, y, o, g\})$ be the array which corresponds to $A$. It is left to show that $I(A) - N(A) = E(\alpha)$ and $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{C_j(\alpha)-1}$. In the latter equality take the product over $i$ of the left hand side: $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n (n-i)A_{ij}}$. We wish to show $C_j(\alpha) - 1 = \sum_{i=1}^n (n - i)A_{ij}$. $C_j(\alpha)$ equals the number of entries of $\alpha$ with value $j$, so $C_j(\alpha)-1$ equals the number of entries of $\alpha$ with value $j$ not counting the $j$ in the 0th column. Using the bijection between ASMs and the arrays $Y_n(\{b, y, o, g\})$ we see that the number of $j$s in columns 1 through $n - 1$ of $\alpha$ equals the number of 1s in column $j$ of $A$ plus the number of zeros in column $j$ of $A$ which are south of a 1 with no $-1$s in between. This is precisely what $\sum_{i=1}^n (n - i)A_{ij}$ counts by taking a positive contribution from every 1 and every entry below that 1 in column $j$ and then subtracting one for every $-1$ and every entry below that $-1$ in column $j$. Thus $C_j(\alpha) - 1 = \sum_{i=1}^n (n - i)A_{ij}$ so that $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{C_j(\alpha)-1}$.

Now we wish to show that $I(A) - N(A) = E(\alpha)$. Recall that the inversion number of an ASM is defined as $I(A) = \sum A_{ij}A_{k\ell}$ where the sum is over all $i, j, k, \ell$ such that $i > k$ and $j < l$. This definition extends the usual notion of inversion in a permutation matrix. Fix $i$, $j$, and $\ell$ and consider $\sum_{k<i} A_{ij}A_{k\ell}$. Let $k'$ be the row of the southernmost nonzero entry in column $\ell$ such that $k' < i$. If there exists no such $k'$ (that is, $A_{k\ell} = 0 \;\forall\; k < i$) or if $A_{k'\ell} = -1$ then $\sum_{k>i} A_{ij}A_{k\ell} = 0$ since there must be an even number of nonzero entries in $\{A_{k\ell}, k < i\}$ half of which are 1 and half of which are $-1$. If $A_{k'\ell} = 1$ then $\sum_{k<i} A_{ij}A_{k\ell} = A_{ij}$. Thus $I(A) = \sum_{i,j} \alpha_{ij}A_{ij}$ where $\alpha_{ij}$ equals the number of columns east of column $j$ such that $A_{k'\ell}$ with $k' > i$ exists and equals 1. Let column $\ell'$ be one of the columns counted by $\alpha_{ij}$. Then $A_{i\ell'}$ cannot equal 1, otherwise $A_{k'\ell'}$ would either not exist or equal $-1$. If $A_{i\ell'} = 0$ then in $\alpha$ there is a corresponding diagonal equality. If $A_{i\ell'} = -1$ then there is no diagonal equality in $\alpha$. Thus $I(A) = E(\alpha) + N(A)$. □

Recall that the left-hand side of (2.6.2) is the generating function for tournaments where the power of $\lambda$ equals the number of upsets and the power of $x_k$ equals the number of wins of $k$. If we set $\lambda = 1$ and take the principle specialization of the $x_k$'s this yields the generating function of Theorem 2.2.8 multiplied by a power of $q$. Furthermore, if we set $\lambda = 1$ and $x_i = 1$ for all $i$ we have the following corollary.

**Corollary 2.6.3.**

$$2^{\binom{n}{2}} = \sum_{A \in A_n} 2^{N(A)}. \tag{2.6.3}$$

That is, $2^{\binom{n}{2}}$ is the 2–enumeration of ASMs with respect to the number of $-1$s. This result appears in [27] and [35]. The connection between this result and Aztec diamonds is shown in [15].

Many people have wondered what the TSSCPP analogue of the $-1$ in an ASM may be. The following theorem does not give a direct analogue, but rather expands the left–hand side of (2.6.2) as a sum over TSSCPPs instead of ASMs.

**Theorem 2.6.4.** *The generating function for tournaments on $n$ vertices can be expanded as a sum over the TSSCPP arrays $Y_n^+(\{b, r, (g), y\})$ in the following way.*

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{\alpha \in Y_n^+(\{b,r,(g),y\})} \lambda^{E(\alpha)} \prod_{i=1}^{n-1} x_i^{n-i-E_i(\alpha)} \sum_{\textit{row shuffles } \alpha' \textit{ of } \alpha} \prod_{j=1}^{n-1} x_j^{E^j(\alpha')} \tag{2.6.4}$$

$$(1 + \lambda)^{\binom{n}{2}} = \sum_{\alpha \in Y_n^+(\{b,r,(g),y\})} \lambda^{E(\alpha)} \prod_{1 \le i \le k \le n-1} \binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)} \tag{2.6.5}$$

*where a row shuffle $\alpha'$ of $\alpha \in Y_n^+(\{b, r, (g), y\}$ is an array obtained by reordering the entries in the rows of $\alpha$ in such a way that $\alpha' \in Y_n^+(\{b, r, (g)\}$.*

*Proof.* The idea of the proof is to begin with the set $Y_n^+(\{b, r, (g), y\})$ and remove the inequality restriction corresponding to the color yellow to obtain the arrays $Y_n^+(\{b, r, (g)\})$. We wish to prove that all the arrays in $Y_n^+(\{b, r, (g)\})$ can be obtained uniquely by the rearrangement of the rows of the arrays in $Y_n^+(\{b, r, (g), y\})$.

First we need to see how $\prod_{1 \leq i < j \leq n}(x_i + \lambda x_j)$ is the generating function of the arrays $Y_n^+(\{b, r, (g)\})$. Recall the bijection of Theorem 2.4.4 between $Y_n^+(\{b, r, (g)\})$ and tournaments on $n$ vertices. Let each entry $\alpha_{i,j}$ with $j \geq 1$ of $\alpha \in Y_n^+(\{b, r, (g)\})$ contribute a weight $x_i$ if $\alpha_{i,j} = \alpha_{i+1,j-1} - 1$ and $\lambda x_{j+i}$ if $\alpha_{i,j} = \alpha_{i+1,j-1}$. Thus $\prod_{1 \leq i < j \leq n}(x_i + \lambda x_j)$ is the generating function of $Y_n^+(\{b, r, (g)\})$.

Now we give an algorithm for turning any element $\alpha \in Y_n^+(\{b, r, (g)\})$ into an element of $Y_n^+(\{b, r, (g), y\})$ thus grouping all the elements of $Y_n^+(\{b, r, (g)\})$ into fibers over the elements of $Y_n^+(\{b, r, (g), y\})$. Assume all the rows of $\alpha$ below row $i$ are in increasing order. Thus $\alpha_{i+1,j} \leq \alpha_{i+1,j+1}$. If $\alpha_{i+1,j} < \alpha_{i+1,j+1}$ then $\alpha_{i,j+1} \leq \alpha_{i+1,j+2}$ since $\alpha_{i,j+1} \in \{\alpha_{i+1,j}, \alpha_{i+1,j} - 1\}$ and $\alpha_{i,j+2} \in \{\alpha_{i+1,j+1}, \alpha_{i+1,j+1} - 1\}$ by the inequalities corresponding to red and blue. So the only entries which may be out of order in row $i$ are those for which their southwest neighbors are equal. If $\alpha_{i+1,j} = \alpha_{i+1,j+1}$ but $\alpha_{i,j+1} > \alpha_{i,j+2}$ it must be that $\alpha_{i,j+1} = \alpha_{i+1,j}$ and $\alpha_{i,j+2} = \alpha_{i+1,j+1} - 1$. So we may swap $\alpha_{i,j+1}$ and $\alpha_{i,j+2}$ along with their entire respective northeast diagonals while not violating the red and blue inequalities. By completing this process for all rows we obtain an array with weakly increasing rows, thus this array is actually in $Y_n^+(\{b, r, (g), y\})$.

Now we do a weighted count of how many arrays in $Y_n^+(\{b, r, (g)\})$ are mapped to a given array in $Y_n^+(\{b, r, (g), y\})$. Again we rely on the fact that entries in a row can be reordered only when their southwest neighbors are equal. Thus to find the weight of all the $Y_n^+(\{b, r, (g)\})$ arrays corresponding to a single $Y_n^+(\{b, r, (g), y\})$ array we simply need to find the set of diagonals containing equalities. The diagonal equalities give a weight dependent on which diagonal they are in, whereas the diagonal inequalities give a weight according to their row (which remains constant). Thus if we are keeping track of the $x_i$ weight we can do no better than to write this as a sum over all the allowable (in the sense of not violating the $b$ or $r$ inequalities) shuffles of the rows of $\alpha$ with the $x$ weight of the diagonal equalities dependent on the position. Thus we have Equation (2.6.4). See Figure 2.20 for an example when $n = 3$.

If we set $x_i = 1$ for all $i$ and only keep track of the $\lambda$ we can make a more precise statement. The above proof shows that the $\lambda$'s result from the diagonal equalities, and the number of different reorderings of the rows tell us the number of different elements of $Y_n^+(\{b, r, (g)\})$ which correspond to a given element of $Y_n^+(\{b, r, (g), y\})$. We count this number of allowable reorderings as a product over all rows $i$ and all array values $k$ as $\binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)}$. This yields Equation (2.6.5). $\square$

$$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 & \\ 3 & & \end{array}$$

$$x_1^2 x_2 \qquad \lambda x_1 x_2 (x_2 + x_3) \qquad \lambda x_1^2 x_3 \qquad \lambda^2 x_1 x_3^2$$

$$\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{array}$$

$$\lambda^2 x_2 x_2 x_3 \qquad \lambda^2 x_1 x_2 x_3 \qquad \lambda^3 x_2 x_3^2$$

Figure 2.20: The arrays $Y_3^+(\{b, r, (g), y\})$ and their corresponding contribution toward the right hand side of Equation (2.6.4)

If we set $\lambda = 1$ in Equation (2.6.5) we obtain the following corollary.

**Corollary 2.6.5.**

$$2^{\binom{n}{2}} = \sum_{\alpha \in Y_n^+(\{b,r,(g),y\})} \prod_{1 \le i \le k \le n-1} \binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)}. \tag{2.6.6}$$

Corollary 2.6.5 is quite different from Corollary 2.6.3. In Corollary 2.6.3 the fibers over ASMs can be only powers of 2, whereas the fibers over TSSCPPs in Corollary 2.6.5 have no such restriction. Thus there is no direct way to match up ASMs and TSSCPPs through all the tournaments in the fibers, but it may be possible to find a system of distinct representatives of tournaments which would show us how to map a fiber above a TSSCPP to a fiber above an ASM.

The difference in the weighting of ASMs and TSSCPPs in Theorems 2.6.2 and 2.6.4 is also substantial. For ASMs the more complicated part of the formula arises in the power of $\lambda$ and for TSSCPPs the complication comes from the $x$ variables. These theorems are also strangely similar. From these theorems we see that the tournament generating function can be expanded as a sum over either ASMs or TSSCPPs, but we still have no direct reason why the number of summands should be the same. The combination of Theorems 2.6.2 and 2.6.4 may eventually contribute toward finding a bijection between ASMs and TSSCPPs, but also shows why a bijection is not obvious.

The last thing we wish to do in this section is to describe which subsets of tournaments correspond to TSSCPPs and ASMs. First we show that for a certain choice of TSSCPP poset inside $T_n$, TSSCPPs can be directly seen to be subsets of tournaments, refining the bijection of Theorem 2.4.4.

**Theorem 2.6.6.** *TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with tournaments on vertices labeled $1, 2, \ldots, n$ which satisfy the following condition on the upsets: if vertex $v$ has $k$ upsets with vertices in $\{u, u+1, \ldots, v-1\}$ then vertex $v-1$ has at most $k$ upsets with vertices in $\{u, u+1, \ldots, v-2\}$.*

*Proof.* We have seen in Theorem 2.4.4 the bijection between the order ideals of $T_n(\{r, b, (g)\})$ and tournaments on $n$ vertices. Recall that the inequalities corresponding to the colors red and blue for $\alpha \in Y_n^+(\{r, b, (g)\})$ place conditions on the diagonals such that there are exactly two choices for any entry $\alpha_{i,j}$ not in column 0. Either $\alpha_{i,j} = \alpha_{i+1,j-1}$ or $\alpha_{i,j} = \alpha_{i+1,j-1} - 1$. In the bijection with tournaments, $\alpha_{i,j}$ tells the outcome of the game between $i$ and $i + j$ in the tournament. The outcome is an upset if $\alpha_{i,j} = \alpha_{i+1,j-1}$ and not an upset otherwise. Thus if we consider the TSSCPP arrays $Y_n^+(\{r, b, (g), y\})$ we need only find an interpretation for the color yellow in terms of tournaments.

Recall that the color yellow corresponds to a weak increase across the rows of $\alpha$. In order for weak increase across the rows of $\alpha$ to be satisfied, for each choice of $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n-i-1\}$ the number of diagonal equalities to the southwest of $\alpha_{i,j}$ must be less than or equal to the number of diagonal equalities to the southwest of $\alpha_{i,j+1}$. So in terms of tournaments, the number of upsets between $i + j + 1$ and vertices greater than or equal to $i$ must be greater than or equal to the number of upsets between $i + j$ and vertices greater than or equal to $i$. $\square$

Note that every integer array/tournament pair in Figure 2.13 satisfies the condition of this theorem and is thus in bijection with a TSSCPP except the pair shown below in Figure 2.21. This is because vertex 3 has zero upsets with vertices in $\{1, 2\}$ but vertex 2 has one upset with vertices in $\{1\}$.

$$\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 2 & \\ 3 & & \end{array}$$

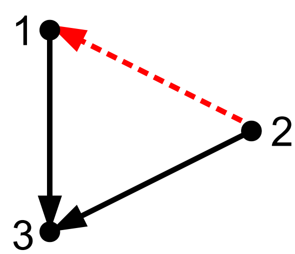



Figure 2.21: The only tournament on 3 vertices (and corresponding array in $Y_3^+(\{b, r, (g)\})$) which does not correspond to a TSSCPP

Next we discuss a previously known way of viewing ASMs as a subset of tournaments, but this viewpoint does not come from the poset perspective. In the paper [18], Hamel and King give a weight–preserving bijection between objects called primed shifted semistandard tableaux (PST) of staircase shape which encode the 2–enumeration of ASMs and primed shifted tableaux (PD) of staircase shape which encode tournaments. Both PSTs and PDs have entries in the set $\{1, 2', 2, 3', 3, \ldots, n', n\}$. The bijection between PSTs and PDs proceeds using jeu de taquin on the primed entries. If one removes the primes from all entries of a PST, one is left with arrays equivalent to $Y_n^+\{b, y, o, g\}$ which are in bijection with $n \times n$ ASMs. Now for any entry of a PST which represents a $-1$ in the corresponding ASM, we have a choice as to whether that entry should be primed or not in the PST. Thus given an ASM, if we fix some rule concerning how to choose which entries of the ASM array should be primed in the inclusion into PSTs, we can map ASMs into tournaments by way of the bijection between PSTs and PDs. Since the bijection between PSTs and PDs uses jeu de taquin, though, it remains difficult to say which subset of tournaments is in the image of this map. Also, since PDs encode tournaments, they are in direct bijection with the tournament tableaux of Sundquist (see Theorem 2.4.5). It would be interesting to look at the bijection from PSTs to PDs to tournament tableaux to SSYT to see if any interesting connections can be made.

Thus far viewing both ASMs and TSSCPPs as subsets of tournaments (or SSYT) has not produced the elusive bijection between ASMs and TSSCPPs. There is still hope, however, that further study of ASMs and TSSCPPs from the poset perspective may shed light on this area.

## 2.7 The five color posets—intersections of ASMs and TSSCPPs

Alternating sign matrices and totally symmetric self–complementary plane partitions appear to be very different from one another. ASMs are square matrices of zeros, ones, and negative ones. TSSCPPs are highly symmetric stack of cubes in a corner. Thus it is strange to talk about objects in the intersection of ASMs and TSSCPPs. But when we view ASMs and TSSCPPs as order ideals in tetrahedral posets, that is subsets of the same set of elements subject to different constraints, we see that they are not so dissimilar after all. If we view them furthermore as staircase shape arrays subject to the inequality conditions corresponding to the edge colors as in Propositions 2.5.2 and 2.5.3 it is more transparent how one might take their intersection—simply by imposing the inequalities from both sets of arrays simultaneously.

There are two different ways to obtain nonisomorphic posets by choosing five of the six colors of edges in $T_n$. One way corresponds to the intersection of ASMs and TSSCPPs and the other corresponds to the intersection of two different sets of TSSCPPs inside $T_n$. The arrays $Y_n(\{r, b, o, (y), (g)\})$ can be thought of as the intersection of the arrays $Y_n(\{b, y, o, g\})$ corresponding to ASMs and the arrays $Y_n(\{r, b, y, (g)\})$ corresponding to TSSCPPs. The dual five color poset $Y_n(\{s, b, o, (y), (g)\})$ which has silver edges instead of red can be thought of similarly: as the intersection of $Y_n(\{b, y, o, g\})$ (ASMs) with $Y_n(\{s, b, y, (g)\})$ (TSSCPPs). We have not found a nice product formula for the number of order ideals of these posets, but one may yet exist. We have calculated the number of order ideals $|J(T_n(\{r, b, o, (y), (g)\}))| = |J(T_n(\{s, b, o, (y), (g)\}))|$ for $n = 1$ to 7 as $1, 2, 6, 26, 162, 1450, 18626$.

The other five color poset is $T_n(\{r, b, s, y, (g)\})$. The ASM poset $T_n(\{b, y, o, g\})$ is not a subset of this poset since the color orange is not included. So we can think of $T_n(\{r, b, s, y, (g)\})$ as an intersection of two manifestations of TSSCPPs inside $T_n$, but not as an intersection of ASMs and TSSCPPs. In fact, removing any of the colors except green from $T_n(\{r, b, s, y, (g)\})$ yields a TSSCPP poset, thus $T_n(\{r, b, s, y, (g)\})$ is the intersection of any two of those four TSSCPP posets. We have not found a nice product formula for the number of order ideals of this poset either, but we have calculated the number of order ideals $|J(T_n(\{r, b, s, y, (g)\}))|$ for $n = 1$ to 6 as $1, 2, 6, 28, 202, 2252$.

## 2.8 The full tetrahedral poset and TSPPs

Throughout this chapter we have seen how to begin with $\binom{n+1}{3}$ poset elements and add ordering relations between them in certain ways until we have built the full tetrahedron $T_n$. In this section we prove Theorem 2.2.10 by showing that the order ideals of $T_n$ are in bijection with totally symmetric plane partitions inside an $(n-1)\times(n-1)\times(n-1)$ box. We will then discuss connections with the conjectured $q$–enumeration of TSPPs and also a way to view TSPPs as an intersection of ASMs and TSSCPPs.

Totally symmetric plane partitions are plane partitions which are symmetric with respect to all permutations of the $x, y, z$ axes. Thus we can take as a fundamental domain the wedge where $x \geq y \geq z$. Then if we draw the lattice points in this wedge (inside a fixed bounding box of size $n-1$) as a poset with edges in the $x$, $y$, and $z$ directions, we obtain the poset $T_n$ where the $x$ direction corresponds to the red edges of $T_n$, the $y$ direction to the orange edges, and the $z$ direction to the silver edges. All other colors of edges in $T_n$ are induced by the colors red, silver, and orange. Thus TSPPs inside an $(n-1)\times(n-1)\times(n-1)$ box are in bijection with the order ideals of $T_n$ (see Figure 2.8).

**Theorem 2.2.10.**

$$|J(T_n)| = \prod_{1\leq i\leq j\leq n-1} \frac{i+j+n-2}{i+2j-2} = \prod_{1\leq i\leq j\leq k\leq n-1} \frac{i+j+k-1}{i+j+k-2}. \tag{2.8.1}$$

*Proof.* As discussed above, the order ideals of $T_n$ are in bijection with totally symmetric plane partitions inside an $(n-1)\times(n-1)\times(n-1)$ box. Thus by the enumeration of TSPPs in [38] the number of order ideals $|J(T_n)|$ is given by the above formula. $\square$

Stembridge enumerated TSPPs in [38], but no one has completely proved the $q$-version of this formula. If one could prove that $\prod_{1\leq i\leq j\leq k\leq n-1} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}$ is the rank-generating function of $J(T_n)$, this would solve the last long-standing problem in the enumeration of the ten symmetry classes of plane partitions. Okada has written

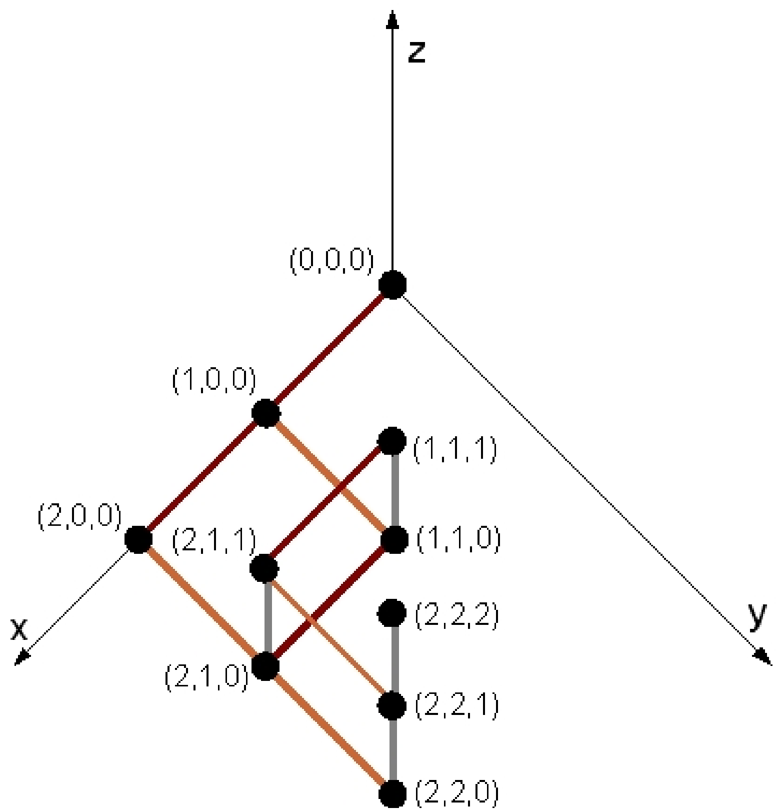



Figure 2.22: The correspondence between the TSPP fundamental domain $x \geq y \geq z$ and the tetrahedron $T_4$.

the $q$–enumeration of TSPPs as a determinant using lattice path techniques [31], but this determinant has proved difficult to evaluate. Recently Kauers, Koutschan, and Zeilberger outlined a method for evaluating this determinant if given a large amount of time on the supercomputer (but did not actually do the computation) [21]. Many people would be more satisfied with a proof not requiring such a massive number of computations; time will tell whether such a proof is attainable.

Another implication of the poset perspective is that TSPPs can be seen as the intersection of the arrays corresponding to ASMs and TSSCPPs for a particular choice of the TSSCPP arrays.

**Proposition 2.8.1.** *Totally symmetric plane partitions inside an* $(n-1) \times (n-1) \times (n-1)$ *box are in bijection with integer arrays corresponding to the intersection of* $n \times n$ *alternating sign matrices with totally symmetric self–complementary plane partitions inside a* $2n \times 2n \times 2n$ *box.*

*Proof.* TSPPs inside an $(n-1)\times(n-1)\times(n-1)$ box are in bijection with the integer arrays $Y_n(\{r, s, o, (g), (y), (b)\})$ which are the intersection of the arrays $Y_n(\{b, y, o, g\})$ with either $Y_n(\{r, b, (g), s\})$ or $Y_n(\{r, y, (g), s\})$. $Y_n(\{b, y, o, g\})$ is in bijection with $n \times n$ ASMs while $Y_n(\{r, b, (g), s\})$ and $Y_n(\{r, y, (g), s\})$ are both in bijection with TSSCPPs inside a $2n \times 2n \times 2n$ box. □

It is very strange to think of the set of totally symmetric plane partitions as being contained inside the set of totally symmetric self–complementary plane partitions, since TSSCPPs are TSPPs with the additional condition of being self–complementary. We must keep in mind that we are thinking about TSPPs in an $(n-1)\times(n-1)\times(n-1)$ box which is much smaller than the $2n \times 2n \times 2n$ box in which we are considering TSSCPPs.

## 2.9 Extensions to trapezoids

Zeilberger's original proof of the enumeration of ASMs in [43] proved something more general. He proved that the number of gog trapezoids equals the number of magog trapezoids, which in our notation means that the number of distinct arrays obtained by cutting off the first $k$ southwest to northeast diagonals of the ASM arrays $Y_n(\{b, y, o, g\})$ equals the number of distinct arrays obtained by cutting off the first $k$ southwest to northeast diagonals of the TSSCPP arrays $Y_n(\{r, y, o, g\})$. This was first conjectured by Mills, Robbins, and Rumsey [8]. This proof, however, did not enumerate gog and magog trapezoids, but only proved they are equinumerous. A counting formula for these trapezoids is still unknown.

Cutting off the first $k$ southwest to northeast diagonals of any set of arrays $Y(S)$ for $S$ an admissible set of colors corresponds to cutting off a corner of the tetrahedral poset $T_n$. In particular, recall from Section 2.2 that $T_n$ can be thought of as the poset which results from beginning with the poset $P_n$, overlaying the posets $P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ successively, and connecting each $P_i$ to $P_{i-1}$ by the orange, yellow, and silver edges. Also recall from Proposition 2.2.12 that the chains made up of the green edges of the copy of $P_j$ inside $T_n$ determine the entries on the $(j-1)$st diagonal of the arrays $X_n(S)$ and likewise the arrays $Y_n(S)$. Therefore cutting off the first $k$ diagonals of the arrays in $Y_n(S)$ corresponds to removing the posets $P_2$, $P_3$, $\ldots$, $P_{k+1}$ and the edges adjacent to them from the Hasse diagram of $T_n$. Call the poset obtained by this process the trapezoidal poset and denote this poset as $T_n^k$. For $S$ an admissible subset of the colors, let $T_n^k(S)$ be $T_n^k$ with only the colors of edges in $S$ (thus $T_n^0(S) = T_n(S)$).

A natural question, then, is what the trapezoidal version of the theorems on the tetrahedral poset may be. Zeilberger's result proves the following theorem relating the trapezoidal ASM and TSSCPP arrays.

**Theorem 2.9.1.**

$$|J(T_n^k(\{b, y, o, g\}))| = |J(T_n^k(\{r, y, o, g\}))|. \tag{2.9.1}$$

Besides this theorem, nothing else is known about $T_n^k(S)$ for other admissible subsets of colors $S$. Perhaps the attempt to generalize the results from this chapter to the trapezoidal case will find relationships between more combinatorial objects or help to find a bijection between ASMs and TSSCPPs.

# Chapter 3

# The polytope perspective

In this chapter we study alternating sign matrices from the polytope perspective. After a brief review of polytope terminology in Section 3.1, we define the ASM polytope in Section 3.2 and discuss its properties throughout the rest of the chapter, making many comparisons with the well–known Birkhoff polytope.

## 3.1 Background on polytopes

A *polytope* is a generalization to $\mathbb{R}^n$ of the familiar concept of a polygon in $\mathbb{R}^2$. One can define a polytope in two ways. The first definition is that a polytope is the convex hull of a finite set of points in $\mathbb{R}^n$. That is, if $K = \{v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^n$ (and $K \neq \emptyset$) then $conv(K) := \{\lambda_1 v_1 + \ldots + \lambda_k v_k : \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\}$ is a polytope. The second definition is that a polytope is the bounded intersection of finitely many closed halfspaces in $\mathbb{R}^n$. This means that $\{v \in \mathbb{R}^n : a_1 \cdot v \leq z_1, a_2 \cdot v \leq z_2, \ldots, a_k \cdot v \leq z_k\}$ for some $\{a_1, a_2, \ldots, a_k\} \subset \mathbb{R}^n$ and $\{z_1, z_2, \ldots, z_k\} \subset \mathbb{R}$ is a polytope. The main theorem for polytopes resolves the apparent ambiguity of this double definition by stating that these two definitions are equivalent; every polytope can be given an equivalent description in terms of points or inequalities [44].

## 3.2 ASM polytope definition

The Birkhoff polytope, which we will denote as $B_n$, has been extensively studied and generalized. It is defined as the convex hull of the $n \times n$ permutation matrices as vectors in $\mathbb{R}^{n^2}$. Many analogous polytopes have been studied which are subsets of $B_n$ (see e.g. [9]). In contrast, we study a polytope containing $B_n$. We begin with the following definition.

**Definition 3.2.1.** The $n$th alternating sign matrix polytope, which we will denote as $ASM_n$, is the convex hull in $\mathbb{R}^{n^2}$ of the $n \times n$ alternating sign matrices.

Permutation matrices are the alternating sign matrices whose entries are nonnegative. Thus $B_n$ is contained in $ASM_n$. In this chapter we find analogues for $ASM_n$ of the following theorems about the Birkhoff polytope (see the discussion in [9] and [42]).

- $B_n$ consists of the $n \times n$ nonnegative doubly stochastic matrices (square matrices with nonnegative real entries whose rows and columns sum to 1).
- The dimension of $B_n$ is $(n-1)^2$.
- $B_n$ has $n!$ vertices.
- $B_n$ has $n^2$ facets (for $n \geq 3$) where each facet is made up of all nonnegative doubly stochastic matrices with a 0 in a specified entry.
- $B_n$ projects onto the permutohedron.
- There exists a nice characterization of its face lattice in terms of elementary bipartite graphs [5].

As we shall see in Theorem 3.4.1, the row and column sums of every matrix in $ASM_n$ must equal 1. Thus the dimension of $ASM_n$ is $(n-1)^2$ because, just as for the Birkhoff polytope, the last entry in each row and column is determined to be precisely what is needed to make that row or column sum equal 1. In Section 3.5 we prove that

$ASM_n$ has $4[(n-2)^2+1]$ facets and its vertices are the alternating sign matrices. We also prove analogous theorems about the inequality description of $ASM_n$ (Section 3.4), the face lattice (Section 3.6), and the projection to the permutohedron (Section 3.5).

The alternating sign matrix polytope was independently defined in [4] in which the authors also study the integer points in the $r$th dilate of $ASM_n$ calling them *higher spin alternating sign matrices*.

## 3.3 Birkhoff Polytope

The main theorem about the Birkhoff polytope is the following theorem of Birkhoff [7] and von Neumann [41] which says that the Birkhoff polytope can be described not only as the convex hull of the permutation matrices but equivalently as the set of all nonnegative doubly stochastic matrices (real square matrices with row and column sums equaling 1 whose entries are nonnegative). In this section we outline von Neumann's proof of this result in [41]. In the next section we use the idea of this proof to prove the inequality description of the ASM polytope.

**Theorem 3.3.1** (Birkhoff–von Neumann)**.** *The convex hull of the $n\times n$ permutation matrices consists of all $n\times n$ real matrices $X$ satisfying:*

$$x_{ij} \geq 0 \qquad \forall\ 1\leq i,j\leq n. \tag{3.3.1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall\ 1\leq j\leq n. \tag{3.3.2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall\ 1\leq i\leq n. \tag{3.3.3}$$

*Such matrices are called nonnegative doubly stochastic matrices.*

*Proof.* It is easy to check that the convex hull of the permutation matrices is contained in the set of nonnegative doubly stochastic matrices. It is left, then, to show that any nonnegative doubly stochastic matrix can be written as a convex combination of permutation matrices.

Let $X$ be an $n \times n$ nonnegative doubly stochastic matrix. Using von Neumann's terminology, we call a real number $\alpha$ *inner* if $0 < \alpha < 1$. We construct a circuit of inner entries in $X$. Begin at any inner entry $x_{ij}$; if no such entry exists, then $X$ is permutation matrix. Since $x_{ij}$ is inner there exists another inner entry $x_{i'j}$ in column $j$ since the sum of all the entries in the column equals 1 and each matrix entry is nonnegative. Then since $x_{i'j}$ is inner, there exists another inner entry $x_{i'j'}$ in row $i'$. This process continues creating a path in $X$ with corner entries that are all inner. Since $X$ is of finite size, the path eventually reaches an entry in the same row or column as a previous entry yielding a circuit in $X$ whose corner entries are all inner. Using this circuit we can write $X$ as a convex combination of two nonnegative doubly stochastic matrices, each with at least one more 0 entry, in the following way.

Label the corners in the circuit alternately $(+)$ and $(-)$. Set $k'$ equal to the minimum of the entries labeled $(-)$ in the circuit. Thus $0 < k' \leq \frac{1}{2}$ since the corner entries are all inner and the rows and columns sum to 1. Subtract $k'$ from the entries labeled $(-)$ and add $k'$ to the entries labeled $(+)$. Subtracting and adding $k'$ in this way preserves the row and column sums and keeps all the entries weakly between 0 and 1, so the result of this process yields another nonnegative doubly stochastic matrix $X'$ with at least one more 0 entry than $X$ (in the position of the minimum $(-)$ corner entry).

Now give opposite labels to the corners in the circuit in $X$ and set $k''$ equal to the minimum of the corner entries now labeled $(-)$. Then subtract and add $k''$ in a similar way to obtain another nonnegative doubly stochastic matrix $X''$ with at least one more 0 entry than $X$. Then $X$ is a convex combination of $X'$ and $X''$, namely $X = \frac{k''}{k'+k''}X' + \frac{k'}{k'+k''}X''$. Therefore, by repeatedly applying this procedure, $X$ can be written as a convex combination of permutation matrices (i.e. matrices with no inner entries). $\square$

## 3.4 The inequality description of the ASM polytope

The inequality description of the alternating sign matrix polytope is similar to that of the Birkhoff polytope. It consists of the subset of doubly stochastic matrices (now allowed to have negative entries) whose partial sums in each row and column are between 0 and 1. The proof uses the idea of the proof of the inequality description of the Birkhoff polytope from [41] outlined in the previous section.

Note that in [4] Behrend and Knight approach the equivalence of the convex hull definition and the inequality description of $ASM_n$ in the opposite manner, defining the alternating sign matrix polytope in terms of inequalities and then proving that the vertices are the alternating sign matrices.

**Theorem 3.4.1.** *The convex hull of $n \times n$ alternating sign matrices consists of all $n \times n$ real matrices $X = \{x_{ij}\}$ such that:*

$$0 \leq \sum_{i=1}^{i'} x_{ij} \leq 1 \qquad \forall\ 1 \leq i' \leq n, 1 \leq j \leq n. \tag{3.4.1}$$

$$0 \leq \sum_{j=1}^{j'} x_{ij} \leq 1 \qquad \forall\ 1 \leq j' \leq n, 1 \leq i \leq n. \tag{3.4.2}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall\ 1 \leq j \leq n. \tag{3.4.3}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall\ 1 \leq i \leq n. \tag{3.4.4}$$

*Proof.* Call the subset of $\mathbb{R}^{n^2}$ given by the above inequalities $P(n)$. It is easy to check that the convex hull of the alternating sign matrices is contained in the set $P(n)$. It remains to show that any $X \in P(n)$ can be written as a convex combination of alternating sign matrices.

Let $X \in P(n)$. Let $r_{ij} = \sum_{j'=1}^{j} x_{ij}$ and $c_{ij} = \sum_{i'=1}^{i} x_{ij}$. Thus the $r_{ij}$ are the row partial sums and the $c_{ij}$ are the column partial sums. It follows from (3.4.1) and

(3.4.2) that $0 \leq r_{ij}, c_{ij} \leq 1$ for all $1 \leq i, j \leq n$. Also, from (3.4.3) and (3.4.4) we see that $r_{in} = c_{nj} = 1$ for all $1 \leq i, j \leq n$. If we set $r_{i0} = c_{0j} = 0$ we see that every entry $x_{ij} \in X$ satisfies $x_{ij} = r_{ij} - r_{i,j-1} = c_{ij} - c_{i-1,j}$. Thus

$$r_{ij} + c_{i-1,j} = c_{ij} + r_{i,j-1}. \tag{3.4.5}$$

Recall the terminology from the last section: a real number $\alpha$ is called *inner* if $0 < \alpha < 1$. As in the previous proof, we construct a circuit in $X$, but instead of the condition that the corner matrix entries be inner, we now require that between adjacent matrix entries in the circuit be an inner partial sum. So we rewrite the matrix $X$ with the partial sums between entries as shown below.

$$\begin{pmatrix}
 & c_{01} & & c_{02} & & & c_{0,n-1} & & c_{0n} & \\
r_{10} & x_{11} & r_{11} & x_{12} & r_{12} & & x_{1,n-1} & r_{1,n-1} & x_{1n} & r_{1n} \\
 & c_{11} & & c_{12} & & \cdots & c_{1,n-1} & & c_{1n} & \\
r_{20} & x_{21} & r_{21} & x_{22} & r_{22} & & x_{2,n-1} & r_{2,n-1} & x_{2n} & r_{2n} \\
 & & & & & & & & & \\
 & & \vdots & & & & & \vdots & & \\
 & c_{n-1,1} & & c_{n-1,2} & & & c_{n-1,n-1} & & c_{n-1,n} & \\
r_{n0} & x_{n1} & r_{n1} & x_{n2} & r_{n2} & \cdots & x_{n,n-1} & r_{n,n-1} & x_{nn} & r_{nn} \\
 & c_{n1} & & c_{n2} & & & c_{n,n-1} & & c_{nn} &
\end{pmatrix}$$

Begin at the vertex to the left or above any inner partial sum; if no such partial sum exists, then $X$ is an alternating sign matrix. Then there exists an adjacent inner partial sum by (3.4.5). By repeated application of (3.4.5) to each new inner partial sum, a path can then be formed by moving from entry to entry of $X$ along inner partial sums. Since $X$ is of finite size and all the boundary partial sums are 0 or 1 (i.e. non–inner), the path eventually reaches an entry in the same row or column as a previous entry yielding a circuit in $X$ whose partial sums are all inner. Using this circuit we can write $X$ as a convex combination of two matrices in $P(n)$, each with at least one more non–inner partial sum, in the following way.

$$
\begin{pmatrix}
0 & .4 & .5 & .1 & 0 \\
.4 & -.4 & .5 & 0 & .5 \\
.6 & .4 & -.3 & -.1 & .4 \\
0 & .3 & -.3 & .9 & .1 \\
0 & .3 & .6 & .1 & 0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & 0 & 0 & .4 & \mathbf{.4} & .5 & \mathbf{.9} & .1 & 1 & 0 & 1 \\
 & 0 & & \mathbf{.4} & & .5 & & \mathbf{.1} & & 0 & \\
0 & .4 & \mathbf{.4} & -.4 & 0 & .5 & .5 & 0 & .5 & .5 & 1 \\
 & \mathbf{.4} & & 0 & & 1 & & \mathbf{.1} & & .5 & \\
0 & .6 & \mathbf{.6} & .4 & 1 & -.3 & \mathbf{.7} & -.1 & .6 & .4 & 1 \\
 & 1 & & \mathbf{.4} & & \mathbf{.7} & & 0 & & .9 & \\
0 & 0 & 0 & .3 & \mathbf{.3} & -.3 & 0 & .9 & .9 & .1 & 1 \\
 & 1 & & .7 & & .4 & & .9 & & 1 & \\
0 & 0 & 0 & .3 & .3 & .6 & .9 & .1 & 1 & 0 & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 &
\end{pmatrix}
$$

Figure 3.1: A matrix in $P(n)$ along with the matrix rewritten with the partial sums between the entries and a circuit of inner partial sums shown in boldface red

Label the corner matrix entries in the circuit alternately $(+)$ and $(-)$. Define

$$k' = \min(r_{ij}, 1 - r_{i'j'}, c_{i''j''}, 1 - c_{i''',j'''})$$

where $r_{ij}$, $r_{i'j'}$, $c_{i''j''}$, and $c_{i''',j'''}$ are taken over respectively the row partial sums to the right of a $(-)$ corner along the circuit, the row partial sums to the right of a $(+)$ corner, the column partial sums below a $(-)$ corner, and the column partial sums below a $(+)$ corner. Subtract $k'$ from the entries labeled $(-)$ and add $k'$ to the entries labeled $(+)$. Subtracting and adding $k'$ in this way preserves the row and column sums and keeps all the partial sums weakly between 0 and 1 (satisfying (3.4.1)–(3.4.4)), so the result is another matrix $X'$ in $P(n)$ with at least one more non–inner partial sum than $X$.

Now give opposite labels to the corners in the circuit in $X$ and subtract and add another constant $k''$ in a similar way to obtain another matrix $X''$ in $P(n)$ with at least one more non–inner partial sum than $X$. Then $X$ is a convex combination of $X'$ and $X''$, namely $X = \frac{k''}{k'+k''}X' + \frac{k'}{k'+k''}X''$. Therefore, by repeatedly applying this procedure, $X$ can be written as a convex combination of alternating sign matrices (i.e. matrices of $P(n)$ with no inner partial sums). $\square$

## 3.5 Properties of the ASM polytope

Now that we can describe the alternating sign matrix polytope in terms of inequalities, let us use this inequality description to examine some of the properties of $ASM_n$, namely, its facets, its vertices, and its projection to the permutohedron.

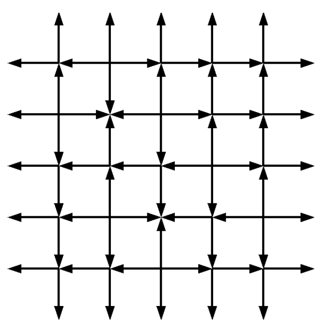
To make the proofs of the next two theorems more transparent, we introduce modified square ice configurations called *simple flow grids* which will be used more extensively in Section 3.6.

Consider a directed graph with $n^2 + 4n$ vertices: $n^2$ 'internal' vertices $(i, j)$ and $4n$ 'boundary' vertices $(i, 0)$, $(0, j)$, $(i, n + 1)$, and $(n + 1, j)$ where $i, j = 1, \ldots, n$. These vertices are naturally depicted in a grid in which vertex $(i, j)$ appears in row $i$ and column $j$. Define the *complete flow grid* $C_n$ to be the directed graph on these vertices with edge set $\{((i, j), (i, j \pm 1)), ((i, j), (i \pm 1, j))\}$ for $i, j = 1, \ldots, n$. So $C_n$ has directed edges pointing in both direction between neighboring internal vertices in the grid, and also directed edges from internal vertices to neighboring border vertices.

**Definition 3.5.1.** A *simple flow grid* of order $n$ is a subgraph of $C_n$ consisting of all the vertices of $C_n$ for which four edges are incident to each internal vertex: either four edges directed inward, four edges directed outward, or two horizontal edges pointing in the same direction and two vertical edges pointing in the same direction.

**Proposition 3.5.2.** *There exists an explicit bijection between simple flow grids of order $n$ and $n \times n$ alternating sign matrices.*

*Proof.* Given an ASM $A$, we will define a corresponding directed graph $g(A)$ on the $n^2$ internal vertices and $4n$ boundary vertices arranged on a grid as described above. Let each entry $a_{ij}$ of $A$ correspond to the internal vertex $(i, j)$ of $g(A)$. For neighboring vertices $v$ and $w$ in $g(A)$ let there be a directed edge from $v$ to $w$ if the partial sum from the border of the matrix to the entry corresponding to $v$ in the direction pointing toward $w$ equals 1. By the definition of alternating sign matrices, there will be exactly one directed edge between each pair of neighboring internal vertices and also a directed edge from an internal vertex to each neighboring border vertex. Vertices of $g(A)$ corresponding to 1's are sources and vertices corresponding to $-1$'s are sinks. The directions of the rest of the edges in $g(A)$ are determined by the placement of the 1's and $-1$'s, in that there is a series of directed edges emanating from the 1's and continuing until they reach a sink or a border vertex. Thus $g(A)$ is a simple flow grid. Also, given a simple flow grid we can easily find the corresponding ASM by replacing all the sources with 1's and all the sinks with $-1$'s. Thus simple flow grids are in one-to-one correspondence with ASMs (see Figure 3.2). $\square$



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 3.2: The simple flow grid—ASM correspondence

Simple flow grids are, in fact, almost the same as configurations of the six-vertex model of square ice with domain wall boundary conditions (see the discussion in [8]), the only difference being that the horizontal arrows point in the opposite direction.

Recall that for $n \geq 3$ the Birkhoff polytope has $n^2$ facets (faces of dimension one less than the polytope itself). ($B_2 = ASM_2$ is simply a line segment, so the number of facets equals the number of vertices which is 2.) Each facet of the Birkhoff polytope consists of all nonnegative doubly stochastic matrices with a zero in a fixed entry, that is, where one of the defining inequalities is made into an equality. The analogous theorem for $ASM_n$ is the following.

**Theorem 3.5.3.** *$ASM_n$ has $4[(n-2)^2+1]$ facets, for $n \geq 3$.*

*Proof.* Note that the $4n^2$ defining inequalities for $X \in ASM_n$ given in (3.4.1) and (3.4.2) can be restated as

$$\sum_{i'=1}^{i} x_{i'j} \geq 0 \qquad \sum_{j'=1}^{j} x_{ij'} \geq 0$$

$$\sum_{i'=i}^{n} x_{i'j} \geq 0 \qquad \sum_{j'=j}^{n} x_{ij'} \geq 0$$

for $i, j = 1, \dots, n$. We have rewritten the statement that the row and column partial sums from the left or top must be less than or equal to 1 as the row and column partial sums from the right and bottom must be greater than or equal to 0. By counting these defining inequalities, one sees that there could be at most $4n^2$ facets, each determined by making one of the above inequalities an equality. It is left to determine how many of these equalities determine a face of dimension less than $(n-1)^2 - 1$.

By symmetry we can determine the number of facets coming from the inequalities $\sum_{i'=1}^{i} x_{i'j} \geq 0$ for $i, j = 1, \dots n$ and then multiply by 4. Since the full row and column sums always equal 1, the equalities such as $\sum_{i'=1}^{n} x_{i'j} = 0$ yield the empty face ($i = n$). Also, $\sum_{i'=1}^{n-1} x_{i'j} \geq 0$ is implied from the fact that $x_{nj} \geq 0$ ($i = n-1$). The inequalities $x_{i'1} \geq 0$ for all $i'$, i.e. the entries in the first column are nonnegative, imply that $\sum_{i'=1}^{i} x_{i'1} \geq 0$ for $2 \leq i \leq n-1$ ($j = 1$), thus each of these sets is a face of dimension less than $(n-1)^2 - 1$, and similarly for $\sum_{i'=1}^{i} x_{i'n} \geq 0$ for $2 \leq i \leq n-1$ ($j = n$) the partial sums of the last column.

So we are left with the $(n-2)^2$ inequalities $\sum_{i'=1}^{i} x_{i'j} = 0$ for $i = 1, \dots n-2$ and $j = 2, \dots, n$ along with the inequality $x_{11} \geq 0$. For our symmetry argument to work, we do not include $x_{n1} \geq 0$ in our count since $x_{n1} \geq 0$ is also an inequality of the form $\sum_{j'=1}^{j} x_{ij'} \geq 0$.

Thus $ASM_n$ has at most $4[(n-2)^2+1]$ facets, given explicitly by the $4(n-2)^2+4$ sets of all $X \in ASM_n$ which satisfy one of the following:

$$\sum_{i'=1}^{i-1} x_{i'j} = 0, \sum_{j'=1}^{j-1} x_{ij'} = 0, \sum_{i'=i+1}^{n} x_{i'j} = 0, \sum_{j'=j+1}^{n} x_{ij'} = 0, i, j \in \{2, \ldots, n-1\}, \quad (3.5.1)$$

$$x_{11} = 0, \; x_{1n} = 0, \; x_{n1} = 0, \text{ or } x_{nn} = 0. \quad (3.5.2)$$

They are facets (not just faces) since each equality determines exactly one more entry of the matrix, decreasing the dimension by one.

Recall that a directed edge in a simple flow grid $g(A)$ represents a location in the corresponding ASM $A$ where the partial sum equals 1, thus a directed edge missing from $g(A)$ represents a location in $A$ where the partial sum equals 0. Thus we can represent each of the $4(n-2)^2$ facets of (3.5.1) as subgraphs of the complete flow grid $C_n$ from which a single directed edge has been removed: $((i \pm 1, j), (i, j))$ or $((i, j \pm 1), (i, j))$ with $i, j \in \{2, \ldots, n-1\}$. We can represent the facets of (3.5.2) as subgraphs of $C_n$ from which two directed edges have been removed: $((1,1),(1,2))$ and $((1,1),(2,1))$, $((1,n),(1,n-1))$ and $((1,n),(2,n))$, $((n,1),(n-1,1))$ and $((n,1),(n,2))$, or $((n,n),(n,n-1))$ and $((n,n),(n-1,n))$.

Now given any two facets $F_1$ and $F_2$, it is easy to exhibit a pair of ASMs $\{X_1, X_2\}$ such that $X_1$ lies on $F_1$ and not on $F_2$. Include the directed edge(s) corresponding to $F_2$ but not the directed edge(s) corresponding to $F_1$ in $g(X_1)$, then do the opposite for $X_2$. Thus each of the $4[(n-2)^2+1]$ equalities gives rise to a unique facet. $\square$

**Corollary 3.5.4.** *For $n \geq 3$, the number of facets of $ASM_n$ on which an ASM $A$ lies is given by $2(n-1)(n-2) +$ (number of corner 1's in $A$).*

*Proof.* Each 0 around the border of $A$ represents one facet. Thus the number of facets corresponding to border zeros of $A$ equals $4(n-1) -$ (# 1's around the border of $A$). Then there are $2(n-2)(n-3)$ facets represented by directed edges pointing in the opposite directions to the directed edges in the $(n-2) \times (n-2)$ interior array of $g(A)$. The sum of these numbers gives the above count. $\square$

Even though $ASM_n$ is defined as the convex hull of the ASMs, it requires some proof that each ASM is actually an extreme point of $ASM_n$.

**Theorem 3.5.5.** *The vertices of $ASM_n$ are the $n \times n$ alternating sign matrices.*

*Proof.* Fix an $n \times n$ ASM $A$. In order to show that $A$ is a vertex of $ASM_n$, we need to find a hyperplane with $A$ on one side and all the other ASMs on the other side. Then since $ASM_n$ is the convex hull of $n \times n$ ASMs, $A$ would necessarily be a vertex.

Consider the simple flow grid corresponding to $A$. In any simple flow grid there are, by definition, $2n(n+1)$ directed edges, where for each entry of the corresponding ASM there is a directed edge whenever the partial sum in that direction up to that point equals 1. Since the total number of directed edges in a simple flow grid is fixed, $A$ is the only ASM with all of those partial sums equaling 1. Thus the hyperplane where the sum of those partial sums equals $2n(n+1) - \frac{1}{2}$ will have $A$ on one side and all the other ASMs on the other. Thus the $n \times n$ ASMs are the vertices of $ASM_n$. $\square$

Another interesting property of the ASM polytope is its relationship to the permutohedron. For a vector $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with distinct entries, define the permutohedron $P_z$ as the convex hull of all vectors obtained by permuting the entries of $z$. That is,

$$P_z = \text{conv}\{(z_{\omega(1)}, z_{\omega(2)}, \ldots, z_{\omega(n)}) \mid \omega \in S_n\}. \tag{3.5.3}$$

Also, for such a vector $z$, let $\phi_z$ be the mapping from the set of $n \times n$ real matrices to $\mathbb{R}^n$ defined by

$$\phi_z(X) = zX, \text{for any } n \times n \text{ real matrix } X.$$

It is well known, and follows immediately from the definitions, that $P_z$ is the image of the Birkhoff polytope under the projection $\phi_z$.

**Proposition 3.5.6.** *Let $B_n$ be the Birkhoff polytope and $z$ be a vector in $\mathbb{R}^n$ with distinct entries. Then*

$$\phi_z(B_n) = P_z. \tag{3.5.4}$$

The next theorem states that when the same projection map is applied to $ASM_n$, the image is the same permutohedron whenever $z$ is a decreasing vector. For the proof of this theorem we will need the concept of majorization [26].

**Definition 3.5.7.** Let $u$ and $v$ be vectors of length $n$. Then $u \preceq v$ (that is $u$ is *majorized* by $v$) if

$$\begin{cases} \sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]}, & \text{for } 1 \leq k \leq n-1 \\ \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i \end{cases} \tag{3.5.5}$$

where the vector $(u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ is obtained from $u$ by rearranging its components so that they are in decreasing order, and similarly for $v$.

**Theorem 3.5.8.** *Let $z$ be a decreasing vector in $\mathbb{R}^n$ with distinct entries. Then*

$$\phi_z(ASM_n) = P_z. \tag{3.5.6}$$

*Proof.* It follows from Proposition 3.5.6 and $B_n \subseteq ASM_n$ that $P_z \subseteq \phi_z(ASM_n)$. Thus it only remains to be shown that $\phi_z(ASM_n) \subseteq P_z$.

Let $z$ be a decreasing $n$–vector (so that $z_i = z_{[i]}$) and $X = \{x_{ij}\}$ an $n \times n$ ASM. Then there is a proposition of Rado which states that for vectors $u$ and $v$ of length $n$, $u \preceq v$ if and only if $u$ lies in the convex hull of the $n!$ permutations of the entries of $v$ [32]. Therefore the proof will be completed by showing $zX \preceq z$. By Definition 3.5.7 we need to show

$$\sum_{j=1}^{k} (zX)_{[j]} \leq \sum_{j=1}^{k} z_j, \text{ for } 1 \leq k \leq n-1 \tag{3.5.7}$$

$$\sum_{j=1}^{n} (zX)_j = \sum_{j=1}^{n} z_j \tag{3.5.8}$$

where the $j$th component $(zX)_j$ of $zX$ is given by $\sum_{i=1}^{n} z_i x_{ij}$.

To verify (3.5.8) note that since $\sum_{j=1}^{n} x_{ij} = 1$,

$$\sum_{j=1}^{n} (zX)_j = \sum_{j=1}^{n} \sum_{i=1}^{n} z_i x_{ij} = \sum_{i=1}^{n} z_i \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{n} z_i.$$

To prove (3.5.7) we will show that $\sum_{j\in J}(zX)_j \leq \sum_{j=1}^{|J|} z_j$ given any $J \subseteq \{1,\ldots,n\}$, so that in particular $\sum_{j=1}^{|J|}(zX)_{[j]} \leq \sum_{j=1}^{|J|} z_j$.

We will need to verify the following:

$$\sum_{i=1}^{m}\sum_{j\in J} x_{ij} \leq \min(m,|J|) \qquad \forall\, m \in \{1,\ldots,n\}. \tag{3.5.9}$$

$$\sum_{i=1}^{n}\sum_{j\in J} x_{ij} = |J|. \tag{3.5.10}$$

To prove (3.5.9) note that

$$\sum_{i=1}^{m}\sum_{j\in J} x_{ij} = \sum_{j\in J}\sum_{i=1}^{m} x_{ij} \leq |J|$$

since $\sum_{i=1}^{m} x_{ij} \leq 1$. But also, since $\sum_{i=1}^{m} x_{ij} \geq 0$ and $\sum_{j=1}^{n} x_{ij} = 1$ we have that

$$\sum_{j\in J}\sum_{i=1}^{m} x_{ij} \leq \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = m.$$

To prove (3.5.10) observe,

$$\sum_{i=1}^{n}\sum_{j\in J} x_{ij} = \sum_{j\in J}\sum_{i=1}^{n} x_{ij} = \sum_{j\in J} 1 = |J|$$

since the columns of $X$ sum to 1. Therefore using (3.5.9) and (3.5.10) we see that

$$\begin{aligned}
\sum_{i=1}^{n} z_i x_{ij} &= \sum_{j\in J}\sum_{i=1}^{n} z_i x_{ij} = \sum_{i=1}^{n} z_i \sum_{j\in J} x_{ij} = \sum_{k=1}^{n-1}(z_k - z_{k+1})\sum_{i=1}^{k}\sum_{j\in J} x_{ij} + z_n \sum_{i=1}^{n}\sum_{j\in J} x_{ij} \\
&= \sum_{k=1}^{n-1}(z_k - z_{k+1})\sum_{i=1}^{k}\sum_{j\in J} x_{ij} + z_n|J| \qquad \text{(by (3.5.10))} \\
&= \sum_{k=1}^{|J|-1}(z_k - z_{k+1})\sum_{i=1}^{k}\sum_{j\in J} x_{ij} + \sum_{k=|J|}^{n-1}(z_k - z_{k+1})\sum_{i=1}^{k}\sum_{j\in J} x_{ij} + z_n|J| \\
&\leq \sum_{k=1}^{|J|-1}(z_k - z_{k+1})k + \sum_{k=|J|}^{n-1}(z_k - z_{k+1})|J| + z_n|J| \qquad \text{(by (3.5.9))} \\
&\leq \sum_{k=1}^{|J|-1} z_k - z_{|J|}(|J|-1) + (z_{|J|} - z_n)|J| + z_n|J| \\
&\leq \sum_{k=1}^{|J|} z_k.
\end{aligned}$$

Thus $zX \preceq z$ and so $zX$ is contained in the convex hull of the permutations of $z$. Therefore $\phi_z(ASM_n) = P_z$. □

## 3.6 The face lattice of the ASM polytope

Another nice result about the Birkhoff polytope is the structure of its face lattice [5]. Associate to each permutation matrix $X$ a bipartite graph with vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ where there is an edge connecting $u_i$ and $v_j$ if and only if there is a 1 in the $(i, j)$ position of $X$. Such a graph will be a perfect matching on the complete bipartite graph $K_{n,n}$. A graph $G$ is called elementary if every edge is a member of some perfect matching of $G$.

**Theorem 3.6.1** (Billera–Sarangarajan)**.** *The face lattice of the Birkhoff polytope is isomorphic to the lattice of elementary subgraphs of $K_{n,n}$ ordered by inclusion.*

This lattice structure was first identified by Billera and Sarangarajan in [5] and [6], but the set of faces itself was first characterized and studied extensively by Brualdi and Gibson in [10] and [11] using certain 0-1 matrices which correspond trivially to elementary subgraphs of $K_{n,n}$. Other relevant results were also obtained by Balinski and Russakoff in [2] and [3].

A similar statement can be made about the face lattice of the ASM polytope using simple flow grids (see Definition 3.5.1) in place of perfect matchings, the complete flow grid $C_n$ instead of the complete bipartite graph $K_{n,n}$, and *elementary flow grids* in place of elementary graphs.

**Definition 3.6.2.** An *elementary flow grid $G$* is a subgraph of the complete flow grid $C_n$ such that the edge set of $G$ is the union of the edge sets of simple flow grids.

Now for any face $F$ of $ASM_n$ define the grid corresponding to the face, $g(F)$, to be the union over all the vertices of $F$ of the simple flow grids corresponding to the vertices. That is,

$$g(F) = \bigcup_{vertices\ A \in F} g(A).$$

Thus $g(F)$ is an elementary flow grid since its edge set is the union of the edge sets of simple flow grids.

Now we wish to define the converse, that is, given an elementary flow grid $G$ we would like to know the corresponding face $f(G)$ of $ASM_n$. Define $f(G)$ to be the convex hull of the vertices of $ASM_n$ whose corresponding simple flow grids are contained in the elementary flow grid $G$. So let

$$f(G) = \text{conv}\{\text{vertices } A \in ASM_n \mid g(A) \subseteq G\}.$$

Recall that we can represent each of the facets of $ASM_n$ either as subgraphs of the complete flow grid $C_n$ from which one of the directed edges in the set $S = \{((i \pm 1, j), (i, j)), ((i, j \pm 1), (i, j)) \mid i, j \in \{2, \ldots, n-1\}\}$ has been removed or from which one of the pairs of directed edges in the set $T = \{\{((1,1)(1,2)), ((1,1),(2,1))\}, \{((1,n),(1,n-1)), ((1,n),(2,n))\}, \{((n,1),(n-1,1)), ((n,1),(n,2))\}, \{((n,n),(n,n-1)), ((n,n),(n-1,n))\}\}$ has been removed.

Thus each of the directed edges in $S$ and the first of each pair of directed edges in $T$ that are not in $G$ represent facets that contain $f(G)$. Let the collection of these directed edges be called $\{e_1, e_2, \ldots, e_k\}$ and their corresponding facets $\{F_1, F_2, \ldots, F_k\}$. Let $I = \bigcap_{j=1}^{k} F_j$ be the intersection of these facets. Thus $I$ is a face of $ASM_n$ and $f(G) \subseteq I$.

We wish to show that $f(G)$ equals $I$. So suppose $f(G) \subsetneq I$. Then since $I$ is a face of $ASM_n$ and $f(G)$ is defined as the convex hull of vertices of $ASM_n$ there exists an additional vertex $B \in I$ of $ASM_n$ such that $B \notin f(G)$. But $g(B)$ must be missing the directed edges $e_1, e_2, \ldots, e_k$ since $B \in I$, thus all the directed edges of $g(B)$ must be in $G$. Therefore $g(B) \subseteq G$ so that $B \in f(G)$ which is a contradiction. So $f(G) = I$. Thus $f(G)$ is a face of $ASM_n$ since it is the intersection of faces of $ASM_n$.

It can easily be seen that $f(g(F)) = F$ and $g(f(G)) = G$. Also if $F_1$ and $F_2$ are faces of $ASM_n$ then $F_1 \subseteq F_2$ if and only if $g(F_1) \subseteq g(F_2)$.

Thus elementary flow grids are in bijection with the faces of $ASM_n$ (if we also regard the empty grid as an elementary flow grid). Elementary flow grids can be made into a lattice by inclusion, where the join is the union of the edge sets and the meet is the largest elementary flow grid made up of the directed edges from the intersection of the edges sets.

This discussion yields the following theorem:

**Theorem 3.6.3.** *The face lattice of $ASM_n$ is isomorphic to the lattice of all $n \times n$ elementary flow grids (or equivalently all $n \times n$ square ice configurations with domain wall boundary conditions) ordered by inclusion.*

The dimension of any face of $ASM_n$ can be determined by looking at $g(F)$ as in the following theorem. The characterization of edges of $ASM_n$ is analogous to the result for the Birkhoff polytope which states that the graphs representing edges of $B_n$ are the elementary subgraphs of $K_{n,n}$ which have exactly one cycle [3] [5] [11].

Given an elementary flow grid $G$, define a *doubly directed region* as a collection of cells in $G$ completely bounded by double directed edges but containing no double directed edges in the interior (see Figure 3.3). Let $\alpha(G)$ denote the number of doubly directed regions in $G$.

**Theorem 3.6.4.** *The dimension of a face $F$ of $ASM_n$ is the number of doubly directed regions in the corresponding elementary flow grid $g(F)$. In particular, the edges of $ASM_n$ are represented by elementary flow grids containing exactly one cycle of double directed edges.*

*Proof.* We proceed by induction on the dimension of the face of $ASM_n$. The simple flow grid corresponding to any ASM $A$ has no double directed edges, thus $\alpha(g(A)) = 0$. Now suppose for every $m$–dimensional face of $ASM_n$, the number of doubly directed

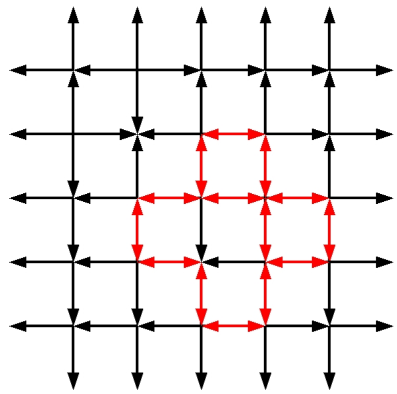


Figure 3.3: An elementary flow grid containing 3 doubly directed regions which corresponds by Theorem 3.6.4 to a 3-dimensional face of $ASM_5$

regions of the elementary flow grid corresponding to the face equals $m$. Let $F$ be an $(m+1)$–dimensional face of $ASM_n$ and $F'$ an $m$–dimensional subface of $F$. We assume $\alpha(g(F')) = m$ and wish to show that $\alpha(g(F)) = m + 1$.

Now $g(F)$ is the elementary flow grid whose edge set is the union of the edge sets of $g(F')$ and $g(A)$ over all ASMs $A$ in $F - F'$. Every vertex in a simple flow grid must have even indegree and even outdegree. Therefore, if we wish to obtain $g(A)$ from $g(A')$, where $A'$ is an ASM in $F'$, by reversing some directed edges, the number of directed edges reversed at each vertex must be even. Thus taking the union of the directed edges of $g(A')$ with the directed edges of $g(A)$ forms one or more circuits of double directed edges, where at least one of the double directed edges is not in $g(F')$. Therefore $g(F)$ has at least one more doubly directed region than $g(F')$, so $\alpha(g(F)) \geq m + 1$. Then since $g(ASM_n)$ equals the complete flow grid $C_n$, we have that $\alpha(g(ASM_n)) = \alpha(C_n) = (n-1)^2 = \dim(ASM_n)$. Therefore moving up the face lattice one rank increases the number of doubly directed regions by exactly one, so $\alpha(g(F)) = m + 1$. $\square$

See Figure 3.4 for the elementary flow grid representing the edge in $ASM_5$ between

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

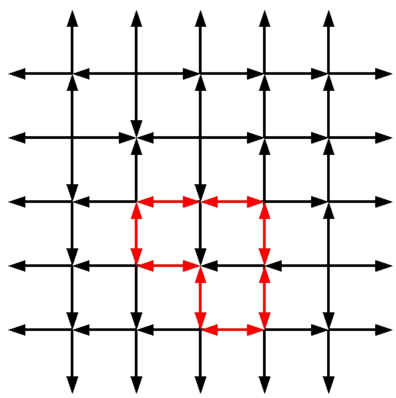


Figure 3.4: The elementary flow grid representing an edge in $ASM_5$

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