Two Essays in Financial Economics

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To my Loving Parents

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Abstract

This dissertation consists of two essays. In the first essay I investigate the optimal consumption-portfolio choice of an agent who exerts effort to earn labor income and has the option to retire from work. The decision to retire is irreversible: the agent cannot join the labor force again, once the retirement option has been exercised. I show that the optimal time to retire is when the wealth level reaches a threshold. As the wealth level approaches the threshold, the investor decreases consumption and increases investment in the risky assets. The investment in the risky assets jumps down at retirement. The threshold level of wealth is higher when the agent can choose how much effort to exert than when a constant amount of effort has to be exerted.

The issue of using margin policies as a tool to control stock volatility is highly debated among academics and practitioners. In particular, the empirical literature is divided on the issues of the effects of margin requirements on asset prices and volatility. In the second essay I investigate the effects of portfolio constraints such as margin requirements on stock price, stock returns volatility and interest rate. I consider a pure exchange, continuous time economy with portfolio constraints and investors with heterogenous risk aversion. The investor with portfolio constraints is assumed to have logarithmic utility with constant risk aversion while the unconstrained investor has time varying risk aversion. I show that in equilibrium, equity Sharpe ratio is higher, interest rate and stock returns volatility are lower in an economy with portfolio constraints as compared to an economy without constraints. I perform welfare analysis and show that unconstrained agent is better off while constrained agent is worse off when the constraint binds.

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Chapter 1

Introduction

”The market crash has erased more than \$3 trillion in retirement savings in 2008. But working a little past your planned retirement date can go a long way toward repairing the damage. For every extra year of work, retirement income from Social Security and investment returns can rise as much as 6.4%. Social Security benefits are less generous if you take them before full retirement age, but you get a bonus for every year you wait after that.” (Wall Street Journal, December 7, 2008)

The recent downturn in the financial markets has put a huge toll on the retirement savings of American workers. The fluctuations in the stock market affects the households primarily through the retirement savings. During the boom period of 1995-2000, when stock market was at its helm, the retirement savings increased sufficiently to enable many workers to opt for early retirement. In fact, one of the motivations for households to own stocks is to increase the retirement savings and bring retirement closer. This strategy works well in good times, when the stock market is soaring. The obvious downside of investing in the stock market is that, in bad times like the recent turmoil in the financial markets, retirement savings can hit the bottom if not wiped out completely. With a planned retirement date looming ahead, this can be a matter of serious concern for the households because there is not enough time to repair the damage. Once the worker retires, labor income is completely lost. Therefore the only way to increase the post-retirement wealth is to invest in the financial markets judiciously. The above quote lays down an alternative strategy. Instead of a planned retirement date, if the worker keeps the retirement date flexible then there is a possibility that the loss in retirement savings can be recouped. Thus, when the economy is not performing well, it may be optimal to defer the retirement date and keep the working hours more flexible. The main purpose of this paper is to investigate the optimal consumption, savings and investment choices when the worker has the option to retire and can choose the amount of effort to be exerted.

The decision to retire may be considered as one of the landmark decisions that an individual ever makes in a lifetime. This decision has tremendous economic impact on the standard of living of the individual after retirement. Moreover, retirement of the working labor force has a significant effect on the economy as a whole. Therefore it is important to understand what economic factors trigger voluntary retirement and how the possibility to retire voluntarily affects the consumption and portfolio choices of an agent. Intuitively, the decision to retire from work comes at a time when the agent has accumulated sufficient wealth to finance future consumption, and the benefit of the utility from leisure far more outweighs the cost of labor income that is forgone. So retirement would occur early if there are retirement benefits like the pension funds or Social Security benefits. On the other hand, retirement decisions may be delayed if there is expectation that labor income would grow in the future. The inclusion of optimal retirement time as another decision variable to the standard portfolio choice problem would provide different and useful insights on the investing behavior of the agents.

In the second chapter, we study the consumption and portfolio choice behavior of an agent who can choose the labor supply, albeit at a cost, and can make voluntary retirement decision. The agent generates income by exerting effort while employed. The amount of income earned while working is proportional to the amount of effort that the agent chooses to exert and also on the skill or productivity of the agent. A worker with higher productivity need to exert less amount of effort to earn the same income than a worker with lower productivity. The agent can choose the amount of effort to exert. Although a higher effort level will generate more labor income, exerting effort is costly to the agent. While working, the agent is losing the enjoyment of leisure and the higher is the effort level, the higher is the opportunity cost of leisure. It is intuitive that the agent will choose to work in the early stage of life and accumulate wealth by exerting effort. Once the wealth level reaches a threshold, the individual may want to retire and enjoy leisure only. The analysis of the option to retire is the main focus of this paper. In particular, we study the optimal portfolio and consumption policies of the agent before and after retirement assuming that retirement is an irreversible decision. That is, the agent is not allowed to re-join the labor force after retirement.

We assume that the agent has preferences represented by constant relative risk aversion utility function. The agent derives utility from consumption and can choose the amount of effort to exert. Once the agent decides to retire, he stops exerting effort for the remaining lifetime. But if he decides to work, he has to exert at least a minimum amount of effort. This assumption is realistic because every employed person must work for a minimum number of hours every day for the employer. We also assume that the maximum amount of effort that can be exerted by the agent is exogenously specified. Again this assumption is realistic because every employed person has only a maximum number of hours to work every day. With this specification of the effort level, we analyze the optimal time for the agent to retire. We show that there exists a threshold level of wealth such that the agent decides to retire once the wealth level exceeds this threshold. This threshold level depends on the minimum and maximum level of effort that the agent can exert and also on the productivity of the worker. For a highly skilled worker, the threshold level will be higher. The threshold will also be higher if the maximum limit on the effort level is increased or if the minimum limit is decreased.

We compare the optimal portfolio policies of an agent who has the option to retire with that of an agent who works forever. We do this by computing the optimal policies of both agents at the same level of wealth. We find that before retirement, the agent who has the option to retire invests more in the risky security than the agent with no option to retire. The threshold level is the level of wealth at which it is optimal for the agent to retire. As the wealth level approaches the threshold, the agent with option to retire increases his portfolio holdings. Once the wealth level reaches the threshold, the agent exercises the option to retire and the portfolio holdings jump down below the portfolio holdings of the agent who has no option to retire. At wealth levels above the threshold, the retired agent will invest less in the risky security than the agent who keeps working. The reason is intuitive. The agent chooses to retire once sufficient amount of wealth has been accumulated. In order to retire early, the agent is willing to take more risk and invests aggressively in the risky security than the agent who does not retire. Moreover, as the retirement time approaches, the agent knows that labor income will stop once the retirement option is exercised. This creates incentive to take on even more risk in portfolio holdings just before retirement, so that the agent can retire early with a greater amount of wealth. After retirement, the agent invests less in the risky security.

Next, we compare the consumption policies at the same level of wealth. We find that the agent with retirement option consumes less than the agent who does not retire. As the threshold level approaches, the agent who is about to retire consumes less and invests more of his wealth in the risky security compared to the agent who keeps working.

Finally, we consider a model where the agent has the option to retire but no flexibility in choosing labor supply. The agent has to exert a constant amount of effort which is costly and choose a retirement date after which he cannot work. We also consider the effects of borrowing constraints on the optimal consumption and portfolio policies. The agent is more cautious in investing in the stock when subject to borrowing constraints because a downturn in the financial markets will make the constraints bind sooner if most of the wealth is invested in the stock. Moreover, the threshold at which the agent retires is lower in the presence of borrowing constraints. The constraint inhibits the agent to choose the amount of effort that would be optimal, and at any given level of financial wealth, the human capital is lower in the presence of the constraint than without. The inability to work optimally when borrowing constraints are present results in a lower proportion of total wealth invested in the stock, compared to the case when there are no borrowing constraints.

In the third chapter, we analyze the effects of portfolio constraints on stock returns and volatility. The effects of portfolio constraints such as margin requirements has received considerable attention in the empirical literature following the stock market crash in October 1987. The issue of using margin policies as a tool to control stock volatility is highly debated. Among the prominent policy makers, the former Chairman of Federal Reserve Bank Alan Greenspan did not support the hypothesis that tighter margins can reduce stock price variability. A consensus was also lacking among the academics. The influential papers by Hardouvelis [27], [28] shortly after the crash furnished evidence that there is a negative relationship between margin requirements and stock market volatility. Subsequent empirical papers such as Ferris and Chance [25], Schwert [49] and Hsieh and Miller [33] argued that Hardouvelis' methodology is flawed and asserted that there is no support in the data to believe that a negative relationship exists between margin and volatility. Recently, Hardouvelis [29] has used sophisticated econometric methods to reaffirm his conclusions made more than a decade ago. But the issue is not settled yet. Among the theoretical work, Kupiec and Sharpe [39] developed an equilibrium model where the presence of irrational traders can induce excess volatility in the stock market. They claimed that there is no definite relation between margin requirement and stock market volatility: depending upon the heterogeneity of the agents, it is equally possible that margin requirements decrease or increase volatility.

The purpose of the third chapter is to investigate the effects of portfolio constraints such as margin requirements on stock price, stock returns volatility and interest rate. As mentioned above, the empirical literature is divided on the issues of the effects of margin requirements on the asset prices and volatility. We analyze such effects in a pure exchange, continuous time economy with portfolio constraints where agents are heterogeneous in their risk aversion. We assume there are two types of agents with different risk aversion. The first agent has logarithmic preferences in the difference between consumption and a habit level, the latter evolving over time. The second agent has standard logarithmic preferences over consumption. The dynamics of evolution of the habit level of consumption is similar to the dynamics of habit formation as in the models of Campbell and Cochrane [12] and Menzly, Santos and Veronesi [43]. The preferences of the first agent generates a time varying counter-cyclical risk aversion that increases during recessions and decreases during booms.

We make the assumption that the first agent is constrained in portfolio investment while the second agent is unconstrained. We find that in equilibrium, the stock Sharpe ratio is higher, the interest rate and stock returns volatility are lower in an economy with portfolio constraints compared to an economy without portfolio constraints. The constrained agent cannot invest more than a stipulated portion of his wealth in the stock market. To clear the stock market, the unconstrained agent has to buy the stock freed up by the constrained agent. To induce the unconstrained agent to absorb the excess supply of stock three things can happen: the stock provides a higher risk premium to make them more attractive; the interest rate decreases so that the unconstrained agent can finance the purchase of the stock by borrowing at a low rate; the stock returns volatility decrease to make the stock less risky. In the constrained economy, the constrained agent becomes a net lender, taking short position in the bond. To induce the unconstrained agent to be a borrower, the equilibrium interest rate decreases. The Sharpe ratio also increases and this boosts up the unconstrained agent's demand for the stock. The stock price therefore rises and stock returns fall. The increase in the Sharpe ratio is brought about by simultaneous decrease in the the stock returns volatility.

Lastly, we compare the agents' welfare in the constrained economy with that in the unconstrained economy. We find that the unconstrained agent's welfare increases and the constrained agent's welfare decreases when the constraint binds.

Chapter 2

Retirement Option with Flexible Labor Supply and Borrowing Constraints

2.1 Introduction

The optimal portfolio choice problem in continuous time with one risk-free and one risky asset is now well understood. Since the seminal studies of Samuelson [48] and Merton [44, 45], there has been a vast volume of work undertaken on this topic. One of the important conclusions from this literature is that a constant fraction of wealth must be invested in the risky asset. This conclusion is obtained in models where investors have preferences represented by constant relative risk aversion (CRRA) utility functions and there is no labor income. Wealth is generated only through investment in the risky asset and a risk-free money market account. In reality, people work while they are young to generate labor income. Thus in the presence of labor income, the wealth of the agent has to be identified as either financial wealth, which is the wealth accumulated through investment in the financial markets, and human capital wealth which is accumulated through exerting effort and earning labor income. It is expected that in the presence of human capital wealth, the portfolio choices may be different from those predicted by the earlier studies which ignored labor income.

Indeed, a number of recent papers, for example, Ameriks and Zeldes [1], Heaton and Lucas [31], Campbell, Cocco, Gomes, and Maenhout [11], Cocco, Gomes, and Maenhout [15], and Benzoni, Collin-Dufresne, and Goldstein [5] have concluded that the portfolio holdings of an agent who lives for a finite time period follow a hump-shaped pattern over their life-cycle. These studies show the importance of incorporating labor income in a standard portfolio choice model.

Generally, agents derive utility from both leisure and consumption. Leisure can be obtained only at the cost of forgoing labor income. Since labor income constitutes a major portion of the agent's wealth, it affects the consumption and portfolio policies in a significant way. Thus, modeling the choice of labor supply may provide richer insights into the consumption and investing behavior of the agent. One of the early studies that investigated the implications of endogenous choice of leisure and labor is Bodie, Merton, and Samuelson [9]. The authors assumed that the agent can choose their labor supply continuously. They showed that the agent takes more risk in investing in risky securities when labor supply can be chosen flexibly than without flexible labor choice. As the authors mention in their paper, the assumption that workers can adjust their labor supply without cost is less realistic. In this paper, we extend their model to consider labor supply that is costly to the worker and also the ability of workers to decide when to retire from work.

The paper by Cvitanic, Goukasian and Zapatero [17] investigates the portfolio choice problem of an investor who can choose to supply labor. These authors make the assumptions that the amount of effort that the agent can exert is bounded both above and below. Their main conclusion is that in a finite horizon model, the optimal allocation to the risky asset as a function of the investment horizon is very different when the investor maximizes utility from terminal wealth than when she maximizes utility from inter-temporal consumption. The authors did not consider the problem with the retirement option. One of the early papers to study the optimal consumption and portfolio choice of an agent in the presence of flexible labor supply is by Bodie, Merton, and Samuelson [9]. The authors assume that the agent can choose their labor supply continuously. They show that the agent takes more risk in investing in risky securities when labor supply can be chosen flexibly. Bodie et.al. [8] have analyzed the effects of habit formation on the optimal consumption and portfolio choice of the agent who has to retire at a fixed date. But these authors did not consider retirement as an option for the agent which is the main focus of our paper. Sundaresan and Zapatero [52] study the consumption-portfolio choice of the agent who accumulates wealth and chooses the optimal time to retire; at the retirement date the agent receives a one-time retirement payment. But the authors did not consider flexible labor supply as we do in our model. Karatzas and Wang [36] used the martingale method to study the optimal consumption-portfolio choice problem of an agent who has the discretion to stop consuming and investing in the markets at or before a pre-specified time. The authors did not consider labor income in their model. Farhi and Panageas [24] used the methodology of Karatzas and Wang [36] to study the problem of an agent whose utility is non-separable in leisure and consumption. The authors show that agents who want to retire early increase their savings and take more risk in financial investments. The authors also show that as the retirement age approaches, the agent increases the proportion of financial wealth invested in the risky assets. Dybvig and Liu [22] also study a similar model with the option to retire. Lachance [40] and Choi and Shim [13] uses the dynamic programming method to solve the optimal retirement problem of an agent whose utility is separable in leisure and consumption. Choi, Shim and Shin [14] studied the optimal portfolio, consumption and retirement problem of an infinitely lived agent with CES (constant elasticity of substitution) utility function. In these papers, the authors assume that the agent can either work or decide to retire. Thus, the agent does not choose how much labor to supply: the agent can only choose whether to work full time or retire. Moreover, these papers do not consider that labor supply is costly which is the main focus of our paper.

In contrast to these papers, the agent in our model has to decide whether to work or to retire.

If he decides to work, then he can choose the amount of effort to be exerted and there is a cost to exerting effort. Moreover, the labor supply in our model is flexible only when it is between an upper bound L_2 and a lower bound L_1. An agent who wants to exert effort less than L_1 has only two options: either to keep exerting L_1 amount of effort or to retire. On the other hand, the agent is not allowed to work and exert more than L_2 amount of effort. Liu and Neis [42] investigate the optimal consumption and portfolio choice problem when workers can choose their labor supply endogenously and assume that the stock price can never fall below a positive level. Besides this strong assumption, these authors did not consider the cost of exerting effort. In a continuous-time general equilibrium framework, Basak [3] studies the effect of labor income on asset pricing.

2.2 Model

We consider a continuous time, infinite-horizon economy with a single agent. Consider a complete probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of states of nature, $\mathcal{F}$ a σ-algebra of observable events and P a probability measure on $(\Omega, \mathcal{F})$. A one-dimensional Brownian motion process in on $(\Omega, \mathcal{F}, P)$, denoted by W. The flow of information is given by the filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$ of σ-algebras of $\mathcal{F}$ generated by W. We use the notation E_t for $E^P(.|\mathcal{F}_t)$, the conditional expected value under P.

The agent can invest in two financial securities. The first security is the money market account which is instantaneously riskless, pays no dividends, and sells for $M_t = M_0 e^{\int_0^t r ds}$ at time t, where r is the positive, constant instantaneously risk-free rate and M_0 is a strictly positive real number. The instantaneous rate of return on this asset is

$$\frac{dM_t}{M_t} = rdt \tag{2.2.1}$$

The agent can also invest in a risky asset, namely a stock. The stock pays dividend at the rate of D_t and sells for S_t per share at time t. We assume that the price process for the stock plus its dividend follows a geometric Brownian motion.

$$dS_t + dD_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2.2}$$

where $\mu > 0, \sigma > 0$ and W_t is a one-dimensional standard Brownian motion on a complete probability space (Ω, F, P). The market price risk (Sharpe ratio) of the stock is defined as $\theta = \frac{\mu - r}{\sigma}$.

The financial market is frictionless and investment in the financial securities is unconstrained. The wealth of the agent at time t is denoted by X_t. The agent invests π_t amount of dollars in the risky asset at time t. $X_t - \pi_t$ is then the amount invested in the money market. π_t is $\mathcal{F}_t$-measurable such that $\int_0^t \pi_s^2 ds < \infty$ for all $t \geq 0$, almost surely (a.s.). At time t the agent consumes an amount c_t. The agent's consumption process c_t is progressively measurable with respect to $\mathcal{F}_t$, $c_t \geq 0$ for all $t \geq 0$ a.s. and $\int_0^t c_t ds < \infty$ for all $t \geq 0$ a.s.

In addition to investing in the financial markets, the agent can choose to work and earn labor income. We denote the amount of effort exerted by the agent at the time t by y_t. The agent gets disutility by exerting effort but she is compensated by the wage she receives. We assume that the wage rate is fixed and normalized to one. At time t, if the agent exerts y_t amount of effort, then the cost of exerting this effort is $\frac{1}{2}y_t^2$ whereas the agent receives income of the amount δy_t dollars. The constant parameter δ represents the skill or productivity of the agent. By exerting the same amount of effort, the agent with high δ can earn more income than the agent with low δ. We assume that

$$0 < L_1 \leq y_t \leq L_2 \tag{2.2.3}$$

The motivation of this constraint on the amount of effort can be understood as follows. The agent has a maximum number of 24 hours to work, so there is an upper limit on the amount of effort that can be exerted by the agent, the maximum limit being L_2. On the other hand, the agent values leisure because effort is costly and provides disutility. But enjoying only leisure is not possible for a person who is employed, who has to work for a minimum number of hours in the office.The minimum level of effort that has to be exerted by the agent in order to remain the labor force is represented by L_1.

The main feature of our model is that the agent can choose whether she wants to be employed, in which case she has to exert an amount of effort $y_t \in [L_1, L_2]$; or she can choose to retire and exert no effort. The decision to retire is irreversible. That is, once the agent decides to retire she cannot join the labor force later. While employed, the agent can choose the amount of effort y_t she wants to exert such that $L_1 \leq y_t \leq L_2$. From the instant she chooses to get retired, she does not exert any effort and does not receive any labor income. The agent chooses her retirement time τ which is assumed to be an $\mathcal{F}$-stopping time.

We assume that the agent is initially endowed with an amount of wealth $x \geq -\frac{\delta L_2}{r}$. $\frac{\delta L_2}{r}$ is the present value of future income of the agent if she exerts the maximum level of effort L_2. When the agent is working, the wealth process X_t is defined by the initial condition $X_0 = x$ and evolves according to the following dynamics:

$$dX_t = \delta y_t dt + \frac{\pi_t}{S_t}(dS_t + D_t dt) + \frac{(X_t - \pi_t)}{M_t} dM_t \tag{2.2.4}$$

The first term on the right-hand side is the amount of income earned by exerting y_t units of effort. π_t is the amount of dollars invested in the stock; $\frac{\pi_t}{S_t}$ is the number of shares of stock held by the investor. So the change in wealth due to capital gains or loss and the continuously paying dividends of the stock is expressed by this second term. $X_t - \pi_t$ is the amount of dollars invested in the money market and earns a constant interest rate of $r > 0$. Substituting the dynamics (2.2.1) and (3.2.3), the wealth process during the working period, follows the dynamic budget constraint:

$$dX_t \;=\; \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t) r dt + (\delta y_t - c_t) dt \tag{2.2.5}$$

The consumption-portfolio effort plan (c_t, π_t, y_t) is called admissible until the stopping time τ, if starting from the initial wealth $X_0 = x$, the corresponding wealth process X_t of (2.2.5) satisfies

$X_t \geq -\frac{\delta L_2}{r}$ for $0 \leq t < \tau$ and $X_\tau \geq 0$. After retirement, the wealth process X_t satisfies $X_t \geq 0$ for all $t \geq \tau$ a.s. and follows the dynamics

$$dX_t = \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt - c_t dt$$

2.2.1 Preferences and Optimization Problem

The agent has utility function of the form

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \;\; \gamma > 0$$

where c_t is per period consumption. The agent has to choose the optimal processes c_t, π_t, y_t and the stopping time τ that maximize expected utility:

$$\underset{c_t,\pi_t,y_t,\tau}{Max}\, E\left[\int_0^\tau e^{-\beta t}\left(u(c_t) - \frac{1}{2}y_t^2\right)dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}u(c_t)dt\right] \tag{2.2.6}$$

subject to

$$\begin{aligned}
dX_t &= \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt + (\delta y_t - c_t)dt \;\; \forall \;\; t \geq 0 \\
L_1 \leq \;\; y_t &\leq L_2 \;\; 0 \leq t < \tau \\
y_t &= 0 \;\; t \geq \tau \\
X_t &\geq -\frac{\delta L_2}{r}, \;\; 0 \leq t < \tau \\
X_t &\geq 0 \;\; \forall \;\; t \geq \tau
\end{aligned}$$

where $\beta > 0$ is the discount factor. We solve this problem backwards. At time τ, when the agent retires with wealth X_τ, and solves the optimization problem:

$$U(X_\tau) = \underset{c_t,\pi_t}{Max}\, E_\tau\left[\int_\tau^\infty e^{-\beta(t-\tau)}u(c_t)dt\right]$$

subject to

$$\begin{aligned}
dX_t &= \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt - c_t dt, \quad t \geq \tau \\
X_t &\geq 0 \;\; \forall \;\; t \geq \tau
\end{aligned}$$

$U(X_\tau)$ is then the value function once the agent retires and X_τ is the wealth at retirement. As shown by Merton [45], Karatzas et.al. [34]

$$U(X_\tau) = \frac{X_\tau^{1-\gamma}}{K^\gamma(1-\gamma)} \tag{2.2.7}$$

where

$$K = r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2$$

Assumption: We assume throughout the paper that $K > 0$.

By the principle of dynamic programming, the agent's optimization problem (2.2.6) can be rewritten as

$$\underset{c_t,\pi_t,y_t,\tau}{Max}\ E\left[\int_0^\tau e^{-\beta t}\left\{u(c_t) - \frac{1}{2}y_t^2\right\}dt + e^{-\beta\tau}U(X_\tau)\right] \tag{2.2.8}$$

subject to

$$dX_t = \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt + (\delta y_t - c_t)dt \quad 0 \le t \le \tau \tag{2.2.9}$$

$$L_1 \le \ y_t \ \le L_2 \ \ 0 \le t < \tau \tag{2.2.10}$$

$$y_\tau = 0 \tag{2.2.11}$$

$$X_t \ge -\frac{\delta L_2}{r}, \ \ 0 \le t < \tau \tag{2.2.12}$$

$$X_\tau \ge 0 \tag{2.2.13}$$

2.3 Solution of the Optimization Problem

We begin by characterizing the optimal threshold level of wealth $\bar{X}$. If the wealth of the agent is below the threshold level, it is optimal for the agent to keep working. When the wealth of the agent exceeds $\bar{X}$, it is optimal for the agent to stop working and retire. The threshold level of wealth, $\bar{X}$ depends, in particular, on L_1, L_2 and δ, but is independent of the initial wealth of the agent. To derive $\bar{X}$, we start by defining the state-price density process ξ_t which follows the dynamics

$$d\xi_t \quad = \quad -\xi_t \left(rdt + \theta dW_t \right)$$

and the state variable

$$Z_t \quad = \quad \lambda e^{\beta t} \xi_t$$

where $\lambda > 0$ is a constant that depends on the initial wealth X_0 such that

$$e^{-\beta t} u^{'}(c_t) = e^{-\beta t} c_t^{-\gamma} = \lambda \xi_t \tag{2.3.1}$$

Z_t is always positive. The above equation (2.3.1) shows that the marginal utility from consumption is higher in the expensive states where ξ is higher. The level of wealth X_t increases when the state variable Z_t decreases and vice-versa. Using the new state variable Z_t we reduce the optimization problem (2.2.8) subject to the constraints (2.2.9), (2.2.10), (2.2.11), (2.2.12), and (2.2.13) as an optimal stopping time problem. When the agent is working, there is the cost of exerting effort which is quadratic in the amount of effort exerted. The benefit of exerting effort comes from the labor income which is proportional to the amount of effort exerted, the proportionality constant being the productivity of the worker, δ. At each time t, the agent chooses optimal consumption c_t, portfolio investment π_t, and the amount of effort y_t. These optimal policies depend only on the wealth level at time t, namely X_t. This is the result of the Markov structure of the problem and the principle of dynamic programming.

In Corollary A.0.11 in Appendix A, we show that X_t, the financial wealth of the agent, is a decreasing function of the state variable Z_t. The state variable Z_t is always positive, whereas the financial wealth X_t may be negative before retirement, but is constrained to be greater than $-\frac{\delta L_2}{r}$.

For high values of the state variable Z_t, the wealth of the agent, X_t is low and the agent works more because the economy is in a bad state and the returns to portfolio investment is low. Therefore it is optimal to exert more effort and accumulate wealth through labor income. Moreover, the marginal value of wealth is high in bad states and the marginal benefit of remaining in labor force is higher than the marginal cost of exerting effort. So it is not optimal for the agent to enter retirement for high values of the state variable Z_t. Instead, the agent wants to wait and accumulate more wealth through her labor income. For low values of Z_t, the wealth level X_t is high, the portfolio returns are high and the optimal effort exerted by the agent is low. Since the agent must exert at least a minimum amount of effort L_1 to remain in the labor force, it becomes optimal for the agent to retire. This idea is captured by the state variable Z_t.

As discussed above, for high values of Z_t the agent wants to work more and for low values of Z_t she wants to work less. Correspondingly, there are two values $\tilde{Z}_1 = \frac{L_1}{\delta}$ and $\tilde{Z}_2 = \frac{L_2}{\delta}$, such that $\tilde{Z}_1 < \tilde{Z}_2$. For $Z_t < \tilde{Z}_1$ it is optimal for the agent to exert less than L_1 amount of labor, and for $Z_t > \tilde{Z}_2$ it is optimal for the agent to exert more than L_2 amount of labor. Since the amount of effort y_t exerted at time t must satisfy the constraint $L_1 \leq y_t \leq L_2$, therefore, for $Z_t < \tilde{Z}_1$, $y_t = L_1$ and for $Z_t > \tilde{Z}_2$, $y_t = L_2$. Moreover, as Z_t decreases further below $\tilde{Z}_1$, the agent becomes more wealthy because of higher returns of the portfolio investment and she wants to work even less. In order to remain in the workforce she has to keep exerting L_1 amount of labor which becomes costlier to her. Thus, as Z_t decreases, the cost of exerting effort increases till Z_t reaches the level $\bar{Z} < \tilde{Z}_1$. At this value of Z_t, it is optimal for the agent to exercise the retirement option because the opportunity cost of waiting and exerting L_1 amount of labor becomes higher than the utility obtained by entering retirement. The wealth levels corresponding to $\tilde{Z}_1$, $\tilde{Z}_2$ are denoted as $\tilde{X}_1$, $\tilde{X}_2$ respectively, and the wealth level corresponding to $\bar{Z}$ is $\bar{X}$, the threshold level of wealth. Since X_t is a decreasing function of Z_t, it follows that $\tilde{X}_2 < \tilde{X}_1 < \bar{X}$.

In the propositions below, we provide closed form expressions for the optimal consumption, portfolio and effort levels as function of the current wealth level. Since the current wealth level X_t at time t lies in the interval $\left(-\frac{\delta L_2}{r}, \infty\right)$, we divide this interval into four disjoint intervals: $\left(-\frac{\delta L_2}{r}, \tilde{X}_2\right]$, $\left(\tilde{X}_2, \tilde{X}_1\right)$, $\left[\tilde{X}_1, \bar{X}\right)$ and $\left[\bar{X}, \infty\right)$. When the wealth level $X_t \in \left(-\frac{\delta L_2}{r}, \tilde{X}_2\right]$, the agent wants to exert more than L_2 amount of effort but is forced to exert only L_2 amount of labor, i.e. the optimal level of effort is $y_t = L_2$. For wealth levels $X_t \in \left(\tilde{X}_2, \tilde{X}_1\right)$, the agent can choose the optimal choice of effort y_t which lies in the interval (L_1, L_2). For wealth levels $X_t \in \left[\tilde{X}_1, \bar{X}\right)$ the agent chooses to exert less than L_1 amount of effort but is forced to exert L_1, i.e. the optimal level of effort is $y_t = L_2$. When the wealth level X_t reaches the threshold level $\bar{X}$, it is optimal to enter retirement and thus the effort exerted is zero.

For the following propositions, we define the constants α_1, α_2, K, $\tilde{Z}_1$, $\tilde{Z}_2$, C_1, C_2, D_1, D_2, E_2, and $\hat{C}_1$, $\hat{C}_2$, $\hat{D}_1$, $\hat{D}_2$, $\hat{E}_2$ as in (A.0.43), (A.0.44), (A.0.42), (A.0.51), (A.0.52), (A.0.53), (A.0.54),(A.0.55),(A.0.56)(A.0.57) and (A.0.103), (A.0.105), (A.0.104), (A.0.106), (A.0.107) in Appendix A.

Lemma 2.3.1. *There exists a unique solution $\bar{Z}$ to the equation:*

$$\left(\frac{1-\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{\delta L_1}{r}\right) Z^{1-\alpha_1} + \left(\frac{\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{L_1^2}{2\beta}\right) Z^{-\alpha_1} + C_1 = 0 \tag{2.3.2}$$

such that

$$\bar{Z} < \frac{\alpha_1 - 1}{2\alpha_1}\tilde{Z}_1 \tag{2.3.3}$$

Define the wealth levels:

$$\bar{X} = \frac{\bar{Z}^{-\frac{1}{\gamma}}}{K} \tag{2.3.4}$$

$$\tilde{X}_1 = -\alpha_1 C_1 \tilde{Z}_1^{\alpha_1 - 1} - \alpha_2 C_2 \tilde{Z}_1^{\alpha_2 - 1} + \frac{\tilde{Z}_1^{-\frac{1}{\gamma}}}{K} - \frac{\delta L_1}{r} \tag{2.3.5}$$

$$\tilde{X}_2 = -\alpha_2 E_2 \tilde{Z}_2^{\alpha_2 - 1} + \frac{\tilde{Z}_2^{-\frac{1}{\gamma}}}{K} - \frac{\delta L_2}{r} \tag{2.3.6}$$

(i) For $-\frac{\delta L_2}{r} < X_t \leq \tilde{X}_2$*, there exists a unique solution* Z_t *to the equation*

$$\alpha_2 E_2 Z^{\alpha_2 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_2}{r} + X_t = 0 \tag{2.3.7}$$

(ii) For $\tilde{X}_2 < X_t < \tilde{X}_1$*, there exists a unique solution* Z_t *to the equation*

$$\alpha_1 D_1 Z^{\alpha_1 - 1} + \alpha_2 D_2 Z^{\alpha_2 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K} - \frac{\delta^2 Z}{(\beta + \theta^2 - 2r)} + X_t = 0 \tag{2.3.8}$$

(iii) For $\tilde{X}_1 \leq X_t < \bar{X}$*, there exists a unique solution* Z_t *to the equation*

$$\alpha_1 C_1 Z^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_t = 0 \tag{2.3.9}$$

Proof: See Appendix A.

The above Lemma derives the threshold level of wealth $\bar{X}$ in terms of $\bar{Z}$. There is no closed-form solution for $\bar{Z}$, but it is shown to solve equation (2.3.2). The value of $\bar{Z}$ depends on the parameters of the economy, especially on δ, L_1 and L_2. It is shown in Appendix A that $\frac{\partial \bar{Z}}{\partial L_1} > 0$, $\frac{\partial \bar{Z}}{\partial L_2} < 0$, and $\frac{\partial \bar{Z}}{\partial \delta} < 0$. Moreover, we provide an upper bound for $\bar{Z}$ as given in (2.3.3). The upper bound depends positively on L_1 and negatively on δ. Lastly, the above lemma provides expressions for the financial wealth X_t in terms of the state variable Z_t as given in equations (2.3.7), (2.3.8) and (2.3.9). It is shown in Appendix A that X_t is a strictly decreasing function of Z_t, i.e. $\frac{\partial X_t}{\partial Z_t} < 0$, for all $Z_t > 0$. This means that we can use the financial wealth level X_t as the state variable instead of Z_t. In the analysis that will follow, it will be more convenient to use X_t as the state variable.

2.3.1 Optimal Retirement time

Now we provide the characterization of the optimal retirement time in the following proposition.

Proposition 2.3.2. *If the initial wealth is* $X_0 = x$*, then the optimal retirement time is*

$$\tau^* \quad = \quad inf\{t \geq 0 : Z^* e^{\beta t} \xi_t = \bar{Z}\}$$

where Z^* is the unique solution of the equation

$$\alpha_1 C_1 Z^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_0 = 0$$

Furthermore, the optimal retirement time can be expressed as

$$\tau^* \;\; = \;\; inf\{t \geq 0 : X_t = \bar{X}\}$$

where $\bar{X}$ is defined in (2.3.4).

Proof: See Appendix A.

The optimal retirement time is the first time that the wealth of the agent reaches the threshold level $\bar{X}$. After retirement, the agent does not exert effort. The above proposition can be understood as follows. As shown in Appendix A, Z_t follows

$$\frac{dZ_t}{Z_t} \;\; = \;\; (\beta - r)dt - \theta dW_t$$

where β is the discount factor, r is the risk-free interest rate and θ is the Sharpe ratio. Suppose the individual is working at time t, and have to decide whether to retire or not. The decision to retire depends on whether the marginal benefit from retiring exceeds the marginal cost of remaining in the labor force. It turns out that that the benefit of retiring is given by $\frac{\gamma}{K(1-\gamma)} Z_t^{1-\frac{1}{\gamma}}$, which is the post-retirement utility. The benefit accrued from remaining in the labor force is given by $\frac{\gamma}{K(1-\gamma)} Z_t^{1-\frac{1}{\gamma}} + \delta L_1 Z_t - \frac{1}{2} L_1^2$, if $Z_t \leq \tilde{Z}_1$. The first term is the utility obtained from the consumption process, the second term denotes the amount of income earned by exerting labor, while the third term signifies the loss in utility for exerting labor. Therefore, for $Z_t \leq \tilde{Z}_1$, the net benefit from retirement is given by $\frac{1}{2} L_1^2 - \delta L_1 Z_t$. Similarly, for $\tilde{Z}_1 < Z_t < \tilde{Z}_2$, the net benefit from retirement is given by $\frac{\gamma}{K(1-\gamma)} Z_t^{1-\frac{1}{\gamma}} - \frac{\gamma}{K(1-\gamma)} Z_t^{1-\frac{1}{\gamma}} - \frac{1}{2}\delta^2 Z_t^2 = -\frac{1}{2}\delta^2 Z_t^2$. And, for $Z_t \geq \tilde{Z}_2$, the net benefit from retirement is $\frac{1}{2} L_2^2 - \delta L_2 Z_t$. If we define the function $NB(Z_t)$ as the net benefit of retiring, then to obtain the optimal retirement time, the individual solves the optimal stopping problem

$$\sup_{\tau} E\left[NB(Z_\tau)\right]$$

Clearly, the individual will not retire when $\tilde{Z}_1 < Z_t < \tilde{Z}_2$, because the net benefit from retirement is negative. Since $\tilde{Z}_2 = \frac{L_2}{\delta}$, therefore $\frac{1}{2}L_2^2 - \delta L_2 Z_t > 0$ for $Z_t \geq \tilde{Z}_2$. Thus it is not optimal to retire when $Z_t \geq \tilde{Z}_2$. Hence, it is not optimal to retire if $Z_t > \tilde{Z}_1$. The intuition is that for $Z_t > \tilde{Z}_1$, the returns of the stock is low and the savings from the labor income cannot increase the total wealth to the threshold level required for retirement. Since labor can be chosen flexibly, it becomes optimal to continue working and accumulate wealth through labor income. For $Z_t \leq \tilde{Z}_1$, the expression $NB(Z_t) = \frac{1}{2}L_1^2 - \delta L_1 Z_t$ shows that the net benefit from retirement increases as Z_t decreases. There are two reasons for this. First, as Z_t decreases, the stock returns increases and the agent becomes more wealthy. The wealth level approaches the threshold level and retirement option can be exercised sooner. Secondly, as the agent's financial wealth increases, he wants to exert less than L_1 amount of effort, whereas the constraint on y_t requires the individual to exert L_1 amount of effort. Thus the cost of exerting effort, $\frac{1}{2}L_1^2$, begins to dominate the benefit obtained from working, namely $\delta L_1 Z_t$. Therefore, there exists $\bar{Z}$ which solves (2.3.2), such that once Z_t reaches $\bar{Z}$, it becomes optimal to retire immediately.

2.3.2 Optimal Consumption Policy

The next proposition characterizes the optimal consumption policies of the agent before and after retirement.

Proposition 2.3.3. *The optimal consumption c_t^* is given as follows:*

(i) For $-\frac{\delta L_2}{r} < X_t \leq \tilde{X}_2$

$$c_t^* = (Z_t^*)^{-\frac{1}{\gamma}}$$

where Z_t^* is the solution of equation (2.3.7).

(ii) For $\tilde{X}_2 < X_t < \tilde{X}_1$

$$c_t^* = (Z_t^*)^{-\frac{1}{\gamma}}$$

where Z_t^* is the solution of equation (2.3.8).

(iii) For $\tilde{X}_1 \leq X_t < \bar{X}$

$$c_t^* = (Z_t^*)^{-\frac{1}{\gamma}}$$

where Z_t^* is the solution of equation (2.3.9).

(iv) For $X_t \geq \bar{X}$, (i.e. when $t \geq \tau$)

$$c_t^* = KX_t$$

Proof: See Appendix A.

The crossed curve in Figure 2.1 shows how optimal consumption c_t varies with the financial wealth X_t when the agent has the option to retire. According to Proposition 2.3.3, as the financial wealth of the agent increases and approaches the threshold level $\bar{X}$, optimal amount of consumption decreases. The agent retires when the financial wealth reaches $\bar{X}$; thereafter the optimal consumption is a fixed proportion of the financial wealth as in Merton's model. Figure 2.2 shows how the marginal propensity to consume varies with the financial wealth. For low levels of wealth such that $X_t < \tilde{X}_2$, the agent wants to increase the total wealth by exerting more than L_2 amount of effort and earning more labor income. Since L_2 is the maximum limit imposed on the effort level, the agent can only exert L_2 amount of effort. Thus in order to increase the wealth, the agent consumes less and saves more. Hence, the marginal propensity to consume is a decreasing function of financial wealth X_t when $X_t < \tilde{X}_2$. For $\tilde{X}_2 < X_t < \tilde{X}_1$, it is optimal to exert less than L_2 amount of effort. In fact, the agent can now choose the effort level y_t which takes value between L_1 and L_2. Due to the flexibility of labor supply, the marginal propensity to consume is increasing in X_t when $\tilde{X}_2 < X_t < \tilde{X}_2$ a shown in Figure 2.2. For $X_t > \tilde{X}_1$, it is optimal to exert less than L_1 amount of effort, but the lower bound on y_t requires that the agent exert L_1 amount of effort. Therefore as X_t increases and approaches the threshold $\bar{X}$, the cost of supplying labor begins to increase and dominate the benefits of labor income. This induces the agent to consume less and increase the investment in the financial market so that retirement can be attained early. This makes the marginal propensity to consume a decreasing function of X_t when $X_t > \tilde{X}_1$. When X_t reaches $\bar{X}$, the agent retires, and the marginal propensity to consume is constant as in the Merton's model. The constant is equal to K.

2.3.3 Optimal Portfolio Policy

The next proposition characterizes the optimal portfolio policies.

Proposition 2.3.4. *The optimal amount of dollars invested in the stock, π_t^* are given as follows:*

(i) For $-\frac{\delta L_2}{r} < X_t \leq \tilde{X}_2$

$$\pi_t^* = \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_2}{r}\right] + \frac{\theta}{\sigma}\alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)E_2(Z_t^*)^{\alpha_2 - 1}$$

where Z_t^* is the solution of equation (2.3.7).

(ii) For $\tilde{X}_2 < X_t < \tilde{X}_1$

$$\begin{aligned} \pi_t^* &= \frac{\theta}{\gamma\sigma}\left[X_t - \frac{\delta^2 Z_t^*}{\beta + \theta^2 - 2r}\right] \\ &+ \frac{\theta}{\sigma}\left[\alpha_1(\alpha_1 + \frac{1}{\gamma} - 1)D_1(Z_t^*)^{\alpha_1 - 1} + \alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)D_2(Z_t^*)^{\alpha_2 - 1} - \frac{\delta^2 Z_t^*}{\beta + \theta^2 - 2r}\right] \end{aligned}$$

where Z_t^* is the solution of equation (2.3.8).

(iii) For $\tilde{X}_1 \leq X_t < \bar{X}$

$$\pi_t^* = \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\left[\alpha_1(\alpha_1 + \frac{1}{\gamma} - 1)C_1(Z_t^*)^{\alpha_1 - 1} + \alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)C_2(Z_t^*)^{\alpha_2 - 1}\right]$$

where Z_t^* is the solution of equation (2.3.9).

(iv) For $X_t \geq \bar{X}$, (i.e. when $t \geq \tau$)

$$\pi_t^* = \frac{\theta X_t}{\gamma\sigma}$$

Proof: See Appendix A.

The optimization problem of the agent after retirement is similar to the problem of an agent with an infinite horizon, who chooses consumption and financial investment optimally to maximize the expected discounted utility. This problem has been studied by Merton ([44], [45]), Karatzas et.al. [34] among others. From the results of these models, we know that after retirement the agent consumes a fixed proportion of her financial wealth and invests a fixed proportion of her financial wealth in the risky asset. Before retirement, the optimal consumption and portfolio policies are quite different. Figure 2.3 shows the optimal amount of dollars invested in the risky security as a function of the financial wealth. Instead of investing a fixed proportion of wealth in the stock, the agent with retirement option increases investment in the stock as financial wealth X_t increases and approaches the threshold level $\bar{X}$. This is because getting closer to retirement provides an incentive to retire early. However, once retirement option is exercised, the agent decreases her investment in the stock and there is a downward jump in her optimal portfolio policy. After retirement the optimal portfolio policy is to invest a constant fraction of her financial wealth in the stock, as in Merton's model.

Figure 2.4 shows the proportion of financial wealth invested in the risky stock as a function of the financial wealth. At low levels of financial wealth, a large proportion of the financial wealth is invested in the stock so that wealth accumulates faster. The investment gradually decrease as wealth increases. The interesting feature is that, beyond a certain level of wealth, the agent again begins to increase investment in the stock till he retires. At retirement the investment in the stock jumps down. This feature of the optimal portfolio policy contradicts the popular recommendations of financial advisers. Typically, the financial professionals advise the young investors (who are far from retirement) to invest more proportion of their financial wealth in the risky asset and the old investors (who are nearing retirement) to invest less. According to Figure 2.4, old persons who are approaching retirement increases their investment in the risky asset before retirement.

2.3.4 Human Capital

When the agent exerts effort to earn labor income, the human capital is the present value of the future labor income. The sum of the financial wealth and human capital is the total wealth. After retirement, the agent has no income from exerting effort, so the human capital is zero. The next proposition characterizes the optimal effort policies and the human capital wealth.

Proposition 2.3.5. *The optimal effort level y_t^* and the present value of future labor income H_t are given as follows:*

(i) For $-\frac{\delta L_2}{r} \le X_t \le \tilde{X}_2$

$$\begin{aligned} y_t^* &= L_2 \\ H_t &= \delta \left[\hat{E}_2 (Z_t^*)_t^{\alpha_2 - 1} + \frac{L_2}{r} \right] \end{aligned}$$

where Z_t^* is the unique solution of equation (2.3.7)

(ii) For $\tilde{X}_2 < X_t < \tilde{X}_1$

$$\begin{aligned} y_t^* &= \delta Z_t^* \\ H_t &= \delta \left[\hat{D}_1 (Z_t^*)_t^{\alpha_1 - 1} + \hat{D}_2 (Z_t^*)_t^{\alpha_2 - 1} - \frac{\delta Z_t^*}{\beta + \theta^2 + 2r} \right] \end{aligned}$$

where Z_t^* is the unique solution of equation (2.3.8).

(iii) For $\tilde{X}_1 \le X_t < \bar{X}$

$$\begin{aligned} y_t^* &= L_1 \\ H_t &= \delta \left[\hat{C}_1 (Z_t^*)_t^{\alpha_1 - 1} + \hat{C}_2 (Z_t^*)_t^{\alpha_2 - 1} + \frac{L_1}{r} \right] \end{aligned}$$

where Z_t^* is the unique solution of equation (2.3.9).

(iv) For $X_t \geq \bar{X}$,

$$\begin{aligned} y_t^* &= 0 \\ H_t &= 0 \end{aligned}$$

Proof: See Appendix A.

When there is an option to retire, human capital is a larger proportion of total wealth at low levels of financial wealth. This is because at low levels of financial wealth, the agent is far from the retirement; so the present value of future labor income is high. As the financial wealth increases and approaches the critical level of wealth at which it is optimal to retire, future labor income becomes smaller. Therefore, human capital is a decreasing function of the financial wealth as stated in the following proposition.

Proposition 2.3.6. *In the model with the option to retire, the present value of future income of the agent H is decreasing in the financial wealth X. That is, at any time $'t'$ prior to retirement, $H(X_t) < H(X_t^{'})$ if $X_t > X_t^{'}$*

Proof: See Appendix A.

In Figure 2.5 we graph the human capital H_t as a function of the financial wealth X_t and observe that H_t is a decreasing function of X_t as in Proposition 2.3.6. Figure 2.6 shows the proportion of total wealth invested in the stock as a function of the financial wealth. It shows two wealth levels at which the graph is discontinuous. Starting with a negative amount of financial wealth, the agent increases the proportion of wealth allocated to the risky asset. The agent have negative wealth in expensive states when the financial market is weak and the returns to the risky asset are low. It is optimal for the agent to exert a large amount of effort to increase the human capital. Since there is an upper limit to the amount of effort that the agent can possibly exert, this limits the amount of human capital that can be accumulated and hence the agent invests a small fraction of her total wealth in the financial security. In better states the agent can choose the amount of effort that can be exerted. So, even if the financial wealth is negative, in better states the proportion of total wealth invested in the risky asset is high. A higher investment in the stock enables the transfer of financial wealth to good states in the future. As the graph shows, the amount of investment in the stock as a proportion of the total wealth exhibits a U-shaped pattern. Due to labor supply flexibility, the agent can choose to accumulate human capital and invest more in the stock. At low levels of financial wealth i.e. when the agent is far from retirement, the human capital is a large fraction of the total wealth as shown in Figure 2.5. Since human capital decreases as financial wealth increases, the proportion of total wealth invested in the stock decreases when the financial wealth is negative. As the financial wealth approaches the critical level, human wealth gradually becomes a smaller fraction of the total wealth. Therefore, proportion of total wealth allocated in the stock increases until the agent exercises the retirement option and then jumps downward to a fixed level when the retirement option is exercised.

Figure 2.7 shows how the threshold level of wealth varies with the relative risk aversion coefficient of the agent. The threshold level increases as the agent becomes less risk averse and decreases as the risk aversion increases. During booms, when stock returns are high, less risk averse agents invest more of their financial wealth in the stock than more risk averse agents. Thus the less risk-averse agents can accumulate more wealth in their retirement savings and retire early.

The following proposition provides comparative statics results of the threshold level $\bar{X}$ with respect to the parameters L_1, L_2 and δ. It also shows that a necessary condition for the optimal retirement time to be finite is $L_1 > 0$ and $L_2 < \infty$. If either $L_1 = 0$ or $L_2 = \infty$, then the agent does not retire.

Proposition 2.3.7. *The threshold level of wealth $\bar{X}$ at which the agent decides to retire increases with L_2, decreases with L_1, and increases with δ. In particular, $\bar{X} \to \infty$ as $L_1 \to 0$ or as $L_2 \to \infty$.*

Proof: See Appendix A.

L_2 is the maximum amount of effort that the agent can exert. By increasing L_2, the individual can work more hours and retire with a higher amount of wealth. L_1 is the minimum amount of effort that the agent must exert to remain employed. As L_1 increases, the agent is forced to work more than it would be optimal. Since there is a cost to exerting labor, at some level of wealth the cost of exerting at least L_1 amount of labor dominates the benefit of receiving the human capital. Therefore, it becomes optimal to retire at lower wealth levels.

The productivity of the worker is denoted by δ. A worker with higher value of δ can earn more income than a worker with a lower value, even though the number of hours worked is same. Therefore, a worker with a higher value of δ can accumulate more wealth at retirement. As the individual becomes more productive or skilled, her retirement is triggered at a higher level of wealth.

The last part of the proposition implies that the agent delays retirement as L_1 decreases or as L_2 increases. In the extreme cases, when $L_1 = 0$ or when $L_2 = \infty$, the agent does not retire. Let us first consider the case when $L_1 = 0$ and $L_2 > 0$. In this case, if the agent does not want to work, he can choose to exert zero amount of effort. At a later time, the agent can again choose to exert a positive level of effort. Thus he does not choose to retire because the option to retire is an irreversible decision. Next, suppose that $L_1 > 0$ and $L_2 = \infty$. Since there is no upper bound on the amount of effort that can be exerted, it is not optimal for the agent to choose retirement. The reason is that if the agent retires, he is forgoing the benefits of future labor income. With no upper bound on the amount of effort that can be exerted, the present value of future labor income plus the utility from consumption always exceeds the post-retirement utility. Hence the agent will not opt to retire.

In the following subsection, we solve for the optimal consumption and portfolio policies of an agent who has no retirement option. The agent can still choose how much effort to supply, y_t, where $L_1 \leq y_t \leq L_2$.

2.3.5 Benchmark Case: No Retirement Option

We refer to the optimization problem of an agent with no option to retire as the benchmark case. When the agent does not have the option to retire, he solves the following problem:

$$\underset{c_t,\pi_t,y_t}{Max}\ E\left[\int_0^{\infty} e^{-\beta t}\left\{u(c_t)-\frac{1}{2}y_t^2\right\}dt\right]$$

subject to

$$\begin{aligned} dX_t &= \pi_t(\mu dt+\sigma dW_t)+(X_t-\pi_t)rdt+(\delta y_t-c_t)dt\forall t\geq 0 \\ L_1 \leq\ y_t &\leq L_2\forall t\geq 0 \\ X_t &\geq -\frac{\delta L_2}{r}\forall t\geq 0 \end{aligned}$$

The following proposition states the optimal policies.

Proposition 2.3.8. *Let the coefficients α_1, α_2, K, C_1, C_2, D_1, D_2 and E_2 be defined as in (A.0.43)-(A.0.42) and (A.0.53)-(A.0.57). Define the wealth levels*

$$\begin{aligned} \tilde{X}_1^N &= -\alpha_1 C_1\tilde{Z}_1^{\alpha_1-1}+\frac{\tilde{Z}_1^{-\frac{1}{\gamma}}}{K}-\frac{\delta L_1}{r} \\ \tilde{X}_2^N &= -\alpha_2(E_2-C_2)\tilde{Z}_2^{\alpha_2-1}+\frac{\tilde{Z}_2^{-\frac{1}{\gamma}}}{K}-\frac{\delta L_2}{r} \end{aligned}$$

The optimal policies c_t^*, y_t^*, π_t^* are given as follows:

(i) For $-\frac{\delta L_2}{r}<X_t\leq\tilde{X}_2^N$

$$\begin{aligned} c_t^N &= (Z_t^N)^{-\frac{1}{\gamma}} \\ y_t^N &= L_2 \\ \pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t+\frac{\delta L_2}{r}\right]+\frac{\theta}{\sigma}\alpha_2(\alpha_2+\frac{1}{\gamma}-1)(E_2-C_2)(Z_t^N)^{\alpha_2-1} \end{aligned}$$

where Z_t^N is the unique solution of the equation

$$\alpha_2(E_2-C_2)Z^{\alpha_2-1}-\frac{Z^{-\frac{1}{\gamma}}}{K}+\frac{\delta L_2}{r}+X_t=0$$

(ii) For $\tilde{X}_2^N < X_t < \tilde{X}_1^N$

$$\begin{aligned}
c_t^N &= (Z_t^N)^{-\frac{1}{\gamma}} \\
y_t^N &= \delta Z_t^N \\
\pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta^2 Z_t^N}{\beta+\theta^2-2r}\right] \\
&+ \frac{\theta}{\sigma}\left[\alpha_1(\alpha_1+\frac{1}{\gamma}-1)D_1(Z_t^N)^{\alpha_1-1} + \alpha_2(\alpha_2+\frac{1}{\gamma}-1)(D_2-C_2)(Z_t^N)^{\alpha_2-1}\right] \\
&- \frac{\theta\delta^2 Z_t^N}{\sigma(\beta+\theta^2-2r)}
\end{aligned}$$

where Z_t^N *solves*

$$\alpha_1 D_1 Z^{\alpha_1-1} + \alpha_2(D_2-C_2)Z^{\alpha_2-1} - \frac{Z^{-\frac{1}{\gamma}}}{K} - \frac{\delta^2 Z}{\beta+\theta^2-2r} + X_t = 0$$

(iii) For $X_t \geq \tilde{X}_1^N$

$$\begin{aligned}
c_t^N &= (Z_t^N)^{-\frac{1}{\gamma}} \\
y_t^N &= L_1 \\
\pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\alpha_1(\alpha_1+\frac{1}{\gamma}-1)C_1(Z_t^N)^{\alpha_1-1}
\end{aligned}$$

where Z_t^N *solves*

$$\alpha_1 C_1 Z^{\alpha_1-1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_t = 0$$

Proof. This is the problem considered in Merton [44], [45], except that in the present model the agent can choose how much labor to supply. The proof is standard and hence omitted. □

Using the results in Proposition 2.3.8, we can now compare the optimal consumption and portfolio policies in the benchmark model and the model with retirement option.

Proposition 2.3.9. *(i) For* $\tilde{X}_1 < X_t < \bar{X}$

$$c_t(X_t) < c_t^N(X_t)$$

$$\pi(X_t) > \pi^N(X_t)$$

and for $X_t \geq \bar{X}$

$$c_t(X_t) < c_t^N(X_t)$$
$$\pi(X_t) < \pi^N(X_t)$$

Proof: See Appendix A.

That is, prior to retirement, at any given wealth level, optimal investment in the stock is higher compared to the benchmark case. Post retirement, at any given level of wealth, the optimal investment in stock is lower than the benchmark case.

2.4 Retirement Option with Constant Labor Supply

In this section we assume that the agent have to exert a fixed amount of effort, that is, he does not choose the labor supply. The amount of effort exerted is assumed to be $L > 0$. The productivity of the worker is δ. The agent has to choose the optimal consumption and portfolio processes c_t, π_t, and the stopping time τ to solve the optimization problem:

$$\underset{c_t,\pi_t,\tau}{Max} E\left[\int_0^\tau e^{-\beta t}\left(u(c_t) - \frac{1}{2}L^2\right)dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}u(c_t)dt\right] \tag{2.4.1}$$

subject to

$$\begin{aligned} dX_t &= \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt + (\delta L - c_t)dt \quad 0 \leq t < \tau \\ dX_t &= \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt - c_t dt \quad t \geq \tau \end{aligned}$$

and

$$\begin{aligned} X_t &\geq -\frac{\delta L}{r}, \;\; 0 \leq t < \tau \\ X_t &\geq 0 \;\; \forall t \geq \tau \end{aligned}$$

X_τ is the wealth at retirement, $U(X_\tau)$ is the value function once the agent retires, and is given by (2.2.7). By the principle of dynamic programming, the agent's optimization problem (2.4.1) can be rewritten as

$$\underset{c_t,\pi_t,\tau}{Max}\ E\left[\int_0^\tau e^{-\beta t}\left\{u(c_t)-\frac{1}{2}L^2\right\}dt+e^{-\beta\tau}U(X_\tau)\right]$$

subject to

$$\begin{aligned} dX_t &= \pi_t(\mu dt+\sigma dW_t)+(X_t-\pi_t)rdt+(\delta L-c_t)dt \quad 0\le t<\tau \\ dX_t &= \pi_t(\mu dt+\sigma dW_t)+(X_t-\pi_t)rdt-c_t dt \quad t\ge\tau \end{aligned}$$

and

$$\begin{aligned} X_t &\ge -\frac{\delta L}{r},\ \ 0\le t<\tau \\ X_\tau &\ge 0, \end{aligned}$$

To characterize the optimal consumption and portfolio policies, we first find the threshold level of wealth at which it is optimal to enter retirement. We denote the threshold level of wealth when labor supply is constant by $\bar{X}^C$. The following proposition characterized the optimal policies.

Proposition 2.4.1. *The constants K, α_1, α_2 are as in (A.0.42), (A.0.43), (A.0.44) in Appendix A and let*

$$\bar{Z}^C = \frac{rL}{2\beta\delta}\frac{\alpha_2}{\alpha_2-1} \tag{2.4.2}$$

$$F_2 = -\frac{\delta L}{\alpha_2 r}(\bar{Z}^C)^{1-\alpha_2}>0 \tag{2.4.3}$$

$$\bar{X}^C = \frac{\left(\bar{Z}^C\right)^{-\frac{1}{\gamma}}}{K} \tag{2.4.4}$$

(a) The optimal retirement time is

$$\begin{aligned} \tau^* &= inf\{s\ge t: Z^* e^{\beta(s-t)}\frac{\xi_s}{\xi_t}=\bar{Z}^C\} \\ &= inf\{s\ge t: X_s=\bar{X}^C\} \end{aligned}$$

where Z^* is the unique solution of

$$\alpha_2 F_2 Z^{\alpha_2 - 1} + \frac{Z^{-\frac{1}{\gamma}}}{K} - \frac{\delta L}{r} + X_t = 0 \tag{2.4.5}$$

(b) The present value of future labor income H_t and optimal policies are given by c_t^*, π_t^*) such that

(i) For $-\frac{\delta L}{r} < X_t < \bar{X}^C$

$$\begin{aligned} c_t^* &= (Z_t^*)^{-\frac{1}{\gamma}} \\ \pi_t^* &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L}{r}\right] + \frac{\theta}{\sigma}\alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)F_2(Z_t^*)^{\alpha_2 - 1} \\ H_t &= \frac{\delta L}{r}\left[1 - \left(\frac{\bar{Z}^C}{Z_t^*}\right)^{1-\alpha_2}\right] \end{aligned}$$

where Z_t^* is the unique solution of the equation (2.4.5).

(iii) For $X_t \geq \bar{X}^C$ (i.e. when $t \geq \tau$), the optimal policies c_t^*, π_t^* and the present value of future labor income are given by

$$\begin{aligned} c_t^* &= KX_t \\ \pi_t^* &= \frac{\theta X_t}{\gamma\sigma} \\ H_t &= 0 \end{aligned}$$

Proof: See Appendix A.

Figure 2.8 shows the amount of consumption as a function of the financial wealth while Figure 2.9 shows how consumption as a fraction of total wealth varies with the financial wealth. Consumption is a fixed proportion of the total wealth when labor supply is constant. Figure 2.10 shows the amount of dollars invested in the stock as a function of the financial wealth, whereas Figure 2.12 shows the amount of dollars invested in the stock as a fraction of the total wealth. The explanation of these graphs is analogous to the explanation for the graphs in Section 2.3. The following proposition compares the threshold level of wealth when labor supply is flexible with the threshold level of wealth when labor supply is constant.

Proposition 2.4.2. *Let $\bar{X}^{constant}, \bar{X}^{Flexible}$ denote the threshold wealth levels in the models with constant labor supply and flexible labor supply respectively. If L is the constant amount of effort exerted in the model with constant labor supply and L_1 is the minimum level of effort in the model with flexible labor supply, then*

$$\bar{X}^{Flexible} > \left(\frac{L}{L_1}\right)^{\frac{1}{\gamma}} \bar{X}^{Constant}$$

In particular, for $L \geq L_1$, $\bar{X}^{Flexible} > \bar{X}^{Constant}$.

Proof: See Appendix A.

The above proposition shows that that the threshold level of wealth $\bar{X}^{Flexible}$ in the model with flexible labor supply is higher than the threshold level of wealth $\bar{X}^{Constant}$ in the model with constant supply of labor, provided that the fixed amount of labor supplied (L) is higher than L_1. The intuition behind this result is as follows. The cost of exerting effort when labor supply is constant is $\frac{1}{2}L^2$, whereas the cost of exerting effort when labor supply is flexible is y_t^2, where y_t is the optimally chosen effort level. In the model with flexible supply, retirement takes place when the cost of exerting effort becomes higher than the benefits. This happens when the agent want to exert less than L_1 amount of effort but is forced to exert L_1 amount of effort. Similarly, in the model with constant labor supply, retirement occurs when the agent wants to work for less than L hours but is forced to work L hours. Now, $\frac{1}{2}L^2 > \frac{1}{2}L_1^2$ if $L > L_1$, so that the retirement occurs early in the model with constant supply of labor than with flexible supply of labor. In other words, $\bar{X}^{Flexible} > \bar{X}^{Constant}$.

2.4.1 Borrowing Constraints

In this section, we consider the model with constant labor supply with the additional restriction that the agent cannot borrow against future labor income. Specifically, the agent always has to maintain a positive level of financial wealth: that is, $X_t \geq 0, \text{ for } t \geq 0$. This is the liquidity constraint or borrowing constraint.

The optimization problem of the agent is to choose c_t, π_t, and the stopping time τ that maximize expected utility:

$$\underset{c_t,\pi_t,\tau}{Max} E\left[\int_0^\tau e^{-\beta t}\left(u(c_t)-\frac{1}{2}L^2\right)dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}u(c_t)dt\right] \tag{2.4.6}$$

subject to

$$\begin{aligned} dX_t &= \pi_t(\mu dt+\sigma dW_t)+(X_t-\pi_t)rdt+(\delta L-c_t)dt \quad 0\le t<\tau \\ dX_t &= \pi_t(\mu dt+\sigma dW_t)+(X_t-\pi_t)rdt-c_t dt \quad t\ge\tau \end{aligned}$$

and

$$X_t \;\ge\; 0 \;\; \forall t\ge 0$$

Proposition 2.4.3. *Define the constants K, α_1, α_2 as defined in (A.0.42), (A.0.43), (A.0.44) of Appendix A. Define the functions f and g as*

$$\begin{aligned} f(\eta) &= \frac{n(\eta)}{d(\eta)} \\ g(\eta) &= \frac{\delta LK}{r}\frac{(\alpha_1-1)\eta^{\alpha_2}+(1-\alpha_2)\eta^{\alpha_1}-(\alpha_1-\alpha_2)\eta}{\eta^{\alpha_2}(\alpha_1+\frac{1}{\gamma}-1)+\eta^{\alpha_1}(1-\frac{1}{\gamma}-\alpha_2)} \\ h(\eta) &= g(\eta)^{-\gamma}-f(\eta) \end{aligned}$$

where

$$\begin{aligned} n(\eta) &= \alpha_1\alpha_2 Lr\left[\eta^{\alpha_1}(1-\frac{1}{\gamma}-\alpha_2)+\eta^{\alpha_2}(\alpha_1+\frac{1}{\gamma}-1)\right] \\ d(\eta) &= 2\beta\delta\left[\alpha_2(\alpha_1-1)(1-\frac{1}{\gamma}-\alpha_2)\eta^{1+\alpha_1}-\alpha_1(1-\alpha_2)(\alpha_1-1+\frac{1}{\gamma})\eta^{1+\alpha_2}\right] \\ &+ 2\beta\delta\frac{\alpha_1-\alpha_2}{\gamma}\eta^{\alpha_1+\alpha_2} \end{aligned}$$

(a) Then there exists a unique number $\eta^*\in(0,1)$ such that $h(\eta^*)=0$.

Let

$$\begin{aligned}\hat{Z} &= f(\eta^*)\\ \bar{Z}^{CL} &= \eta^*\hat{Z} = \eta^* f(\eta^*)\end{aligned}$$

Define

$$\begin{aligned}G_1 &= -\frac{\delta L}{r}\frac{1-\alpha_2}{\alpha_1-\alpha_2}(\bar{Z}^{CL})^{1-\alpha_1} - \frac{\alpha_2}{\alpha_1-\alpha_2}\frac{L^2}{2\beta}(\bar{Z}^{CL})^{-\alpha_1}\\ G_2 &= -\frac{\delta L}{r}\frac{\alpha_1-1}{\alpha_1-\alpha_2}(\bar{Z}^{CL})^{1-\alpha_2} + \frac{\alpha_1}{\alpha_1-\alpha_2}\frac{L^2}{2\beta}(\bar{Z}^{CL})^{-\alpha_2}\end{aligned}$$

Define the wealth levels

$$\begin{aligned}\bar{X}^{CL} &= \frac{\left(\bar{Z}^{CL}\right)^{-\frac{1}{\gamma}}}{K}\\ \hat{X} &= -\alpha_1 G_1 \hat{Z}^{\alpha_1-1} - \alpha_2 G_2 \hat{Z}^{\alpha_2-1} + \hat{Z}^{-\frac{1}{\gamma}} - \frac{\delta L}{r}\end{aligned}$$

Then $\hat{X} > \bar{X}^{CL}$.

b) There exists a positive decreasing process N_s^* *with* $N_t^* = 1$ *so that the retirement time is*

$$\begin{aligned}\hat{\tau} &= inf\{s \geq t : X_s = \bar{X}^{CL}\}\\ &= inf\{s \geq t : Z^* e^{\beta(s-t)} N_s^* \frac{\xi_s}{\xi_t} = \bar{Z}^{CL}\}\end{aligned}$$

where Z^* *is the solution of the equation*

$$\alpha_1 G_1 Z^{\alpha_1-1} + \alpha_2 G_2 Z^{\alpha_2-1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L}{r} + X_t = 0$$

c) If $0 \leq X_t < \bar{X}$, *then the optimal consumption* c_t^*, *stock holdings* π_t^* *and the present value of* *future income are given by*

$$\begin{aligned}
c_t^* &= (Z_t^*)^{-\frac{1}{\gamma}} \\
\pi_t^* &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L}{r}\right] \\
&+ \frac{\theta}{\sigma}\left[\alpha_1(\alpha_1 + \frac{1}{\gamma} - 1)G_1(Z_t^*)^{\alpha_1 - 1} + \alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)G_2(Z_t^*)^{\alpha_2 - 1}\right] \\
H_t &= \frac{\delta L}{r}\left[1 - \left(\frac{\bar{Z}^{CL}}{Z_t^*}\right)^{1-\alpha_2}\right]
\end{aligned}$$

where Z_t^* is the unique solution of the equation

$$\alpha_1 G_1 Z^{\alpha_1 - 1} + \alpha_2 G_2 Z^{\alpha_2 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K} + \frac{\delta L}{r} + X_t = 0$$

d) If $X_t \geq \bar{X}$, it is optimal for the agent to retire and the optimal policies and present value of future income are given by

$$\begin{aligned}
c_t^* &= KX_t \\
\pi_t^* &= \frac{\theta X_t}{\gamma\sigma} \\
H_t &= 0
\end{aligned} \tag{2.4.7}$$

Proof: See Appendix A.

The comparison of the optimal policies with and without borrowing constraints are shown in Figures 2.8 - 2.11. There is no qualitative difference in optimal policies when borrowing constraints are present. Quantitatively, as Figure 2.8 shows, in the presence of borrowing constraints the agent consumes less than when there is no borrowing constraints. The optimal portfolio policy is shown in Figure 2.10. The threshold level of wealth is lower in the presence of borrowing constraints and therefore the human capital is also lower. This is shown in Figure 2.11. The agent retires at lower threshold level because the borrowing constraint is effective when the agent receives labor income. Thus while working, the agent's utility is lower in the presence of the borrowing constraint. Once retirement option is exercised, the post-retirement utility $U(X_\tau)$ is same for the constrained agent who retires with wealth X_τ and also for the unconstrained agent who retires with the same level of wealth X_τ. Therefore, in the presence of the borrowing constraint, the agent decides to retire with a lower level of wealth. As Figure 2.12 illustrates, for the same level of financial wealth, the optimal investment in the stock as a fraction of total wealth is much lower when the agent is subject to borrowing constraints.

2.5 Conclusion

In this paper we considered a model to study the effects of flexible labor supply on the optimal consumption and portfolio policies of an agent who can choose when to retire. In our model, the agent can choose how much effort to exert but the amount of effort is constrained to lie within a specified interval. Exerting effort to earn labor income is assumed to be costly to the agent. The joint effects of flexible labor supply and the option to retire have been analyzed. It is shown that consumption decreases and portfolio holdings jump down when the agent retires. There exists a threshold level of wealth at which it is optimal to retire. The threshold level increases with the productivity of the agent and also depends on the minimum and maximum level of effort that the agent can exert. As retirement approaches, the agent finds it optimal to consume less and invest more of her wealth in the risky asset to reach the critical level of wealth early. Thus the marginal propensity to consume out of financial wealth increases at low levels of financial wealth (when retirement is far) but increases as the financial wealth approaches the critical level. We also find that the proportion of financial wealth invested in the stocks falls as the financial wealth increases, but again rises as retirement approaches. In other words, young investors whose human capital is a larger portion of total wealth invests gradually decreases their investment in the stocks a their financial wealth increases, but again increases their investment once their financial wealth approaches the critical level of wealth. Finally, we consider a model with the option to retire and constant labor supply. In this model, the agent has to exert a constant amount of effort before he decides to retire. The effort is costly to the agent as in the previous model. Within this setup of constant labor supply and option to retire, we study the effects of borrowing constraints on the optimal consumption and portfolio policies and the human capital. We show that the agent is more cautious in investing in the stock when subject to borrowing constraints because a downturn in the financial markets will make the constraints binding sooner if most of the wealth is invested in the stock. Moreover, the threshold level of wealth at which the agent retires is lower in the presence of borrowing constraints.

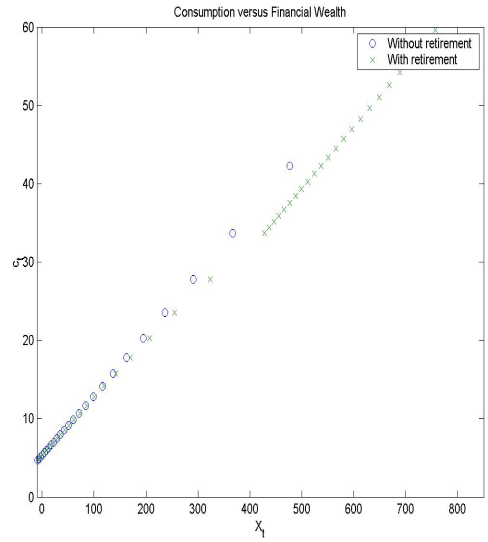

Figure 2.1: Consumption given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

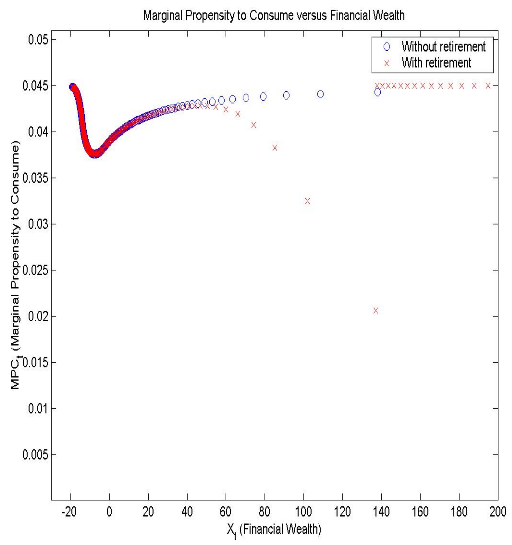

Figure 2.2: Marginal Propensity to Consume given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

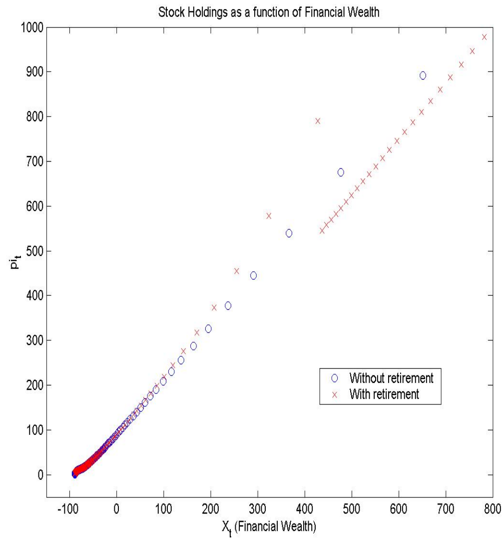

Figure 2.3: Stock Investment in Dollars given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

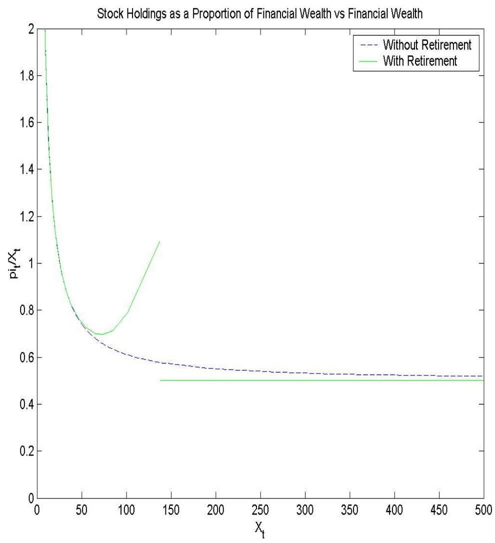

Figure 2.4: Stock Investment as a Fraction of Financial Wealth given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

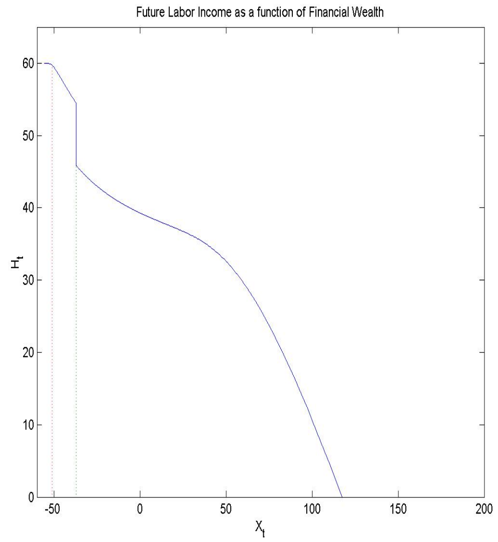

Figure 2.5: Human Capital given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

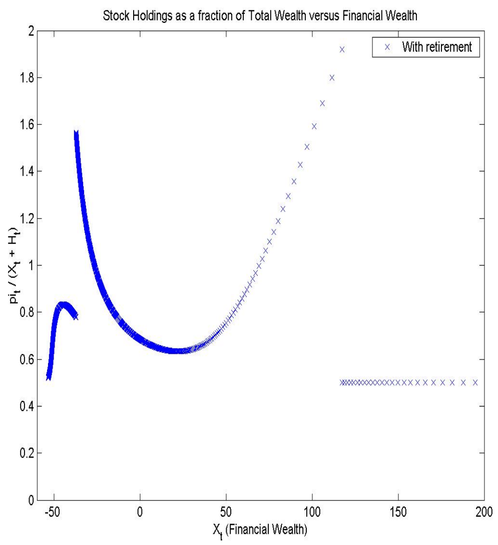

Figure 2.6: Stock Investment as a Fraction of Total Wealth given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$, $\gamma = 2$.

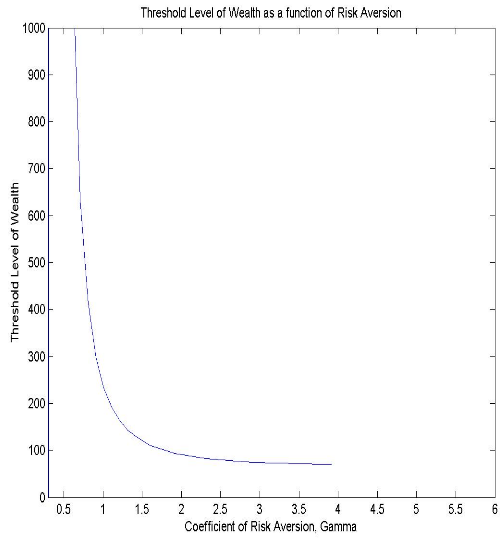

Figure 2.7: Threshold Level given Financial Wealth. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L_1 = 0.8$, $L_2 = 3$.

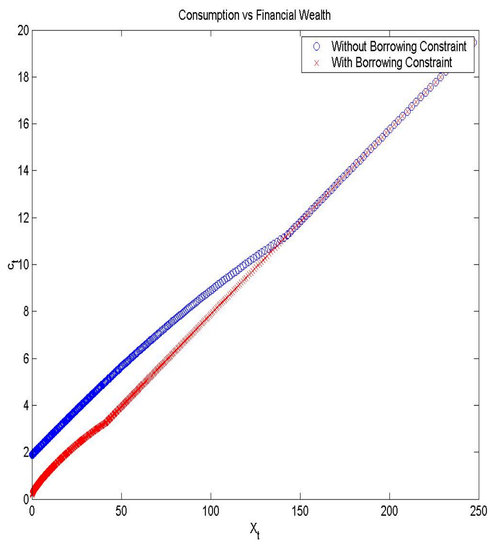

Figure 2.8: Consumption given Financial Wealth for constant labor supply. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L = 0.8$, $\gamma = 0.8$.

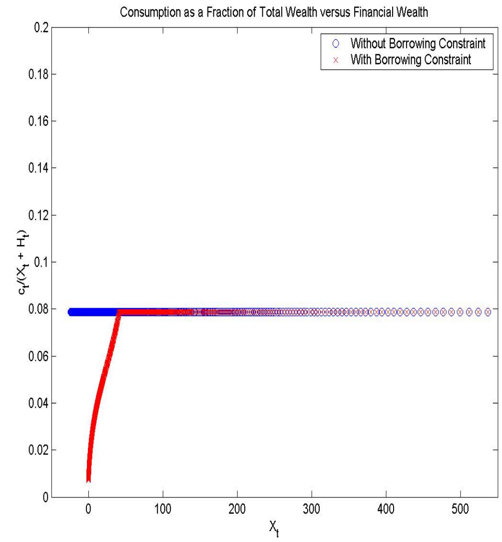

Figure 2.9: Consumption as a fraction of Total wealth given Financial Wealth for constant labor supply. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L = 0.8$, $\gamma = 0.8$.

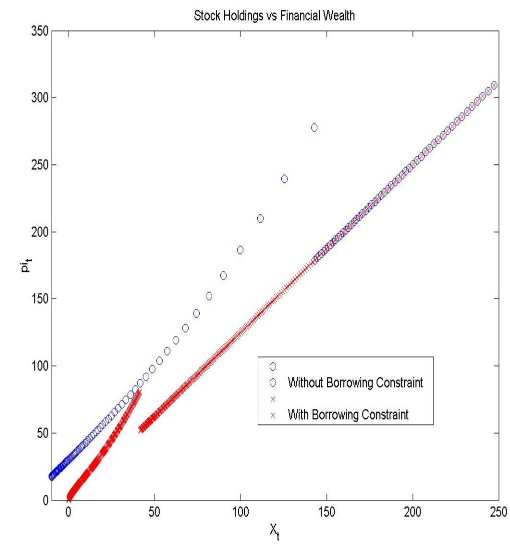

Figure 2.10: Stock Holdings given Financial Wealth for constant labor supply. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L = 0.8$, $\gamma = 0.8$.

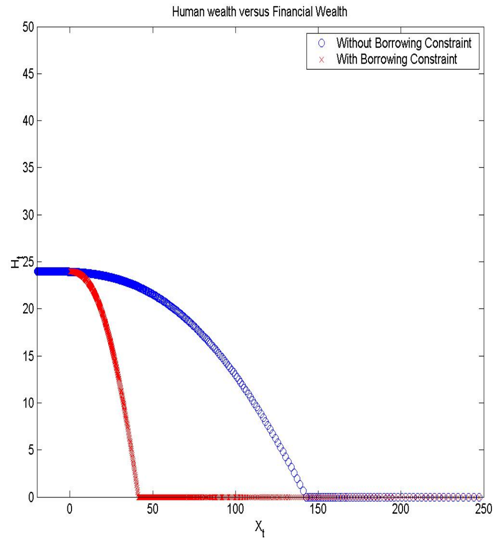

Figure 2.11: Present Value of Future Labor Income given Financial Wealth, for constant labor supply. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L = 0.8$, $\gamma = 0.8$.

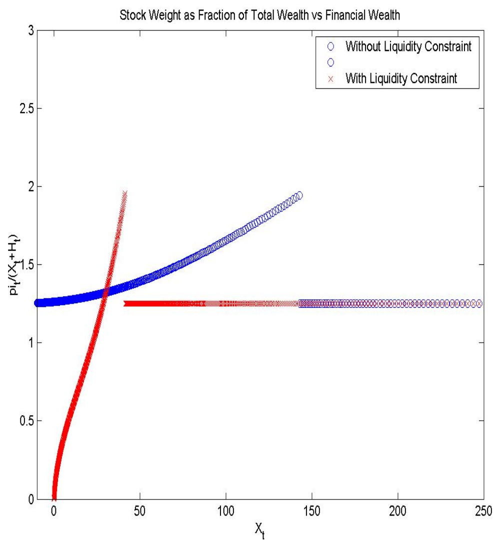

Figure 2.12: Proportion of Total Wealth invested in Stocks given Financial Wealth, for constant labor supply. Parameters: $\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\delta = 0.3$, $L = 0.8$, $\gamma = 0.8$.

Chapter 3

Asset Pricing with Heterogeneous Risk Aversion and Portfolio Constraints

3.1 Introduction

In the aftermath of the stock market crash of October 1987, considerable amount of empirical literature focussed on studying the effects of margin requirements on stock returns volatility. The issue of using margin policies as a tool to control stock volatility has been highly debated. The Katzenbach Report (1987) of NYSE analyzing the stock market crash as well as the Securities and Exchange Commission's Report (1988) recommended tightening of margin requirements to reduce the leverage of speculative traders. On the other hand several economists including the former Chairman of Federal Reserve Bank Alan Greenspan did not support the hypothesis that tighter margins can reduce stock price variability. A consensus was also lacking among the academics. The influential papers by Hardouvelis [27], [28] shortly after the crash furnished evidence that there is a negative relationship between margin requirements and stock market volatility. Subsequent empirical papers such as Ferris and Chance [25], Schwert [49] and Hsieh and Miller [33] argued that Hardouvelis' methodology is flawed and asserted that there is no support in the data to believe that a negative relationship exists between margin and volatility. Recently, Hardouvelis [29] has used sophisticated econometric methods to reaffirm his conclusions made more than a decade ago. But the issue is not settled yet. Among the theoretical work, Kupiec and Sharpe [39] developed an equilibrium model where the presence of irrational traders can induce excess volatility in the stock market. They claimed that there is no definite relation between margin requirement and stock market volatility: depending upon the heterogeneity of the agents, it is equally possible that margin requirements decrease or increase volatility.

The purpose of this paper is to investigate the effects of portfolio constraints such as margin requirements on stock price, stock returns volatility and interest rate. In particular, we are interested in knowing whether stock returns volatility increases or decreases in the presence of portfolio constraints. As mentioned above, the empirical literature is divided on the effects of margin requirements on the asset prices and volatility. We analyze such effects in a pure exchange, continuous time economy with agents having different levels of relative risk aversion.

Detemple and Murthy [19] study the effect of portfolio constraints on asset prices. They consider a model with two agents both having time separable logarithmic utility functions. In their model the aggregate consumption growth rate is unobservable to the agents, one of them having higher prior belief about the growth rate than the other. Since they differ in their inferences about the dividend growth rate, they engage in trading and the leverage constraint binds on the agent who is more optimistic while the short sale constraint binds on the pessimistic agent. Unlike their model, the growth rate of aggregate consumption is observable by both agents in our model but the agents trade because they have different risk aversion coefficients. Our model has the flexibility to study the impact of portfolio constraints on asset prices and volatility. In a model with logarithmic agents, the stock returns volatility is equal to the volatility of the underlying dividend process. Thus the portfolio constraints do not affect the stock price or its volatility. Hence they are not suitable for the purpose of studying the effects of portfolio constraints on the stock returns volatility. Bhamra and Uppal [6] have studied the effects of introducing a non-redundant derivative on the volatilities of the stock market returns and the (locally) risk-free interest rate. The authors consider a general equilibrium model with two agents with CRRA utility functions who differ in their risk aversion coefficients.

Other papers which study the effect of portfolio constraints are Hollifield and Gallmeyer [32] (short-sale restrictions) and Detemple and Serrat [20] (liquidity constraints). Basak and Cuoco [4] studies equilibrium in an economy where one class of agents face information or trading costs that prevents them from participating in the stock market (restricted stock market participation). Kogan and Uppal [38] has analyzed the effects of portfolio constraints in a general equilibrium framework with one logarithmic agent and one agent with CRRA utility function.

3.2 The Economy

We consider a continuous time, finite horizon $[0, T]$ pure exchange economy. The uncertainty is represented by a complete probability space $(\Omega, \{\mathcal{F}_t\}_{0\leq t\leq 0}, \mathcal{P})$ on which is defined a one-dimensional real-valued Brownian motion Z. The filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ represents complete information and the probability measure P is the reference measure. All the processes used in the sequel are understood to be progressively measurable with respect to $\mathcal{F}$ and all equalities involving random variables or processes are assumed to hold $\mathcal{P}$ a.s. For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ for $t \geq 0$ is the realization or trajectory of the process X associated with ω. For any two processes X and Y, X_t is said to be increasing in Y_t if there exists a set $N \in \mathcal{F}$ such that $\mathcal{P}(N) = 0$ and for all $t \in [0, T]$, for all $\omega, \omega' \in \Omega \setminus N$, we have $X_t(\omega) > X_t(\omega')$ whenever $Y_t(\omega) > Y_t(\omega')$.

There is a single perishable good used as the numeraire. c is a consumption process if it is positive, $\{\mathcal{F}_t\}$-progressively measurable and in $\mathcal{L}^2(\mathcal{P})$. The set of consumption processes is denoted by $\mathcal{C}$.

(A process f is in $\mathcal{L}^2(\mathcal{P})$ if $E\left[\int_0^T |f(t)|^2 dt\right] < \infty$)

3.2.1 Securities Market

The investment opportunities are represented by a locally risk-free bond and one risky stock. The bond earns an instantaneous interest rate r_t and is in zero net supply. The bond price process follows the dynamics

$$dB_t = r_t B_t dt \tag{3.2.1}$$

The stock is a claim to an exogenously given strictly positive dividend process δ_t which follows the dynamics

$$d\delta_t = \delta_t[\mu_\delta dt + \sigma_\delta dZ_t] \tag{3.2.2}$$

where the mean growth rate of dividends μ_δ and σ_δ are positive constants. The stock is in net supply of one unit. Let S_t denote the ex-dividend stock price process. We restrict attention to prices which are Ito processes in equilibrium:

$$dS_t + \delta_t dt = S_t[\mu_{S,t} dt + \sigma_t dZ_t] \tag{3.2.3}$$

The interest rate process r, the expected stock returns μ_S and the stock returns volatility σ are assumed to be $\{\mathcal{F}_t\}$-progressively measurable with r, μ_S in $\mathcal{L}^2(\mathcal{P})$ and σ bounded above and below away from zero. All these prices are to be determined endogenously in equilibrium.

3.2.2 Preferences and Endowments

There are two types of agents in the economy. We will call them agents 1 and 2. Agents' preferences are defined on $\mathcal{C}$ and have the following utility representations:

$$\begin{aligned} U_1(c^1) &= E\left[\int_0^T e^{-\beta t} log(c_t^1 - X_t)dt\right], c_t^1 > X_t, \; \forall t \in [0,T] \qquad (3.2.4) \\ U_2(c^2) &= E\left[\int_0^T e^{-\beta t} log(c_t^2)dt\right] \qquad (3.2.5) \end{aligned}$$

where β is a constant subjective discount factor which is identical across agents. X_t denotes the habit level of agent 1 which is external to agent 1. This means that X_t does not depend on agent 1's individual consumption but is determined by the aggregate consumption δ_t.

Agent 1 is endowed at time $t = 0$ with 1 share of stock and a short position in b shares of bond. Agent 2 is endowed with a long position of b shares of bond. Thus, at the initial time $t = 0$, agent 1 is in debt. The initial wealths of the agents are thus: $W_0^1 = S_0 - b$ and $W_0^2 = b$ where S_0 is the stock price and $B_0 = 1$ is the price of the bond at $t = 0$. In order that equilibrium exists, we have to make the assumption stated in equation (3.3.2). It says that agent 1 can repay his debt from the dividend payments obtained from the stock.

3.2.3 Consumption-Portfolio Strategies

Trading takes place continuously. At time t an agent invests π_t proportion of his wealth in the stock and the remaining proportion $(1 - \pi_t)$ of his wealth in bonds.

A portfolio trading strategy π is admissible if it is $\{\mathcal{F}_t\}$-progressively measurable and satisfies

$$\int_0^T \left(|(1-\pi_t)r_t| + |\pi_t \mu_{S,t}| + |\pi_t \sigma_t|^2\right) dt < \infty \qquad (3.2.6)$$

The set of admissible portfolio trading strategies is denoted by Π.

A portfolio trading strategy $\pi \in \Pi$ is said to finance a consumption plan $c \in \mathcal{C}$ if the corresponding wealth process W_t satisfies the dynamic budget constraint:

$$dW_t = [r_t(1-\pi_t)W_t + \mu_{S,t}\pi_t W_t - c_t]dt + \pi_t W_t \sigma_t dZ_t \tag{3.2.7}$$

A consumption plan is feasible if it is financed by an admissible portfolio trading strategy.

An arbitrage opportunity is a non-zero consumption plan $c \in \mathcal{C}$ that can be financed by some admissible portfolio trading strategy $\pi \in \Pi$ with zero initial wealth. As shown by Dybvig and Huang [21], arbitrage opportunities can be ruled out by imposing a non-negative wealth constraint, $W_t \geq 0$ a.s. $\forall t \in [0, T]$ on the class of admissible trading strategies.

We denote by $\mathcal{A}$ the set of consumption portfolio processes (c, π) such that $c \in \mathcal{C}$ is financed by the admissible portfolio trading strategy $\pi \in \Pi$.

We assume that agent 1 is unrestricted in his trading strategy while agent 2's trading strategy is subject to the following constraint:

$$\pi_t^2 \leq \bar{\pi} \; a.s. \; \forall t \in [0, T] \tag{3.2.8}$$

where $\bar{\pi}$ is a positive constant. We denote by

$$\mathcal{A}^c = \{(c, \pi) \in \mathcal{A} : \pi_t \leq \bar{\pi} \; a.s. \; \forall t \in [0, T]\} \tag{3.2.9}$$

the set of feasible consumption processes and admissible portfolio trading strategies subject to the constraint (3.2.8). $\bar{\pi}$ is the maximum proportion of wealth that agent 2 can invest in the stock. Such portfolio constraint can arise due to margin requirements. For example, when $\bar{\pi} > 1$, agent 2 cannot borrow more than $\frac{\bar{\pi}-1}{\bar{\pi}}$ of the amount of wealth invested in the stock (this follows from the fact that $(1-\pi_t)W_t$ is the amount of wealth invested in the bond, $\pi_t W_t$ is the amount of wealth invested in the stock and $\pi_t \leq \bar{\pi}$). A special case is the constraint corresponding to $\bar{\pi} = 1$. When $\bar{\pi} = 1$, agent 2 is prohibited from borrowing.

3.2.4 Equilibrium

An equilibrium is a price process (B, S) and a set $(\hat{c}^i, \hat{\pi}^i)$ for $i = 1, 2$ of consumption and admissible trading strategies for the two agents such that

(i) $\hat{\pi}^i$ finances $\hat{c}^i$ for $i = 1, 2$;

(ii) $\hat{c}^1$ maximizes U^1 over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $\pi \in \Pi$ with initial wealth $W_0^1 = S_0 - b$;

(iii) $\hat{c}^2$ maximizes U^2 over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $\pi \in \Pi$ with initial wealth $W_0^2 = b$ and $\pi_t \leq \bar{\pi}$ a.s. $\forall t \in [0, T]$;

(iv) Goods and security markets clear; that is, almost surely $\forall t \in [0, T]$

$$\hat{c}_t^1 + \hat{c}_t^2 = \delta_t \tag{3.2.10}$$

$$\hat{\pi}_t^1 W_t^1 + \hat{\pi}_t^2 W_t^2 = S_t \tag{3.2.11}$$

$$(1 - \hat{\pi}_t^1)\hat{W}_t^1 + (1 - \hat{\pi}_t^2)\hat{W}_t^2 = 0 \tag{3.2.12}$$

3.3 Specification of the Process for Habit Level

The main motive of this paper is to find the effects of portfolio constraints on the stock returns volatility. In the absence of constraints, it has been well documented that stock returns volatility (see Schwert [50]) and risk premium (see Fama and French [23] and Ferson and Harvey [26]) are related to the business cycle: they are counter-cyclical, high when stock price is low and low when stock price is high. Moreover, risk-premia and stock returns volatility do not vary much in rising markets but fluctuate more in falling markets.

There are two types of models that are employed to explain these empirical facts. The first type of models introduces time varying risk-premia as their state variable whereas the second type of models use time-varying expected dividend growth as the state variable. Examples of models which generate time-varying risk premia are the habit formation models of Campbell and Cochrane [12], Menzly, Santos and Veronesi [43] among others. These models relate the risk premia, interest rates, and returns volatility to the preferences of the agents. It is found that models with external habit formation lead to counter-cyclical volatility. Models of the second type that make more specific assumptions about the underlying dividend process include Veronesi [53], Brennan and Xia [10], and Bansal and Yaron [2] among others. These models also predict counter-cyclical volatility.

We follow the external habit formation models of Campbell and Cochrane [12] and Menzly, Santos and Veronesi [43]. These authors introduce a time-varying habit level X_t to the standard power utility function as in (3.2.4). Instead of working with X_t directly, we follow Menzly, Santos and Veronesi [43] and define the habit level X_t via the relation $Y_t = \frac{\delta_t}{\delta_t - X_t}$. Therefore the dynamics of Y_t has to be specified in such a way that the economically sensible properties of X_t are satisfied. The most desirable properties of X_t are the following:

(i) X_t must be bounded between 0 and the aggregate consumption δ_t to ensure that the marginal utility of agent 1 remains finite. If we specify a dynamics of X_t independent of the specification of δ_t, then there is always a non-zero probability that X_t will exceed δ_t at some time t. Since Y_t is a function of δ_t and the difference $\delta_t - X_t$, we would specify a dynamics for Y_t such that Y_t always remains above zero, i.e. X_t never exceeds δ_t.

(ii) An individual agent's habit level X_t must "catch up" and be positively correlated with the aggregate consumption δ_t. Since the aggregate consumption process δ_t has been specified to be a random walk with drift as in (3.2.2), X_t must also follow the same stochastic process. A deterministic dynamics of X_t would not be able to capture the "catching up with the Joneses" idea. For example, when the aggregate consumption suffers a negative shock, the habit level must be adjusted downwards and in case of a positive shock it should be adjusted upward. A deterministic specification of X_t will not suffice for this purpose as aggregate consumption can fall below the habit level with such a specification. Again, by assuming that Y_t follows a stochastic process, we can formalize the idea of "catching up with the Joneses".

(iii) X_t must adjust slowly in response to the change in aggregate consumption. As Campbell and Cochrane [12] puts it, this captures a fundamental feature of human psychology : "repetition of a stimulus diminishes the perception of the stimulus and responses to it". For example, when the economy has been in booms for some time, the habit level becomes very high. A slight downturn in the economy now makes the agent far more worse than he would have been if the same magnitude of downturn was experienced in periods of recession. That is, when the recession strikes after periods of boom, the agent takes time to adjust his habit level downwards. This feature cannot be captured by specifying the process for X_t alone; we should consider the dynamics of a variable which is a function of the difference $\delta_t - X_t$ and δ_t. The variable Y_t serves this purpose.

The above properties of X_t are translated into the following properties of Y_t:

(i) Y_t is bounded below by some positive number so that the restriction $\delta_t > X_t$ is always satisfied;

(ii) Y_t is negatively perfectly correlated with the innovations to the aggregate consumption process δ_t;

(iii) Y_t is mean-reverting;

(iv) Y_t has a time-varying volatility.

Following Menzly, Santos, Veronesi [43] we assume the following process for Y_t:

$$dY_t = k(\bar{Y} - Y_t)dt - \alpha(Y_t - \bar{\lambda})\sigma_\delta dZ_t \tag{3.3.1}$$

where $\bar{Y}$ is the long-run mean of Y_t, k is the speed of the mean reversion, $\alpha > 0$ and $\bar{Y} > \bar{\lambda}$. This process for Y_t ensures that $0 < X_t < \delta_t$ almost surely for $t \in [0, T]$. With a positive value of α, a negative shock to the aggregate consumption δ_t leads to an increase in Y_t so that property (ii) is satisfied. The parameter $\bar{\lambda} \geq 1$ ensures a lower bound for Y_t: once Y_t reaches this lower bound, the diffusion term is zero and the drift term is positive (since $\bar{Y} > \bar{\lambda}$) and Y_t begins to increase and remain above $\bar{\lambda}$.

We make the following assumption which says that it is possible for agent 1 to repay his debt from the future dividend payoff from the stock.

Assumption:

$$b < \frac{1 - e^{-\beta T}}{\beta} \frac{\delta_0}{Y_0} \tag{3.3.2}$$

3.4 Unconstrained Economy

We begin by setting up the optimization problems of the agents when both of them are unconstrained in their trading strategies. We will call the economy studied in this section as the unconstrained economy. In Section 3.5, we consider the economy where agent 2 is constrained in his portfolio holdings. We will call that economy the constrained economy.

We begin with agent 1's optimization problem. His lifetime consumption-portfolio problem is

$$\underset{(c^1,\pi^1)\in\mathcal{A}}{Max} E\left[\int_0^T e^{-\beta t} log(c_t^1 - X_t)dt\right] \tag{3.4.1}$$

subject to the dynamic budget constraint

$$dW_t^1 = [r_t(1 - \pi_t^1)W_t^1 + \mu_{S,t}\pi_t^1 W_t^1 - c_t^1]dt + \pi_t^1 W_t^1 \sigma_t dZ_t \tag{3.4.2}$$

with $c_t^1 > X_t$, $W_t^1 \geq 0 \; \forall t \in [0, T]$ and $W_0^1 = S_0 - b$.

Agent 2's optimization problem is

$$\underset{(c^2,\pi^2)\in\mathcal{A}}{Max} E\left[\int_0^T e^{-\beta t} log(c_t^2)dt\right] \tag{3.4.3}$$

subject to the dynamic budget constraint

$$dW_t^2 = \left[r_t(1 - \pi_t^2)W_t^2 + \mu_{S,t}\pi_t^2 W_t^2 - c_t^2\right] dt + \pi_t^2 W_t^2 \sigma_t dZ_t \tag{3.4.4}$$

with $W_t^2 \geq 0$ for $t \in [0, T]$ and $W_0^2 = b$.

We solve for the equilibrium quantities in two steps. First, following Cox and Huang [16] and Karatzas, Lehoczky and Shreve [35] we transform the above optimization problems with dynamic budget constraints into optimization problems with static budget constraints and find the optimal consumption sharing rule of the agents. In the second step, we find the prices which support the optimal allocations.

For the price processes (B, S), markets are complete if the stock price volatility σ is non-zero. We solve the model with the assumption that $\sigma_t \neq 0$ a.s. $\forall t \in [0, T]$ and then verify that the volatility is indeed non-zero (in fact strictly positive) in equilibrium, so that markets are complete in equilibrium. This implies the existence of a unique non-negative state price density process ξ_t which satisfies the following dynamics

$$d\xi_t = -\xi_t[r_t dt + \kappa_t dZ_t], \; \xi_0 = 1 \tag{3.4.5}$$

where the market price of risk κ_t (Sharpe ratio) is defined as

$$\kappa_t = \frac{\mu_{S,t} - r_t}{\sigma_t}$$

Following Cox and Huang [16], the optimization problem (3.4.1) subject to (3.4.2) of agent 1 can be rewritten as a consumption-portfolio choice problem with static budget constraint as follows:

$$\underset{\{c_t^1 : c_t^1 > X_t\}}{Max} \; E\left[\int_0^T e^{-\beta t} log(c_t^1 - X_t) dt\right] \tag{3.4.6}$$

subject to

$$E\left[\int_0^T \xi_t c_t^1 dt\right] \leq W_0^1 \tag{3.4.7}$$

where $W_0^1 = S_0 - b$.

The Lagrangian of this problem is

$$\mathcal{L}^1 = E\left[\int_0^T e^{-\beta t} log(c_t^1 - X_t) dt\right] - \lambda_1 \left[E\left[\int_0^T \xi_t c_t^1 dt\right] - W_0^1\right]$$

Maximizing point-wise with respect to c_t^1, the optimal consumption of agent 1 is given by

$$\hat{c}_t^1 = X_t + \frac{e^{-\beta t}}{\lambda_1 \xi_t} \tag{3.4.8}$$

The Lagrange multiplier λ_1 satisfies the static budget constraint (3.4.7) with equality:

$$E\left[\int_0^T \xi_t \hat{c}_t^1 dt\right] = W_0^1 = S_0 - b \tag{3.4.9}$$

Similar to agent 1's problem, we can restate the optimization problem (3.4.3, 3.4.4) of agent 2 as a consumption-portfolio choice problem with a static budget constraint:

$$\underset{\{c_t^2\}}{Max} E\left[\int_0^T e^{-\beta t} log(c_t^2) dt\right] \tag{3.4.10}$$

subject to

$$E\left[\int_0^T \xi_t c_t^2 dt\right] \leq W_0^2 \tag{3.4.11}$$

where $W_0^2 = b$.

The Lagrangian of this problem is

$$\mathcal{L}^2 = E\left[\int_0^T e^{-\beta t} log(c_t^2) dt\right] - \lambda_2 \left[E\left[\int_0^T \xi_t c_t^2 dt\right] - W_0^2\right]$$

Maximizing point-wise with respect to c_t^2, the optimal consumption of agent 2 is given by

$$\hat{c}_t^2 = \frac{e^{-\beta t}}{\lambda_2 \xi_t} \tag{3.4.12}$$

The Lagrange multiplier λ_2 satisfies the static budget constraint (3.4.11) with equality:

$$E\left[\int_0^T \xi_t \hat{c}_t^2 dt\right] = W_0^2 = b \tag{3.4.13}$$

The following proposition provides expressions for the equilibrium consumption allocations and the state price density in terms of the state variables of the economy, viz. δ_t and Y_t. Let us define the variable $Y_t = \frac{\delta_t}{\delta_t - X_t}$.

Proposition 3.4.1. *There exists an equilibrium in the unconstrained economy. In equilibrium, the optimal consumption allocations, state price density and stock price satisfy the following equations:*

$$\hat{c}_t^1 = \left[1 - \frac{\lambda_1}{(\lambda_1+\lambda_2)Y_t}\right]\delta_t \tag{3.4.14}$$

$$\hat{c}_t^2 = \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\delta_t}{Y_t} \tag{3.4.15}$$

$$\xi_t = e^{-\beta t}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\frac{Y_t}{\delta_t} \tag{3.4.16}$$

$$S_t = \frac{1}{\xi_t}E_t\left[\int_t^T \xi_u \delta_u du\right] \tag{3.4.17}$$

where the Lagrangian multipliers λ_1 and λ_2 are positive constants given by:

$$\lambda_1 = \left[\frac{\delta_0}{Y_0} - \frac{\beta b}{1-e^{-\beta T}}\right]^{-1} > 0 \tag{3.4.18}$$

$$\lambda_2 = \frac{1-e^{-\beta T}}{\beta b} > 0 \tag{3.4.19}$$

Proof: See Appendix B.

The curvature of the utility function of agent 1 at the optimal consumption policy $\hat{c}_t^1$ is

$$-\frac{\hat{c}_t^1 u_{cc}(\hat{c}_t^1)}{u_c(\hat{c}_t^1)} = \frac{\hat{c}_t^1}{\hat{c}_t^1 - X_t} \tag{3.4.20}$$

Using the expression for $\hat{c}_t^1$ from (3.4.14), the risk aversion coefficient of agent 1 at the optimal consumption policy $\hat{c}_t^1$ is

$$-\frac{\hat{c}_t^1 u_{cc}(\hat{c}_t^1)}{u_c(\hat{c}_t^1)} = \left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)Y_t + \frac{\lambda_1}{\lambda_2} \tag{3.4.21}$$

which is increasing in Y_t. Y_t increases when the difference between the aggregate consumption δ_t and the habit level X_t gets smaller. Thus, when the aggregate consumption δ_t is close to the habit level X_t, agent 1 becomes more risk averse. The optimal consumption processes of the agents monotonically increase with the aggregate consumption δ_t. When δ_t increases, the stock pays higher dividends and hence the consumption of the agents increase.

Equations (3.4.14) and (3.4.15) show that with decrease in Y_t the optimal consumption of agent 1 decreases while that of agent 2 increases. During booms, the dividend process δ_t receives a positive shock and increases further away from the habit level X_t. The habit level X_t takes time to adjust upwards in response to the increased aggregate consumption δ_t. Hence, the difference $\delta_t - X_t$ becomes large. Thus, $Y_t = \frac{\delta_t}{\delta_t - X_t}$ is low during recession. By (3.4.21), agent 1 becomes less risk averse and invests more of his wealth in the stock market while decreasing consumption $\hat{c}_t^1$. The goods market clearing condition then requires that agent 2 increase his consumption $\hat{c}_t^2$ in the good states. During recessions, the dividend process δ_t suffers a negative shock and declines towards the habit level X_t. Since it takes time to adjust the habit level X_t downwards in response to the reduced consumption level, the difference $\delta_t - X_t$ becomes small. Thus, Y_t is high during recession. According to (3.4.21), agent 1 becomes more risk averse during recession. This leads to a decrease in investment in stocks and increase in the consumption level $\hat{c}_t^1$. Agent 2, on the other hand, decreases his consumption $\hat{c}_t^2$ while increasing his investment in the stock.

In Proposition 1 we derived the optimal consumption allocations and the state price density. The next proposition characterizes the equilibrium prices that support the optimal consumption allocations. Let us define the following quantities:

$$\rho_t = \int_t^T e^{-\beta(u-t)} du = \frac{1 - e^{-\beta(T-t)}}{\beta} \tag{3.4.22}$$

$$\chi_t = \int_t^T e^{-(\beta+k)(u-t)} du = \frac{1 - e^{-(\beta+k)(T-t)}}{\beta + k} \tag{3.4.23}$$

It is easy to verify that $\rho_t > \chi_t$ for all $t \in [0, T]$.

Proposition 3.4.2. *In equilibrium the price processes are given as follows.*

(i) The equilibrium stock price-dividend ratio satisfies

$$\frac{S_t}{\delta_t} = \left(1 - \frac{\bar{Y}}{Y_t}\right) \chi_t + \frac{\rho_t \bar{Y}}{Y_t} \tag{3.4.24}$$

(ii) The equilibrium stock returns volatility satisfies

$$\sigma_t = \sigma_\delta + \frac{\alpha\sigma_\delta(Y_t - \bar{\lambda})}{Y_t}\frac{(\rho_t - \chi_t)\bar{Y}}{(Y_t - \bar{Y})\chi_t + \rho_t\bar{Y}} > \sigma_\delta \tag{3.4.25}$$

(iii) The equilibrium interest rate is given by

$$r_t = \beta + \mu_\delta - \sigma_\delta^2 + k\left(1 - \frac{\bar{Y}}{Y_t}\right) - \alpha\sigma_\delta^2\left(1 - \frac{\bar{\lambda}}{Y_t}\right) \tag{3.4.26}$$

(iv) The equilibrium market price of risk is given by

$$\kappa_t = \sigma_\delta + \alpha\sigma_\delta\left(1 - \frac{\bar{\lambda}}{Y_t}\right) \tag{3.4.27}$$

(v) The equilibrium stock returns is given by

$$\mu_{S,t} = \beta + \mu_\delta + k\left(\frac{Y_t - \bar{Y}}{Y_t}\right) + \sigma_\delta^2\left[1 + \alpha\left(\frac{Y_t - \bar{\lambda}}{Y_t}\right)\right]\frac{\alpha(\rho_t - \chi_t)(Y_t - \bar{Y})\bar{Y}}{Y_t\left((\rho_t - \chi_t)\bar{Y} + \chi_t Y_t\right)} \tag{3.4.28}$$

(vi) In equilibrium, the optimal wealth processes of the agents satisfy

$$\hat{W}_t^1 = \frac{\delta_t}{Y_t}\left[(\rho_t - \chi_t)\bar{Y} + \chi_t Y_t - \frac{\lambda_1}{\lambda_1 + \lambda_2}\int_t^T e^{-\beta(u-t)}du\right] \tag{3.4.29}$$

$$\hat{W}_t^2 = \frac{\delta_t}{Y_t}\left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\int_t^T e^{-\beta(u-t)}du\right] \tag{3.4.30}$$

(vii) The optimal portfolio holdings are given by

$$\hat{\pi}_t^1 = 1 - \frac{\alpha\chi_t(Y_t - \bar{\lambda})}{(\rho_t - \chi_t)\bar{Y}\left[1 + \frac{\alpha(Y_t - \bar{\lambda})}{Y_t}\right] + \chi_t Y_t}\frac{W_t^2}{W_t^1} \tag{3.4.31}$$

$$\hat{\pi}_t^2 = 1 + \frac{\alpha\chi_t(Y_t - \bar{Y})}{(\rho_t - \chi_t)\bar{Y}\left[1 + \frac{\alpha(Y_t - \bar{\lambda})}{Y_t}\right] + \chi_t Y_t} \tag{3.4.32}$$

Proof: See Appendix B.

We now discuss the characterization of equilibrium prices.

(i) *Stock Price:* Since $\rho_t > \chi_t$, the expression for the stock price in (3.4.24) shows that it is monotonically decreasing in Y_t. An increase in Y_t increases the risk aversion of agent 1 who decreases his investment in the stock. Since agent 2's risk aversion does not change, the net demand for stock is reduced. Therefore the stock price decreases to induce agents to hold more stocks in equilibrium. Similarly, when Y_t decreases, agent 1's risk aversion decreases which induces him to increase his investment in the stock. The net demand for the stock increase and thus the stock price also increases.

At any time $t \in [0, T]$ the stock price is maximum at $S_t = \rho_t \delta_t$ when $Y_t = \bar{Y}$ and then continuously decreases with increase in Y_t. $Y_t = \bar{Y}$ corresponds to the situation when both agents have logarithmic preferences. Therefore, in the economy with heterogeneous preferences, the stock price is always lower compared to the economy with homogeneous preferences.

(ii) *Stock Price Volatility:* Equation (3.4.25) shows that the stock price volatility σ_t is always greater than the aggregate consumption volatility σ_δ if $Y_t > \bar{Y}$. Since both agents have logarithmic utility functions, $\sigma_t = \sigma_\delta$ at $Y_t = \bar{Y}$. The stock price volatility increases with Y_t till Y_t reaches $\bar{Y} + \sqrt{\frac{\rho_t}{\chi_t}}\bar{Y}$; thereafter it decreases. Since the stock price is strictly decreasing in Y_t, in the range $(\bar{Y}, \bar{Y} + \sqrt{\frac{\rho_t}{\chi_t}}\bar{Y})$, volatility increases as stock price decreases. The increase in the volatility is quite steep. The volatility reaches its maximum at $Y_t = \bar{Y} + \sqrt{\frac{\rho_t}{\chi_t}}\bar{Y}$, and then decreases to σ_δ the decrease being slow.

The negative correlation between the stock price and its volatility in the range $(\bar{Y}, \bar{Y} + \sqrt{\frac{\rho_t}{\chi_t}}\bar{Y})$ can be explained as follows. Suppose there is a negative shock to the dividends so that Y_t increases. In equilibrium, this makes the stock less attractive for investment and its price decreases. If both agents in the economy had logarithmic utility functions with coefficient of relative risk aversion equal to unity, then the decrease in the stock price would be in the same proportion as the increase in dividends. The volatility of the stock price would then be equal to the volatility of the dividend process (this happens when $Y_t = \bar{Y}$). In our economy, agent 1 has time varying, counter-cyclical risk aversion coefficient always exceeding unity while agent 2's coefficient of relative risk aversion equals unity. Therefore, agent 1 becomes more risk averse when the stock price is low and is less willing to invest in the stock. To clear the securities market, the stock price has to decrease by a proportion higher than the decrease in the dividends. This makes the stock price more volatile than the volatility of the dividend process. Similarly, a positive shock to dividends decreases Y_t. Due to counter-cyclical risk aversion, agent 1 becomes less risk averse and invests more wealth in the stock. The net demand for the stock increases which results in an increase in the stock price. The increase in stock price is proportionately more compared to the increase in dividends. Therefore, the stock price exhibits excess volatility in the economy with heterogeneous preferences.

This is consistent with the well known fact documented by Shiller [51]: stock prices are more volatile compared to the dividend process to which they are claims. Several authors like Black [7], and Schwert [50] have found that volatility rises following a decline in stock prices and falls following a rise in stock prices. As discussed above, when the state variable Y_t is in the range $\bar{Y} + \sqrt{\frac{\rho_t}{\chi_t}}\bar{Y}$, the stock price behavior is consistent with the empirical findings of these authors.

(iii) *Interest Rate:* The first three terms in the expression for the interest rate (3.4.26) are as in the standard model with a logarithmic agent. With higher impatience (β high), agents would like to increase present consumption so that the interest rate has to be high to induce them to save. With a high dividend growth (high μ_δ) agents increase present consumption, saving less today, thus raising the interest rate in equilibrium. Finally, increased volatility in aggregate consumption (high σ_δ) induces precautionary savings; agents worry about the low state of consumption tomorrow and save more today; the interest rate decreases to counter this effect. To smooth consumption, agents increase today's consumption and saves less (or increase borrowing) This increases the equilibrium interest rate. The term $(k - \alpha\sigma_\delta^2)\left(\frac{Y_t - \bar{Y}}{Y_t}\right)$ appears due to the time varying risk aversion of agent 1. The first component represents the inter-temporal substitution effect: when Y is high, it is expected to decrease in the future due to its mean reverting nature. Aggregate consumption δ is therefore expected to increase in the future. To smooth consumption, agents increase today's consumption and saves less (or increase borrowing). This increases the equilibrium interest rate. This effect increases with k which is the speed at which Y_t reverts from a high level to a low level. The second component arises because of precautionary savings motive. When Y_t is high, agents worry that aggregate consumption is close to the habit level; therefore they save more today to increase future consumption and decrease the probability that consumption falls to the habit level in the future. This precautionary savings motive decreases the equilibrium interest rate. This effect is increasing in α because a high value of α can magnify the effect of a negative shock in aggregate consumption on Y_t and induce agents to save more.

(iv) *Sharpe Ratio:* We note that the market price of risk κ_t is also the volatility of the stochastic discount factor ξ_t. If both agents have logarithmic utility preference (this would happen if $Y_t = \bar{Y} = \bar{\lambda} = 1$), the market price of risk would be only σ_δ, volatility of the aggregate consumption. For $Y_t > \bar{Y}$, agent 1 becomes more risk averse than the logarithmic agent so that net demand for the stock falls. Since the stock is in fixed supply, the Sharpe ratio increases to clear the stock market. Therefore, in the economy with heterogeneous preference the market price of risk responds to the risk aversion of the non-logarithmic agent and hence higher than the volatility of the aggregate consumption.

(v) *Optimal Portfolio Weight $\hat{\pi}_t^2$:* The expressions for the optimal portfolio weights (3.4.31) and (3.4.32) show that when $Y_t > \bar{Y}$, agent 1 takes a long position in the bond and invests less than 100% of his wealth in the stock. On the other hand, agent 2 takes a short position in the bond and invests more than 100% of his wealth in the stock. Thus, in equilibrium, agent 1 is the lender market and agent 2 is the borrower. When $Y_t = \bar{Y} = \bar{\lambda}$, both the agents have logarithmic preferences and invest all their wealth in the stock market so that $\hat{\pi}_t^1 = \hat{\pi}_t^2 = 1$.

Differentiating (3.4.32) with respect to Y_t and using the fact that $Y_t > \bar{Y} > \bar{\lambda}$, it can be verified that agent 2's optimal demand for stock $\hat{\pi}_t^2$ is increasing in Y_t. In particular, agent 2 reduces his investment in the stock as Y_t decreases. The reason is that with a decrease in Y_t, agent 1 becomes less risk averse and increases his portfolio holdings in the stock. Due to excess demand for the stock, its price rises. Since agent 1 is a net lender, the fraction of his wealth invested in the stock market tends to increase; on the other hand, since agent 2 is a net borrower, his fraction of wealth invested in the stock tends to decrease. (Note that $W_t^1 = \theta_{S,t}S_t + \theta_{B,t}B_t$ where $\theta_{S,t}$ and $\theta_{B,t}$ are the amount of stock and bonds held by agent 1. Then, $\pi_t^1 = \frac{\theta_{S,t}S_t}{\theta_{S,t}S_t+\theta_{B,t}B_t}$. Now, $\theta_{B,t}$ is positive since agent 1 is a net lender so that as S_t increases, π_t^1 increases. For agent 2, $\theta_{B,t}$ is negative and hence with an increase in S_t, π_t^2 decreases.) Similarly, $\hat{\pi}_t^2$ increases with increase in Y_t. Therefore, if agent 2 faces a portfolio constraint of the form $\pi_t^2 \leq \bar{\pi}$, the constraint is likely to bind when Y_t is high. For example, the borrowing constraint corresponding to $\bar{\pi} = 1$ will bind whenever $Y_t > \bar{Y}$, whereas constraints corresponding to $\bar{\pi} > 1$ will bind only for sufficiently high values of Y_t depending on how large $\bar{\pi}$ is. We will analyze the equilibrium asset prices in the presence of borrowing constraints in the next section.

3.5 Constrained Economy

Our objective in this section is to find the effects of portfolio constraints on the stock price, stock returns, stock returns volatility and the interest rate. We assume that agent 1 is unrestricted in his trading strategy while agent 2's trading strategy is subject to the following constraint:

$$\pi_t^2 \leq \bar{\pi} \; a.s. \; \forall t \in [0, T]$$

where $\bar{\pi}$ is a non-negative constant. To characterize the equilibrium in the constrained economy we first consider the consumption portfolio choice problems of individual agents. Agent 1's optimization problem is exactly the same as in the unconstrained economy i.e. he maximizes (3.4.1) subject to the dynamic budget constraint (3.4.2). Agent 2, however, maximizes (3.4.3) subject to (3.4.4) and the portfolio constraint $\pi_t^2 \leq \bar{\pi}$. Unlike the unconstrained economy where both agents had a common state price density process ξ_t, agents in the constrained economy will have different state price densities. This is because agents can perfectly share risk in the unconstrained economy by trading in the stock market which is not possible when agent 2 is constrained in his investment strategy.

Agent 1's investment opportunity set is given by (3.2.1) and (3.2.3). Given the price processes (B, S), a state price density process ξ_t^1 can be constructed satisfying the following dynamics

$$d\xi_t^1 = -\xi_t^1[r_t dt + \kappa_t^1 dZ_t],\ \xi^1(0) = 1 \tag{3.5.1}$$

where the market price of risk κ_t^1 for agent 1 is defined as

$$\kappa_t^1 = \frac{\mu_{S,t} - r_t}{\sigma_t}$$

Agent 1 does not face portfolio constraints. Therefore, the optimization problem of agent 1 is (3.4.1) subject to (3.4.2). The optimal consumption of agent 1 is then given by

$$\hat{c}_t^1 = X_t + \frac{e^{-\beta t}}{\hat{\lambda}_1^c \xi_t^1} \tag{3.5.2}$$

The Lagrange multiplier $\hat{\lambda}_1^c$ satisfies the static budget constraint with equality:

$$E\left[\int_0^T \xi_t^1 \hat{c}_t^1 dt\right] = W_0^1$$

Next, we turn to the optimization problem of agent 2 who faces portfolio constraints. Agent 2's optimization problem is

$$\underset{(c^2,\pi^2)\in\mathcal{A}^c}{Max} E\left[\int_0^T e^{-\beta t} log(c_t^2) dt\right] \tag{3.5.3}$$

subject to the dynamic budget constraint

$$dW_t^2 = \left[r_t(1-\pi_t^2)W_t^2 + \mu_{S,t}\pi_t^2 W_t^2 - c_t^2\right] dt + \pi_t^2 W_t^2 \sigma_t dZ_t \tag{3.5.4}$$

with $W_t^2 \geq 0$ a.s. $\forall t \in [0, T]$ and $W_0^2 = b$. The method for solving the constrained agent's problem has been developed by Cvitanic and Karatzas [18]. Let $\mathbf{H}$ denote the space of processes $\nu = \{\nu_t :$

$0 \leq t \leq T\}$ that are $\{\mathcal{F}_t\}$-progressively measurable, with values in $\mathbf{R}$ such that $E\left[\int_0^{\mathrm{T}} \nu_t^2 dt\right] < \infty$. For all $t \in [0, T]$, let $K^{\bar{\pi}}$ be the non-empty, closed, convex subset of $\mathbf{R}$ defined as

$$K^{\bar{\pi}} \quad = \quad \{\pi_t^2 \in \mathbf{R} : \pi_t^2 \leq \bar{\pi}\}$$

The support function of $-K^{\bar{\pi}}$ is the function $g^{\bar{\pi}} : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ defined as

$$g^{\bar{\pi}}(x) = \underset{\pi_t^2 \in K^{\bar{\pi}}}{sup} - (\pi_t^2 x) = \left\{ \begin{array}{cc} -\bar{\pi}x & \text{if} x \in \tilde{K}^{\bar{\pi}} \\ \infty & \text{otherwise} \end{array} \right\}$$

where $\tilde{K}^{\bar{\pi}}$ denotes the effective domain of the support function:

$$\tilde{K}^{\bar{\pi}} = \{x \in \mathbf{R} : g^{\bar{\pi}}(x) < \infty\} = \{x \in \mathbf{R} : x \leq 0\} = \mathbf{R}_{-}$$

Let $\mathcal{Q} = \{\nu \in \mathbf{H} : E\left[\int_0^{\mathrm{T}} g^{\bar{\pi}}(\nu_t) dt\right] < \infty\}$. Then $\nu \in \mathcal{Q}$ implies that $\nu_t \in \tilde{K}^{\bar{\pi}}$ a.s. $\forall t \in [0, T]$. For any given $\nu \in \mathcal{Q}$ consider the new financial market $\mathcal{M}_\nu$ in which the bond and stock prices satisfy:

$$dB_t^\nu = \left(r_t + g^{\bar{\pi}}(\nu_t)\right) B_t^\nu dt \tag{3.5.5}$$

$$dS_t^\nu \quad = \quad S_t^\nu[(\mu_{S,t} + \nu_t + g^{\bar{\pi}}(\nu_t))dt + \sigma_t dZ_t] \tag{3.5.6}$$

In $\mathcal{M}_\nu$, agent 2 is unconstrained, i.e. he can borrow/lend at the interest rate $r_t^\nu = r_t + g^{\bar{\pi}}(\nu_t)$ and invest in the stocks whose returns are $\mu_{S,t}^\nu = \mu_{S,t} + \nu_t + g^{\bar{\pi}}(\nu_t)$.

The wealth process $W_{\nu,t}^2$ corresponding to a given portfolio-consumption process (π_t^2, c_t^2) in $\mathcal{M}_\nu$ satisfies

$$dW_{\nu,t}^2 \quad = \quad \left[(r_t + g^{\bar{\pi}}(\nu_t)) + \pi_t^2(\mu_{S,t} - r_t + \nu_t)\right] W_{\nu,t}^2 dt - c_t^2 dt + \pi_t^2 \sigma_t W_{\nu,t}^2 dZ_t \tag{3.5.7}$$

with $W_{\nu,0}^2 = b$, $W_{\nu,t}^2 \geq 0$ a.s. $\forall t \in [0, T]$.

Agent 2's *unconstrained* optimization problem in $\mathcal{M}_\nu$ is then:

$$\underset{\{c^2, \pi^2\}}{Max}\, E\left[\int_0^T e^{-\beta t} log(c_t^2) dt\right] \tag{3.5.8}$$

subject to the dynamic budget constraint (3.5.7).

Given the price processes (3.5.5) and (3.5.6), the state price density $\xi^2_{\nu,t}$ in the financial market M_ν satisfies the following dynamics

$$d\xi^2_{\nu,t} = -\xi^2_{\nu,t}\left[(r_t + g^{\bar{\pi}}(\nu_t))dt + \kappa^2_{\nu,t}dZ_t\right] \tag{3.5.9}$$

with $\xi^2_{\nu,0} = 1$ and

$$\begin{aligned}\kappa^2_{\nu,t} &= \frac{\mu^\nu_t - r^\nu_t}{\sigma_t} = \frac{\mu_{S,t} - r_t + \nu_t}{\sigma_t} \\ &= \frac{\mu_{S,t} - r_t}{\sigma_t} + \frac{\nu_t}{\sigma_t}\end{aligned} \tag{3.5.10}$$

Cvitanic and Karatzas [18] showed that the optimization problem (3.5.3), (3.5.4) can be solved by considering the following dual minimization problem associated with the constraint set $K^{\bar{\pi}}$:

$$\underset{\lambda^c_2 \in \mathbf{R}_+, \nu \in \mathcal{Q}}{min} E\left[\int_0^T e^{-\beta t}\tilde{u}_2(\lambda^c_2\xi^2_{\nu,t})dt + \lambda^c_2 W^2(0)\right] \tag{3.5.11}$$

where

$$\tilde{u}_2(y) = \underset{x>0}{Max}(u_2(x) - yx)$$

Since $u_2(c^2_t) = log(c^2_t)$, (3.5.11) becomes

$$\underset{\lambda^c_2 \in \mathbf{R}_+, \nu \in \mathcal{Q}}{min} E\left[-\int_0^T e^{-\beta t}(1 + \beta t + log(\lambda_2\xi^2_{\nu,t}))dt + \lambda^c_2 W^2(0)\right] \tag{3.5.12}$$

Performing the minimization with respect to λ^c_2 yields

$$\hat{\lambda}^c_2 = \frac{1 - e^{-\beta T}}{\beta W^2(0)} \tag{3.5.13}$$

Next, using (3.5.9), (3.5.13) and minimizing the expression inside the expectation operator in (3.5.12) point-wise with respect to ν_t gives

$$\begin{aligned}\nu^*_t &= \underset{\nu_t \in \tilde{K}^{\bar{\pi}}}{argmin}\left[2g^{\bar{\pi}}(\nu_t) + \left(\frac{\mu_{S,t} - r_t}{\sigma_t} + \frac{\nu_t}{\sigma_t}\right)^2\right] \\ &= \underset{\nu_t \in \tilde{K}^{\bar{\pi}}}{argmin}\left[-2\bar{\pi}\nu_t + \left(\frac{\mu_{S,t} - r_t}{\sigma_t} + \frac{\nu_t}{\sigma_t}\right)^2\right] \\ &= Min\left\{\sigma_t\left[\bar{\pi}\sigma_t - \frac{\mu_{S,t} - r_t}{\sigma_t}\right], 0\right\}\end{aligned} \tag{3.5.14}$$

For subsequent use, we define

$$\hat{\nu}_t = -\frac{\nu_t^*}{\sigma_t} = Max\left\{\left[\frac{\mu_{S,t} - r_t}{\sigma_t} - \bar{\pi}\sigma_t\right], 0\right\} \geq 0 \tag{3.5.15}$$

Then $\xi^2_{\hat{\nu},t}$ follows the dynamics

$$d\xi^2_{\hat{\nu},t} = -\xi^2_{\hat{\nu},t}\left[(r_t + \bar{\pi}\hat{\nu}_t\sigma_t)dt + \kappa^2_{\hat{\nu},t}dZ_t\right] \tag{3.5.16}$$

where

$$\kappa^2_{\hat{\nu},t} = \frac{\mu_{S,t} - r_t}{\sigma_t} - \hat{\nu}_t \tag{3.5.17}$$

Theorems 9.1 and 10.1 in Cvitanic and Karatzas [18] establish that the optimal consumption-portfolio policies in the auxiliary market $\mathcal{M}_{\hat{\nu}}$ are also optimal for the constrained optimization problem (3.5.3), (3.5.4). The optimal consumption policy of agent 2 is given by

$$\hat{c}_t^2 = (e^{\beta t}\xi^2_{\hat{\nu},t}\hat{\lambda}_2^c)^{-1} \tag{3.5.18}$$

where $\hat{\lambda}_2^c$ is given by (3.5.13).

In the following proposition we derive the equilibrium state price densities and optimal consumption allocations.

Proposition 3.5.1. *In equilibrium, the agents' state price densities satisfy*

$$\xi_t^1 = e^{-\beta t}\left(\frac{1+\eta_t}{\hat{\lambda}_1^c}\right)\frac{Y_t}{\delta_t} \tag{3.5.19}$$

$$\xi^2_{\hat{\nu},t} = e^{-\beta t}\left(\frac{1+\eta_t}{\hat{\lambda}_2^c\eta_t}\right)\frac{Y_t}{\delta_t} \tag{3.5.20}$$

where $\eta_t = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}\frac{\xi_t^1}{\xi^2_{\hat{\nu},t}}$ *follows the dynamics*

$$d\eta_t = -\eta_t\hat{\nu}_t dZ_t \tag{3.5.21}$$

with

$$\hat{\nu}_t = Max\left\{\left[\frac{\mu_{S,t} - r_t}{\sigma_t} - \bar{\pi}\sigma_t\right], 0\right\} \geq 0 \tag{3.5.22}$$

and

$$\eta(0) = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c} = \left[\left(\frac{1 - e^{-\beta T}}{\beta b}\right)\frac{\delta_0}{Y_0} - 1\right]^{-1} \tag{3.5.23}$$

$$\hat{\lambda}_2^c = \frac{1 - e^{-\beta T}}{\beta b} \tag{3.5.24}$$

The equilibrium consumption allocations are

$$\hat{c}_t^1 = \left[1 - \frac{\eta_t}{(1 + \eta_t)Y_t}\right]\delta_t \tag{3.5.25}$$

$$\hat{c}_t^2 = \frac{\eta_t}{1 + \eta_t}\frac{\delta_t}{Y_t} \tag{3.5.26}$$

Conversely, if there exist processes ξ_t^1, $\xi_{\hat{\nu},t}^2$, η_t and $\hat{\nu}_t$ satisfying equations (3.5.19)-(3.5.22), the associated optimal consumption and portfolio policies satisfy all market clearing conditions.

Proof: See Appendix B.

Having derived the state price densities and optimal consumption allocation, we now characterize the equilibrium prices in the constrained economy.

Proposition 3.5.2. *In equilibrium, the price processes are described as follows:*

(i) The equilibrium stock price-dividend ratio satisfies the equation

$$\frac{S_t}{\delta_t} = \frac{E_t\left[\int_t^T e^{-\beta(u-t)}(1 + \eta_u)Y_u du\right]}{(1 + \eta_t)Y_t} \tag{3.5.27}$$

(ii) The equilibrium interest rate is given by

$$r_t = \beta + \mu_\delta - \sigma_\delta^2 + k\left(1 - \frac{\bar{Y}}{Y_t}\right) - \alpha\sigma_\delta^2\left(1 - \frac{\bar{\lambda}}{Y_t}\right) - \frac{\eta_t\hat{\nu}_t}{1 + \eta_t}\left(1 + \frac{\alpha(Y_t - \bar{\lambda})}{Y_t}\right)\sigma_\delta \tag{3.5.28}$$

(iii) The equilibrium market price of risk is given by

$$\frac{\mu_{S,t} - r_t}{\sigma_t} = \kappa_t^1 = \sigma_\delta + \alpha \left(1 - \frac{\bar{\lambda}}{Y_t}\right) \sigma_\delta + \frac{\eta_t \hat{\nu}_t}{1 + \eta_t} \tag{3.5.29}$$

(iv) The stock returns volatility σ_t and the shadow cost of the constraint $\hat{\nu}_t$ satisfy the following equations:

$$\begin{aligned} \sigma_t &= \sigma_\delta + \alpha\sigma_\delta \left(1 - \frac{\bar{\lambda}}{Y_t}\right) - \frac{\hat{\nu}_t}{1 + \eta_t} \\ &+ \frac{E_t\left[\int_t^T e^{-\beta(u-t)} \left\{\left(1 + \frac{1}{\eta_u}\right) \mathcal{D}_t Y_u - \frac{Y_u}{\eta_u^2} \mathcal{D}_t \eta_u\right\} du\right]}{E_t\left[\int_t^T e^{-\beta(u-t)} \left(1 + \frac{1}{\eta_u}\right) Y_u du\right]} \qquad (3.5.30) \\ \hat{\nu}_t &= Max\left[\kappa_t^1 - \bar{\pi}\sigma_t, 0\right] \qquad (3.5.31) \end{aligned}$$

where $\mathcal{D}_t Y_u$ is the Malliavin derivative for the process Y_u and $\mathcal{D}_t \eta_u$ is the Malliavin derivative for the process η_u and are derived in the proof.

Proof: See Appendix B.

Remark: Existence of equilibrium requires that there is a solution to the equation (3.5.21) subject to the initial condition (3.5.23) and (3.5.29), (3.5.30), (3.5.31).

3.5.1 A Special Case

We now consider the special case when agent 2 cannot invest more than 100 percent of his wealth in the stock, i.e. $\bar{\pi} = 1$.

Proposition 3.5.3. *When $\bar{\pi} = 1$, the equilibrium stock returns volatility σ_t and the shadow cost of the constraint $\hat{\nu}_t$ are given by:*

$$\begin{aligned} \sigma_t &= \sigma_\delta \qquad (3.5.32) \\ \hat{\nu}_t &= (1 + \eta_t)\, \alpha\sigma_\delta \left(1 - \frac{\bar{\lambda}}{Y_t}\right) \qquad (3.5.33) \end{aligned}$$

Proof: See Appendix B.

Comparing the stock returns volatility in (3.5.32) with the volatility (3.4.25) in the unconstrained economy, we see that the volatility is unambiguously reduced when $\bar{\pi} = 1$ i.e. when the constrained agent cannot borrow from the unconstrained agent.

3.5.2 Analysis of the Effects of the Constraint

We discuss the implications of Proposition 3.5.2 and 3.5.3 in this section. The above propositions characterize the equilibrium asset prices for $Y_t > \bar{Y}$. We make the following observations from the above Proposition.

(i) *Interest rate is lower in the constrained economy compared to the unconstrained economy.*

(ii) *The market price of risk (Sharpe ratio) is higher in the constrained economy compared to the unconstrained economy.*

The above two conclusions are immediate from the expressions for the market prices of risk (3.4.27), (3.5.29) and from the expressions for the interest rates (3.4.26), (3.5.28). Therefore, the Sharpe ratio can increase and the interest rate can decrease simultaneously in our economy where one agent is portfolio-constrained. The intuition behind this result is as follows. The constrained agent's investment in the stock is less when compared to the portfolio investment if there were no constraints. Since wealth is either invested in the stock or the bond, the constrained agent has to invest positive amount of wealth in the bond. This means that the constrained agent is the lender and the unconstrained agent is the borrower in the constrained economy. As the constrained agent is unable to invest more than $\bar{\pi}$ proportion of his wealth in the stocks, the stock market clearing condition implies that the unconstrained agent must buy the excess supply of stocks. Since the unconstrained agent is more risk averse than the constrained agent, the securities market will clear only if the market price of risk is high and the cost of borrowing is low. Thus a higher Sharpe ratio and lower interest rate result in the constrained economy as compared to an unconstrained economy.

3.6 Welfare Analysis

Proposition 3.6.1. *Let $\hat{c}^{iu}$ and $\hat{c}^{ic}$ denote the optimal consumption choices of agent i in the unconstrained and constrained economies. Then*

$$U_1(\hat{c}^{1c}) \geq U_1(\hat{c}^{1u}) \tag{3.6.1}$$

$$U_2(\hat{c}^{2c}) \leq U_2(\hat{c}^{2u}) \tag{3.6.2}$$

where

$$\begin{aligned} U_1(c) &= E\left[\int_0^T e^{-\beta t} log(c_t - X_t)dt\right] \\ U_2(c) &= E\left[\int_0^T e^{-\beta t} log(c_t)dt\right] \end{aligned}$$

The inequalities in (3.6.1) and (3.6.2) will be strict if there exists a non-degenerate time interval $\mathcal{T} \subset [0, T]$ such that $\hat{\nu}_t > 0$ with positive probability for all $t \in \mathcal{T}$.

Since agent 2 is endowed with long position in bonds only, $\hat{\nu}_0 = 0$. We say that the constraint $\hat{\pi}_t \leq \bar{\pi}$ binds on agent 2 in the event (ω, t) if $\hat{\nu}_t(\omega) > 0$.

In the constrained economy, the unconstrained agent is the borrower while the constrained agent is the lender. Moreover, compared to the unconstrained economy, the unconstrained agent invests a higher proportion of his wealth in the stock in the constrained economy. Therefore, the unconstrained agent is better off due to the fact that equilibrium interest rate is lower and the Sharpe ratio is higher in the constrained economy compared to the unconstrained economy.

3.7 Conclusion

This chapter analyzes the effects of portfolio constraints on asset returns and volatility. Portfolio constraints may arise due to minimum capital requirement regulations, margin requirements or leverage constraints on portfolio managers. We analyze how heterogeneity in preferences affect the equilibrium stock returns and volatility in the presence of portfolio constraints. We show that portfolio constraints can simultaneously produce high Sharpe ratio and low interest rates in equilibrium. Moreover, the stock returns volatility is lower in the constrained economy compared to the unconstrained economy. We perform welfare analysis and show that the unconstrained agent is made better off while the constrained is worse off when the constraint binds.

Appendix A

Agent's Optimization Problem With Option to Retire

The state price density ξ_t follows the dynamics

$$d\xi_t \quad = \quad -\xi_t[rdt + \theta dW_t]$$

where $\theta = \frac{\mu - r}{\sigma}$ is the Sharpe ratio.

For $0 \leq t < \tau$, applying Ito's formula to the product $\xi_t X_t$, we get

$$d(\xi_t X_t) \quad = \quad \xi_t(\delta y_t - c_t)dt + (\pi_t \sigma - X_t \theta)\xi_t dW_t$$

Integrating, and using $\xi_0 = 1$, $X_0 = x$ ($X_0 = x$ is the wealth of the agent at time $t = 0$), we get

$$\xi_t X_t - \xi_0 X_0 = \int_0^t \xi_s(\delta y_s - c_s)ds + \int_0^t \xi_s(\pi_s \sigma - \theta X_s)dW_s$$

For $0 < t \leq \tau$ and an admissible plan (c_t, π_t, y_t), $\int_0^t \xi_s(\pi_s\sigma - \theta X_s)dW_s$ is a continuous P-local martingale. By Fatou's Lemma, it is a super-martingale since it is bounded below. The optional sampling theorem implies

$$E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] \leq x \tag{A.0.1}$$

for all $\tau \in \mathcal{S}$ where $\mathcal{S}$ denotes the set of all $\mathcal{F}$-stopping time τ's. Thus, in a dynamically complete markets, the dynamic budget constraint (2.2.5) is reduced to a single inter-temporal budget constraint of the type (A.0.1). When the agent retires, $y_t = 0$ for all $t \geq \tau$ and (A.0.1) becomes

$$E_\tau \left[\int_\tau^t \xi_s c_s dt + \xi_t X_t \right] \leq \xi_\tau X_\tau \ \ \forall t \geq \tau$$

where X_τ is the wealth of the agent at retirement. Let $\mathcal{A}(x)$ denote the set of consumption-portfolio-effort strategy (c, π, y, τ) available at the initial wealth $X_0 = x > 0$, such that for $\tau \in \mathcal{S}$ and $0 < t \leq \tau$, the wealth process X_t follows

$$dX_t = \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t) r dt + (\delta y_t - c_t) dt \quad 0 \leq t < \tau \tag{A.0.2}$$

and

$$X_t \geq -\frac{\delta L_2}{r}, \ \ 0 \leq t < \tau \tag{A.0.3}$$

$$X_\tau > 0 \tag{A.0.4}$$

$$0 < L_1 \leq y_t \leq L_2 \tag{A.0.5}$$

The agent's problem is to maximize the expected discounted utility:

$$J(x; c, \pi, y, \tau) = E \left[\int_0^\tau e^{-\beta t} \left\{ u(c_t) - \frac{1}{2} y_t^2 \right\} dt + e^{-\beta \tau} U(X_\tau) \right]$$

subject to (A.0.2), (A.0.3), (A.0.4) and (A.0.5) where

$$U(X_\tau) = \underset{c_t, \pi_t}{Max} \, E_\tau \left[\int_\tau^\infty e^{-\beta(t-\tau)} u(c_t) dt \right]$$

subject to

$$dX_t = \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t) r dt - c_t dt, \quad t \geq \tau$$

and

$$X_t \geq 0 \ \forall t \geq \tau$$

Merton ([44], [45]) and Karatzas et.al. ([34]) have shown that

$$U(X_\tau) \quad = \quad \frac{X_\tau^{1-\gamma}}{K^\gamma(1-\gamma)} \tag{A.0.6}$$

where $K = r + \frac{\beta-\gamma}{\gamma} + \frac{\gamma-1}{2\gamma^2}\theta^2$.

Define the value function by

$$V(x) = \underset{(c,\pi,y,\tau)\in\mathcal{A}(x)}{sup} J(x;c,\pi,y,\tau), \quad x \in \left(-\frac{\delta L_2}{r}, \infty\right) \tag{A.0.7}$$

The admissible policy $(c^*, \pi^*, y^*, \tau^*) \in \mathcal{A}(x)$ is optimal for (A.0.7) if $V(x) = J\left(x; c^*, \pi^*, y^*, \tau^*\right)$. Following Karatzas and Wang [36], we use the martingale approach to solve for $V(x)$. Let $\mathcal{S}$ denote the set of all $\mathcal{F}$-stopping time τ's. For a fixed stopping time $\tau \in \mathcal{S}$, let $\Pi_\tau(x)$ be the set of consumption-portfolio-effort strategy (c, π, y) for which $(c, \pi, y, \tau) \in \mathcal{A}(x)$. Define the following maximization problem

$$V_\tau(x) \quad = \quad \underset{(c,\pi,y)\in\Pi_\tau(x)}{sup} J(x;c,\pi,y,\tau) \tag{A.0.8}$$

Then

$$V(x) \quad = \quad \underset{\tau}{sup} V_\tau(x) \tag{A.0.9}$$

Now, for any $(c, \pi, y, \tau) \in \mathcal{A}(x)$ and any real number $\lambda > 0$,

$$\begin{aligned} J(x;c,\pi,y,\tau) \quad &= \quad E\left[\int_0^\tau e^{-\beta t}(u(c_t) - \frac{1}{2}y_t^2)dt + e^{-\beta\tau}U(X_\tau)\right] \\ &= \quad E\left[\int_0^\tau e^{-\beta t}(u(c_t) - \frac{1}{2}y_t^2 - \lambda e^{\beta t}\xi_t c_t + \lambda e^{\beta t}\xi_t \delta y_t)dt\right] \\ &+ \quad E\left[e^{-\beta\tau}(U(X_\tau) - \lambda e^{\beta\tau}\xi_\tau X_\tau)\right] + \lambda E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] \\ &= \quad E\left[\int_0^\tau e^{-\beta t}(u(c_t) - \lambda e^{\beta t}\xi_t c_t)dt\right] + E\left[\int_0^\tau e^{-\beta t}(\lambda e^{\beta t}\xi_t \delta y_t - \frac{1}{2}y_t^2)dt\right] \\ &+ \quad E\left[e^{-\beta\tau}(U(X_\tau) - \lambda e^{\beta\tau}\xi_\tau X_\tau)\right] + \lambda E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] \\ &\leq \quad E\left[\int_0^\tau e^{-\beta t}\left[\tilde{u}(\lambda e^{\beta t}\xi_t) + \tilde{g}(\lambda e^{\beta t}\xi_t)\right]dt + e^{-\beta\tau}\tilde{U}(\lambda e^{\beta\tau}\xi_\tau)\right] + \lambda x \end{aligned} \tag{A.0.10}$$

where the dual functions $\tilde{u}$, $\tilde{g}$ and $\tilde{U}$ are defined as follows:

$$\tilde{u}(z) = \underset{c\geq 0}{Max}\,[u(c) - zc] \tag{A.0.11}$$

$$\tilde{g}(z) = \underset{L_1\leq y\leq L_2}{Max}\left[zy - \frac{1}{2}y^2\right] \tag{A.0.12}$$

$$\tilde{U}(z) = \underset{X\geq 0}{Max}\,[U(X) - zX] \tag{A.0.13}$$

Thus,

$$\tilde{u}(\lambda e^{\beta t}\xi_t) = \underset{c_t\geq 0}{Max}\left[u(c_t) - \lambda e^{\beta t}\xi_t c_t\right] = \frac{\gamma}{1-\gamma}\left(\lambda e^{\beta t}\xi_t\right)^{1-\frac{1}{\gamma}} \tag{A.0.14}$$

$$\begin{aligned}
\tilde{g}(\lambda e^{\beta t}\xi_t) &= \underset{L_1\leq y_t\leq L_2}{Max}\left[\lambda\delta e^{\beta t}y_t\xi_t - \frac{1}{2}y_t^2\right] \\
&= \frac{1}{2}\delta^2(\lambda e^{\beta t}\xi_t)^2 1_{\{\frac{L_1}{\delta}<\lambda e^{\beta t}\xi_t<\frac{L_2}{\delta}\}} + \left[\delta L_1\lambda e^{\beta t}\xi_t - \frac{1}{2}L_1^2\right]1_{\{\lambda e^{\beta t}\xi_t\leq\frac{L_1}{\delta}\}} \\
&+ \left[\delta L_2\lambda e^{\beta t}\xi_t - \frac{1}{2}L_2^2\right]1_{\{\lambda e^{\beta t}\xi_t\geq\frac{L_2}{\delta}\}}
\end{aligned} \tag{A.0.15}$$

$$\tilde{U}(\lambda e^{\beta t}\xi_\tau) = \underset{X_\tau\geq 0}{Max}\left[U(X_\tau) - \lambda e^{\beta\tau}\xi_\tau X_\tau\right] = \frac{1}{K}\frac{\gamma}{1-\gamma}(\lambda e^{\beta\tau}\xi_\tau)^{\frac{\gamma-1}{\gamma}} \tag{A.0.16}$$

The inequality (A.0.10) follows from the definitions of the dual functions and the intertemporal budget constraint $E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] \leq x$. Then

$$\begin{aligned}
V_\tau(x) &= \sup_{(c,\pi,y)\in\Pi_\tau(x)} J(x;c,\pi,y,\tau) \\
&\leq E\left[\int_0^\tau e^{-\beta t}\left[\tilde{u}(\lambda e^{\beta t}\xi_t) + \tilde{g}(\lambda e^{\beta t}\xi_t)\right]dt + e^{-\beta\tau}\tilde{U}(\lambda e^{\beta\tau}\xi_\tau)\right] + \lambda x
\end{aligned} \tag{A.0.17}$$

The inequality (A.0.17) becomes an equality if and only if

$$X_\tau = \frac{(\lambda e^{\beta\tau}\xi_\tau)^{-\frac{1}{\gamma}}}{K} \tag{A.0.18}$$

$$c_t = (\lambda e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} \quad \forall\ 0\leq t\leq\tau \ \ a.s. \tag{A.0.19}$$

$$y_t = (\lambda\delta e^{\beta t}\xi_t)1_{\{\frac{L_1}{\delta}<\lambda e^{\beta t}\xi_t<\frac{L_2}{\delta}\}} + L_1 1_{\{\lambda e^{\beta t}\xi_t\leq\frac{L_1}{\delta}\}} + L_2 1_{\{\lambda e^{\beta t}\xi_t\geq\frac{L_2}{\delta}\}} \ \text{ for } 0<t<\tau \tag{A.0.20}$$

and

$$E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] \quad = \quad x \tag{A.0.21}$$

hold. Define

$$\tilde{J}(\lambda;\tau) = E\left[\int_0^\tau e^{-\beta t}\left(\tilde{u}(\lambda e^{\beta t}\xi_t) + \tilde{g}(\lambda e^{\beta t}\xi_t)\right)dt + e^{-\beta\tau}\tilde{U}(\lambda e^{\beta\tau}\xi_\tau)\right] \tag{A.0.22}$$

Then

$$V_\tau(x) \quad \leq \quad \underset{\lambda>0}{inf}\left[\tilde{J}(\lambda;\tau) + \lambda x\right] \quad \forall \tau \in \mathcal{S}$$

with equality if and only if (A.0.18), (A.0.19), (A.0.20), and (A.0.21) holds. Thus

$$V(x) = \underset{\tau\in\mathcal{S}}{sup} \quad V_\tau(x) \leq \underset{\tau\in\mathcal{S}}{sup} \ \underset{\lambda>0}{inf} \quad \left[\tilde{J}(\lambda;\tau) + \lambda x\right] \tag{A.0.23}$$

The following proposition is analogous to Proposition 6.5 in Karatzas and Wang [36] and shows that given the initial financial wealth x, the consumption-effort policies in (A.0.19), (A.0.20) and the retirement wealth in (A.0.18), there exists a portfolio policy π_t such that $(c_t, \pi_t, y_t) \in \Pi_\tau(x)$ and is an optimal policy for the problem (A.0.8). In particular, it shows that the inequality in (A.0.23) is actually an equality because the portfolio policy π_t can finance the consumption-effort policies (c_t, y_t) and the retirement wealth X_τ.

Proposition A.0.1. *For any $\tau \in \mathcal{S}$, there exists λ^* such that*

$$V_\tau(x) = \underset{\lambda>0}{inf}\left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \tilde{J}(\lambda^*;\tau) + \lambda^* x$$

and the optimal solution to (A.0.8) is given by (A.0.18), (A.0.19) and (A.0.20). The value function (A.0.7) is given by

$$V(x) \quad = \quad \underset{\tau}{sup} V_\tau(x) = \underset{\tau}{sup}\ \underset{\lambda>0}{inf}\left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \underset{\tau}{sup}\ \left[\tilde{J}(\lambda^*;\tau) + \lambda^* x\right] \tag{A.0.24}$$

Proof. First, we show that there exists a $\lambda^* > 0$ that satisfies $f(\lambda^*) = x$, for $x \in \left(\frac{-\delta L_2}{r}, \infty\right)$ where

$$f(\lambda) \;=\; E\left[\int_0^\tau \xi_t \left[(\lambda e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} - \delta y_t\right] dt + \xi_\tau \frac{(\lambda e^{\beta\tau}\xi_\tau)^{-\frac{1}{\gamma}}}{K}\right]$$

and y_t is as in (A.0.20). By differentiating with respect to λ, it is straightforward to check that $f^{'}(\lambda) < 0$. Therefore, f is a continuous, strictly decreasing function mapping $(0, \infty)$ onto itself with $f(0+) = \infty$ and $f(\infty) \geq -\frac{\delta L_2}{r}$; thus there exists a λ^* satisfying $f(\lambda^*) = x$. This proves that

$$E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau X_\tau\right] = x, \;\; x \in \left(\frac{-\delta L_2}{r}, \infty\right) \tag{A.0.25}$$

with

$$X_\tau \;=\; \frac{(\lambda^* e^{\beta\tau}\xi_\tau)^{-\frac{1}{\gamma}}}{K} \tag{A.0.26}$$

$$c_t \;=\; (\lambda^* e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} \;\; \forall \;\, 0 \leq t \leq \tau \;\; a.s. \tag{A.0.27}$$

$$y_t \;=\; (\lambda^* \delta e^{\beta t}\xi_t) 1_{\{\frac{L_1}{\delta} < \lambda^* e^{\beta t}\xi_t < \frac{L_2}{\delta}\}} + L_1 1_{\{\lambda^* e^{\beta t}\xi_t \leq \frac{L_1}{\delta}\}} + L_2 1_{\{\lambda^* e^{\beta t}\xi_t \geq \frac{L_2}{\delta}\}} \;\; \text{for} \;\; 0 < t < \tau \tag{A.0.28}$$

Next, we prove that given the initial financial wealth x, the consumption-effort policies (c_t, y_t) in (A.0.27), (A.0.28) and the retirement wealth X_τ in (A.0.26), there exists a portfolio policy π_t such that the intertemporal budget constraint (A.0.25) holds. The existence of such a portfolio policy is proved in Lemma A.0.2 below.

Lastly, from the derivations of c_t and y_t in (A.0.19), (A.0.20), it follows that (A.0.27), (A.0.28) are optimal for the problem (A.0.8). This completes the proof of the proposition. $\square$

Lemma A.0.2. *For any $\tau \in \mathcal{S}$ and $\mathcal{F}_\tau$-measurable random variable B with $P[B > 0] = 1$, and any progressively measurable process $c_t \geq 0$ and $y_t \in [L_1, L_2]$ that satisfy*

$$E\left[\int_0^\tau \xi_t(c_t - \delta y_t)dt + \xi_\tau B\right] = x, \;\; x \in \left(\frac{-\delta L_2}{r}, \infty\right)$$

there exists a portfolio π_t such that, a.s. $X_t \geq -\frac{\delta L_2}{r}, \;\; 0 \leq t < \tau$ and $X_\tau = B$

Proof. Define

$$X_t = \frac{1}{\xi_t} E_t \left[\int_{t\wedge\tau}^{\tau} \xi_s (c_s - \delta y_s) ds + \xi_\tau B \right]$$

Then

$$E \left[\int_0^{\tau} \xi_s (c_s - \delta y_s) ds + \xi_\tau B \right] = x$$

and $X_\tau = \frac{1}{\xi_\tau} E_\tau[\xi_\tau B] = B$. Also, for $0 \le t < \tau$,

$$X_t = \frac{1}{\xi_t} E_t \left[\int_{t\wedge\tau}^{\tau} \xi_s (c_s - \delta y_s) ds + \xi_\tau B \right] - \delta E_t \left[\int_{t\wedge\tau}^{\tau} \frac{\xi_s}{\xi_t} y_s ds \right]$$

But, $E_t \left[\int_{t\wedge\tau}^{\tau} \frac{\xi_s}{\xi_t} y_s ds \right] \le \frac{L_2}{r}$. Therefore,

$$X_t \ge -\frac{\delta L_2}{r} \quad \forall 0 \le t < \tau$$

Define

$$M_t = \xi_t X_t + \int_0^t \xi_s (c_s - \delta y_s) ds + \frac{L_2}{r} \tag{A.0.29}$$

Then M_t is a non-negative P-martingale with $M_0 = x + \frac{\delta L_2}{r} > 0$. By the Martingale-representation theorem,

$$M_t = x + \frac{\delta L_2}{r} + \int_0^t \varphi(s) dW_s, \ \ 0 \le t \le \tau$$

for some F-adapted process $\varphi(.)$ that satisfies $\int_0^{\tau} ||\varphi_s||^2 ds < \infty$ a.s. Define

$$\pi_t = \sigma^{-1} \left(\frac{\varphi_t}{\xi_t} + \theta X_t \right)$$

Then

$$\varphi_t = \xi_t \sigma \pi_t - \theta X_t \xi_t$$

and

$$dM_t = \varphi(t) dW_t \tag{A.0.30}$$

From the definition of M_t in (A.0.29)

$$dM_t = d(\xi_t X_t) + \xi_t(c_t - \delta y_t)dt \qquad \text{(A.0.31)}$$

Combining (A.0.30) and (A.0.31), we get

$$\begin{aligned} d(\xi_t X_t) &= \xi_t(\delta y_t - c_t)dt + \varphi_t dW_t \\ &= \xi_t(\delta y_t - c_t)dt + \xi_t(\sigma\pi_t - \theta X_t)dW_t \end{aligned}$$

Then, an application of Ito's Lemma gives

$$dX_t = \pi_t(\mu dt + \sigma dW_t) + (X_t - \pi_t)rdt + (\delta y_t - c_t)dt \quad 0 \leq t < \tau$$

which is the dynamics of the wealth process as in (A.0.2). This proves the existence of the portfolio process π_t that finances the consumption stream $c_t \geq 0$, the effort policy $y_t \in [L_1, L_2]$ and the wealth X_τ. □

Therefore, the utility maximization problem of the agent with option to retire can be solved in two steps. First, fix the stopping time and determine the optimal consumption, portfolio and effort policies. Second, determine the value function and maximize over all stopping times. As Karatzas and Wang [36] notes, the problem with this method is that it is not clear how the λ^*'s are related to each other for different stopping times. Instead, Karatzas and Wang [36] converts the original problem (A.0.7) into a family of pure optimal stopping problem as follows. From (A.0.24) in Proposition A.0.1, for any $\lambda > 0$ we have

$$V(x) = \underset{\tau}{sup}\, \underset{\lambda>0}{inf} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] \leq \underset{\tau}{sup} \left[\tilde{J}(\lambda;\tau) + \lambda x\right]$$

Taking the infimum over all $\lambda > 0$,

$$V(x) = \underset{\tau}{sup}\, \underset{\lambda>0}{inf} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] \leq \underset{\lambda>0}{inf}\, \underset{\tau}{sup} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \underset{\lambda>0}{inf} \left[\tilde{V}(\lambda) + \lambda x\right] \qquad \text{(A.0.32)}$$

with

$$\tilde{V}(\lambda) = \underset{\tau}{sup}\tilde{J}(\lambda;\tau) = \underset{\tau}{sup}\; E\left[\int_0^\tau e^{-\beta t}\left[\tilde{u}(\lambda e^{\beta t}\xi_t) + \tilde{g}(\lambda e^{\beta t}\xi_t)\right]dt + e^{-\beta\tau}\tilde{U}(\lambda e^{\beta\tau}\xi_\tau)\right] \tag{A.0.33}$$

Now, (A.0.33) is a pure stopping time problem which can be solved explicitly in our model.

Define $Z_t = \lambda e^{\beta t}\xi_t$. By Ito's formula

$$dZ_t = Z_t\left[(\beta - r)dt - \theta dB_t\right] \tag{A.0.34}$$

Z_t is a unique strong solution of (A.0.34) with initial value $Z_0 = \lambda$. We can then rewrite $\tilde{V}(\lambda)$ in (A.0.33) as

$$\tilde{V}(\lambda) = \underset{\tau}{sup}E\left[\int_0^\tau e^{-\beta t}\left[\tilde{u}(Z_t) + \tilde{g}(Z_t)\right]dt + e^{-\beta\tau}\tilde{U}(Z_\tau)\right] \tag{A.0.35}$$

Consider the following optimal stopping problem

$$\phi(t, Z) \quad = \quad \underset{\tau > t}{sup}\; E_t\left[\int_t^\tau e^{-\beta s}\left[\tilde{u}(Z_s) + \tilde{g}(Z_s)\right]ds + e^{-\beta\tau}\tilde{U}(Z_\tau)|Z_t = Z\right] \tag{A.0.36}$$

Then $\tilde{V}(\lambda) = \phi(0, \lambda)$ and finding a solution for $\phi(t, Z)$ will provide a solution for $\tilde{V}(\lambda)$ and the value function $V(x)$ in (A.0.7). To this end, we consider the differential operator $\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r)Z\frac{\partial}{\partial t} + \frac{1}{2}\theta^2 Z^2\frac{\partial^2}{\partial Z^2}$ acting on a mapping $\psi : (0, \infty) \times \mathcal{R}^+ \to \mathcal{R}^+$. As shown by Dybvig and Liu [22], Farhi and Panageas [24], Choi and Shim [13], and Choi, Shim and Shin [14]. a solution to the optimal stopping problem (B.0.60) is a solution to the following free boundary value problem.

Variational Inequality: Find a free boundary $\bar{Z} > 0$ and a continuously differentiable function $\tilde{\phi}$ satisfying, for all $t > 0$:

$$\mathcal{L}\tilde{\phi} + e^{-\beta t}\left[\tilde{u}(Z) + \tilde{g}(Z)\right] = 0, \quad Z > \bar{Z} \tag{A.0.37}$$

$$\mathcal{L}\tilde{\phi} + e^{-\beta t}\left[\tilde{u}(Z) + \tilde{g}(Z)\right] \leq 0, \quad 0 < Z \leq \bar{Z} \tag{A.0.38}$$

$$\tilde{\phi}(t, Z) \geq e^{-\beta t}\tilde{U}(Z), \qquad Z \geq \bar{Z} \tag{A.0.39}$$

$$\tilde{\phi}(t, Z) = e^{-\beta t}\tilde{U}(Z), \quad 0 < Z \leq \bar{Z} \tag{A.0.40}$$

We first prove the following Lemmas and then provide a solution to the Variational Inequality in Theorem A.0.6 below.

Lemma A.0.3. *Let α_1 and α_2 be the roots of the equation*

$$\frac{1}{2}\theta^2\alpha^2 + (\beta - r - \frac{1}{2}\theta^2)\alpha - \beta \quad = \quad 0 \tag{A.0.41}$$

Assume that

$$K \quad = \quad r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0 \tag{A.0.42}$$

Then

$$\alpha_1 \quad = \quad \frac{-(\beta - r - \frac{1}{2}\theta^2) + \sqrt{(\beta - r - \frac{1}{2}\theta^2)^2 + 2\theta^2\beta}}{\theta^2} \tag{A.0.43}$$

$$\alpha_2 \quad = \quad \frac{-(\beta - r - \frac{1}{2}\theta^2) - \sqrt{(\beta - r - \frac{1}{2}\theta^2)^2 + 2\theta^2\beta}}{\theta^2} \tag{A.0.44}$$

and

$$\begin{aligned}
\alpha_1+\alpha_2 &= -\frac{2\beta-2r-\theta^2}{\theta^2} && \text{(A.0.45)}\\
\alpha_1\alpha_2 &= -\frac{2\beta}{\theta^2} && \text{(A.0.46)}\\
(1-\alpha_1)(1-\alpha_2) &= -\frac{2r}{\theta^2} && \text{(A.0.47)}\\
\beta+\theta^2-2r &= \frac{\theta^2}{2}(2-\alpha_1)(2-\alpha_2) && \text{(A.0.48)}\\
1< \ \alpha_1 &< 2 && \text{(A.0.49)}\\
\alpha_2 &< 0 && \\
\alpha_2+\frac{1}{\gamma}-1 &< 0 && \text{(A.0.50)}
\end{aligned}$$

Proof. The above relations can be verified by direct computations. □

Lemma A.0.4. *Define the constants*

$$\begin{aligned}
\tilde{Z}_1 &= \frac{L_1}{\delta} && \text{(A.0.51)}\\
\tilde{Z}_2 &= \frac{L_2}{\delta} && \text{(A.0.52)}\\
C_1 &= \frac{2\delta^2[\tilde{Z}_2^{2-\alpha_1}-\tilde{Z}_1^{2-\alpha_1}]}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} && \text{(A.0.53)}\\
D_1 &= \frac{2\delta^2\tilde{Z}_2^{2-\alpha_1}}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} && \text{(A.0.54)}\\
C_2 &= -\frac{\delta^2\tilde{Z}_1\bar{Z}^{1-\alpha_2}}{\theta^2(\alpha_1-\alpha_2)}\left[\frac{\tilde{Z}_1}{\alpha_2\bar{Z}}+\frac{2}{1-\alpha_2}\right] && \text{(A.0.55)}\\
D_2 &= C_2+\frac{2\delta^2\tilde{Z}_1^{2-\alpha_2}}{\theta^2\alpha_2(1-\alpha_2)(2-\alpha_2)(\alpha_1-\alpha_2)} && \text{(A.0.56)}\\
E_2 &= C_2-\frac{2\delta^2[\tilde{Z}_2^{2-\alpha_2}-\tilde{Z}_1^{2-\alpha_2}]}{\theta^2\alpha_2(1-\alpha_2)(2-\alpha_2)(\alpha_1-\alpha_2)} && \text{(A.0.57)}
\end{aligned}$$

Then the following relations hold: (i) $C_1>0$ *(ii)* $D_1>0$ *(iii)* $C_2>0$ *(iv)* $D_2-C_2<0$ *(v)* $E_2>0$.

(vi) Moreover, there exists a unique solution $\bar{Z}$ of the equation:

$$\left(\frac{1-\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{\delta L_1}{r}\right)Z^{1-\alpha_1}+\left(\frac{\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{L_1^2}{2\beta}\right)Z^{-\alpha_1}+C_1=0 \tag{A.0.58}$$

such that $\bar{Z} < \left(\frac{\alpha_1-1}{2\alpha_1}\right)\tilde{Z}_1$.

Proof. (i), (ii), (iv), and (v) follow from the fact that $1 < \alpha_1 < 2$ and $\alpha_2 < 0$. Proof of (iii) is provided after proving (vi). In the following we show that there exists a unique solution $\bar{Z}$ to the equation (A.0.58). Using the expression for C_1 from (A.0.53), and the relations (A.0.46), (A.0.47) we can rewrite (A.0.58) as

$$\frac{\eta^{1-\alpha_1}}{\alpha_1-1}-\frac{\eta^{-\alpha_1}}{2\alpha_1}+\left[\left(\frac{\tilde{Z}_2}{\tilde{Z}_1}\right)^{2-\alpha_1}-1\right]\left[\frac{1}{\alpha_1(\alpha_1-1)(2-\alpha_1)}\right] = 0 \tag{A.0.59}$$

where $\eta = \frac{Z}{\tilde{Z}_1}$. To show that equation (A.0.59) has a unique solution $\eta \in \left(0, \frac{\alpha_1-1}{2\alpha_1}\right)$, define the function $f(\eta) = \frac{\eta^{1-\alpha_1}}{\alpha_1-1} - \frac{\eta^{-\alpha_1}}{2\alpha_1} + \left[\left(\frac{\tilde{Z}_2}{\tilde{Z}_1}\right)^{2-\alpha_1}-1\right]\left[\frac{1}{\alpha_1(\alpha_1-1)(2-\alpha_1)}\right]$. Differentiating $f(\eta)$ with respect to η gives

$$f'(\eta) = \eta^{-\alpha_1-1}\left(\frac{1}{2}-\eta\right) > 0$$

for $\eta \in \left(0, \frac{\alpha_1-1}{2\alpha_1}\right)$. Therefore $f(\eta)$ is strictly increasing in $\left(0, \frac{\alpha_1-1}{2\alpha_1}\right)$. Now,

$$f\left(\frac{\alpha_1-1}{2\alpha_1}\right) = \left[\left(\frac{\tilde{Z}_2}{\tilde{Z}_1}\right)^{2-\alpha_1}-1\right]\frac{1}{\alpha_1(\alpha_1-1)(2-\alpha_1)} > 0$$

since $\tilde{Z}_2 > \tilde{Z}_1$. Also, $f(\eta) = \frac{\eta^{-\alpha_1}}{\alpha_1-1}\left(\eta - \frac{\alpha_1-1}{2\alpha_1}\right) + \frac{\left[\left(\frac{\tilde{Z}_2}{\tilde{Z}_1}\right)^{2-\alpha_1}-1\right]}{\alpha_1(\alpha_1-1)(2-\alpha_1)} \to -\infty$ as $\eta \to 0$. Therefore there exists $\bar{\eta} \in \left(0, \frac{\alpha_1-1}{2\alpha_1}\right)$ such that $f(\bar{\eta}) = 0$. Define $\bar{Z} = \bar{\eta}\tilde{Z}_1$. Then $\frac{\bar{Z}}{\tilde{Z}_1} = \bar{\eta} < \frac{\alpha_1-1}{2\alpha_1}$ as claimed. Finally, we prove that $C_2 > 0$. Since $\alpha_2 < 0$, therefore $C_2 > 0$ if $\frac{\tilde{Z}_1}{\alpha_2\bar{Z}} + \frac{2}{1-\alpha_2} < 0$ i.e. if $\frac{\bar{Z}}{\tilde{Z}_1} < \frac{1-\alpha_2}{(-2\alpha_2)}$. Since $\alpha_1 > 0 > \alpha_2$, therefore $\frac{\alpha_1-1}{2\alpha_1} < \frac{1-\alpha_2}{(-2\alpha_2)}$. Therefore, using the result in (vi), we get $\frac{\bar{Z}}{\tilde{Z}_1} < \frac{\alpha_1-1}{2\alpha_1} < \frac{1-\alpha_2}{(-2\alpha_2)}$, which proves that $C_2 > 0$. □

Lemma A.0.5. *For $Z \in \left(\bar{Z}, \infty\right)$, define the function $f(Z)$ as follows:*

$$f(Z) = \alpha_1 C_1 Z^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} + \frac{\delta L_1}{r} \quad \textit{for} \quad \bar{Z} < Z < \tilde{Z}_1 \tag{A.0.60}$$

$$f(Z) = \alpha_1 D_1 Z^{\alpha_1 - 1} + \alpha_2 D_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} - \frac{\delta^2 Z}{\beta + \theta^2 - 2r} \quad \textit{for} \quad \tilde{Z}_1 < Z < \tilde{Z}_2 \tag{A.0.61}$$

$$f(Z) = \alpha_2 E_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} + \frac{\delta L_2}{r} \quad \textit{for} \quad Z \geq \tilde{Z}_2 \tag{A.0.62}$$

Then $f'(Z) > 0$ for all $Z \in \left(\bar{Z}, \infty\right)$.

Proof. (i) For $\bar{Z} < Z \leq \tilde{Z}_1$

$$\frac{\partial f(Z)}{\partial Z} = \alpha_1(\alpha_1 - 1) C_1 Z^{\alpha_1 - 2} + \alpha_2(\alpha_2 - 1) C_2 Z^{\alpha_2 - 2} + \frac{Z^{-1-\frac{1}{\gamma}}}{K\gamma} > 0$$

since $\alpha_2 < 0$, $\alpha_1 > 1$, $C_1 > 0$ and $C_2 > 0$.

(ii) For $\tilde{Z}_1 < Z < \tilde{Z}_2$

$$\frac{\partial f(Z)}{\partial Z} = \alpha_1(\alpha_1 - 1) D_1 Z^{\alpha_1 - 2} + \alpha_2(\alpha_2 - 1) D_2 Z^{\alpha_2 - 2} + \frac{Z^{-1-\frac{1}{\gamma}}}{K\gamma} - \frac{\delta^2}{\beta + \theta^2 - 2r}$$

We claim that

$$\alpha_1(\alpha_1 - 1) D_1 Z^{\alpha_1 - 2} + \alpha_2(\alpha_2 - 1) D_2 Z^{\alpha_2 - 2} + \frac{Z^{-1-\frac{1}{\gamma}}}{K\gamma} - \frac{\delta^2}{\beta + \theta^2 - 2r} > 0 \tag{A.0.63}$$

Using the expression of D_1 from (A.0.54) together with $1 < \alpha_1 < 2$ and $\tilde{Z}_1 < Z < \tilde{Z}_2$, we get

$$\begin{aligned} \alpha_1(\alpha_1 - 1) D_1 Z^{\alpha_1 - 2} &= \alpha_1(\alpha_1 - 1)\left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta + \theta^2 - 2r)} + \frac{\alpha_2}{2\beta}\right] \frac{\delta^2 \tilde{Z}_2^{2-\alpha_1} Z^{\alpha_1 - 2}}{(\alpha_1 - \alpha_2)} \\ &> \alpha_1(\alpha_1 - 1)\left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta + \theta^2 - 2r)} + \frac{\alpha_2}{2\beta}\right] \frac{\delta^2}{(\alpha_1 - \alpha_2)} \end{aligned}$$

Using the expression of D_2 from (A.0.56), we have

$$\begin{aligned} \alpha_2(\alpha_2 - 1) D_2 Z^{\alpha_2 - 2} &= \alpha_2(\alpha_2 - 1) C_2 Z^{\alpha_2 - 2} \\ &- \alpha_2(\alpha_2 - 1)\left[\frac{2-\alpha_1}{2(\beta + \theta^2 - 2r)} - \frac{\alpha_1 - 1}{r} + \frac{\alpha_1}{2\beta}\right] \frac{\delta^2 \tilde{Z}_1^{2-\alpha_2} Z^{\alpha_2 - 2}}{(\alpha_1 - \alpha_2)} \end{aligned}$$

Since $C_2 > 0$ and $\alpha_2 < 0$, it follows that $\alpha_2(\alpha_2 - 1)C_2 Z^{\alpha_2 - 2} > 0$. Also, $\tilde{Z}_1 < Z < \tilde{Z}_2$ and $\alpha_2 < 2$ imply that $\tilde{Z}^{\alpha_2-2} > Z^{\alpha_2-2}$ i.e. $\tilde{Z}^{\alpha_2-2} Z^{\alpha_2-2} < 1$. Hence,

$$\alpha_2(\alpha_2 - 1)D_2 Z^{\alpha_2-2} > -\alpha_2(\alpha_2 - 1)\left[\frac{2-\alpha_1}{2(\beta+\theta^2-2r)} - \frac{\alpha_1 - 1}{r} + \frac{\alpha_1}{2\beta}\right]\frac{\delta^2}{(\alpha_1 - \alpha_2)}$$

Using (A.0.46), (A.0.47), (A.0.48) and the fact that $\frac{Z^{-1-\frac{1}{\gamma}}}{K\gamma} > 0$ we get

$$\begin{aligned}
\text{LHS of (A.0.63)} \quad &> \quad \alpha_1(\alpha_1 - 1)\left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta}\right]\frac{\delta^2}{(\alpha_1-\alpha_2)} \\
&- \quad \alpha_2(\alpha_2 - 1)\left[-\frac{\alpha_1 - 1}{r} + \frac{2-\alpha_1}{2(\beta+\theta^2-2r)} + \frac{\alpha_1}{2\beta}\right]\frac{\delta^2}{(\alpha_1-\alpha_2)} \\
&- \quad \frac{\delta^2}{\beta+\theta^2-2r} \\
&> \quad 0
\end{aligned}$$

which proves the claim that inequality (A.0.63) holds.

(iii) For $Z \geq \tilde{Z}_2$

$$\frac{\partial f(Z)}{\partial Z} = \alpha_2(\alpha_2 - 1)E_2 Z^{\alpha_2-2} + \frac{Z^{-1-\frac{1}{\gamma}}}{K\gamma} > 0$$

The inequality follows since $\alpha_2 < 0$ and $E_2 > 0$. This completes the proof of the lemma. □

Proof of Lemma 2.3.1:

Proof. The proof follows from Lemma A.0.4 and A.0.5. □

Theorem A.0.6. *Consider the function $v(Z)$ defined on $(0, \infty)$ as follows:*

$$v(Z) = \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} \quad \textit{for} \quad 0 < Z \leq \bar{Z} \tag{A.0.64}$$

$$v(Z) = C_1 Z^{\alpha_1} + C_2 Z^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_1}{r} Z - \frac{1}{2}\frac{L_1^2}{\beta} \quad \textit{for} \quad \bar{Z} < Z < \tilde{Z}_1 \tag{A.0.65}$$

$$v(Z) = D_1 Z^{\alpha_1} + D_2 Z^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}} - \frac{\delta^2 Z^2}{2(\beta+\theta^2-2r)} \quad \textit{for} \quad \tilde{Z}_1 < Z < \tilde{Z}_2 \tag{A.0.66}$$

$$v(Z) \;=\; E_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_2}{r} Z - \frac{1}{2}\frac{L_2^2}{\beta} \quad \textit{for} \quad Z \geq \tilde{Z}_2 \qquad \text{(A.0.67)}$$

Then $\tilde{\phi}(t, Z) = e^{-\beta t} v(Z)$ is a solution to the Variational Inequality.

Proof. Consider the first partial differential equation (A.0.37) of Variational Inequality:

$$\frac{\partial \tilde{\phi}}{\partial t} + (\beta - r) Z \frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2}\theta^2 Z^2 \frac{\partial^2 \tilde{\phi}}{\partial Z^2} + e^{\beta t}\left[\tilde{u}(Z) + \tilde{g}(Z)\right] = 0, \;\; Z > \bar{Z} \qquad \text{(A.0.68)}$$

with the boundary condition given in (A.0.40):

$$\tilde{\phi}(t, \bar{Z}) = e^{-\beta t} \tilde{U}(\bar{Z})$$

We divide the interval $(\bar{Z}, \infty)$ into three intervals $(\bar{Z}, \tilde{Z}_1), (\tilde{Z}_1, \tilde{Z}_2)$ and $(\tilde{Z}_2, \infty)$ and solve (A.0.68) in each interval.

(i) For $\bar{Z} < Z \leq \tilde{Z}_1$

$$\begin{aligned} \tilde{u}(Z) &= \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} \\ \tilde{g}(Z) &= \delta L_1 Z - \frac{1}{2} L_1^2 \end{aligned}$$

Therefore (A.0.68) becomes

$$\frac{\partial \phi}{\partial t} + (\beta - r) Z \frac{\partial \phi}{\partial Z} + \frac{1}{2}\theta^2 Z^2 \frac{\partial^2 \phi}{\partial Z^2} + e^{-\beta t}\left[\frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \delta L_1 Z - \frac{1}{2} L_1^2\right] = 0$$

We guess a solution of the form $\phi(t, Z) = e^{-\beta t} v(Z)$. Then $v(Z)$ satisfies

$$-\beta v(Z) + (\beta - r) Z v'(Z) + \frac{1}{2}\theta^2 Z^2 v''(Z) + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \delta L_1 Z - \frac{1}{2} L_1^2 = 0$$

with the solution

$$v(Z) \;=\; C_1 Z^{\alpha_1} + C_2 Z^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_1}{r} Z - \frac{1}{2}\frac{L_1^2}{\beta} \qquad \text{(A.0.69)}$$

where α_1 and α_2 are roots of the equation:

$$\frac{1}{2}\alpha^2\theta^2 + (\beta - r - \frac{1}{2}\theta^2)\alpha - \beta = 0$$

which is equation (A.0.41). Hence the roots are given by (A.0.43) and (A.0.44).

(ii) For $\tilde{Z}_1 < Z < \tilde{Z}_2$,

$$\begin{aligned}\tilde{u}(Z) &= \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} \\ \tilde{g}(Z) &= \frac{1}{2}\delta^2 Z^2\end{aligned}$$

Therefore $v(Z)$ solves

$$-\beta v(Z) + (\beta - r)Zv'(Z) + \frac{1}{2}\theta^2 Z^2 v''(Z) + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \frac{1}{2}\delta^2 Z^2 = 0$$

Hence the solution is:

$$v(Z) = D_1 Z^{\alpha_1} + D_2 Z^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}} - \frac{\delta^2 Z^2}{2(\beta + \theta^2 - 2r)} \tag{A.0.70}$$

(iii) For $Z \geq \tilde{Z}_2$

$$\begin{aligned}\tilde{u}(Z) &= \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} \\ \tilde{g}(Z) &= \delta L_2 Z - \frac{1}{2} L_2^2\end{aligned}$$

Then $v(Z)$ solves

$$-\beta v(Z) + (\beta - r)Zv'(Z) + \frac{1}{2}\theta^2 Z^2 v''(Z) + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \delta L_2 Z - \frac{1}{2} L_2^2 = 0$$

with solution

$$v(Z) = E_1 Z^{\alpha_1} + E_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_2}{r} Z - \frac{1}{2}\frac{L_2^2}{\beta} \tag{A.0.71}$$

According to equation (A.0.40) in the Variational Inequality

$$\tilde{\phi}(t, Z) = e^{-\beta t}\tilde{U}(Z) \ \ for \ \ 0 < Z \leq \bar{Z}$$

so that

$$v(Z) = \tilde{U}(Z) = \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} \ \ for \ \ 0 < Z \leq \bar{Z} \tag{A.0.72}$$

We have to determine the coefficients $C_1, C_2, D_1, D_2, E_1, E_2$ and the optimal exercise point $\bar{Z}$ using the value-matching and smooth-pasting conditions. Since $v(Z)$ has to be bounded in the region $(\tilde{Z}_2, \infty)$ and $\alpha_1 > 0$, we require that

$$E_1 = 0$$

From (A.0.70) and (A.0.71), value-matching and smooth-pasting conditions at $Z = \tilde{Z}_2$ give respectively:

$$E_2\tilde{Z}_2^{\alpha_2} + \frac{\delta L_2}{r}\tilde{Z}_2 - \frac{L_2^2}{2\beta} = D_1\tilde{Z}_2^{\alpha_1} + D_2\tilde{Z}_2^{\alpha_2} - \frac{\delta^2}{2(\beta+\theta^2-2r)}\tilde{Z}_2^2 \tag{A.0.73}$$

$$\alpha_2 E_2\tilde{Z}_2^{\alpha_2-1} + \frac{\delta L_2}{r} = D_1\alpha_1\tilde{Z}_2^{\alpha_1-1} + D_2\alpha_2\tilde{Z}_2^{\alpha_2-1} - \frac{\delta^2\tilde{Z}_2}{\beta+\theta^2-2r} \tag{A.0.74}$$

Solving (A.0.73) and (A.0.74) for D_1 gives

$$D_1 = \left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\tilde{Z}_2^{2-\alpha_1}$$

Using (A.0.46), (A.0.47) and (A.0.48), we get

$$\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta} = \frac{2}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)} \tag{A.0.75}$$

Therefore,

$$D_1 = \frac{2\delta^2\tilde{Z}_2^{2-\alpha_1}}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} \tag{A.0.76}$$

Since $1 < \alpha_1 < 2$ and $\alpha_2 < 0$, it follows that $D_1 > 0$. Next, solving (A.0.73) and (A.0.74) for E_2 and D_2 yields

$$E_2 = D_2 + \left[-\frac{\alpha_1-1}{r} + \frac{2-\alpha_1}{2(\beta+\theta^2-2r)} + \frac{\alpha_1}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\tilde{Z}_2^{2-\alpha_2} \tag{A.0.77}$$

Once we can find D_2, E_2 would be known from (A.0.77). To find D_2, we use the value-matching and smooth pasting condition at $Z = \tilde{Z}_1$. Then from (A.0.69) and (A.0.70) we get

$$C_1\tilde{Z}_1^{\alpha_1} + C_2\tilde{Z}_1^{\alpha_2} + \frac{\delta L_1}{r}\tilde{Z}_1 - \frac{L_1^2}{2\beta} = D_1\tilde{Z}_1^{\alpha_1} + D_2\tilde{Z}_1^{\alpha_2} - \frac{\delta^2}{2(\beta+\theta^2-2r)}\tilde{Z}_1^2 \tag{A.0.78}$$

$$\alpha_1 C_1 \tilde{Z}_1^{\alpha_1-1} + \alpha_2 C_2 \tilde{Z}_1^{\alpha_2-1} + \frac{\delta L_1}{r} = D_1 \alpha_1 \tilde{Z}_1^{\alpha_1-1} + D_2 \alpha_2 \tilde{Z}_1^{\alpha_2-1} - \frac{\delta^2 \tilde{Z}_1}{\beta+\theta^2-2r}$$

Solving (A.0.78) and (A.0.79) for C_1 we get:

$$C_1 = D_1 - \tilde{Z}_1^{2-\alpha_1}\left[\frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{1-\alpha_2}{r} + \frac{\alpha_2}{2\beta}\right]\frac{\delta^2}{(\alpha_1-\alpha_2)}$$

Using the expression of D_1 from (A.0.76) and the relation in (A.0.75), we get

$$C_1 = \frac{2\delta^2[\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}]}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} \tag{A.0.79}$$

Since $\tilde{Z}_2 > \tilde{Z}_1$ and $1 < \alpha_1 < 2$, it follows that $C_1 > 0$. From (A.0.78) and (A.0.79) we also get

$$D_2 = C_2 - \left[-\frac{\alpha_1-1}{r} + \frac{2-\alpha_1}{2(\beta+\theta^2-2r)} + \frac{\alpha_1}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\tilde{Z}_1^{2-\alpha_2}$$

Using (A.0.46), (A.0.47) and (A.0.48), we get

$$\left[-\frac{\alpha_1-1}{r} + \frac{2-\alpha_1}{2(\beta+\theta^2-2r)} + \frac{\alpha_1}{2\beta}\right] = -\frac{2}{\theta^2\alpha_2(1-\alpha_2)(2-\alpha_2)} > 0 \tag{A.0.80}$$

since $\alpha_2 < 0$. Therefore

$$D_2 = C_2 + \frac{2}{\theta^2\alpha_2(1-\alpha_2)(2-\alpha_2)}\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\tilde{Z}_1^{2-\alpha_2} \tag{A.0.81}$$

If C_2 is known, then D_2 is found from (A.0.81) and E_2 is found from (A.0.77). To find C_2 we apply the value-matching and smooth pasting conditions at $Z = \bar{Z}$. From (A.0.69) and (A.0.72) we get

$$C_1\bar{Z}^{\alpha_1} + C_2\bar{Z}^{\alpha_2} + \frac{\gamma}{K(1-\gamma)}\bar{Z}^{1-\frac{1}{\gamma}} + \frac{\delta L_1}{r}\bar{Z} - \frac{L_1^2}{2\beta} = \frac{\gamma}{K(1-\gamma)}\bar{Z}^{1-\frac{1}{\gamma}} \tag{A.0.82}$$

$$\alpha_1 C_1\bar{Z}^{\alpha_1-1} + \alpha_2 C_2\bar{Z}^{\alpha_2-1} - \frac{1}{K}\bar{Z}^{-\frac{1}{\gamma}} + \frac{\delta L_1}{r} = \frac{1}{K}\bar{Z}^{-\frac{1}{\gamma}} \tag{A.0.83}$$

Solving (A.0.82) and (A.0.83) for C_1, we get

$$C_1 + \frac{1-\alpha_2}{\alpha_1-\alpha_2}\frac{\delta L_1}{r}\bar{Z}^{1-\alpha_1} + \frac{\alpha_2}{\alpha_1-\alpha_2}\frac{L_1^2}{2\beta}\bar{Z}^{-\alpha_1} = 0 \tag{A.0.84}$$

But C_1 is known from (A.0.79). Therefore $\bar{Z}$ is obtained by solving (A.0.84). We do not have a closed form solution for $\bar{Z}$. By part (vi) of Lemma A.0.4, there exists a unique solution $\bar{Z}$ of

(A.0.84) such that $\bar{Z} < \left(\frac{\alpha_1 - 1}{2\alpha_1}\right)\tilde{Z}_1$. Having obtained $\bar{Z}$, equations (A.0.82) and (A.0.83) can be solved for the coefficient C_2:

$$C_2 \;=\; -\left(\frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}\right)\left(\frac{\delta L_1}{r}\right)\bar{Z}^{1-\alpha_2} + \left(\frac{\alpha_1}{\alpha_1 - \alpha_2}\right)\left(\frac{L_1^2}{2\beta}\right)\bar{Z}^{-\alpha_2}$$

Using (A.0.46), (A.0.47), and (A.0.48), we get

$$C_2 = -\frac{\delta^2 \tilde{Z}_1 \bar{Z}^{1-\alpha_2}}{\theta^2(\alpha_1 - \alpha_2)}\left[\frac{\tilde{Z}_1}{\alpha_2 \bar{Z}} + \frac{2}{1-\alpha_2}\right] \tag{A.0.85}$$

which is strictly positive by (iii) of Lemma A.0.4. Thus, D_2 is obtained from (A.0.81) and E_2 from (A.0.77).

In the following, we verify condition (A.0.39) in the Variational Inequality.

(i) For $\bar{Z} < Z < \tilde{Z}_1$, we have to show that

$$C_1 Z^{\alpha_1} + C_2 Z^{\alpha_2} + \frac{\delta L_1}{r} Z - \frac{L_1^2}{2\beta} \;\geq\; 0 \tag{A.0.86}$$

Define

$$f(Z) = C_1 Z^{\alpha_1} + C_2 Z^{\alpha_2} + \frac{\delta L_1}{r} Z - \frac{L_1^2}{2\beta} \tag{A.0.87}$$

From the value-matching and smooth-pasting conditions at $\bar{Z}$ as given in equations (A.0.82), (A.0.83), we get $f(\bar{Z}) = 0$ and $f'(\bar{Z}) = 0$. If we can show that $f'(\bar{Z}) \geq 0$ for $Z > \bar{Z}$, then it would follow that $f(Z) \geq 0$ for $Z \geq \bar{Z}$. To this end, we compute

$$f'(Z) = \alpha_1 C_1 Z^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} + \frac{\delta L_1}{r} \tag{A.0.88}$$

Since $C_1 > 0$, $C_2 > 0$, $\alpha_1 > 1$ and $\alpha_2 < 0$, we have that, for $Z > \bar{Z}$, $\frac{Z}{\bar{Z}} > 1$ and $\left(\frac{Z}{\bar{Z}}\right)^{\alpha_1 - 1} > \left(\frac{Z}{\bar{Z}}\right)^{\alpha_2 - 1}$. This implies that

$$\begin{aligned} f'(Z) \;&=\; \alpha_1 C_1 \bar{Z}^{\alpha_1 - 1}\left(\frac{Z}{\bar{Z}}\right)^{\alpha_1 - 1} + \alpha_2 C_2 \bar{Z}^{\alpha_2 - 1}\left(\frac{Z}{\bar{Z}}\right)^{\alpha_2 - 1} + \frac{\delta L_1}{r} \\ &>\; \left(\frac{Z}{\bar{Z}}\right)^{\alpha_2 - 1}\left[\alpha_1 C_1 \bar{Z}^{\alpha_1 - 1} + \alpha_2 C_2 \bar{Z}^{\alpha_2 - 1}\right] + \frac{\delta L_1}{r} \\ &=\; \left[1 - \left(\frac{Z}{\bar{Z}}\right)^{\alpha_2 - 1}\right]\left(\frac{\delta L_1}{r}\right) > 0 \end{aligned}$$

where the last inequality follows since $\left(\frac{Z}{\bar{Z}}\right)^{\alpha_2 - 1} < 1$ for $Z > \bar{Z}$ and $\alpha_2 < 0$. Therefore $f(Z) \geq 0$ for $Z \geq \bar{Z}$. This proves (A.0.86).

(ii) Next, we have to prove that for $\tilde{Z}_1 < Z < \tilde{Z}_2$,

$$D_1 Z^{\alpha_1} + D_2 Z^{\alpha_2} - \frac{\delta^2 Z^2}{2(\beta + \theta^2 - 2r)} + \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}} \geq \frac{\gamma}{(1-\gamma)K} Z^{1-\frac{1}{\gamma}}$$

i.e.

$$D_1 Z^{\alpha_1} + D_2 Z^{\alpha_2} - \frac{\delta^2 Z^2}{2(\beta + \theta^2 - 2r)} \quad \geq \quad 0 \tag{A.0.89}$$

Note that from (A.0.75) and (A.0.80),

$$\begin{aligned}
\left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right) &> 0 \\
\left[\frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_1}{2\beta} - \frac{\alpha_1 - 1}{r}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right) &> 0
\end{aligned}$$

Since $\alpha_2 < 0$ and $\bar{Z} < Z$, therefore $\bar{Z}^{1-\alpha_2} < Z^{1-\alpha_2}$.

Also, $Z > \tilde{Z}_1$ implies that $\tilde{Z}_1 \bar{Z}^{1-\alpha_2} < Z^{2-\alpha_2}$ i.e. $\tilde{Z}_1 \bar{Z}^{1-\alpha_2} Z^{\alpha_2} < Z^{2-\alpha_2} Z^{\alpha_2} = Z^2$. Hence

$$-\left(\frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}\right)\left(\frac{\delta^2}{r}\right)\tilde{Z}_1 \bar{Z}^{1-\alpha_1} Z^{\alpha_2} \quad > \quad -\left(\frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}\right)\left(\frac{\delta^2}{r}\right) Z^2 \tag{A.0.90}$$

Also, $\alpha_2 < 2$ and $\tilde{Z}_1 < Z$ implies that $\tilde{Z}_1^{2-\alpha_2} < Z^{2-\alpha_2}$ i.e.

$$Z^{\alpha_2} \tilde{Z}_1^{2-\alpha_2} \quad < \quad Z^2 \tag{A.0.91}$$

Using inequality (A.0.91) we get

$$\begin{aligned}
D_1 Z^{\alpha_1} &= \left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\tilde{Z}_2^{2-\alpha_1} Z^{\alpha_1} \\
&> \left[\frac{1-\alpha_2}{r} + \frac{2-\alpha_2}{2(\beta+\theta^2-2r)} + \frac{\alpha_2}{2\beta}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right) Z^2
\end{aligned}$$

Next, using (A.0.85), (A.0.81), (A.0.90) and (A.0.91) we get,

$$\begin{aligned} D_2Z^{\alpha_2} &= \frac{\alpha_1\delta^2}{2\beta(\alpha_1-\alpha_2)}\tilde{Z}_1^2\bar{Z}^{-\alpha_2}Z^{\alpha_2} - \left(\frac{\alpha_1-1}{\alpha_1-\alpha_2}\right)\left(\frac{\delta^2}{r}\right)\tilde{Z}_1\bar{Z}^{1-\alpha_2}Z^{\alpha_2} \\ &- \left[\frac{2-\alpha_1}{2(\beta+\theta^2-2r)}+\frac{\alpha_1}{2\beta}-\frac{\alpha_1-1}{r}\right]\left(\frac{\delta^2}{r}\right)\bar{Z}^{2-\alpha_2}Z^{\alpha_2} \\ &> \frac{\alpha_1\delta^2}{2\beta(\alpha_1-\alpha_2)}\tilde{Z}_1^2\bar{Z}^{-\alpha_2}Z^{\alpha_2} - \left(\frac{\alpha_1-1}{\alpha_1-\alpha 2}\right)\left(\frac{\delta^2}{r}\right)Z^2 \\ &- \left[\frac{2-\alpha_1}{2(\beta+\theta^2-2r)}+\frac{\alpha_1}{2\beta}-\frac{\alpha_1-1}{r}\right]\left(\frac{\delta^2}{r}\right)Z^2 \end{aligned}$$

Therefore

$$D_1Z^{\alpha_1}+D_2Z^{\alpha_2}-\frac{\delta^2Z^2}{2(\beta+\theta^2-2r)} > \frac{\alpha_1\delta^2}{2\beta(\alpha_1-\alpha_2)}+\frac{\delta^2Z^2}{(\alpha_1-\alpha_2)}\left[\frac{1-\alpha_2}{r}-\frac{\alpha_1-\alpha_2}{2\beta}\right]$$

The first term in the right hand side of the inequality is positive since $\alpha_1 > \alpha_2$. We claim that

$$\frac{1-\alpha_2}{r}-\frac{\alpha_1-\alpha_2}{2\beta} > 0 \tag{A.0.92}$$

Using (A.0.46) and (A.0.47) we get $\frac{1-\alpha_2}{r}-\frac{\alpha_1-\alpha_2}{2\beta} = -\frac{2}{\theta^2(1-\alpha_1)}+\frac{\alpha_1-\alpha_2}{\theta^2\alpha_1\alpha_2} = \frac{1}{\theta^2}\frac{\alpha_1-\alpha_1^2-\alpha_2-\alpha_1\alpha_2}{\alpha_1\alpha_2(1-\alpha_1)}$. Since $\alpha_1 > 1$ and $\alpha_2 < 0$, we have $\alpha_1\alpha_2\,(1-\alpha_1) > 0$. We now show that $\alpha_1-\alpha_1^2-\alpha_2-\alpha_1\alpha_2 > 0$. This will be true if $\alpha_1-\alpha_2-\alpha_1(\alpha_1+\alpha_2) > 0$ i.e. if $1-\frac{\alpha_2}{\alpha_1}-(\alpha_1+\alpha_2) > 0$ i.e. if $-\frac{\alpha_2}{\alpha_1} > \alpha_1+\alpha_2-1 = -\frac{2(\beta-r)}{\theta^2}$. Since $\alpha_2 < 0$ and $\beta > r$, the last inequality is true. Thus we get that $\alpha_1-\alpha_1^2-\alpha_2-\alpha_1\alpha_2 > 0$. Therefore inequality (A.0.92) follows. Hence (A.0.89) holds.

iii) Lastly, we have to prove that for $Z \geq \tilde{Z}_2$,

$$E_2Z^{\alpha_2}+\frac{\gamma}{K(1-\gamma)}Z^{1-\frac{1}{\gamma}}+\frac{\delta L_2}{r}Z-\frac{1}{2}\frac{L_2^2}{\beta} \geq \frac{\gamma}{K(1-\gamma)}Z^{1-\frac{1}{\gamma}}$$

i.e

$$E_2Z^{\alpha_2}+\frac{\delta L_2}{r}Z-\frac{1}{2}\frac{L_2^2}{\beta} \geq 0 \tag{A.0.93}$$

Since $C_2 > 0$, $\tilde{Z}_2 > \tilde{Z}_1$ and $\alpha_2 < 0$, using (A.0.91) we get

$$E_2 = C_2+\left[\frac{2-\alpha_1}{2(\beta+\theta^2-2r)}+\frac{\alpha_1}{2\beta}-\frac{\alpha_1-1}{r}\right]\left(\frac{\delta^2}{\alpha_1-\alpha_2}\right)\left(\tilde{Z}_2^{2-\alpha_2}-\tilde{Z}_1^{2-\alpha_2}\right) > 0$$

Also, $\frac{\delta L_2}{r}Z - \frac{1}{2}\frac{L_2^2}{\beta} = \frac{\delta^2}{r}Z\tilde{Z}_2 - \frac{1}{2}\frac{\delta^2\tilde{Z}_2^2}{\beta} > \delta^2\tilde{Z}_2^2\left(\frac{1}{r} - \frac{1}{2\beta}\right) > 0$, since $\beta > r$ and $Z \geq \tilde{Z}_2$. Thus (A.0.93) holds.

This completes the proof of the theorem. $\square$

Theorem A.0.7. *If the pair* $(\bar{Z}, \tilde{\phi}(t, Z)$ *is a solution to the Variational Inequality, then* $\tilde{\phi}(t, Z) = \phi(t, Z)$ *where* $\phi(t, Z)$ *is defined in (A.0.36), and the optimal stopping time is given by* $\tau = inf\{s > t : Z_s \leq \bar{Z}\}$.

Proof. The proof is standard and follows the proof of Theorem 10.4.1 of Oksendal [47]. $\square$

Corollary A.0.8. *For any* $\lambda > 0$*, let* τ_λ *be the corresponding optimal stopping time such that*

$$\tilde{V}(\lambda) = E\left[\int_0^{\tau_\lambda} e^{-\beta t}\left[\tilde{u}(\lambda e^{\beta t}\xi_t) + \tilde{g}(\lambda e^{\beta t}\xi_t)\right] dt + e^{-\beta\tau_\lambda}\tilde{U}(\lambda e^{\beta\tau_\lambda}\xi_{\tau_\lambda})\right] \quad \text{(A.0.94)}$$

If $\tilde{V}(\lambda)$ *is differentiable, then*

$$\tilde{V}'(\lambda) = -E\left[\int_0^{\tau_\lambda}\left[\xi_t(\lambda e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} - \delta\xi_t y_t\right] dt\right] - E\left[\xi_{\tau_\lambda}X_{\tau_\lambda}\right]$$

where y_t *is given in (A.0.20).*

Proof. The proof follows by differentiating the expressions (A.0.14), (A.0.15) and (A.0.16) with respect to λ and using the expression of y_t in (A.0.20). $\square$

Corollary A.0.9. *The function* $v(Z)$ *defined in (A.0.64), (A.0.65), (A.0.66) and (A.0.67) is continuously differentiable and strictly convex, i.e.* $v''(Z) > 0$ *for all* $Z > 0$*. Moreover,* $v'(Z)$ *maps* $(0, \infty)$ *onto* $\left(-\infty, \frac{\delta L_2}{r}\right)$*. Hence, for any* $x \in \left(-\frac{\delta L_2}{r}, \infty\right)$*, the equation* $v'(Z) = -x$ *has a unique solution in* $(0, \infty)$*.*

Proof. (i) For $0 < Z \leq \bar{Z}$

$$\begin{aligned} v(Z) &= \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} \\ \frac{\partial v(Z)}{\partial Z} &= -Z^{-\frac{1}{\gamma}} < 0 \\ \frac{\partial^2 v(Z)}{\partial Z^2} &= \frac{1}{\gamma} Z^{-1-\frac{1}{\gamma}} > 0 \end{aligned}$$

(ii) For $\bar{Z} < Z \leq \tilde{Z}_1$

$$\begin{aligned} v(Z) &= C_1 Z^{\alpha_1} + C_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_1}{r} Z - \frac{L_1^2}{2\beta} \\ \frac{\partial v(Z)}{\partial Z} &= \alpha_1 C_1 Z^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} + \frac{\delta L_1}{r} \end{aligned}$$

(iii) For $\tilde{Z}_1 < Z < \tilde{Z}_2$

$$\begin{aligned} v(Z_t) &= D_1 Z^{\alpha_1} + D_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} - \frac{\delta^2 Z^2}{2(\beta + \theta^2 - 2r)} \\ \frac{\partial v(Z)}{\partial Z} &= \alpha_1 D_1 Z^{\alpha_1 - 1} + \alpha_2 D_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} - \frac{\delta^2 Z}{\beta + \theta^2 - 2r} \end{aligned}$$

(iv) For $Z \geq \tilde{Z}_2$

$$\begin{aligned} v(Z) &= E_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} + \frac{\delta L_2}{r} Z - \frac{L_2^2}{2\beta} \\ \frac{\partial v(Z)}{\partial Z} &= \alpha_2 E_2 Z^{\alpha_2 - 1} - \frac{1}{K} Z^{-\frac{1}{\gamma}} + \frac{\delta L_2}{r} \end{aligned}$$

By Lemma A.0.5, it follows that $\frac{\partial^2 v(Z)}{\partial Z^2} > 0$ i.e. $v(Z)$ is strictly convex in Z.

Next, we show that $v'(Z) < \frac{\delta L_2}{r}$ for all $Z > 0$. For $Z \geq \tilde{Z}_2$,

$$\begin{aligned} v'(Z) &= \alpha_2 E_2 Z_t^{\alpha_2 - 1} - \frac{Z_t^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_2}{r} \\ &< \frac{\delta L_2}{r} \end{aligned}$$

The inequality follows because $\alpha_2 < 0$, $E_2 > 0$ and $K > 0$. Since $v'(Z)$ is continuous and strictly increasing in Z, therefore $v'(Z) < \frac{\delta L_2}{r}$ for all $Z > 0$. Thus, for $x \in \left(-\frac{\delta L_2}{r}, \infty\right)$ the equation $v'(Z) = -x$ has a solution in $(0, \infty)$. This completes the proof. $\square$

Proofs of Propositions 2.3.2, 2.3.3, 2.3.4 and 2.3.5:

Proof. We will show that

$$\begin{aligned} c_t^* &= \left(Z^* e^{\beta t}\xi_t\right)^{-\frac{1}{\gamma}} 1_{\{0<t\le\tau^*\}} \\ \tau^* &= inf\left\{t \ge 0 : Z^* e^{\beta t}\xi_t = \bar{\lambda}\right\} \\ y_t^* &= \delta Z^* e^{\beta t}\xi_t 1_{\{\frac{L_1}{\delta} < Z^* e^{\beta t}\xi_t < \frac{L_2}{\delta}\}} + L_1 1_{\{Z^* e^{\beta t}\xi_t \le \frac{L_1}{\delta}\}} \\ &+ L_2 1_{\{Z^* e^{\beta t}\xi_t \ge \frac{L_2}{\delta}\}}\ for\ 0 \le t < \tau^* \end{aligned}$$

is an optimal policy for the optimization problem (A.0.7) where Z^* solves (2.3.7), (2.3.8) or (2.3.9) (in those equations X_t must be replaced by x) depending on the initial value x of financial wealth. Consider the function $\tilde{V}(Z) = \phi(0, Z) = v(Z)$ as obtained in Theorem A.0.6. Since τ^* is the optimal stopping time associated with Z^*, therefore

$$\tilde{V}(Z^*) = E\left[\int_0^{\tau^*} e^{-\beta t}\{\tilde{u}(Z^* e^{\beta t}\xi_t) + \tilde{g}(Z^* e^{\beta t}\xi_t)\}dt + e^{-\beta\tau^*}\tilde{U}(Z^* e^{\beta\tau^*}\xi_{\tau^*})\right]$$

and by Corollary A.0.8

$$\tilde{V}'(Z^*) = -E\left[\int_0^{\tau^*}\left[\xi_t(Z^* e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} - \delta\xi_t y_t^*\right]dt\right] - E\left[\xi_{\tau^*}X_{\tau^*}\right]$$

By Corollary A.0.9, $V(Z)$ is continuously differentiable, strictly convex and $V'(Z)$ maps $(0, \infty)$ to $(-\infty, \frac{\delta L_2}{r})$. It can be directly verified from the equations (2.3.7), (2.3.8) or (2.3.9) that Z^* solves the equation $\tilde{V}'(Z^*) = -x$ i.e. Z^* minimizes $\left[\tilde{V}(\lambda) + \lambda x\right]$ over all $\lambda > 0$. Therefore,

$$\begin{aligned} x &= -\tilde{V}'(Z^*) = E\left[\int_0^{\tau^*}\left[\xi_t(Z^* e^{\beta t}\xi_t)^{-\frac{1}{\gamma}} - \delta\xi_t y_t^*\right]dt\right] + E\left[\xi_{\tau^*}X_{\tau^*}\right] \\ &= E\left[\int_0^{\tau^*}\xi_t(c_t^* - \delta y_t^*)dt + \xi_{\tau^*}X_{\tau^*}^*\right] \end{aligned} \tag{A.0.95}$$

which shows that the financial wealth x can finance the consumption-effort policy (c_t^*, y_t^*) and the retirement wealth X_{τ^*}. This proves that the policy is feasible. Moreover,

$$\tilde{V}(Z^*) + Z^* x = \inf_{\lambda>0}\left[\tilde{V}(\lambda) + \lambda x\right] \tag{A.0.96}$$

By (A.0.32),

$$V(x) = \underset{\tau}{sup} \underset{\lambda>0}{inf} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] \leq \underset{\lambda>0}{inf} \underset{\tau}{sup} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \underset{\lambda>0}{inf} \left[\tilde{V}(\lambda) + \lambda x\right] \quad (A.0.97)$$

But

$$\begin{aligned} V(x) &= \underset{\tau}{sup}\, J(x;c,\pi,y,\tau) \\ &\geq J(x;c^*,\pi^*,y^*,\tau^*) \\ &= E\left[\int_0^{\tau^*} e^{-\beta t}\left\{u(c_t^*) - \frac{1}{2}(y_t^*)^2\right\} dt + e^{-\beta\tau^*}U(X_{\tau^*})\right] \\ &= E\left[\int_0^{\tau^*} e^{-\beta t}\{\tilde{u}(Z^* e^{\beta t}\xi_t) + \tilde{g}(Z^* e^{\beta t}\xi_t)\}dt + e^{-\beta\tau^*}\tilde{U}(Z^* e^{\beta\tau^*}\xi_{\tau^*})\right] \\ &+ Z^* E\left[\int_0^{\tau^*} \xi_t(c_t^* - \delta y_t^*)dt + \xi_{\tau^*}X_{\tau^*}\right] \\ &= \tilde{V}(Z^*) + Z^* x = \underset{\lambda>0}{inf}\left[\tilde{V}(\lambda) + \lambda x\right] \end{aligned} \quad (A.0.98)$$

where the third equality follows from (A.0.95) and the definitions of $\tilde{u}$ and $\tilde{g}$ as in (A.0.11) and (A.0.12), the last equality follows from (A.0.96). Combining (A.0.96) and (A.0.98) we get

$$V(x) = J(x;c^*,\pi^*,y^*,\tau^*) \quad (A.0.99)$$

This proves the optimality of the policy (c_t^*, y_t^*, τ^*). The optimal financial wealth at time t is given by

$$X_t^* = \frac{1}{\xi_t}\left[\int_t^{\tau^*} \xi_s\,(c_s^* - \delta y_s^*)\,ds + \xi_{\tau^*}X_{\tau^*}\right]$$

Using Ito's lemma and the expressions of the optimal consumption policies in Proposition 2.3.3, we get the optimal portfolio policies π^* given in Proposition 2.3.4. The results in Proposition 2.3.5 follows from Proposition A.0.12 below. This complete the proofs of the propositions. □

For initial financial wealth $X_0 = x$, the value function $V(x)$ in (A.0.7) is derived in the next proposition.

Proposition A.0.10. *Define the wealth levels:*

$$
\begin{aligned}
\bar{X} &= \frac{\bar{Z}^{-\frac{1}{\gamma}}}{K} \\
\tilde{X}_1 &= -\alpha_1 C_1 \tilde{Z}_1^{\alpha_1-1} - \alpha_2 C_2 \tilde{Z}_1^{\alpha_2-1} + \frac{\tilde{Z}_1^{-\frac{1}{\gamma}}}{K} - \frac{\delta L_1}{r} \\
\tilde{X}_2 &= -\alpha_2 E_2 \tilde{Z}_2^{\alpha_2-1} + \frac{\tilde{Z}_2^{-\frac{1}{\gamma}}}{K} - \frac{\delta L_2}{r}
\end{aligned}
$$

Then the value function $V(x)$ in (A.0.7) is given as follows:

(i) For $-\frac{\delta L_2}{r} < x \leq \tilde{X}_2$

$$
V(x) = E_2 (Z^*)^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} (Z^*)^{1-\frac{1}{\gamma}} + \frac{\delta L_2}{r} Z^* - \frac{1}{2}\frac{L_2^2}{\beta} + Z^* x
$$

where Z^* is the unique solution of the equation

$$
\alpha_2 E_2 (Z^*)^{\alpha_2-1} - \frac{(Z^*)^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_2}{r} + x = 0 \tag{A.0.100}
$$

(ii) For $\tilde{X}_2 < x < \tilde{X}_1$

$$
V(x) = D_1 (Z^*)^{\alpha_1} + D_2 (Z^*)^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} (Z^*)^{1-\frac{1}{\gamma}} - \frac{\delta^2 (Z^*)^2}{2(\beta+\theta^2-2r)} + \lambda x
$$

where λ is the unique solution of the equation

$$
\alpha_1 D_1 (Z^*)^{\alpha_1-1} + \alpha_2 D_2 (Z^*)^{\alpha_2-1} - \frac{(Z^*)^{-\frac{1}{\gamma}}}{K} - \frac{\delta^2 Z^*}{(\beta+\theta^2-2r)} + x = 0 \tag{A.0.101}
$$

(iii) For $\tilde{X}_1 \leq x < \bar{X}$

$$
V(x) = C_1 (Z^*)^{\alpha_1} + C_2 (Z^*)^{\alpha_2} + \frac{\gamma}{(1-\gamma)K} (Z^*)^{1-\frac{1}{\gamma}} + \frac{\delta L_1}{r} Z^* - \frac{1}{2}\frac{L_1^2}{\beta} + Z^* x
$$

where Z^* is the unique solution of the equation

$$
\alpha_1 C_1 (Z^*)^{\alpha_1-1} + \alpha_2 C_2 (Z^*)^{\alpha_2-1} - \frac{(Z^*)^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + x = 0 \tag{A.0.102}
$$

(iv) For $x \geq \bar{X}$

$$
V(x) = \frac{K^{-\gamma}}{1-\gamma} x^{1-\gamma}
$$

Proof. The value function $V(x)$ is given by:

$$V(x) = \underset{\lambda>0}{inf}\left[\tilde{V}(\lambda) + \lambda x\right]$$

where $\tilde{V}(\lambda) = \phi(0,\lambda) = v(\lambda)$ and the expression for $v(\lambda)$ is as obtained in Theorem A.0.6. By Corollary A.0.9, $v(\lambda)$ is strictly convex and $v'(\lambda)$ maps $(0,\infty)$ to $\left(-\infty, \frac{\delta L_2}{r}\right)$. Therefore, there exists a unique $Z^* > 0$ such that $\tilde{V}'(Z^*) = -x$ and $V(x) = \tilde{V}(Z^*) + Z^*x = \underset{\lambda>0}{inf}\left[\tilde{V}(\lambda) + \lambda x\right] = \underset{\lambda>0}{inf}\left[v(\lambda) + \lambda x\right]$. The expressions of $V(x)$ follow directly from the expressions of $v(\lambda)$ in Theorem A.0.6 which also shows that Z^* satisfies the equations (A.0.100), (A.0.101) and (A.0.102) for the corresponding values of x. This completes the proof of the proposition. $\square$

Corollary A.0.11. *The financial wealth X_t is a strictly decreasing function of Z_t and satisfies $X_t \geq -\frac{\delta L_2}{r}$ for all $Z_t > 0$.*

Proof. Since $X_t = -v'(Z_t)$, therefore by Lemma A.0.9 we have that $\frac{\partial X_t}{\partial Z_t} = -v''(Z_t) < 0$ and hence X_t is a decreasing function of Z_t. Next, we prove that $X_t \geq -\frac{\delta L_2}{r}$ for all $Z_t > 0$. For $Z_t \geq \tilde{Z}_2$,

$$\begin{aligned} X_t &= -\left[\alpha_2 E_2 Z_t^{\alpha_2 - 1} - \frac{Z_t^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_2}{r}\right] \\ &> -\frac{\delta L_2}{r} \end{aligned}$$

The inequality follows because $\alpha_2 < 0$, $E_2 > 0$ and $K > 0$. Since X_t is continuous and decreasing in Z_t, therefore $X_t \geq -\frac{\delta L_2}{r}$ for all $Z_t > 0$. This completes the proof. $\square$

The next proposition gives a characterization of the human capital. Define

$$\hat{C}_1 = \frac{2\delta[\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}]}{\theta^2(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} > 0 \tag{A.0.103}$$

$$\hat{D}_1 = \frac{2\delta\tilde{Z}_2^{2-\alpha_1}}{\theta^2(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} > 0 \tag{A.0.104}$$

$$\hat{C}_2 = -\hat{C}_1\bar{Z}^{\alpha_1-\alpha_2} - \frac{L_1}{r}\bar{Z}^{1-\alpha_2} < 0 \tag{A.0.105}$$

$$\hat{D}_2 = \hat{C}_2 + \frac{2\delta\tilde{Z}_1^{2-\alpha_2}}{\theta^2(1-\alpha_2)(2-\alpha_2)(\alpha_1-\alpha_2)} \tag{A.0.106}$$

$$\hat{E}_2 = \hat{C}_2 - \frac{2\delta[\tilde{Z}_2^{2-\alpha_2} - \tilde{Z}_1^{2-\alpha_2}]}{\theta^2(1-\alpha_2)(2-\alpha_2)(\alpha_1-\alpha_2)} < 0 \tag{A.0.107}$$

Proposition A.0.12. *Let $\hat{C}_1$, $\hat{D}_1$, $\hat{C}_2$, $\hat{D}_2$, and $\hat{E}_2$ be defined as in (A.0.103), (A.0.104), (A.0.105), (A.0.106) and (A.0.107). For $Z_t = Z$, the present value of future labor income (human capital) $H(Z)$ is given as follows:*

(i) $H(Z) = 0$ for all $Z \leq \bar{Z}$

(ii) For $\bar{Z} < Z \leq \tilde{Z}_1$

$$H(Z) = \delta\left[\hat{C}_1 Z^{\alpha_1-1} + \hat{C}_2 Z^{\alpha_2-1} + \frac{L_1}{r}\right]$$

(iii) For $\tilde{Z}_1 < Z < \tilde{Z}_2$,

$$H(Z) = \delta\left[\hat{D}_1 Z^{\alpha_1-1} + \hat{D}_2 Z^{\alpha_2-1} - \frac{\delta Z}{\beta+\theta^2-2r}\right]$$

(iv) For $Z \geq \tilde{Z}_2$

$$H(Z) = \delta\left[\hat{E}_2 Z^{\alpha_2-1} + \frac{L_2}{r}\right]$$

$H(Z)$ is increasing in Z and hence decreasing in the financial wealth X.

Proof. For $Z_t = Z$, the present value of future labor income at time t is given by

$$H(Z) = \frac{1}{\xi_t}E_t\left[\int_t^\tau \delta y_s\xi_s ds\right] = \frac{\delta}{\lambda\xi_t}E_t\left[\int_t^\tau e^{-\beta s}y_s Z_s ds | Z_t = Z\right]$$

where τ is the optimal stopping time, $Z_t = \lambda e^{\beta t}\xi_t$, $Z_\tau = \bar{Z}$ and

$$y_t = (\delta Z_t)1_{\{\tilde{Z}_1 < Z_t < \tilde{Z}_2\}} + L_1 1_{\{Z_t \le \tilde{Z}_1\}} + L_2 1_{\{Z_t \ge \tilde{Z}_2\}}$$

Define the function

$$f(Z) = E_t\left[\int_t^\tau e^{-\beta s} y_s Z_s ds | Z_t = Z\right]$$

Then f solves the following equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\theta^2 Z^2 \frac{\partial^2 f}{\partial Z^2} + (\beta - r)\, Z \frac{\partial f}{\partial Z} + e^{-\beta t} y_t Z = 0$$

We guess the solution to be $f(Z) = e^{-\beta t}\phi(Z)$. Then $\phi(Z)$ satisfies the following equations depending on the value of Z.

(i) For $\bar{Z} < Z \le \tilde{Z}_1$

$$-\beta\phi(Z) + \frac{1}{2}\theta^2 Z^2 \phi^{''}(Z) + (\beta - r) Z \phi^{'}(Z) + L_1 Z = 0$$

with solution

$$\phi(Z) = \hat{C}_1 Z^{\alpha_1} + \hat{C}_2 Z^{\alpha_2} + \frac{L_1 Z}{r} \tag{A.0.108}$$

(ii) For $\tilde{Z}_1 < Z < \tilde{Z}_2$

$$-\beta\phi(Z) + \frac{1}{2}\theta^2 Z^2 \phi^{''}(Z) + (\beta - r) Z \phi^{'}(Z) + \delta Z^2 = 0$$

with solution

$$\phi(Z) = \hat{D}_1 Z^{\alpha_1} + \hat{D}_2 Z^{\alpha_2} - \frac{\delta Z^2}{\beta + \theta^2 - 2r} \tag{A.0.109}$$

(iii) For $Z \ge \tilde{Z}_2$

$$-\beta\phi(Z) + \frac{1}{2}\theta^2 Z^2 \phi^{''}(Z) + (\beta - r) Z \phi^{'}(Z) + L_2 Z = 0$$

with solution

$$\phi(Z) = \hat{E}_1 Z^{\alpha_1} + \hat{E}_2 Z^{\alpha_2} + \frac{L_2 Z}{r} \tag{A.0.110}$$

We use value-matching and smooth-pasting conditions to determine the coefficients $\hat{C}_1$, $\hat{C}_2$, $\hat{D}_1$, $\hat{D}_2$, $\hat{E}_1$ and $\hat{E}_2$. For $Z \to \infty$, $H(Z)$ must remain finite. This requires that

$$\hat{E}_1 \;=\; 0$$

Using equations (A.0.108) and (A.0.109), value-matching and smooth-pasting condition at $Z = \tilde{Z}_1$ gives

$$\hat{C}_1 \tilde{Z}_1^{\alpha_1} + \hat{C}_2 \tilde{Z}_1^{\alpha_2} + \frac{L_1 \tilde{Z}_1}{r} \;=\; \hat{D}_1 Z_1^{\alpha_1} + \hat{D}_2 Z_1^{\alpha_2} - \frac{\delta \tilde{Z}_1^2}{\beta + \theta^2 - 2r} \tag{A.0.111}$$

$$\alpha_1 \hat{C}_1 \tilde{Z}_1^{\alpha_1 - 1} + \alpha_2 \hat{C}_2 \tilde{Z}_1^{\alpha_2 - 1} + \frac{L_1}{r} \;=\; \alpha_1 \hat{D}_1 Z_1^{\alpha_1 - 1} + \alpha_2 \hat{D}_2 Z_1^{\alpha_2 - 1} - \frac{2\delta \tilde{Z}_1}{\beta + \theta^2 - 2r} \tag{A.0.112}$$

Moreover, $\phi(\bar{Z}) = 0$ since the agent retires at time $t = \tau$ i.e. when $Z = \bar{Z}$. Therefore from (A.0.108),

$$\hat{C}_1 \bar{Z}^{\alpha_1} + \hat{C}_2 \bar{Z}^{\alpha_2} + \frac{L_1 \bar{Z}}{r} \;=\; 0 \tag{A.0.113}$$

Solving equations (A.0.111), (A.0.112) and (A.0.113) for $\hat{C}_1$, $\hat{D}_1$ and $\hat{D}_2$ gives

$$\hat{D}_1 \;=\; \frac{(2-\alpha_2)\delta}{(\alpha_1 - \alpha_2)(\beta + \theta^2 - 2r)} \tilde{Z}_2^{2-\alpha_1} + \frac{(1-\alpha_2)L_2}{(\alpha_1 - \alpha_2) r} \tilde{Z}_2^{1-\alpha_1} \tag{A.0.114}$$

$$\hat{C}_1 \;=\; \hat{D}_1 - \frac{(1-\alpha_2)L_1}{(\alpha_1 - \alpha_2) r} \tilde{Z}_1^{1-\alpha_1} - \frac{(2-\alpha_2)\delta}{(\alpha_1 - \alpha_2)(\beta + \theta^2 - 2r)} \tilde{Z}_1^{2-\alpha_1} \tag{A.0.115}$$

$$\hat{D}_2 \;=\; \hat{C}_2 + \frac{(\alpha_1 - 1)L_1}{(\alpha_1 - \alpha_2) r} \tilde{Z}_1^{1-\alpha_2} - \frac{(2-\alpha_1)\delta}{(\alpha_1 - \alpha_2)(\beta + \theta^2 - 2r)} \tilde{Z}_1^{2-\alpha_2} \tag{A.0.116}$$

$$\hat{C}_2 \;=\; -\hat{C}_1 \bar{Z}^{\alpha_1 - \alpha_2} - \frac{L_1}{r} \bar{Z}^{1-\alpha_2} \tag{A.0.117}$$

Lastly, using equations (A.0.109) and (A.0.110), value-matching and smooth-pasting conditions at $Z = \tilde{Z}_2$ give

$$\hat{D}_1 Z_2^{\alpha_1} + \hat{D}_2 Z_2^{\alpha_2} - \frac{\delta \tilde{Z}_2^2}{\beta + \theta^2 - 2r} \;=\; \hat{E}_2 \tilde{Z}_2^{\alpha_2} + \frac{L_2 \tilde{Z}_2}{r} \tag{A.0.118}$$

$$\alpha_1 \hat{D}_1 Z_2^{\alpha_1 - 1} + \alpha_2 \hat{D}_2 Z_2^{\alpha_2 - 1} - \frac{2\delta \tilde{Z}_2}{\beta + \theta^2 - 2r} \;=\; \alpha_2 \hat{E}_2 \tilde{Z}_2^{\alpha_2 - 1} + \frac{L_2}{r} \tag{A.0.119}$$

Solving equations (A.0.118) and (A.0.119) for $\hat{E}_2$ gives

$$\hat{E}_2 \;=\; \hat{D}_2 + \frac{(2-\alpha_1)\delta}{(\alpha_1-\alpha_2)(\beta+\theta^2-2r)}\tilde{Z}_2^{2-\alpha_2} - \frac{(\alpha_1-1)L_2}{(\alpha_1-\alpha_2)r}\tilde{Z}_2^{1-\alpha_2} \qquad \text{(A.0.120)}$$

Equations (A.0.114), (A.0.115), (A.0.116), (A.0.117) and (A.0.120) form a system of equations which can be solved to get the coefficients as in (A.0.103), (A.0.104), (A.0.105), (A.0.106), and (A.0.107). Thus, the human capital is given by

$$\begin{aligned} H(Z) &= \frac{\delta}{Z}\left[\hat{C}_1 Z^{\alpha_1} + \hat{C}_2 Z^{\alpha_2} + \frac{L_1 Z}{r}\right], \;\; \bar{Z} < Z \le \tilde{Z}_1 \\ H(Z) &= \frac{\delta}{Z}\left[\hat{D}_1 Z^{\alpha_1} + \hat{D}_2 Z^{\alpha_2} + \frac{\delta Z^2}{\beta+\theta^2-2r}\right], \;\; \tilde{Z}_1 < Z < \tilde{Z}_2 \\ H(Z) &= \frac{\delta}{Z}\left[\hat{E}_2 Z^{\alpha_2} + \frac{L_2 Z}{r}\right], \;\; Z \ge \tilde{Z}_2 \end{aligned}$$

This completes the proof of the proposition. □

Proof of Proposition 2.3.6:

Proof. Since $\frac{\partial H_t}{\partial X_t} = \frac{\partial H_t}{\partial Z_t} / \frac{\partial X_t}{\partial Z_t}$ and $\frac{\partial X_t}{\partial Z_t} < 0$, therefore it suffices to prove that $\frac{\partial H_t}{\partial Z_t} > 0$. For $Z_t \le \bar{Z}$, $H(Z_t) = 0$. Therefore we prove that $\frac{\partial H_t}{\partial Z_t} > 0$ for $Z_t > \bar{Z}$.

(i) For $\bar{Z} < Z \le \tilde{Z}_1$

$$\begin{aligned} H(Z_t) &= \delta\left[\hat{C}_1 Z_t^{\alpha_1-1} + \hat{C}_2 Z_t^{\alpha_2-1} + \frac{L_1}{r}\right] \\ \frac{\partial H(Z_t)}{\partial Z_t} &= \delta\left[(\alpha_1-1)\hat{C}_1 Z_t^{\alpha_1-2} + (\alpha_2-1)\hat{C}_2 Z_t^{\alpha_2-2}\right] \end{aligned}$$

Since $\alpha_1 > 1, \alpha_2 < 0, \hat{C}_1 > 0, \hat{C}_2 < 0$ therefore $\frac{\partial H(Z_t)}{\partial Z_t} > 0$

(ii) For $\tilde{Z}_1 < Z_t < \tilde{Z}_2$,

$$\begin{aligned}
H(Z_t) &= \delta\left[\hat{D}_1 Z_t^{\alpha_1-1} + \hat{D}_2 Z_t^{\alpha_2-1} - \frac{\delta Z_t}{\beta+\theta^2-2r}\right] \\
\frac{\partial H(Z_t)}{\partial Z_t} &= \delta\left[(\alpha_1-1)\hat{D}_1 Z_t^{\alpha_1-2} + (\alpha_2-1)\hat{D}_2 Z_t^{\alpha_2-2} - \frac{\delta}{\beta+\theta^2-2r}\right] \\
&= \delta(\alpha_2-1)\hat{C}_2 Z_t^{\alpha_2-2} \\
&\quad + \frac{2\delta^2}{\theta^2(\alpha_1-\alpha_2)}\left[\frac{\tilde{Z}_2^{2-\alpha_1} Z_t^{\alpha_1-2}}{2-\alpha_1} - \frac{\tilde{Z}_1^{2-\alpha_2} Z_t^{\alpha_2-2}}{2-\alpha_2}\right] \\
&\quad - \frac{\delta^2}{\beta+\theta^2-2r}
\end{aligned}$$

Since $\alpha_2 < 0$ and $\hat{C}_2 < 0$, therefore $\delta(\alpha_2-1)\hat{C}_2 Z_t^{\alpha_2-2} > 0$. Since $\alpha_2 < 0$ and $\tilde{Z}_1 < Z_t < \tilde{Z}_2$, therefore $Z_t^{\alpha_2-2} > \tilde{Z}_2^{\alpha_2-2} \Rightarrow Z_t^{\alpha_2-2} > \tilde{Z}_2^{2-\alpha_1} > 1$ and $\tilde{Z}_1^{2-\alpha_2} < Z_t^{2-\alpha_2} \Rightarrow \tilde{Z}_1^{2-\alpha_2} Z_t^{\alpha_2-2} < 1$. Therefore,

$$\begin{aligned}
\frac{\partial H(Z_t)}{\partial Z_t} &> \frac{2\delta^2}{\theta^2(\alpha_1-\alpha_2)}\left[\frac{1}{2-\alpha_1} - \frac{1}{2-\alpha_2}\right] - \frac{\delta^2}{\beta+\theta^2-2r} \\
&= \frac{2\delta^2}{\theta^2(\alpha_1-\alpha_2)}\frac{\alpha_1-\alpha_2}{(2-\alpha_1)(2-\alpha_2)} - \frac{\delta^2}{\beta+\theta^2-2r} \\
&= \frac{2\delta^2}{\theta^2(2-\alpha_1)(2-\alpha_2)} - \frac{\delta^2}{\beta+\theta^2-2r} = 0
\end{aligned}$$

The last equality follows from the fact that $\beta+\theta^2-2r = \frac{\theta^2}{2}(2-\alpha_1)(2-\alpha_2)$. Thus, for $\tilde{Z}_1 < Z_t < \tilde{Z}_2$, $\frac{\partial H(Z_t)}{\partial Z_t} > 0$.

(iii) Finally, for $Z_t \geq \tilde{Z}_2$

$$\begin{aligned}
H(Z_t) &= \delta\left[\hat{E}_2 Z_t^{\alpha_2-1} + \frac{L_2}{r}\right] \text{ where} \\
\frac{\partial H(Z_t)}{\partial Z_t} &= \delta(\alpha_2-1)\hat{E}_2 Z_t^{\alpha_2-2}
\end{aligned}$$

Since $\hat{E}_2 < 0$ and $\alpha_2 < 0$, therefore $\frac{\partial H(Z_t)}{\partial Z_t} > 0$.

This proves the proposition. □

Lemma A.0.13. *Let C_1 be as defined in (A.0.53):*

$$C_1 = \frac{2\delta^2[\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}]}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)}$$

where $\tilde{Z}_1 = \frac{L_1}{\delta}$, $\tilde{Z}_2 = \frac{L_2}{\delta}$. *Then (i)* $\frac{\partial C_1}{\partial L_1} < 0$ *(ii)* $\frac{\partial C_1}{\partial L_2} > 0$ *(iii)* $\frac{\partial C_1}{\partial \delta} > 0$

Proof. We write $C_1 = k\delta^2[\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}]$, where $k = \frac{2}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)(\alpha_1-\alpha_2)} > 0$. Differentiating C_1 with respect to L_1, L_2, and δ we get

$$\begin{aligned}\frac{\partial C_1}{\partial L_1} &= -(2-\alpha_1)\tilde{Z}_1^{1-\alpha_1}k\delta < 0\\ \frac{\partial C_1}{\partial L_2} &= -(2-\alpha_1)\tilde{Z}_2^{1-\alpha_1}k\delta > 0\\ \frac{\partial C_1}{\partial L_2} &= \alpha_1 k\delta(\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}) > 0\end{aligned}$$

where the inequalities follow from the fact that $\alpha_1 < 2$ and $\tilde{Z}_2 > \tilde{Z}_1$. This proves the lemma. □

Lemma A.0.14. *Let* $\bar{Z}$ *be the solution to the equation*

$$\left(\frac{1-\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{\delta L_1}{r}\right)Z^{1-\alpha_1} + \left(\frac{\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{L_1^2}{2\beta}\right)Z^{-\alpha_1} + C_1 = 0$$

Then (i) $\frac{\partial \bar{Z}}{\partial L_1} > 0$ *(ii)* $\frac{\partial \bar{Z}}{\partial L_2} < 0$ *(iii)* $\frac{\partial \bar{Z}}{\partial \delta} < 0$

Proof. (i) Using the expression of C_1 in (A.0.53), $\bar{Z}$ is the solution to the equation

$$\frac{(1-\alpha_1)\delta L_1}{r}\bar{Z}^{1-\alpha_1} + \frac{\alpha_2 L_1^2}{2\beta}\bar{Z}^{-\alpha_1} + \frac{2[\tilde{Z}_2^{2-\alpha_1} - \tilde{Z}_1^{2-\alpha_1}]\delta^2}{\theta^2\alpha_1(\alpha_1-1)(2-\alpha_1)} = 0 \qquad \text{(A.0.121)}$$

Differentiating (A.0.121) with respect to L_1 we get,

$$\begin{aligned}&\left(\frac{\partial \bar{Z}}{\partial L_1}\right)\frac{\bar{Z}^{-\alpha-1}}{(\alpha_1-\alpha 2)}\left[\frac{(1-\alpha_1)(1-\alpha_2)}{r}\delta L_1\bar{Z} - \frac{\alpha_1\alpha_2 L_1^2}{2\beta}\right]\\ &\quad + \left(\frac{\bar{Z}^{-\alpha}}{\alpha_1-\alpha 2}\right)\left[\frac{(1-\alpha_2)\delta}{r}\bar{Z} + \frac{\alpha_2 L_1}{\beta}\right] + \frac{\partial C_1}{\partial L_1} = 0\end{aligned}$$

From Lemma A.0.13 we have $\frac{\partial C_1}{\partial L_1} < 0$. Also, $\frac{(1-\alpha_1)(1-\alpha_2)}{r}\delta L_1\bar{Z} - \frac{\alpha_1\alpha_2 L_1^2}{2\beta} = \frac{L_1^2}{\theta^2}\left(1 - \frac{2\bar{Z}}{\tilde{Z}_1}\right) > 0$, since $\frac{\bar{Z}}{\tilde{Z}_1} < \frac{\alpha_1-1}{2\alpha_1}$ and $\frac{\alpha_1-1}{2\alpha_1} < \frac{1}{2}$. Moreover, $\frac{(1-\alpha_2)\delta\bar{Z}}{r} + \frac{\alpha_2 L_1}{\beta} = -\frac{2\delta}{\theta^2}\left[\frac{\tilde{Z}_1}{\alpha_1} - \frac{\bar{Z}}{\alpha_1-1}\right] < 0$, since $\bar{Z} < \frac{\alpha_1-1}{2\alpha_1}\tilde{Z}_1 < \frac{\alpha_1-1}{\alpha_1}\tilde{Z}_1$. Combining these results, we get

$$\frac{\partial \bar{Z}}{\partial L_1} > 0$$

(ii) Differentiating (A.0.121) with respect to L_2, we get

$$\left(\frac{\partial \bar{Z}}{\partial L_2}\right)\left(\frac{\bar{Z}^{-\alpha_1}}{\theta^2}\right)\left(\frac{\delta L_1}{\alpha_1-\alpha_2}\right)\left(\frac{\tilde{Z}_1}{\bar{Z}}-2\right)+\frac{\partial C_1}{\partial L_2}=0$$

From Lemma A.0.13, we have $\frac{\partial C_1}{\partial L_2}>0$. Also, $\frac{\tilde{Z}_1}{\bar{Z}}>\frac{2\alpha_1}{\alpha_1-1}>2$ since $\alpha_1>1$. Therefore

$$\frac{\partial \bar{Z}}{\partial L_2}<0$$

(iii) Equation (A.0.121) can be rewritten as

$$\frac{\eta^{1-\alpha_1}}{\alpha_1-1}-\frac{\eta^{-\alpha_1}}{2\alpha_1}+\left[\left(\frac{L_2}{L_1}\right)^{2-\alpha_1}-1\right]\frac{1}{\alpha_1(\alpha_1-1)(2-\alpha_1)}=0$$

where $\eta=\frac{\bar{Z}}{\tilde{Z}_1}=\frac{\bar{Z}\delta}{L_1}$.

Differentiating with respect to δ gives

$$\eta^{-\alpha_1-1}\left(\frac{1}{2}-\eta\right)\frac{\partial \eta}{\partial \delta} \quad = \quad 0$$

Since $\eta<\frac{1}{2}$, this implies

$$\frac{\partial \eta}{\partial \delta} \quad = \quad 0$$

But

$$\frac{\partial \eta}{\partial \delta}=\frac{\bar{Z}}{L_1}+\frac{\delta}{L_1}\frac{\partial \bar{Z}}{\partial \delta}$$

Therefore

$$\frac{\partial \bar{Z}}{\partial \delta}=-\frac{\bar{Z}}{\delta}<0$$

This completes the proof of the lemma. □

Proof of Proposition 2.3.7:

Proof. Since

$$\bar{X} = \frac{\bar{Z}^{-\frac{1}{\gamma}}}{K}$$

Differentiating with respect to $\bar{Z}$, we get

$$\frac{\partial \bar{X}}{\partial \bar{Z}} = \frac{\bar{Z}^{-1-\frac{1}{\gamma}}}{\gamma K}$$

Using the results in Lemma A.0.14, we get

$$\begin{aligned}
\frac{\partial \bar{X}}{\partial L_1} &= \frac{\partial \bar{X}}{\partial \bar{Z}} \frac{\partial \bar{Z}}{\partial L_1} < 0 \\
\frac{\partial \bar{X}}{\partial L_2} &= \frac{\partial \bar{X}}{\partial \bar{Z}} \frac{\partial \bar{Z}}{\partial L_2} > 0 \\
\frac{\partial \bar{X}}{\partial \delta} &= \frac{\partial \bar{X}}{\partial \bar{Z}} \frac{\partial \bar{Z}}{\partial \delta} > 0
\end{aligned}$$

This proves the first part of the proposition. To prove the second part, let us first consider the case $L_1 = 0$ and $L_2 < \infty$. From (A.0.53), $0 < C_1 < \infty$. Suppose there exists a finite positive solution $\bar{Z}$ to the equation (A.0.58), i.e. $0 < \bar{Z} < \infty$. Then the first two terms on the left hand side of equation (A.0.58) is zero while the third term is $C_1 > 0$. This is a contradiction. Therefore $\bar{Z}$ must be either 0 or ∞. Since $1 - \alpha_1 < 0$, $\bar{Z} = \infty$ leads to a contradiction. Thus either $\bar{Z} = 0$ or there is no solution to (A.0.58). If $\bar{Z} = 0$, then $\tau = \infty$ and the agent never retires. If there is no solution to (A.0.58), it means that the agent has no option to retire as in the benchmark case (section 2.3.5). Now, consider the case $L_1 > 0$ but $L_2 = \infty$. Then $C_1 = \infty$. Again, since $\alpha_1 > 1$ and $\alpha_2 < 0$, it must be that either $\bar{Z} = 0$ or there is no solution to (A.0.58). In either case we conclude that the agent does not retire. This completes the proof of the proposition. □

Proof of Proposition 2.3.9:

Proof. From Propositions 2.3.3, 2.3.4 and 2.3.8, we have

(i) For $\tilde{X}_1 \le X_t < \bar{X}$

$$\begin{aligned} c_t &= Z_t^{-\frac{1}{\gamma}} \\ \pi_t &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\left[\alpha_1(\alpha_1 + \frac{1}{\gamma} - 1)C_1 Z_t^{\alpha_1 - 1} + \alpha_2(\alpha_2 + \frac{1}{\gamma} - 1)C_2 Z_t^{\alpha_2 - 1}\right] \end{aligned}$$

where Z_t solves the equation

$$\alpha_1 C_1 (Z_t)^{\alpha_1 - 1} + \alpha_2 C_2 Z^{\alpha_2 - 1} - \frac{(Z_t)^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_t = 0$$

and

$$\begin{aligned} c_t^N &= (Z_t^N)^{-\frac{1}{\gamma}} \\ \pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\alpha_1(\alpha_1 + \frac{1}{\gamma} - 1)C_1 (Z_t^N)^{\alpha_1 - 1} \end{aligned}$$

where Z_t^N solves the equation

$$\alpha_1 C_1 (Z_t^N)^{\alpha_1 - 1} - \frac{(Z_t^N)^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_t = 0$$

We prove that $Z_t > Z_t^N$. Define

$$f(Z) = \alpha_1 C_1 Z^{\alpha_1 - 1} - \frac{Z^{-\frac{1}{\gamma}}}{K}$$

Then $f'(Z) = \alpha_1(\alpha_1 - 1)C_1 Z^{\alpha_1 - 2} + \frac{Z^{-\frac{1}{\gamma}-1}}{\gamma K} > 0$, since $\alpha_1 > 1$ and $C_1 > 0$. Therefore,

$$f(Z_t) - f(Z_t^N) = -\alpha_2 C_2 Z_t^{\alpha_2 - 1} > 0$$

since $\alpha_2 < 0$ and $C_2 > 0$. Therefore $Z_t > Z_t^N$ which implies $Z_t^{-\frac{1}{\gamma}} < (Z_t^N)^{-\frac{1}{\gamma}}$ i.e. $c_t < c_t^N$.

Now we prove that $\pi_t > \pi_t^N$. Observe that

$$\alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 Z_t^{\alpha_1 - 1} > \alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 \left(Z_t^N\right)^{\alpha_1 - 1}$$

since $\alpha_1 > 1$ and $C_1 > 0$. Since $\alpha_2 < 0$, $C_2 > 0$, by (A.0.50) we have $\alpha_2\left(\alpha_2 + \frac{1}{\gamma} - 1\right) C_2 > 0$.

Thus,

$$\alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 Z_t^{\alpha_1 - 1} + \alpha_2\left(\alpha_2 + \frac{1}{\gamma} - 1\right) C_2 Z_t^{\alpha_1 - 1} > \alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 \left(Z_t^N\right)^{\alpha_1 - 1}$$

and hence $\pi_t > \pi_t^N$.

(ii) For $X_t \geq \bar{X}$,

$$\begin{aligned} c_t &= KX_t \\ \pi_t &= \frac{\theta X_t}{\gamma\sigma} \end{aligned}$$

and

$$\begin{aligned} c_t^N &= (Z_t^N)^{-\frac{1}{\gamma}} \\ \pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 \left(Z_t^N\right)^{\alpha_1 - 1} \end{aligned}$$

where

$$\alpha_1 C_1 \left(Z_t^N\right)^{\alpha_1 - 1} - \frac{(Z_t^N)^{-\frac{1}{\gamma}}}{K} + \frac{\delta L_1}{r} + X_t = 0$$

Therefore,

$$c_t^N = KX_t + \frac{\delta L_1 K}{r} + \alpha_1 K C_1 \left(Z_t^N\right)^{\alpha_1 - 1} > KX_t$$

since $\alpha_1 > 0$, $C_1 > 0$, and $K > 0$. Thus $c_t^N > c_t$ when $X_t \geq \bar{X}$

Also,

$$\begin{aligned} \pi_t^N &= \frac{\theta}{\gamma\sigma}\left[X_t + \frac{\delta L_1}{r}\right] + \frac{\theta}{\sigma}\alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right) C_1 \left(Z_t^N\right)^{\alpha_1 - 1} \\ &= \frac{\theta X_t}{\gamma\sigma} + \frac{\theta\delta L_1}{\gamma\sigma r} + \frac{\theta\alpha_1\left(\alpha_1 + \frac{1}{\gamma} - 1\right)}{\sigma} C_1 \left(Z_t^N\right)^{\alpha_1 - 1} \end{aligned}$$

The second term in the right-hand-side is positive. Since $\alpha_1 > 1$ and $C_1 > 0$, the last term on the right-hand-side is positive. Therefore $\pi_t^N > \pi_t$. This proves the proposition. □

Proof of Proposition 2.4.1:

Proof. The proof follows the same steps as in the proofs of Propositions 2.3.2, 2.3.3, 2.3.4 and 2.3.5, with $L_1 = L_2 = L$. □

Proof of Proposition 2.4.2:

Proof. From equations (2.3.4)and (2.4.4)

$$\begin{aligned}\bar{X}^{Flexible} &= \frac{(\bar{Z}^{Flexible})^{-\frac{1}{\gamma}}}{K} \\ \bar{X}^{Constant} &= \frac{(\bar{Z}^{Constant})^{-\frac{1}{\gamma}}}{K}\end{aligned}$$

We prove that $\bar{Z}^{Constant} > (\frac{L}{L_1})\bar{Z}^{Flexible}$, from which the claim in the proposition follows. From (2.4.2), we have

$$\bar{Z}^{Constant} = \frac{\alpha_2}{\alpha_2 - 1}\frac{rL}{2\beta\delta} \tag{A.0.122}$$

and from (A.0.84), $\bar{Z}^{Flexible}$ satisfies the equation

$$\left(\frac{1-\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{\delta L_1}{r}\right)(\bar{Z}^{Flexible})^{1-\alpha_1} + \left(\frac{\alpha_2}{\alpha_1-\alpha_2}\right)\left(\frac{L_1^2}{2\beta}\right)(\bar{Z}^{Flexible})^{-\alpha_1} + C_1 = 0 \tag{A.0.123}$$

where C_1 is defined in (A.0.53). Using equation (A.0.122), we can rewrite (A.0.123) as

$$-\frac{\alpha_2 L_1^2}{2\beta}\left[\left(\frac{L}{L_1}\right)\left(\frac{\bar{Z}^{Flexible}}{\bar{Z}^{Constant}}\right) - 1\right] + C_1(\alpha_1-\alpha_2)(\bar{Z}^{Flexible})^{\alpha_1} = 0$$

Since $\alpha_2 < 0$, $C_1 > 0$, and $\alpha_1 > \alpha_2$, the above equation is satisfied if

$$\left(\frac{L}{L_1}\right)\left(\frac{\bar{Z}^{Flexible}}{\bar{Z}^{Constant}}\right) - 1 < 0$$

which implies

$$\bar{Z}^{Constant} > \left(\frac{L}{L_1}\right)\bar{Z}^{Flexible}$$

Therefore,

$$(\bar{Z}^{Constant})^{-\frac{1}{\gamma}} < \left(\frac{L}{L_1}\right)^{-\frac{1}{\gamma}}(\bar{Z}^{Flexible})^{-\frac{1}{\gamma}}$$

Hence

$$\begin{aligned}
\bar{X}^{Flexible} = \frac{(\bar{Z}^{Flexible})^{-\frac{1}{\gamma}}}{K} \quad &> \quad \left(\frac{L}{L_1}\right)^{\frac{1}{\gamma}} \frac{(\bar{Z}^{Constant})^{-\frac{1}{\gamma}}}{K} \\
&= \quad \left(\frac{L}{L_1}\right)^{\frac{1}{\gamma}} \bar{X}^{Constant}
\end{aligned}$$

This completes the proof of the proposition. □

A.1 Borrowing Constraints and proof of Proposition 2.4.3

We follow He and Pages [30] in the following analysis. Examples of other papers which study the consumption-portfolio choice under borrowing constraints are Karoui and Jeanblanc [37], Detemple and Serrat [20], Farhi and Pangeas [24], and Lim and Shin [41]. However, these papers do not consider the retirement option when labor supply is costly. Since the investor cannot borrow against future income, the wealth level should always be non-negative, that is, $X_t \geq 0, \ \text{ for } \ t \geq 0$. This is the borrowing constraint. The optimal wealth process X_t satisfies

$$X_t \quad = \quad \frac{1}{\xi_t} \ E_t \left[\xi_\tau X_\tau + \int_t^\tau \xi_s (c_s - \delta L) ds \right] \quad \text{for all} \quad 0 \leq t \leq \tau$$

So the borrowing constraint $X_t \geq 0$ for all $0 \leq t \leq \tau$ implies

$$E_t \left[\frac{\xi_\tau X_\tau}{\xi_t} + \int_t^\tau \frac{\xi_s}{\xi_t} (c_s - \delta L) ds \right] \geq 0 \quad \text{for all} \quad 0 \leq t \leq \tau$$

For a fixed stopping time τ, define

$$J(x; c, \pi, \tau) \quad = \quad E \left[\int_0^\tau e^{-\beta t} \left[u(c_t) - \frac{1}{2} L^2 \right] dt + e^{-\beta \tau} U(X_\tau) \right]$$

for any admissible pair (c, π) satisfying $X_t \geq 0 \ \forall \ 0 \leq t \leq \tau$ and the budget constraint

$$dX_t = r(X_t - r_t)dt + \pi_t(\mu dt + \sigma dB_t) - c_t dt + \delta L 1_{\{t < \tau\}}$$

Let λN_t be a non-increasing process with $N_0 = 1$ and $\lambda > 0$. Then for any admissible pair (c, π),

$$\begin{aligned} J(x; c, \pi, \tau) &= E\left[\int_0^\tau e^{-\beta t}\left[u(c_t) - \frac{1}{2}L^2\right]dt + e^{-\beta\tau}U(X_\tau)\right] \\ &\leq E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t e^{\beta t}\xi_t)dt + e^{-\beta\tau}\tilde{U}(\lambda N_\tau e^{\beta\tau}\xi_\tau)\right] \\ &\quad +\lambda E\left[N_\tau\xi_\tau X_\tau + \int_0^\tau N_t\xi_t c_t dt\right] - \int_0^\tau \delta N_t \xi_t L dt \end{aligned}$$

where $\tilde{u}(z) = \underset{c\geq 0}{Max}\left[u(c) - \frac{1}{2}L^2 - zc\right] = \frac{\gamma}{1-\gamma}z^{\frac{\gamma-1}{\gamma}} - \frac{1}{2}L^2$ so that

$$\tilde{u}(\lambda N_t e^{\beta t}\xi_t) = \frac{\gamma}{1-\gamma}(\lambda N_t e^{\beta t}\xi_t)^{1-\frac{1}{\gamma}} - \frac{1}{2}L^2$$

Since $N_0 = 1$, integration by parts give

$$\begin{aligned} & E\left[N_\tau\xi_\tau X_\tau + \int_0^\tau N_t\xi_t c_t dt - \int_0^\tau \delta N_t\xi_t L dt\right] \\ &= E\left[N_\tau X_\tau + \int_0^\tau N_t\xi_t(c_t - \delta L)dt\right] \\ &= E\left[\xi_\tau X_\tau + \int_0^\tau \xi_t(c_t - \delta L)dt\right] \\ &+E\left[\int_0^\tau \xi_t \frac{E_t\left[\int_t^\tau \xi_s(c_s - \delta L)ds + \xi_\tau X_\tau\right]}{\xi_t}dN_t\right] \end{aligned}$$

Therefore,

$$\begin{aligned} J(x; c, \pi, \tau) &\leq E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t e^{\beta t}\xi_t)dt + e^{-\beta\tau}\tilde{U}(\lambda N_\tau e^{\beta\tau}\xi_\tau)\right] \\ &+ \lambda E\left[\xi_\tau X_\tau + \int_0^\tau \xi_t(c_t - \delta L)dt\right] \\ &+ \lambda E\left[\int_0^\tau \xi_t \frac{E_t\left[\int_t^\tau \xi_s(c_s - \delta L)ds + \xi_\tau X_\tau\right]}{\xi_t}dN_t\right] \\ &\leq E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t e^{\beta t}\xi_t)dt + e^{-\beta\tau}\tilde{U}(\lambda N_\tau e^{\beta\tau}\xi_\tau)\right] + \lambda x \\ &= E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t e^{\beta t}\xi_t)dt + e^{-\beta\tau}\tilde{U}(\lambda N_\tau e^{\beta\tau}\xi_\tau)\right] + \lambda x \end{aligned}$$

The second inequality follows since $E_t\left[\int_t^\tau \xi_s(c_s - \delta L)ds + \xi_\tau X_\tau\right]dN_t \leq 0$ and $E\left[\int_0^\tau \xi_t(c_t - \delta L)dt + \xi_\tau X_\tau\right] \leq x$.

Let $Z_t = \lambda N_t e^{\beta t}\xi_t$. Then $\tilde{u}(Z_t) = \frac{\gamma}{1-\gamma} Z_t^{1-\frac{1}{\gamma}}$, $U(X\tau) = \frac{1}{K^\gamma}\frac{X_\tau^{1-\gamma}}{1-\gamma}$, and $\tilde{U}(Z_\tau) = \frac{\gamma}{K(1-\gamma)} Z_\tau^{1-\frac{1}{\gamma}}$ Therefore

$$J(x;c,\pi,\tau) \leq E\left[\int_0^\tau e^{-\beta t}\tilde{u}(Z_t)dt + e^{-\beta\tau}\tilde{U}(Z_\tau)\right] + \lambda x$$

and

$$\begin{aligned} V_\tau(x) &= \underset{(c,\pi)\in\pi_\tau(x)}{sup} J(x;c,\pi,\tau) \\ &\leq \underset{\{\lambda>0,N_t>0\}}{inf}\left[\tilde{J}(\lambda,N_t;\tau) + \lambda x\right] \end{aligned} \tag{A.1.1}$$

where

$$\tilde{J}(\lambda,N_t;\tau) = E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t e^{\beta t}\xi_t)dt + e^{-\beta t}\tilde{U}(\lambda N_\tau e^{\beta\tau}\xi_\tau)\right]$$

The above inequality holds as equality if and only if

$$\begin{aligned} c_t &= (\lambda N_t \xi_t e^{\beta t})^{\frac{1}{\gamma}} \\ X_\tau &= (\lambda N_\tau \xi\tau e^{\beta\tau})^{\frac{1}{\gamma}} \quad \forall\, 0 \leq t \leq \tau \\ E\left[\xi_\tau X_\tau + \int_0^\tau \xi_t c_t dt\right] &= x + E\left[\int_0^\tau \delta\xi_t L dt\right] \\ \frac{E_t\left[\int_t^\tau \xi_s(c_s - \delta L)ds + \xi_\tau X_\tau\right]}{\xi_t} dN_t &= 0 \end{aligned}$$

Let

$$\begin{aligned} \tilde{V}(N_t,\lambda) &= \underset{\tau}{sup}\tilde{J}(\lambda,N_t;\tau) \\ &= \underset{\tau}{sup}E\left[\int_0^\tau e^{-\beta t}\tilde{u}(\lambda N_t\xi_t e^{\beta t})dt + e^{-\beta\tau}\tilde{U}(\lambda N_\tau \xi_\tau e^{\beta\tau})\right] \\ \tilde{V}(\lambda) &= \underset{\{N_t>0\}}{inf}\ \tilde{V}(\{N_t\},\lambda) \end{aligned}$$

Then it can be shown that $V(x) = \underset{\tau}{sup} V_\tau(x)$ i.e.

$$\begin{aligned} V(x) &= \underset{\tau}{sup} \underset{\lambda>0,N_t>0}{inf} \left[\tilde{J}(\lambda, N_t; \tau) + \lambda x\right] \\ &= \underset{\lambda>0,N_t>0}{inf} \underset{\tau}{sup} \left[\tilde{J}(\lambda, N_t; \tau) + \lambda x\right] \\ &= \underset{\lambda>0}{inf} \underset{\{N_t>0\}}{inf} \underset{\tau}{sup} \left[\tilde{J}(\lambda, N_t; \tau) + \lambda x\right] \\ &= \underset{\lambda>0}{inf} \left[\tilde{V}(\lambda) + \lambda x\right] \end{aligned}$$

where

$$\tilde{V}(\lambda) = \underset{\tau}{sup} \underset{N_t>0}{inf} E\left[\int_0^\tau e^{-\beta s}\tilde{u}(Z_s)ds + e^{-\beta t}\tilde{U}(Z_\tau)|Z_0 = \lambda\right]$$

Therefore we have to compute $\tilde{V}(\lambda)$ in order to obtain the value function $V(x)$. Define

$$\psi(t, Z) = \underset{\tau>t}{sup} \underset{N_t>0}{inf} E\left[\int_t^\tau e^{-\beta s}\tilde{u}(Z_s)ds + e^{-\beta t}\tilde{U}(Z_\tau)|Z_t = Z\right]$$

where $Z_t = \lambda N_t e^{\beta t}\xi_t$, $Z_0 = \lambda > 0$ and

$$\frac{dZ_t}{Z_t} = \frac{dN_t}{N_t} + (\beta - r)dt - \theta dW_t$$

Following He and Pages [30] let N_t solve the differential equation of the $dN_t = -\chi(t)N_t dt$ for some $\chi(t) \geq 0$. Then we can get the following Bellman equation

$$min\{\mathcal{L}\psi(t, Z) + e^{-\beta t}\tilde{u}(Z), -\frac{\partial\psi}{\delta Z}\} = 0$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r)Z\frac{\partial}{\partial Z} + \frac{1}{2}\theta^2 Z^2\frac{\partial^2}{\partial Z^2}$$

Let N_t^* be the optimal solution of the Bellman equation. Analogous to the proofs in the previous section, it can be shown that $\tilde{V}(\lambda) = \psi(0, \lambda)$ where $\psi(t, Z)$ satisfies the following variational inequality:

Variational Inequality: Find a continuously differentiable function ψ, the free boundary $\bar{Z}^{CL}$ and $\hat{Z}$ such that, for all $t > 0$,

$$
\begin{aligned}
\mathcal{L}\psi + e^{-\beta t}\tilde{u}(Z) &= 0 \text{ for } \bar{Z}^{CL} < Z \leq \hat{Z} \\
\mathcal{L}\psi + e^{-\beta t}\tilde{u}(Z) &\leq 0 \text{ for } 0 < Z \leq \bar{Z}^{CL} \\
\psi(t, Z) &> e^{-\beta t}\tilde{U}(Z) \text{ for } Z > \bar{Z}^{CL} \\
\psi(t, Z) &= e^{-\beta t}\tilde{U}(Z) \text{ for } 0 < Z \leq \bar{Z}^{CL} \\
\frac{\partial \psi}{\partial Z}(t, Z) &\leq 0 \text{ for } 0 < Z \leq \hat{Z} \\
\frac{\partial \psi}{\partial Z}(t, Z) &= 0 \text{ for } Z > \hat{Z}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
\frac{\partial \psi}{\partial Z}(t, \hat{Z}) &= 0 \\
\frac{\partial^2 \psi}{\partial Z^2}(t, \hat{Z}) &= 0
\end{aligned}
$$

Then $\tau^* = inf\{s > t \mid Z_s \leq \bar{Z}^{CL}\} < \infty$ is the optimal stopping time. We guess a solution of the form $\psi(t, Z) = e^{-\beta t}v(Z)$. For $\bar{Z}^{CL} < \hat{Z} < \infty$,

$$
\begin{aligned}
v(Z) &= \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} \text{ for } 0 < Z \leq \bar{Z}^{CL} \\
v(Z) &= G_1 Z^{\alpha_1} + G_2 Z^{\alpha_2} + \frac{\gamma}{K(1-\gamma)} Z^{1-\frac{1}{\gamma}} + \frac{\delta L}{r} Z - \frac{L^2}{2\beta} \text{ for } \bar{Z}^{CL} < Z \leq \hat{Z}
\end{aligned}
$$

$\hat{Z}$ is determined from the boundary conditions $v'(\hat{Z}) = 0$ and $v''(\hat{Z}) = 0$. This implies

$$
\begin{aligned}
\alpha_1 G_1 \hat{Z}^{\alpha_1 - 1} + \alpha_2 G_2 \hat{Z}^{\alpha_2 - 1} - \frac{1}{K}\hat{Z}^{-\frac{1}{\gamma}} + \frac{\delta L}{r} &= 0 \\
(\alpha_1 - 1)\alpha_1 G_1 \hat{Z}^{\alpha_1 - 2} + (\alpha_2 - 1)\alpha_2 G_2 \hat{Z}^{\alpha_2 - 2} + \frac{1}{\gamma K}\hat{Z}^{-1-\frac{1}{\gamma}} &= 0
\end{aligned}
$$

Rearranging the terms and solving for G_1 and G_2 gives

$$G_1 = \frac{1}{K\alpha_1(\alpha_1 - \alpha_2)}\left[1 - \frac{1}{\gamma} - \alpha_2\right]\hat{Z}^{1-\alpha_1-\frac{1}{\gamma}} - \frac{1-\alpha_2}{\alpha_1(\alpha_1 - \alpha_2)}\frac{\delta L}{r}\hat{Z}^{1-\alpha_1} \tag{A.1.2}$$

$$G_2 = \frac{1}{K\alpha_2(\alpha_1 - \alpha_2)}\left[\frac{1}{\gamma} - 1 + \alpha_1\right]\hat{Z}^{1-\alpha_2-\frac{1}{\gamma}} - \frac{\alpha_1 - 1}{\alpha_2(\alpha_1 - \alpha_2)}\frac{\delta L}{r}\hat{Z}^{1-\alpha_2} \tag{A.1.3}$$

Using the value-matching and smooth-pasting conditions at $Z = \bar{Z}^{CL}$, we get

$$\begin{aligned} G_1(\bar{Z}^{CL})^{\alpha_1} + G_2(\bar{Z}^{CL})^{\alpha_2} + \frac{\delta L}{r}\bar{Z}^{CL} - \frac{L^2}{2\beta} &= 0 \\ \alpha_1 G_1(\bar{Z}^{CL})^{\alpha_1 - 1} + \alpha_2 G_2(\bar{Z}^{CL})^{\alpha_2 - 1} + \frac{\delta L}{r} &= 0 \end{aligned}$$

This gives

$$G_1 = -\frac{\delta L}{r}\,\frac{1-\alpha_2}{\alpha_1-\alpha_2}(\bar{Z}^{CL})^{1-\alpha_1} - \frac{\alpha_2}{\alpha_1-\alpha_2}\,\frac{L^2}{2\beta}(\bar{Z}^{CL})^{-\alpha_1} \tag{A.1.4}$$

$$G_2 = -\frac{\delta L}{r}\,\frac{\alpha_1-1}{\alpha_1-\alpha_2}(\bar{Z}^{CL})^{1-\alpha_2} - \frac{\alpha_1}{\alpha_1-\alpha_2}\,\frac{L^2}{2\beta}(\bar{Z}^{CL})^{-\alpha_2} \tag{A.1.5}$$

Eliminating G_1 and G_2 from (A.1.2), (A.1.3) with (A.1.4), (A.1.5) and using $\eta = \frac{\bar{Z}^{CL}}{\hat{Z}}$ we get

$$\begin{aligned} \frac{1}{K}\hat{Z}^{-\frac{1}{\gamma}} - \frac{(1-\alpha_2)}{(1-\frac{1}{\gamma}-\alpha_2)}\frac{\delta L}{r} &= -\frac{\delta L}{r}\frac{(1-\alpha_2)\alpha_1}{(1-\frac{1}{\gamma}-\alpha_2)}\eta^{1-\alpha_1} - \frac{\alpha_1\alpha_2 L^2}{2\beta(1-\frac{1}{\gamma}-\alpha_2)}\frac{\eta^{-\alpha_1}}{\hat{Z}} \\ \frac{1}{K}\hat{Z}^{-\frac{1}{\gamma}} - \frac{(\alpha_1-1)}{(\frac{1}{\gamma}-1+\alpha_1)}\frac{\delta L}{r} &= -\frac{\delta L}{r}\frac{(\alpha_1-1)\alpha_2}{(\frac{1}{\gamma}-1+\alpha_1)}\eta^{1-\alpha_2} + \frac{\alpha_1\alpha_2 L^2}{2\beta(\frac{1}{\gamma}-1+\alpha_1)}\frac{\eta^{-\alpha_2}}{\hat{Z}} \end{aligned}$$

Rearranging terms and after simplification, we get

$$\begin{aligned} \hat{Z}\frac{\delta L}{r}\left[\alpha_2(\alpha_1-1)\left(1-\frac{1}{\gamma}-\alpha_2\right)\eta^{1-\alpha_2} - \alpha_1(1-\alpha_2)\left(\frac{1}{\gamma}-1+\alpha_1\right)\eta^{1-\alpha_1} + \frac{1}{\gamma}(\alpha_1-\alpha_2)\right] \\ = \frac{\alpha_1\alpha_2 L^2}{2\beta}\left[\eta^{-\alpha_1}\left(1-\frac{1}{\gamma}-\alpha_2\right) + \eta^{-\alpha_2}\left(\frac{1}{\gamma}-1+\alpha_1\right)\right] \end{aligned} \tag{A.1.6}$$

which is reduced to the equation

$$g(\eta)^{-\gamma} - f(\eta) = 0$$

where $f(\eta)$ and $g(\eta)$ are given in Proposition 2.4.3. Define $h(\eta) = g(\eta)^{-\gamma} - f(\eta)$. By simple calculation, we can verify that $g(1) = 0$, $f(1) < 0$, so that $G(\eta) \to \infty$ as $\eta \to 1$. On the other hand, $f(\eta) \to \infty$ as $\eta \to 0$ and $g(\eta) \to \frac{\delta LK}{r}\frac{\alpha_1-1}{\frac{1}{\gamma}-1+\alpha_1} > 0$. Thus, $h(\eta) \to -\infty$ as $\eta \to 0$. Therefore, there exists $\eta^* \in (0,1)$ such that $h(\eta^*) = 0$.

From (A.1.6), $\hat{Z} = f(\eta^*)$ and $\bar{Z}^{CL} = \eta^*\hat{Z}$. The rest of the proposition can be proved using the same steps as in the proofs of Propositions 2.3.2 - 2.3.5.

Appendix B

Proofs of Chapter 3

Lemma B.0.1. *Suppose $\sigma_t \neq 0$. Let the gains process and the bond price follows*

$$dS_t + \delta_t dt = S_t \left[\mu_{S,t} dt + \sigma_t dZ_t\right] \tag{B.0.1}$$

$$dB_t = r_t B_t dt \tag{B.0.2}$$

where S_t, B_t are the stock price and bond price at time t.

(i) At asset prices (B_t, S_t), the state price density process ξ_t follows the dynamics

$$d\xi_t = -\xi_t \left[r_t dt + \kappa_t dZ_t\right] \tag{B.0.3}$$

where $\kappa_t = \frac{\mu_{S,t} - r_t}{\sigma_t}$.

(ii) If (c_t, π_t, W_t) satisfies the dynamic budget constraint

$$dW_t = \left[r_t W_t + \pi_t \left(\mu_{S,t} - r_t\right) W_t - c_t\right] dt + \pi_t \sigma_t W_t dZ_t \tag{B.0.4}$$

and

$$W_t \geq 0 \; a.s. \; \forall t \in [0, T] \tag{B.0.5}$$

then c_t satisfies the static budget constraint

$$E\left[\int_0^T \xi_t c_t dt\right] \leq W_0 \tag{B.0.6}$$

(iii) Conversely, if c_t satisfies the static budget constraint (B.0.6), then there exists a portfolio trading strategy π_t and wealth process W_t such that (c_t, π_t, W_t) satisfies the dynamic budget constraint (B.0.4) and the non-negative wealth constraint (B.0.5). Moreover, if (B.0.6) holds with equality for $c_t = \hat{c}_t$, then the corresponding wealth process $\hat{W}_t$ satisfies

$$\hat{W}_t = \frac{1}{\xi_t} E_t \left[\int_t^T \xi_u \hat{c}_u du \right] \tag{B.0.7}$$

Proof. (i) The state price density ξ_t is a Ito process such that the deflated gains process $\left(\xi_t S_t + \int_0^t \xi_u \delta_u du \right)$ is a martingale and hence has zero drift. Suppose

$$d\xi_t = \xi_t \left[\mu_{\xi,t} dt + \sigma_{\xi,t} dZ_t \right] \tag{B.0.8}$$

By Ito's Lemma,

$$d(\xi_t S_t) + \xi_t \delta_t dt = \xi_t S_t \left[(\mu_{S,t} + \mu_{\xi,t} + \sigma_t \sigma_{\xi,t}) dt + (\sigma_t + \sigma_{\xi,t}) dZ_t \right] \tag{B.0.9}$$

Then,

$$\mu_{S,t} + \mu_{\xi,t} + \sigma_t \sigma_{\xi,t} = 0 \tag{B.0.10}$$

Similarly, the process $\xi_t B_t$ has zero drift which yields

$$r_t + \mu_{\xi,t} = 0 \tag{B.0.11}$$

Therefore, $\mu_{\xi,t} = -r_t$ and $\sigma_{\xi,t} = -\frac{\mu_{S,t} - r_t}{\sigma_t}$. Hence the state price density ξ_t follows the dynamics (B.0.3).

(ii) Suppose (c_t, π_t, W_t) satisfies the dynamic budget constraint (B.0.4) and the the non-negative wealth constraint (B.0.5). Using the expression for the state price density ξ_t in (B.0.3) and Ito's Lemma,

$$d(\xi_t W_t) = -\xi_t c_t dt + \xi_t W_t \left(\pi_t \sigma_t - \kappa_t \right) dZ_t$$

which yields

$$\xi_t W_t + \int_0^t \xi_u c_u du \quad = \quad W_0 + \int_0^t \xi_u W_u \left[\pi_u \sigma_{S,u} - \kappa_u\right] dZ_u \tag{B.0.12}$$

The left-hand side of the above equation is a local martingale. Since $W_0 \geq 0$, it is bounded below by a martingale and hence is a supermartingale. By the optional sampling theorem,

$$E\left[\xi_T W_T + \int_0^T \xi_t c_t du\right] \leq W_0 \tag{B.0.13}$$

i.e.

$$E\left[\int_0^T \xi_t c_t du\right] \leq W_0 - E\left[\xi_T W_T\right] \tag{B.0.14}$$

Since $\xi_T \geq 0$ and $W_T \geq 0$, we conclude that

$$E\left[\int_0^T \xi_t c_t dt\right] \leq W_0 \tag{B.0.15}$$

which is the static budget constraint (B.0.6).

(iii) Now suppose that c_t satisfies the static budget constraint (B.0.6). Define the martingale

$$H_t \quad = \quad E_t\left[\int_0^T \xi_u c_u du\right] - E\left[\int_0^T \xi_u c_u du\right] \tag{B.0.16}$$

By the martingale representation theorem,

$$H_t \quad = \quad \int_0^t \phi_u dZ_u \tag{B.0.17}$$

for some progressively measurable R-valued process ϕ_t with $\int_0^T |\phi_t|^2 dt < \infty$. Define the wealth process W_t and the portfolio weight process π_t as

$$W_t \quad = \quad \frac{1}{\xi_t}\left[W_0 - \int_0^t \xi_u c_u du + H_t\right] \tag{B.0.18}$$

$$\pi_t \quad = \quad \frac{1}{\sigma_t}\left(\kappa_t + \frac{\phi_t}{\xi_t W_t}\right) \tag{B.0.19}$$

H_t and W_t can be re-written as

$$\begin{aligned} H_t &= \int_0^t \xi_u W_u \left[\pi_u \sigma_{S,u} - \kappa_u\right] dZ_u \\ W_t &= \frac{1}{\xi_t}\left[W_0 - \int_0^t \xi_u c_u du + \int_0^t \xi_u W_u \left[\pi_u \sigma_{S,u} - \kappa_u\right] dZ_u\right] \end{aligned} \tag{B.0.20}$$

Applying Ito's Lemma to (B.0.20), we get

$$\begin{aligned} dW_t &= \frac{1}{\xi_t}\left[-\xi_t c_t dt + \xi_t W_t \left(\pi_t \sigma_t - \kappa_t\right) dZ_t\right] \\ &+ d\left(\frac{1}{\xi_t}\right)\left[W_0 - \int_0^t \xi_u c_u du + \int_0^t \xi_u W_u \left[\pi_u \sigma_{S,u} - \kappa_u\right] dZ_u\right] \\ &+ d\left(\frac{1}{\xi_t}\right)\left[-\xi_t c_t dt + \xi_t W_t (\pi_t \sigma_t - \kappa_t) dZ_t\right] \end{aligned}$$

Using (B.0.3) and simplifying yields

$$dW_t = \left[r_t W_t + \pi_t W_t(\mu_{S,t} - r_t) - c_t\right] dt + \pi_t W_t \sigma_t dZ_t \tag{B.0.21}$$

which is the dynamic budget constraint (B.0.4).

To show that $W_t \geq 0$ a.s. $\forall t \in [0, T]$, we use the definition of H_t in (B.0.16) to rewrite W_t in (B.0.18) as

$$W_t = \frac{1}{\xi_t}\left[W_0 - \int_0^t \xi_u c_u du + E_t\left[\int_0^T \xi_u c_u du\right] - E\left[\int_0^T \xi_u c_u du\right]\right]$$

Since $E_t\left[\int_0^t \xi_u c_u du\right] = \int_0^t \xi_u c_u du$,

$$W_t = \frac{1}{\xi_t}\left[W_0 + E_t\left[\int_t^T \xi_u c_u du\right] - E\left[\int_0^T \xi_u c_u du\right]\right] \tag{B.0.22}$$

Since c_t satisfies the static budget constraint (B.0.6), $E\left[\int_0^T \xi_u c_u du\right] \leq W_0$. We conclude that $W_t \geq 0$ a.s. $\forall t \in [0, T]$.

Now suppose that the static budget constraint (B.0.6) hold with equality when $c_t = \hat{c}_t$, i.e.

$$E\left[\int_0^T \xi_t \hat{c}_t dt\right] = W_0 \tag{B.0.23}$$

Then from (B.0.22), the associated wealth process $(\hat{W}_t)$ satisfies

$$\begin{aligned}\hat{W}_t &= \frac{1}{\xi_t}\left[W_0 + E_t\left[\int_t^T \xi_u \hat{c}_u du\right] - E\left[\int_0^T \xi_u \hat{c}_u du\right]\right] \\ &= \frac{1}{\xi_t} E_t\left[\int_t^T \xi_u \hat{c}_u du\right] \end{aligned} \tag{B.0.24}$$

This completes the proof. □

Proof of Proposition 1:

Proof. In Lemma 1 we proved that the dynamic budget constraint (B.0.4) is equivalent to the static budget constraint (B.0.6). Therefore agent 1's optimization problem (3.4.1), (3.4.2) is equivalent to the problem (3.4.6), (3.4.7). Similarly, agent 2's optimization problem (3.4.3), (3.4.4) is equivalent to the problem (3.4.10), (3.4.11). Therefore, the consumption allocation $\hat{c}_t^1$ given by (A.0.115) is optimal for the problem (3.4.1), (3.4.2) of agent 1, while the consumption allocation $\hat{c}_t^2$ given by (A.0.117) is optimal for the problem (3.4.3), (3.4.4) of agent 2. To prove the existence of equilibrium we have to show that, for $i = 1, 2$ there exist portfolio strategies $\hat{\pi}_t^i$ and wealth processes $\hat{W}_t^i$ such that $\hat{W}_t^i \geq 0$ a.s. $\forall t \in [0, T]$, $(\hat{c}_t^i, \hat{\pi}_t^i, \hat{W}_t^i)$ satisfy the dynamic budget constraints (3.4.2) and (3.4.4) and all markets clear.

The goods market clearing condition implies

$$\hat{c}_t^1 + \hat{c}_t^2 = \delta_t$$

Using (A.0.115) and (A.0.117), we get

$$\begin{aligned} X_t + \frac{e^{-\beta t}}{\lambda_1 \xi_t} + \frac{e^{-\beta t}}{\lambda_2 \xi_t} &= \delta_t \\ \frac{e^{-\beta t}}{\xi_t}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) &= \delta_t - X_t \end{aligned}$$

By the definition of $Y_t = \frac{\delta_t}{\delta_t - X_t}$, $\delta_t - X_t = \frac{\delta_t}{Y_t}$ and the expression for ξ_t is obtained as in (3.4.16).

Plugging this expression for ξ_t in (A.0.115) and (A.0.117) the expressions for the optimal consumption allocations are as given in (3.4.14) and (3.4.15).

The Lagrangian multipliers λ_1 and λ_2 satisfy the static budget constraints (3.4.9) and (3.4.13) with equality. From (A.0.117), $\xi_t \hat{c}_t^2 = \frac{e^{-\beta t}}{\lambda_2}$. Therefore, λ_2 satisfies

$$E\left[\int_0^T \frac{e^{-\beta t}}{\lambda_2} dt\right] = b$$

which gives

$$\lambda_2 = \frac{1 - e^{-\beta T}}{\beta b} \tag{B.0.25}$$

Equating (A.0.115) and (3.4.14) for consumption at date $t = 0$, we obtain

$$X_0 + \frac{1}{\lambda_1} = \delta_0 - \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\delta_0}{Y_0}$$

Using $X_0 = \delta_0 - \frac{\delta_0}{Y_0}$ and rearranging terms, we get

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{\delta_0}{Y_0} \tag{B.0.26}$$

Using the value of λ_2 from (B.0.25) we get

$$\lambda_1 = \left[\frac{\delta_0}{Y_0} - \frac{\beta b}{1 - e^{-\beta T}}\right]^{-1} \tag{B.0.27}$$

The consumption process $\hat{c}_t^i$ satisfies the static budget constraint with equality, i.e. $E\left[\int_0^T \hat{c}_u^i \xi_u du\right] = \hat{W}_0^i$, where $\hat{W}_0^1 = S_0 - b$ and $\hat{W}_0^2 = b$ are the initial wealths of agents 1 and 2 respectively. By Lemma 1, there exists portfolio strategy $\hat{\pi}_t^i$ and wealth process $\hat{W}_t^i$ such that $(\hat{c}_t^i, \hat{\pi}_t^i, \hat{W}_t^i)$ satisfies

$$d\hat{W}_t^i = \left[r_t \hat{W}_t^i + \hat{\pi}_t^i \left(\mu_{S,t} - r_t\right) \hat{W}_t^i - \hat{c}_t^i\right] dt + \hat{\pi}_t^i \sigma_t \hat{W}_t^i dZ_t \tag{B.0.28}$$

and $\hat{W}_t^i \geq 0$ a.s. $\forall t \in [0, T]$. We will now verify that the portfolio strategy $\hat{\pi}_t^i$ clear the stock and bond markets. We do this in two steps.

Step 1: We first show that $\hat{W}_t^1 + \hat{W}_t^2 = S_t$ a.s. $\forall t \in [0, T]$.

Adding the dynamic budget constraints (B.0.28) over the agents, we get

$$\begin{aligned} d(\hat{W}_t^1 + \hat{W}_t^2) &= r_t[\hat{W}_t^1 + \hat{W}_t^2]dt + [\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2][\mu_{S,t} - r_t]dt \\ &\quad -[\hat{c}_t^1 + \hat{c}_t^2]dt + [\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2]\sigma_t dZ_t \end{aligned} \tag{B.0.29}$$

Applying Ito's Lemma, using (B.0.29) and the dynamics of ξ_t given by (B.0.3) we get

$$d\left[\xi_t(\hat{W}_t^1 + \hat{W}_t^2)\right] = -\xi_t[\hat{c}_t^1 + \hat{c}_t^2]dt + \xi_t\left[(\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2)\sigma_t - \kappa_t(\hat{W}_t^1 + \hat{W}_t^2)\right]dZ_t \tag{B.0.30}$$

Hence, $\xi_t(\hat{W}_t^1 + \hat{W}_t^2) + \int_0^t \xi_u[\hat{c}_u^1 + \hat{c}_u^2]du$ is a martingale satisfying

$$\begin{aligned} \hat{W}_t^1 + \hat{W}_t^2 &= \frac{1}{\xi_t} E_t\left[\int_t^T \xi_u(\hat{c}_u^1 + \hat{c}_u^2)du\right] \\ &= \frac{1}{\xi_t} E_t\left[\int_t^T \xi_u \delta_u du\right] \end{aligned} \tag{B.0.31}$$

where the second equality follows from goods clearing condition. Again, applying Ito's Lemma and using the dynamics of the gains process given in (B.0.1) and the state price density in (B.0.3), we get

$$d(\xi_t S_t) = -\xi_t \delta_t dt + \xi_t S_t(\sigma_t - \kappa_t)dZ_t \tag{B.0.32}$$

implying that $\xi_t S_t + \int_0^t \xi_u \delta_u du$ is a martingale. Thus

$$S_t = \frac{1}{\xi_t} E_t\left[\int_t^T \xi_u \delta_u du\right] \tag{B.0.33}$$

Comparing the right hand sides of (B.0.31) and (B.0.33) we conclude that

$$\hat{W}_t^1 + \hat{W}_t^2 = S_t \; a.s. \; \forall t \in [0, T] \tag{B.0.34}$$

Step 2: We prove that stock markets clear, i.e.

$$\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2 = S_t \tag{B.0.35}$$

This follows by comparing the diffusion coefficients in (B.0.30) and (B.0.32):

$$\left[(\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2)\sigma_t - \kappa_t(\hat{W}_t^1 + \hat{W}_t^2)\right] = S_t(\sigma_t - \kappa_t) \tag{B.0.36}$$

Using (B.0.34) and rearranging terms we deduce

$$\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2 = S_t \tag{B.0.37}$$

Subtracting (B.0.37) from (B.0.34) we conclude that bond markets clear:

$$(1 - \hat{\pi}_t^1)\hat{W}_t^1 + (1 - \hat{\pi}_t^2)\hat{W}_t^2 = 0 \tag{B.0.38}$$

This completes the proof of the proposition. □

Proof of Proposition 2:

Proof. (i) From (3.4.17) and (3.4.16) in proposition 1, the stock price-dividend ratio satisfies

$$\frac{S_t}{\delta_t} = \frac{1}{Y_t} E_t \left[\int_t^T e^{-\beta(t-u)} Y_u du \right]$$

Now, the solution to the stochastic differential equation

$$dY_t = k(\bar{Y} - Y_t)dt - \alpha(Y_t - \bar{\lambda})\sigma_\delta dZ_t$$

is given by

$$\begin{aligned} Y_t - \bar{\lambda} &= \left(Y_0 - \bar{\lambda}\right) exp\left[-\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right) t - \alpha\sigma_\delta Z_t \right] \\ &+ k\left(\bar{Y} - \bar{\lambda}\right) \int_0^t exp\left\{ -\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right) t - \alpha\sigma_\delta Z_t \right\} du \end{aligned}$$

For $0 \leq s \leq t$, we can express Y_t in terms of Y_s as follows:

$$\begin{aligned} Y_t - \bar{\lambda} &= \left(Y_s - \bar{\lambda}\right) exp\left[-\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right)(t-s) - \alpha\sigma_\delta(Z_t - Z_s) \right] \\ &+ k\left(\bar{Y} - \bar{\lambda}\right) \int_s^t exp\left\{ -\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right)(t-s) - \alpha\sigma_\delta(Z_t - Z_u) \right\} du \end{aligned}$$

Therefore

$$E_s[Y_t] \quad = \quad \bar{Y} + \left(Y_s - \bar{Y}\right) e^{-k(t-s)}$$

The stock price-dividend ratio can then be evaluated as follows:

$$\begin{aligned}
\frac{S_t}{\delta_t} &= \frac{1}{Y_t} E_t \left[\int_t^T e^{-\beta(t-u)} Y_u du \right] \\
&= \frac{1}{Y_t} \left[\int_t^T e^{-\beta(t-u)} E_t Y_u du \right] \\
&= \frac{1}{Y_t} \left[\int_t^T e^{-\beta(t-u)} \left[\bar{Y} + \left(Y_t - \bar{Y}\right) e^{-k(t-u)} du \right] \right] \\
&= \frac{1}{Y_t} \left[\rho_t \bar{Y} + (Y_t - \bar{Y}) \chi_t \right] \\
&= \left(1 - \frac{\bar{Y}}{Y_t}\right) \chi_t + \frac{\rho_t \bar{Y}}{Y_t}
\end{aligned}$$

which proves (3.4.24).

(ii) Applying Ito's Lemma to both sides of the equation

$$\frac{S_t Y_t}{\delta_t} \quad = \quad \left(Y_t - \bar{Y}\right) \chi_t + \rho_t \bar{Y}$$

and collecting the coefficient of the diffusion term, we get the expression for the stock returns volatility as in (3.4.25).

(iii), (iv) and (v): To derive the expressions for the interest rate and the market price of risk, apply Ito's lemma to the expression for the state price density

$$\xi_t \delta_t = e^{-\beta t} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) Y_t$$

to get:

$$\xi_t d\delta_t + \delta_t d\xi_t + d\delta_t d\xi_t = e^{-\beta t} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) (dY_t - \beta Y_t dt)$$

Collecting the diffusion terms gives us the expression for the market price of risk κ_t as in (3.4.27). Collecting the drift terms and using the expression for the market price of risk gives us the expression for the interest rate as in (3.4.26). From the expressions for the market price of risk and the interest rate, the expression for the stock returns $\mu_{S,t} = r_t + \sigma_t \kappa_t$ is obtained as in (3.4.28).

(vi) The optimal wealth process of agent 2 is:

$$\begin{aligned}
\hat{W}_t^2 &= \xi_t^{-1} E_t \left[\int_t^T e^{-\beta u} \hat{c}_u^2 \xi_u du \right] \\
&= \frac{1}{\lambda_2 \xi_t} E_t \left[\int_t^T e^{-\beta u} du \right] \\
&= \frac{\delta_t}{Y_t} \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} \int_t^T e^{-\beta(u-t)} du \right]
\end{aligned}$$

The optimal wealth process of agent 1 is:

$$\begin{aligned}
\hat{W}_t^1 &= \xi_t^{-1} E_t \left[\int_t^T e^{-\beta u} \hat{c}_u^1 \xi_u du \right] \\
&= \xi_t^{-1} E_t \left[\int_t^T e^{-\beta u} \left[\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) Y_u - \frac{1}{\lambda_2} \right] du \right] \\
&= \xi_t^{-1} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) E_t \left[\int_t^T e^{-\beta u} Y_u du \right] - W_t^2 \\
&= \frac{\delta_t}{Y_t} \left[(\rho_t - \chi_t) \bar{Y} + \chi_t Y_t - \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_t^T e^{-\beta(u-t)} du \right]
\end{aligned}$$

(vii) The optimal portfolio weight of the logarithmic agent is

$$\hat{\pi}_t^2 = \frac{\mu_{S,t} - r_t}{\sigma_t^2} = \frac{\kappa_t}{\sigma_t}$$

which gives the expression (3.4.32). From the security markets clearing condition,

$$\hat{\pi}_t^1 = 1 - (\hat{\pi}_t^2 - 1) \frac{W_t^2}{W_t^1}$$

from which the expression (3.4.31) follows.

This completes the proof. □

Proof of Proposition 3:

Proof. In equilibrium, the goods market clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = \delta_t$$

Plugging the expressions (3.5.2) and (3.5.18) into the goods market clearing condition and using the definition of $Y_t = \frac{\delta_t}{\delta_t - X_t}$ gives

$$\begin{aligned} \frac{e^{-\beta t}}{\hat{\lambda}_1^c \xi_t^1}\left[1 + \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}\frac{\xi_t^1}{\xi_{\nu,t}^2}\right] &= \frac{\delta_t}{Y_t} \\ \frac{e^{-\beta t}}{\hat{\lambda}_1^c \xi_t^1}\left[1 + \eta_t\right] &= \frac{\delta_t}{Y_t} \\ \xi_t^1 &= e^{-\beta t}\left(\frac{1+\eta_t}{\hat{\lambda}_1^c}\right)\frac{Y_t}{\delta_t} \end{aligned}$$

Then

$$\xi_{\hat{\nu},t}^2 = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}\frac{\xi_t^1}{\eta_t} = e^{-\beta t}\left(\frac{1+\eta_t}{\hat{\lambda}_2^c \eta_t}\right)\frac{Y_t}{\delta_t}$$

The expressions for the optimal consumption policies follow after inserting the expressions for ξ_t^1 and $\xi_{\hat{\nu},t}^2$ in (3.5.2) and (3.5.18) respectively.

Applying Ito's Lemma to the expression $\eta_t = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}\frac{\xi_t^1}{\xi_{\hat{\nu},t}^2}$, using the dynamics of the state price densities as in (3.5.19) and (3.5.20) and equation (3.5.17) gives the dynamics of η_t as in (3.5.21).

The expression for $\eta(0)$ is obtained from the time $t = 0$ consumption demand of agent 1 as follows. From (3.5.2), at $t = 0$ we get

$$\begin{aligned} \hat{c}_0^1 &= X_0 + \frac{1}{\hat{\lambda}_1^c} \\ &= \delta_0 - \frac{\delta_0}{Y_0} + \frac{1}{\eta(0)\hat{\lambda}_2^c} \end{aligned}$$

since $\eta(0) = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}\frac{\xi_0^1}{\xi_{\nu,0}^2} = \frac{\hat{\lambda}_1^c}{\hat{\lambda}_2^c}$. Again, from (3.5.25), at $t = 0$, we get

$$\hat{c}_0^1 = \delta_0 - \frac{\eta(0)\delta_0}{(1+\eta(0))Y_0}$$

Equating these two expressions for $\hat{c}_0^1$ we get,

$$\delta_0 - \frac{\delta_0}{Y_0} + \frac{1}{\eta(0)\hat{\lambda}_2^c} = \delta_0 - \frac{\eta(0)\delta_0}{(1+\eta(0))Y_0}$$

which yields

$$\eta(0) = \left[\frac{\hat{\lambda}_2^c \delta_0}{Y_0} - 1\right]^{-1}$$

From (3.5.13),

$$\hat{\lambda}_2^c = \frac{1-e^{-\beta T}}{\beta b}$$

since $W_0^2 = b$. Therefore,

$$\eta(0) = \left[\left(\frac{1-e^{-\beta T}}{\beta b}\right)\frac{\delta_0}{Y_0} - 1\right]^{-1}$$

Now we prove the converse. The optimal consumption policies are given by (3.5.2) and (3.5.18). Using the expressions for the state price densities as in (3.5.19) and (3.5.20) we get the goods market clearing condition: $\hat{c}_t^1 + \hat{c}_t^2 = \delta_t$.

Next, Theorems (9.1) and (10.1) in Cvitanic and Karatzas (1992) show that there exists a portfolio process $\hat{\pi}^2 \in \mathcal{A}'$ such that $\hat{\pi}_t^2$ is optimal for agent 2 in the original economy and

$$(\bar{\pi} - \hat{\pi}_t^2)\hat{\nu}_t = 0 \; a.s. \; \forall t \in [0,T]$$

Moreover, there exists wealth process $\hat{W}_t^2$ such that $(\hat{c}_t^2, \hat{\pi}_t^2, \hat{W}_t^2)$ satisfies

$$d\hat{W}_t^2 = \left[r_t\hat{W}_t^2 + \hat{\pi}_t^2\left(\mu_{S,t} - r_t\right)\hat{W}_t^2 - \hat{c}_t^2\right]dt + \hat{\pi}_t^2\sigma_t\hat{W}_t^2 dZ_t \tag{B.0.39}$$

and $\hat{W}_t^2 \geq 0$ a.s. $\forall t \in [0,T]$.

Similar to the proof in Proposition 1, there exists wealth process $\hat{W}_t^1$ for agent 1 such that $(\hat{c}_t^1, \hat{\pi}_t^1, \hat{W}_t^1)$ satisfies

$$d\hat{W}_t^1 = \left[r_t\hat{W}_t^1 + \hat{\pi}_t^1\left(\mu_{S,t} - r_t\right)\hat{W}_t^1 - \hat{c}_t^1\right]dt + \hat{\pi}_t^1\sigma_t\hat{W}_t^1 dZ_t \tag{B.0.40}$$

and $\hat{W}_t^1 \geq 0$ a.s. $\forall t \in [0,T]$.

We will now verify that the portfolio process $\hat{\pi}_t^i$ clear the stock and bond markets. We do this in two steps.

Step 1: We first show that $\hat{W}_t^1 + \hat{W}_t^2 = S_t$ a.s. $\forall t \in [0,T]$.

Adding the dynamic budget constraints (B.0.39) and (B.0.40), we get

$$\begin{aligned} d(\hat{W}_t^1 + \hat{W}_t^2) &= r_t[\hat{W}_t^1 + \hat{W}_t^2]dt + [\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2][\mu_{S,t} - r_t]dt \\ &\quad -[\hat{c}_t^1 + \hat{c}_t^2]dt + [\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2]\sigma_t dZ_t \end{aligned} \tag{B.0.41}$$

Applying Ito's Lemma, using (B.0.41) and the dynamics of ξ_t^1 given by (3.5.19) we get

$$d\left[\xi_t^1(\hat{W}_t^1 + \hat{W}_t^2)\right] = -\xi_t^1[\hat{c}_t^1 + \hat{c}_t^2]dt + \xi_t^1\left[(\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2)\sigma_t - \kappa_t^1(\hat{W}_t^1 + \hat{W}_t^2)\right]dZ_t \tag{B.0.42}$$

Hence, $\xi_t^1(\hat{W}_t^1 + \hat{W}_t^2) + \int_0^t \xi_u^1[\hat{c}_u^1 + \hat{c}_u^2]du$ is a martingale satisfying

$$\begin{aligned} \hat{W}_t^1 + \hat{W}_t^2 &= \frac{1}{\xi_t^1}E_t\left[\int_t^T \xi_u^1(\hat{c}_u^1 + \hat{c}_u^2)du\right] \\ &= \frac{1}{\xi_t^1}E_t\left[\int_t^T \xi_u^1 \delta_u du\right] \end{aligned} \tag{B.0.43}$$

where the second equality follows from goods clearing condition. Again, applying Ito's Lemma and using the dynamics of the gains process given in (B.0.1) and the state price density in (3.5.19), we get

$$d(\xi_t^1 S_t) = -\xi_t^1 \delta_t dt + \xi_t^1 S_t(\sigma_t - \kappa_t^1)dZ_t \tag{B.0.44}$$

implying that $\xi_t^1 S_t + \int_0^t \xi_u^1 \delta_u du$ is a martingale. Thus

$$S_t = \frac{1}{\xi_t^1}E_t\left[\int_t^T \xi_u^1 \delta_u du\right] \tag{B.0.45}$$

Comparing the right hand sides of (B.0.43) and (B.0.45) we conclude that

$$\hat{W}_t^1 + \hat{W}_t^2 = S_t \; a.s. \; \forall t \in [0,T] \tag{B.0.46}$$

Step 2: We now prove that stock markets clear, i.e.

$$\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2 \quad = \quad S_t \tag{B.0.47}$$

This follows by comparing the diffusion coefficients in (B.0.42) and (B.0.44):

$$\left[(\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2)\sigma_t - \kappa_t (\hat{W}_t^1 + \hat{W}_t^2) \right] \quad = \quad S_t(\sigma_t - \kappa_t) \tag{B.0.48}$$

Using (B.0.46) and rearranging terms we deduce

$$\hat{\pi}_t^1 \hat{W}_t^1 + \hat{\pi}_t^2 \hat{W}_t^2 \quad = \quad S_t \tag{B.0.49}$$

Subtracting (B.0.49) from (B.0.46) we conclude that bond markets clear:

$$(1 - \hat{\pi}_t^1)\hat{W}_t^1 + (1 - \hat{\pi}_t^2)\hat{W}_t^2 \quad = \quad 0 \tag{B.0.50}$$

This completes the proof of the proposition. □

Proof of Proposition 4 (i) The stock price satisfies

$$S_t \quad = \quad \frac{1}{\xi_t^1} E_t \left[\int_t^T \xi_u^1 \delta_u du \right]$$

Using the expression for ξ_t^1 from (3.5.19),

$$\lambda_1^c e^{\beta t} \xi_t^1 S_t \quad = \quad E_t \left[\int_t^T e^{-\beta(u-t)} \left(1 + \eta_u\right) Y_u du \right]$$

Therefore,

$$\frac{S_t}{\delta_t} \quad = \quad \frac{E_t \left[\int_t^T e^{-\beta(u-t)} (1 + \eta_u) Y_u du \right]}{(1 + \eta_t) Y_t} \tag{B.0.51}$$

(ii) and (iii): To obtain the interest rate and the market price of risk, we apply Ito's Lemma to

$$\lambda_1^c e^{\beta t} \xi_t^1 \delta_t = (1 + \eta_t) Y_t$$

and use (3.3.1), (3.5.21) and (3.2.2). Equating the drift terms gives

$$r_t = \beta + \mu_\delta - \kappa_t^1 \sigma_\delta + \left(1 - \frac{\bar{\lambda}}{Y_t}\right) k$$

Equating the diffusion terms gives

$$\kappa_t^1 = \sigma_\delta + \alpha \left(1 - \frac{\bar{\lambda}}{Y_t}\right) \sigma_\delta + \frac{\eta_t \nu_t}{1 + \eta_t}$$

Using this in the first equation we get the expression for the interest rate.

(iv) First, we state the following Lemma:

Lemma 2 (Clarke-Ocone Formula) Let $F \in D^{1,2}$ where the space $D^{1,2}$ is the closure of the class of smooth random variables with respect to the norm

$$||F||_{1,2} \quad = \quad \left[E(|F|^2) + E(||\mathcal{D}F||^2_{L^2(T)})\right]^{\frac{1}{2}} \tag{B.0.52}$$

Suppose that Z is a one-dimensional Brownian motion. Then

$$F \quad = \quad E(F) + \int_0^T E\left(\mathcal{D}_s F | \mathcal{F}_s\right) dZ_s \tag{B.0.53}$$

and

$$E_t(F) \quad = \quad E(F) + \int_0^t E\left(\mathcal{D}_s F | \mathcal{F}_s\right) dZ_s \tag{B.0.54}$$

Proof: Please see the proof on page 42 in the book by David Nualart [46].

Proof of the stock returns volatility expression: We will derive the stock price volatility from the stock price computed from agent 2's perspective. Using (3.5.20)

$$\begin{aligned}
\frac{S_t^{\hat{\nu}}}{\delta_t} &= E\left[\int_t^T \frac{\xi^2_{\hat{\nu},u}\delta_u}{\xi^2_{\hat{\nu},t}\delta_t} du\right] \\
&= \frac{\delta_t}{\left(1 + \frac{1}{\eta_t}\right) Y_t} E_t\left[\int_t^T e^{-\beta(u-t)}\left(1 + \frac{1}{\eta_u}\right) Y_u du\right]
\end{aligned}$$

Define the $\mathcal{L}^2(\mathcal{P})$ martingale M_t as follows:

$$M_t \quad = \quad E_t\left[\int_0^T e^{-\beta(u-t)}\left(1 + \frac{1}{\eta_u}\right) Y_u du\right] \tag{B.0.55}$$

By the martingale representation theorem, there exists a process ψ such that $E\left[\int_0^T |\psi_u|^2 du\right] < \infty$ and

$$M_t \quad = \quad M_0 + \int_0^T \psi_t dZ_t \tag{B.0.56}$$

Then the value of the stock to agent 2 can be expressed as

$$S_t^{\hat{\nu}} \quad = \quad \frac{\delta_t}{\left(1 + \frac{1}{\eta_t}\right) Y_t} \left[M_t - \int_0^t e^{-\beta(u-t)} \left(1 + \frac{1}{\eta_u}\right) Y_u du\right] \tag{B.0.57}$$

Agent 2's valuation of the stock price has the dynamics

$$dS_t^{\hat{\nu}} + \delta_t dt \quad = \quad S_t^{\hat{\nu}} \left[\mu_t^{\hat{\nu}} dt + \sigma_t dZ_t\right] \tag{B.0.58}$$

where $\mu_t^{\hat{\nu}} = \mu_t + (\bar{\pi} - 1)\hat{\nu}_t \sigma_t$.

Applying Ito's Lemma to (B.0.57) and matching the diffusion coefficient with the dynamics (B.0.58), we get

$$\begin{aligned} \sigma_t \quad &= \quad \sigma_\delta + \alpha \sigma_\delta \left(1 - \frac{\bar{\lambda}}{Y_t}\right) - \frac{\hat{\nu}_t}{1 + \eta_t} \\ &+ \quad \frac{\phi_t}{E_t\left[\int_t^T e^{-\beta(u-t)} \left(1 + \frac{1}{\eta_u}\right) Y_u du\right]} \end{aligned} \tag{B.0.59}$$

Now

$$dM_t \quad = \quad \phi_t dZ_t$$

which implies (by the Clarke-Ocone formula)

$$\begin{aligned} \phi_t \quad &= \quad \frac{E_t\left[\int_t^T e^{-\beta(u-t)} \left\{\left(1 + \frac{1}{\eta_u}\right) \mathcal{D}_t Y_u - \frac{Y_u}{\eta_u^2} \mathcal{D}_t \eta_u\right\} du\right]}{E_t\left[\int_t^T e^{-\beta(u-t)} \left(1 + \frac{1}{\eta_u}\right) Y_u du\right]} \\ &= \quad E_t\left[\int_0^T e^{-\beta(u-t)} \left(1 + \frac{1}{\eta_u}\right) \mathcal{D}_t Y_u du\right] \\ &- \quad E_t\left[\int_0^T e^{-\beta(u-t)} \frac{Y_u}{\eta_u^2} \mathcal{D}_t \eta_u du\right] \end{aligned} \tag{B.0.60}$$

Thus,

$$\begin{aligned}
\sigma_t &= \sigma_\delta + \alpha\sigma_\delta\left(1 - \frac{\bar{\lambda}}{Y_t}\right) - \frac{\hat{\nu}_t}{1+\eta_t} \\
&+ \frac{E_t\left[\int_0^T e^{-\beta(u-t)}\left(1+\frac{1}{\eta_u}\right)\mathcal{D}_t Y_u du\right]}{E_t\left[\int_t^T e^{-\beta(u-t)}\left(1+\frac{1}{\eta_u}\right)Y_u du\right]} \\
&- \frac{E_t\left[\int_0^T e^{-\beta(u-t)}\frac{Y_u}{\eta_u^2}\mathcal{D}_t\eta_u du\right]}{E_t\left[\int_t^T e^{-\beta(u-t)}\left(1+\frac{1}{\eta_u}\right)Y_u du\right]}
\end{aligned} \tag{B.0.61}$$

The Malliavin derivatives $\mathcal{D}_t Y_u$ and $\mathcal{D}_t\eta_u$ are derived as follows. From (3.3.1), for $t > s$

$$Y_t = Y_s + \int_s^t k(\bar{Y} - Y_u)du - \alpha\sigma_\delta\int_s^t (Y_u - \bar{\lambda})dZ_u$$

Applying the Malliavin operator on both sides,

$$\mathcal{D}_s Y_t = \mathcal{D}_s Y_s - k\int_s^t \mathcal{D}_s Y_u du - \alpha\sigma_\delta\int_s^t \mathcal{D}_s Y_u dZ_u$$

Then the Malliavin derivative $\mathcal{D}_s Y_u$ satisfies

$$\frac{d(\mathcal{D}_s Y_t)}{\mathcal{D}_s Y_t} = -kdt - \alpha\sigma_\delta dZ_t$$

Thus

$$\mathcal{D}_s Y_t = \mathcal{D}_t Y_t exp\left[-\int_s^t\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right)du - \int_s^t \alpha\sigma_\delta dZ_u\right]$$

We obtain $\mathcal{D}_t Y_t$ as follows. Since

$$\begin{aligned}
Y_t - \bar{\lambda} &= exp\left[-\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right)t - \alpha\sigma_\delta Z_t\right] \\
\times &\left[Y_0 - \bar{\lambda} + k\left(\bar{Y} - \bar{\lambda}\right)\int_0^t exp\left\{\left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right)u + \alpha\sigma_\delta Z_u\right\}du\right]
\end{aligned}$$

Taking the Malliavin derivative and noting that for $t > u$, $\mathcal{D}_t Z_u = 0$, we get

$$\mathcal{D}_t Y_t = -\alpha\sigma_\delta X_t = -\alpha\sigma_\delta(Y_t - \bar{\lambda})$$

Therefore,

$$\mathcal{D}_s Y_t = -\alpha\sigma_\delta (Y_t - \bar{\lambda}) exp\left[-\int_s^t \left(k + \frac{1}{2}\alpha^2\sigma_\delta^2\right) du - \int_s^t \alpha\sigma_\delta dZ_u\right] < 0$$

Therefore, the third term in (B.0.61) is negative.

We similarly derive the expression for the Malliavin derivative $\mathcal{D}_t \eta_u$. Since

$$d\eta_t = -\eta_t \hat{\nu}_t dZ_t$$

we get

$$\eta_t = \eta_0 exp\left[-\frac{1}{2}\int_0^t (\hat{\nu}_u)^2 du - \int_0^t \hat{\nu}_u dZ_u\right] \quad \text{(B.0.62)}$$

We next use the fact that for any random Riemann integral $\int_0^T h_s(Z) dZ_s$ where the integrand h_s is a progressively measurable process depending on the past trajectory of the Brownian motion Z, such that the integral exists (i.e. $\int_0^T |h_s| ds < \infty$ with probability one), we have for $t \in [0, T]$,

$$\mathcal{D}_t \int_0^T h_s(Z) dZ_s = h_t + \int_t^T \mathcal{D}_t h_s dZ_s \quad \text{(B.0.63)}$$

Taking the Malliavin derivative of (B.0.62) and using the above fact, we get

$$\mathcal{D}_s \eta_t = -\eta_t \left[\int_t^s \hat{\nu}_u (\mathcal{D}_s \hat{\nu}_u) du + \int_s^t (\mathcal{D}_s \hat{\nu}_u) dZ_u + \hat{\nu}_s\right] \quad \text{(B.0.64)}$$

Proof of Proposition 5: When $\bar{\pi} = 1$, both agents invest only in the stock when the constraint binds. This is because when the constraint is binding, agent 2 is investing 100 percent of his wealth in the stock and so he cannot invest in bonds. Since bond markets have to clear, (the net supply of the bond is zero), agent 1 should also invest zero percent of his wealth in bonds and 100 percent of his wealth in stocks. Therefore when the constraint binds, the optimal portfolio weights of the agents are

$$\hat{\pi}_t^1 = 1$$

$$\hat{\pi}_t^2 = 1 \quad \text{(B.0.65)}$$

Since agent 2 has a logarithmic utility function, he is myopic. So the portfolio weight of agent 2 is

$$\begin{aligned}\hat{\pi}_t^2 &= \frac{\mu_t^{\hat{\nu}} - r_t^{\hat{\nu}}}{\sigma_{S,t}^2} \\ &= \frac{\kappa_{\hat{\nu},t}^2}{\sigma_t}\end{aligned}$$

Therefore, for $\bar{\pi} = 1$ and $\hat{\nu}_t > 0$,

$$\kappa_{\hat{\nu},t}^2 = \sigma_t$$

i.e.

$$\kappa_t^1 - \hat{\nu}_t = \sigma_t$$

Using the expression (3.5.29) for the market price of risk of agent 1 and the stock returns volatility in (3.5.30), we get that the last term in (3.5.30) is zero:

$$\frac{E_t\left[\int_t^T e^{-\beta(u-t)}\left\{\left(1+\frac{1}{\eta_u}\right)\mathcal{D}_t Y_u - \frac{Y_u}{\eta_u^2}\mathcal{D}_t\eta_u\right\}du\right]}{E_t\left[\int_t^T e^{-\beta(u-t)}\left(1+\frac{1}{\eta_u}\right)Y_u du\right]} = 0 \tag{B.0.66}$$

which is equivalent to

$$E_t\left[\int_t^T e^{-\beta(u-t)}\mathcal{D}_t\left\{\left(1+\frac{1}{\eta_u}\right)Y_u\right\}du\right] = 0 \tag{B.0.67}$$

This implies together with (B.0.60) in the proof of (iv) of Proposition 4, that $\phi_t = 0$ a.s for all $t \in [0, T]$. From (B.0.56), we get

$$M_t = M_0 \; a.s. \forall t \in [0, T]$$

That is, the martingale M_t is constant in $[0, T]$. In particular, $M_T = M_0$. From the definition of M_t in (B.0.55), we get $M_T = \int_0^T e^{-\beta(u-t)}\left(1+\frac{1}{\eta_u}\right)Y_u du$ which is a constant. This implies that $\left(1+\frac{1}{\eta_t}\right)Y_t$ is a deterministic function.

Applying Ito's Lemma to $\left(1+\frac{1}{\eta_t}\right)Y_t$ and setting the diffusion term to zero, we get

$$\hat{\nu}_t \quad = \quad (1+\eta_t)\,\alpha\sigma_\delta\left(1-\frac{\bar{\lambda}}{Y_t}\right)$$

Therefore, the stock returns volatility in (3.5.30) becomes

$$\sigma_t \quad = \quad \sigma_\delta$$

This completes the proof of the proposition.

Proof of Proposition 6: First note that from (3.4.18), (3.4.19) and (3.5.23),

$$\eta_0 = \frac{\lambda_1^c}{\lambda_2^c} = \frac{\lambda_1}{\lambda_2} = \left[\left(\frac{1-e^{-\beta T}}{\beta b}\right)\frac{\delta_0}{Y_0} - 1\right]^{-1}$$

From (3.4.14) and (3.5.25),

$$\begin{aligned}
\hat{c}_t^{1c} - X_t &= \frac{1}{1+\eta_t}\frac{\delta_t}{Y_t} \\
\hat{c}_t^{1u} - X_t &= \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\delta_t}{Y_t} \\
&= \frac{1}{1+\eta_0}\frac{\delta_t}{Y_t}
\end{aligned}$$

Therefore,

$$\begin{aligned}
U_1(\hat{c}^{1c}) - U_1(\hat{c}^{1u}) &= E\left[\int_0^T e^{-\beta t} log\left(\frac{1}{1+\eta_t}\frac{\delta_t}{Y_t}\right)dt\right] - E\left[\int_0^T e^{-\beta t} log\left(\frac{1}{1+\eta_0}\frac{\delta_t}{Y_t}\right)dt\right] \\
&= \int_0^T e^{-\beta t} log\,(1+\eta_0)\,dt - E\left[\int_0^T e^{-\beta t} log\,(1+\eta_t)\,dt\right]
\end{aligned}$$

Since $f(x) = log(x)$ is increasing and concave, by Jensen's inequality,

$$E\left[\int_0^T e^{-\beta t} log(1+\eta_t)dt\right] \leq \int_0^T e^{-\beta t} log(E(1+\eta_t))dt$$

Since η_t is a non-negative local martingale, it is a supermartingale implying $E\left[1+\eta_t\right] \leq 1+\eta_0$.

Hence,

$$E\left[\int_0^T e^{-\beta t} log(1+\eta_t)dt\right] \leq \int_0^T e^{-\beta t} log(1+\eta_0)dt$$

Therefore,

$$U_1(\hat{c}^{1c}) - U_1(\hat{c}^{1u}) \geq \int_0^T e^{-\beta t} log\,(1+\eta_0)\,dt - \int_0^T e^{-\beta t} log\,(1+\eta_0)\,dt = 0$$

This proves the first inequality (3.6.1) in the proposition. To prove the second inequality, we use (3.4.15) and (3.5.26) to get (noting that $\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\eta_0}{1+\eta_0}$)

$$\begin{aligned} U_2(\hat{c}^{2c}) - U_2(\hat{c}^{2u}) &= E\left[\int_0^T e^{-\beta t} log\left(\frac{\eta_t}{1+\eta_t}\frac{\delta_t}{Y_t}\right) dt\right] - E\left[\int_0^T e^{-\beta t} log\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\delta_t}{Y_t}\right) dt\right] \\ &= E\left[\int_0^T e^{-\beta t} log\left(\frac{\eta_t}{1+\eta_t}\right) dt\right] - \int_0^T e^{-\beta t} log\left(\frac{\eta_0}{1+\eta_0}\right) dt \end{aligned}$$

By Jensen's inequality,

$$E\left[\int_0^T e^{-\beta t} log\left(\frac{\eta_t}{1+\eta_t}\right) dt\right] \leq \int_0^T e^{-\beta t} log\left[E\left(\frac{\eta_t}{1+\eta_t}\right)\right] dt$$

Since the function $f(x) = \frac{x}{1+x}$ is increasing and concave and η_t is a supermartingale,

$$E\left(\frac{\eta_t}{1+\eta_t}\right) \leq \frac{\eta_0}{1+\eta_0}$$

implying

$$E\left[\int_0^T e^{-\beta t} log\left(\frac{\eta_t}{1+\eta_t}\right) dt\right] \leq \int_0^T e^{-\beta t} log\left[\frac{\eta_0}{1+\eta_0}\right] dt$$

Thus,

$$U_2(\hat{c}^{2c}) - U_2(\hat{c}^{2u}) \leq 0$$

proving the second inequality (3.6.2).

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