DIFFERENTIAL INVARIANTS OF EQUI-AFFINE SURFACES

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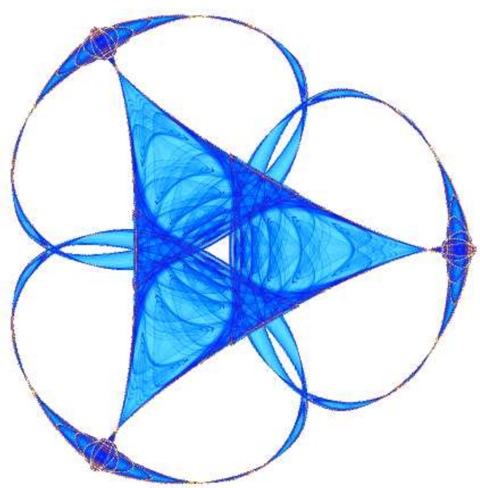

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Differential Invariants of Equi–Affine Surfaces

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Abstract. We show that the algebra of equi-affine differential invariants of a suitably generic surface $S \subset \mathbb{R}^3$ is entirely generated by the third order Pick invariant via invariant differentiation. The proof is based on the new, equivariant approach to the method of moving frames.

The goal of this paper is to prove that, in three-dimensional equi-affine geometry, all higher order differential invariants of suitably nondegenerate surfaces $S \subset \mathbb{R}^3$ are generated by the well-known Pick invariant, [**1**, **5**, **8**, **14**, **15**], through repeated invariant differentiation. Thus, in surprising contrast to Euclidean surface geometry, which requires two generating differential invariants — the Gauss and mean curvatures, [**3**, **9**, **15**] — equi-affine surface geometry is, in a sense, simpler, in that the local geometry, equivalence and symmetry properties of generic surfaces are entirely prescribed by the single Pick differential invariant.

Our proof is based on the equivariant approach to Cartan's method of moving frames that has been developed over the last decade by the author and various collaborators, [**2**, **11**, **12**]. One immediate advantage of the equivariant method is that it is not tied to geometrically-based actions, but can, in fact, be directly applied to *any* transformation group. In geometrical contexts, the equivariant approach mimics the classical moving frame construction, [**3**, **5**], but goes significantly further, in that it supplies us with the

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complete, explicit structure of the underlying algebra of differential invariants through the so-called recurrence relations. Surprisingly, these fundamental relations can be explicitly determined using only the (prolonged) infinitesimal generators of the group action and the moving frame normalization equations. One does *not* need to know the explicit formulas for either the group action, or the moving frame, or even the differential invariants and invariant differential operators themselves, in order to completely understand the differential invariant algebra they generate!

Let us review the basics of the equivariant method of moving frames for finite-dimensional Lie group actions; see [**11**] for a recent review, and the original papers [**2, 12**] for full details. Extensions to infinite-dimensional pseudo-groups can be found in [**13**]. In general, given an r-dimensional Lie group G acting on an m-dimensional manifold M, we are interested in studying its induced action on submanifolds $S \subset M$ of a prescribed dimension, say $p < m$. To this end, we prolong the group action to the (extended) submanifold jet bundles $\mathrm{J}^n = \mathrm{J}^n(M, p)$ of order $n \geq 0$, [**9**]. A *differential invariant* is a (perhaps locally defined) real-valued function $I \colon \mathrm{J}^n \to \mathbb{R}$ that is invariant under the prolonged group action. According to Cartan, the local equivalence and symmetry properties of submanifolds are entirely prescribed by their differential invariants. Any finite-dimensional Lie group action admits an infinite number of functionally independent differential invariants of progressively higher order. Moreover, there always exist $p = \dim S$ linearly independent invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$. The *Fundamental Basis Theorem*, first formulated by Lie, [**7**; p. 760], states that all the differential invariants can be generated from a finite number of low order invariants by repeated invariant differentiation; see, for instance, [**9**] for a modern version of this result.

The most familiar example is when $G = \mathrm{SE}(3)$ is the (special) Euclidean group, consisting of all rigid, orientation-preserving motions of $M = \mathbb{R}^3$, acting on surfaces $S \subset \mathbb{R}^3$. The generating differential invariants are the Gauss and mean curvatures. The invariant differentiations $\mathcal{D}_1, \mathcal{D}_2$ are closely related to the standard covariant derivatives, cf. [**3, 9**]. Any Euclidean surface differential invariant is a function of the iterated invariant derivatives of the Gauss and mean curvatures. The structure of the resulting differential invariant algebra is complicated by the existence of a basic functional relation or *syzygy* among the differentiated curvature invariants, namely the Gauss–Codazzi relation; see [**6**] for details.

The goal of this paper is to establish the analogous result for surfaces $S \subset \mathbb{R}^3$ under the standard action of the *equi-affine group* $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ consisting of all (oriented) volume-preserving affine transformations:

$$g \cdot z = A z + b, \qquad \text{where} \qquad g = (A, B) \in \mathrm{SA}(3), \quad \det A = 1, \quad z = (x, y, u)^T \in \mathbb{R}^3. \quad (1)$$

Our main result can be stated as follows:

Theorem 1. *The algebra of differential invariants for nondegenerate surfaces under the action of the equi-affine group is generated by a single third order differential invariant, known as the Pick invariant, through invariant differentiation.*

In other words, on any suitably generic surface, as specified precisely below, any equi-affine differential invariant can be expressed (locally) as a function

$$I = H(\ \ldots \ \mathcal{D}_I P \ \ldots \ ) \qquad (2)$$

of the Pick invariant P and a finite collection of its successive derivatives

$$\mathcal{D}_I P = \mathcal{D}_{i_1} \mathcal{D}_{i_2} \cdots \mathcal{D}_{i_k} P, \qquad \text{for} \qquad i_\nu \in \{1, 2\}, \tag{3}$$

with respect to the invariant differential operators. Keep in mind that, since $\mathcal{D}_1, \mathcal{D}_2$ do not commute, one must keep track of the order of differentiation in (3). As a consequence, if we use the Pick invariant as the sole generator of the algebra of equi-affine differential invariants, one immediately deduces that there are *no* syzygies in equi-affine surface geometry comparable to the Gauss–Codazzi relation. Further consequences of this result will be explored in forthcoming publications.

Remark: The term "nondegenerate" will be explained in detail during the course of the paper. In particular, surfaces with constant Pick invariant are degenerate, and hence not covered by the result. If *all* the differential invariants are constant, then Cartan's Theorem, [**2**], implies that the surface must be the orbit of a suitable two-parameter subgroup of SA(3), e.g., an ellipsoid or hyperboloid. However, because of the degeneracy, it is possible for a surface to have constant Pick invariant and yet not all of its higher order differential invariants be constant. See [**1**, **5**, **8**, **14**] for details on the classification of surfaces with constant Pick invariant.

We will be working under the assumption that the surface S is locally given by the graph of a function $u = f(x, y)$. But this is purely for computational convenience: All calculations and results are readily be extended to general parametrized surfaces, modulo the action of the infinite-dimensional reparametrization pseudo-group, cf. [**2**]. (The equi-affine action on surfaces with a fixed parametrization leads to a different system of differential invariants, which can also be straightforwardly handled by the equivariant moving frame methodology, but this case will not be investigated here.)

Let $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ denote the n^{th} order surface jet bundle, with the usual induced coordinates $z^{(n)} = (x, y, u, u_x, u_y, u_{xx}, \ldots, u_{jk}, \ldots)$ for $j+k \leq n$, whose fiber coordinates u_{jk} represent the partial derivatives $\partial^{j+k}u/\partial x^j \partial y^k$. The induced action of SA(3) on J^n is obtained by the standard prolongation process, [**9**], (or, more prosaically, by implicit differentiation). The explicit formulas are easily established but, for the present purposes, not required.

According to [**2**], an n^{th} order *right moving frame* is a (locally defined) equivariant map $\rho\colon \mathrm{J}^n \to \mathrm{SA}(3)$, whence $\rho(g^{(n)} \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$ for all $g \in \mathrm{SA}(3)$ and all jets $z^{(n)} \in \mathrm{J}^n$ in the domain of ρ. Classical moving frames, as in [**3**, **5**], can all be interpreted as left equivariant maps to the group, and so can be obtained by composing the right-equivariant version with the group inversion map $g \mapsto g^{-1}$.

The existence of a moving frame requires that the prolonged group action be free and regular, [**2**]. Since

$$\dim \mathrm{SA}(3) = 11, \qquad \text{while} \qquad \dim \mathrm{J}^n = 2 + \binom{n+2}{2} = \tfrac{1}{2} n^2 + \tfrac{3}{2} n + 3,$$

a necessary condition for the existence of an equi-affine moving frame is that the jet order $n \geq 3$. Indeed, the prolonged action of SA(3) is locally free on the dense open subset

$$V_3 = \{\, u_{xx} u_{yy} - u_{xy}^2 \neq 0,\ P \neq 0 \,\} \subset \mathrm{J}^3 \tag{4}$$

of jets of *non-singular*[†] *surfaces*. Here P refers to the third order Pick invariant, to be defined in (12) below.

A moving frame is uniquely prescribed by the choice of a cross-section to the group orbits through Cartan's normalization procedure, [**2**]. Since the n–jet of a function can be identified with its n^{th} order Taylor polynomial, the choice of cross-section normalization is equivalent to specification of a *normal form* for the leading terms in the Taylor expansion of the functional equation $u = f(x,y)$ defining the surface. In the non-singular regime, there are two standard nondegenerate normal forms:

Hyperbolic case: Assuming $u_{xx}u_{yy} - u_{xy}^2 < 0$, we define the cross-section $K \subset V_3$ by the equations

$$\begin{gathered} x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1, \\ u_{xyy} = u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0. \end{gathered} \tag{5}$$

This corresponds to the power series normal form

$$u(x,y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}\,c\,(x^3 + 3\,xy^2) + \ \cdots \tag{6}$$

for the surface at the distinguished point $\mathbf{0} = (0,0,0)$. A hyperbolic surface is *nonsingular* if and only if $c \neq 0$.

Elliptic case: Assuming $u_{xx}u_{yy} - u_{xy}^2 > 0$, we use

$$\begin{gathered} x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = 1, \\ u_{xyy} = -u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0, \end{gathered} \tag{7}$$

to define the cross-section, corresponding to the power series normal form

$$u(x,y) = \tfrac{1}{2}(x^2 + y^2) + \tfrac{1}{6}\,c\,(x^3 - 3\,xy^2) + \ \cdots\ . \tag{8}$$

Non-singularity of the elliptic surface again requires $c \neq 0$.

In both cases, the coefficient c can be identified with the (square root of the) Pick invariant.

Remark: The *parabolic case*, where $u_{xx}u_{yy} - u_{xy}^2 \equiv 0$, requires a higher order moving frame, and the geometric and differential invariant theoretic structure is quite different; for instance, there is no direct analog of the Pick invariant. A detailed analysis and classification of parabolic surfaces can be found in Jensen, [**5**; chapter VI].

Given a cross-section $K \subset \mathrm{J}^n$, the induced right moving frame $\rho\colon \mathrm{J}^n \to \mathrm{SA}(3)$, defined on a suitable open subset $V \subset \mathrm{J}^n$ containing K, is given by $\rho(z^{(n)}) = g \in \mathrm{SA}(3)$, which is the[‡] group element that maps the jet $z^{(n)} \in V$ to the cross-section: $g^{(n)} \cdot z^{(n)} \in K$. The moving frame in turn induces an *invariantization process*, denoted by ι, that maps

[†] The non-degenerate surfaces alluded to above are necessarily non-singular, but require an additional genericity constraint; see equation (32) below.

[‡] Uniqueness requires that G act freely. For a locally free action, there remain discrete ambiguities that are dealt with by further prolongation. See [**10**] for some simple examples.

differential functions to differential invariants, differential forms to invariant differential forms, differential operators to invariant differential operators, and so on. Specifically, the invariantization of any differential function $F \colon \mathrm{J}^n \to \mathbb{R}$ is the unique differential invariant $I = \iota(F)$ that agrees with F when restricted to the cross-section: $I \mid K = F \mid K$. In particular, $\iota(I) = I$ if I is any differential invariant. Thus, invariantization prescribes a morphism that projects the algebra[§] of differential functions to the algebra of differential invariants.

In particular, invariantization of the basic jet coordinates results in the *normalized differential invariants*

$$H_1 = \iota(x) = 0, \qquad H_2 = \iota(y) = 0, \qquad I_{jk} = \iota(u_{jk}), \qquad j, k \geq 0. \tag{9}$$

The invariantizations of the variables appearing in the cross-section equations (5) or (7) will be constant, and are known as *phantom differential invariants*, while the remaining non-constant *basic differential invariants* form a complete system of functionally independent invariants for the prolonged group action. We use

$$I^{(n)} = (I_{00}, I_{10}, I_{01}, I_{20}, I_{11}, \ \ldots \ , I_{0n}) = \iota(u^{(n)}) \tag{10}$$

to denote all the normalized differential invariants, both phantom and basic, of order $\leq n$ obtained by invariantizing the dependent variable u and its derivatives.

To be specific, let us concentrate on the hyperbolic regime from now on, leaving the elliptic modifications until the end of the paper. For the hyperbolic cross-section (5), the phantom differential invariants are

$$H_1 = H_2 = I_{00} = I_{10} = I_{01} = I_{11} = I_{21} = I_{03} = 0, \quad I_{20} = 1, \quad I_{02} = -1, \quad I_{30} - I_{12} = 0. \tag{11}$$

There is one nontrivial independent differential invariant of order 3,

$$P = I_{30} = \iota(u_{xxx}) = I_{12} = \iota(u_{xyy}). \tag{12}$$

which corresponds to the coefficient c in the normalized Taylor expansion (6). To avoid an ambiguous sign, resulting from the fact that the action of $\mathrm{SA}(3)$ on J^3 is only locally free, its square, P^2, is traditionally known as the *Pick invariant*, [**15**], although for brevity, we will often refer to P itself as the Pick invariant.

There are 5 functionally independent basic differential invariants of order 4, which we denote by

$$\begin{gathered} Q_0 = I_{40} = \iota(u_{xxxx}), \qquad Q_1 = I_{31} = \iota(u_{xxxy}), \qquad Q_2 = I_{22} = \iota(u_{xxyy}), \\ Q_3 = I_{13} = \iota(u_{xyyy}), \qquad Q_4 = I_{04} = \iota(u_{yyyy}), \end{gathered} \tag{13}$$

followed by 6 basic differential invariants of order 5, and, in general, $n+1$ independent differential invariants I_{jk} of order $n = j + k$. These can all be identified with the Taylor coefficients in the normalized series expansion (6).

§ More rigorously, since functions are only locally defined, one should use the language of sheaves, [**16**], rather than algebras. But this extra technicality is not required here.

In addition, the two basic invariant differential operators are obtained by invariantizing the total derivatives $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$, or, equivalently, are given as the dual differentiations with respect to the contact-invariant coframe

$$\omega_1 = \iota(dx), \qquad \omega_2 = \iota(dy), \tag{14}$$

fixed by the moving frame. If F is any differential function, then its (horizontal[†]) differential

$$dF = (D_x F)\,dx + (D_y F)\,dy = (\mathcal{D}_1 F)\,\omega_1 + (\mathcal{D}_2 F)\,\omega_2. \tag{15}$$

In particular, the invariant differential operators map any non-phantom differential invariant I to a pair of independent higher order differential invariants $\mathcal{D}_1 I, \mathcal{D}_2 I$.

Since the prolonged equi-affine action is locally free almost everywhere on J^3, a general result in [**2**] implies that all the higher differential invariants can be generated by invariant differentiation of the 5 differential invariants $P, Q_0, \ldots, Q_4$ of order ≤ 4. This fact can also be deduced from the recurrence formulae presented below. Thus, to establish our claimed Theorem 1, we need only show that all the fourth order invariants Q_j can, in fact, be written as functions of the invariant derivatives of the third order Pick invariant P.

In general, the complete structure of the algebra of differential invariants is based on the general *recurrence formulae*, first established in [**2**], that relate the normalized and differentiated invariants. These formulae are explicitly constructed from the prolonged infinitesimal generators of the group action. In our case, the Lie algebra $\mathfrak{sa}(3)$ of infinitesimal generators of the equi-affine group is spanned by the following 11 vector fields:

$$\begin{gathered} \mathbf{v}_1 = x\partial_x - u\partial_u, \qquad \mathbf{v}_2 = y\partial_y - u\partial_u, \\ \mathbf{v}_3 = y\partial_x, \qquad \mathbf{v}_4 = u\partial_x, \qquad \mathbf{v}_5 = x\partial_y, \qquad \mathbf{v}_6 = u\partial_y, \qquad \mathbf{v}_7 = x\partial_u, \qquad \mathbf{v}_8 = y\partial_u, \\ \mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u, \end{gathered} \tag{16}$$

We prolong each of these to the submanifold jet spaces J^n using the standard prolongation formula, [**9**]: The n^{th} prolongation of a vector field

$$\mathbf{v} = \xi(x,y,u)\,\frac{\partial}{\partial x} + \eta(x,y,u)\,\frac{\partial}{\partial y} + \varphi(x,y,u)\,\frac{\partial}{\partial u} \tag{17}$$

on $\mathbb{R}^3$ is the vector field

$$\mathbf{v}^{(n)} = \mathbf{v} + \sum_{1 \leq j+k \leq n} \varphi^{jk}(x,y,u^{(j+k)})\,\frac{\partial}{\partial u_{jk}} \tag{18}$$

on $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$, whose coefficients are given by

$$\varphi^{jk} = D_x^j D_y^k \left(\varphi - \xi\, u_x - \eta\, u_y \right) + \xi\, u_{j+1,k} + \eta\, u_{k,j+1}. \tag{19}$$

[†] The term "horizontal" refers to the fact that we are ignoring any contact forms that appear in the invariantized one-forms, because they do not play a role in our subsequent analysis. The contact components are, however, of importance when studying equi-affine-invariant variational problems. See [**6**] for a complete development.

For conciseness, we do not write out the explicit formulas for the prolonged equi-affine infinitesimal generators (16) here, although they are easily calculated using (19).

Specializing the general moving frame recurrence formulae found in [**2**, **12**] to the present context, we have the following key result:

Theorem 2. *The* recurrence formulae *for the differentiated invariants are*

$$\begin{aligned} \mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa, \\ \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa, \end{aligned} \qquad j+k \geq 1, \tag{20}$$

where R_i^κ are certain differential invariants.

In (20), the coefficients of the R_i^κ denotes the invariantization of the prolonged vector field coefficient φ_κ^{jk}, which is obtained by replacing each jet coordinate $x, y, u, \ldots, u_{il}, \ldots$ by the corresponding differential invariant $H_1 = 0, H_2 = 0, I_{00} = 0, \ldots, , I_{il}, \ldots,$ as in (9).

The differential invariants R_i^κ appearing in (20) arise as the coefficients of the invariant one-forms ω_i, cf. (14), appearing in the invariantized Maurer–Cartan form $\gamma^\kappa = \iota(\mu^\kappa)$ that is dual to the infinitesimal generator $\mathbf{v}_\kappa$, [**2**, **12**]. For this reason, $R_i = (R_i^1, \ \ldots \ , R_i^8)$, $i = 1, 2$, will be collectively known as the *Maurer–Cartan invariants*. A full explanation of this identification would require several paragraphs. Moreover, it is, in fact, not needed when performing the actual computations. Indeed, the explicit formulas for the Maurer–Cartan invariants can be found directly from the recurrence formulas for the phantom differential invariants, irrespective of how they arise from the underlying theory. And so, in the interests of brevity, we refer the reader [**2**, **12**] for the complete story.

Remark: In (20), we have omitted the recurrence formulas for the trivial order zero differential invariants $H_1 = H_2 = I_{00} = 0$, since they only affect the additional Maurer–Cartan invariants associated to the translational generators $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Since these infinitesimal generators have trivial prolongation, their Maurer–Cartan invariants play no role in any of the higher order recurrence formulas (20).

In the hyperbolic regime, using the explicit formulas for the coefficients of the prolonged infinitesimal generators of SA(3), the resulting phantom recurrence formulae are

$$\begin{aligned} &0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, && 0 = \mathcal{D}_2 I_{10} = R_2^7, \\ &0 = \mathcal{D}_1 I_{01} = R_1^8, && 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8, \\ &0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, && 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2, \\ &0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, && 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5, \\ &0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, && 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2, \\ &0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30}R_1^3 - 2I_{30}R_1^5 + R_1^6, && 0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30}R_2^3 - 2I_{30}R_2^5 + R_2^6, \\ &0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30}R_2^3 - 3R_2^6, && 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30}R_2^3 - 3R_2^6. \end{aligned} \tag{21}$$

In addtion, we have the following recurrence formulae for the non-constant third order invariants

$$\begin{aligned}
&\mathcal{D}_1 I_{30} = I_{40} - 4 I_{30} R_1^1 - I_{30} R_1^2 - 3 R_1^4, && \mathcal{D}_2 I_{30} = I_{31} - 4 I_{30} R_2^1 - I_{30} R_2^2 - 3 R_2^4,\\
&\mathcal{D}_1 I_{12} = I_{22} - 2 I_{30} R_1^1 - 3 I_{30} R_1^2 + R_1^4, && \mathcal{D}_2 I_{12} = I_{13} - 2 I_{30} R_2^1 - 3 I_{30} R_2^2 + R_2^4.
\end{aligned} \tag{22}$$

Owing to our normalization condition (12),

$$\mathcal{D}_1 I_{30} = -\mathcal{D}_1 I_{12}, \qquad \mathcal{D}_2 I_{30} = -\mathcal{D}_2 I_{12}. \tag{23}$$

Solving the combined linear system (21–23) produces the explicit forms of the Maurer–Cartan invariants:

$$\begin{aligned}
R_1 &= \left(\tfrac{1}{2} I_{30}, -\tfrac{1}{2} I_{30}, \frac{3 I_{31} + I_{13}}{12 I_{30}}, \tfrac{1}{4} I_{40} - \tfrac{1}{4} I_{22} - \tfrac{1}{2} I_{30}^2, \frac{3 I_{31} + I_{13}}{12 I_{30}}, -\tfrac{1}{4} I_{31} + \tfrac{1}{4} I_{13}, -1, 0 \right)\\
&= \left(\tfrac{1}{2} P, -\tfrac{1}{2} P, \frac{3 Q_1 + Q_3}{12 P}, \tfrac{1}{4} Q_0 - \tfrac{1}{4} Q_2 - \tfrac{1}{2} P^2, \frac{3 Q_1 + Q_3}{12 P}, -\tfrac{1}{4} Q_1 + \tfrac{1}{4} Q_3, -1, 0 \right),\\
R_2 &= \left(0, 0, \frac{3 I_{22} + I_{04}}{12 I_{30}} + \tfrac{1}{2} I_{30}, \tfrac{1}{4} I_{31} - \tfrac{1}{4} I_{13}, \frac{3 I_{22} + I_{04}}{12 I_{30}} - \tfrac{1}{2} I_{30}, 0, -\tfrac{1}{4} I_{22} + \tfrac{1}{4} I_{04} - \tfrac{1}{2} I_{30}^2, 0, 1 \right)\\
&= \left(0, 0, \frac{3 Q_2 + Q_4}{12 P} + \tfrac{1}{2} P, \tfrac{1}{4} Q_1 - \tfrac{1}{4} Q_3, \frac{3 Q_2 + Q_4}{12 P} - \tfrac{1}{2} P, -\tfrac{1}{4} Q_2 + \tfrac{1}{4} Q_4 - \tfrac{1}{2} P^2, 0, 1 \right).
\end{aligned} \tag{24}$$

These expressions are then substituted back into the remaining recurrence formulae for the basic differential invariants, thereby producing the complete system of recurrence relations among the normalized and differentiated invariants.

Our proof of Theorem 1 relies on a detailed analysis of these basic equi-affine recurrence relations. In particular, the recurrence formulae for the third and fourth order differential invariants are

$$\begin{aligned}
\mathcal{D}_1 I_{30} &= I_{40} - 4 I_{30} R_1^1 - I_{30} R_1^2 - 3 R_1^4,\\
\mathcal{D}_2 I_{30} &= I_{31} - 4 I_{30} R_2^1 - I_{30} R_2^2 - 3 R_2^4,\\
\mathcal{D}_1 I_{40} &= I_{50} - 5 I_{40} R_1^1 - I_{40} R_1^2 - 10 I_{30} R_1^4 - 4 I_{31} R_1^5,\\
\mathcal{D}_2 I_{40} &= I_{41} - 5 I_{40} R_2^1 - I_{40} R_2^2 - 10 I_{30} R_2^4 - 4 I_{31} R_2^5,\\
\mathcal{D}_1 I_{31} &= I_{41} - 4 I_{31} R_1^1 - 2 I_{31} R_1^2 - I_{40} R_1^3 - 3 I_{22} R_1^5 - 2 I_{30} R_1^6,\\
\mathcal{D}_2 I_{31} &= I_{32} - 4 I_{31} R_2^1 - 2 I_{31} R_2^2 - I_{40} R_2^3 - 3 I_{22} R_2^5 - 2 I_{30} R_2^6,\\
\mathcal{D}_1 I_{22} &= I_{32} - 3 I_{22} R_1^1 - 3 I_{22} R_1^2 - 2 I_{31} R_1^3 - 2 I_{30} R_1^4 - 2 I_{13} R_1^5,\\
\mathcal{D}_2 I_{22} &= I_{23} - 3 I_{22} R_2^1 - 3 I_{22} R_2^2 - 2 I_{31} R_2^3 - 2 I_{30} R_2^4 - 2 I_{13} R_2^5,\\
\mathcal{D}_1 I_{13} &= I_{23} - 2 I_{13} R_1^1 - 4 I_{13} R_1^2 - 3 I_{22} R_1^3 - I_{04} R_1^5 + 6 I_{30} R_1^6,\\
\mathcal{D}_2 I_{13} &= I_{14} - 2 I_{13} R_2^1 - 4 I_{13} R_2^2 - 3 I_{22} R_2^3 - I_{04} R_2^5 + 6 I_{30} R_2^6,\\
\mathcal{D}_1 I_{04} &= I_{14} - I_{04} R_1^1 - 5 I_{04} R_1^2 - 4 I_{13} R_1^3 + 6 I_{30} R_1^4,\\
\mathcal{D}_2 I_{04} &= I_{05} - I_{04} R_2^1 - 5 I_{04} R_2^2 - 4 I_{13} R_2^3 + 6 I_{30} R_2^4,
\end{aligned} \tag{25}$$

where we now replace the Maurer–Cartan invariants by their explicit formulas (24).

The Maurer–Cartan invariants (24) are all of order ≤ 4. Thus, whenever $n = j+k \geq 4$, the only differential invariant of order $n+1$ appearing on the right hand side of the recurrence formula (20) is the leading term — namely, $I_{j+1,k}$ or $I_{j,k+1}$. This immediately establishes, by a simple induction argument, our earlier claim that all of the differential invariants of order ≥ 5 can be written in terms of (iterated) invariant derivatives of the differential invariants of order 3 and 4, namely P and $Q_0, \ldots, Q_4$.

To find formulas for the fourth order invariants Q_i in terms of derivatives of the Pick invariant P, we proceed as follows. In view of (12, 13) and (24), the first two recurrence formulae (25) are

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4} Q_0 + \tfrac{3}{4} Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4} Q_1 + \tfrac{3}{4} Q_3. \tag{26}$$

Thus, we are already able to generate 2 linear combinations of the fourth order invariants.

Secondly, the invariant differential operators do not commute, but rather satisfy

$$\mathcal{D}_3 = \left[\, \mathcal{D}_1, \mathcal{D}_2 \,\right] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2, \tag{27}$$

for certain differential invariants Y_1, Y_2. Specializing the general commutator formulas established in [**2**, **6**], we find†

$$\begin{aligned} Y_1 &= \sum_{\kappa=1}^{8} \left(\frac{\partial \xi_\kappa}{\partial x}(0,0,0)\, R_2^\kappa - \frac{\partial \xi_\kappa}{\partial y}(0,0,0)\, R_1^\kappa \right) = R_2^1 - R_1^3, \\ Y_2 &= \sum_{\kappa=1}^{8} \left(\frac{\partial \eta_\kappa}{\partial x}(0,0,0)\, R_2^\kappa - \frac{\partial \eta_\kappa}{\partial y}(0,0,0)\, R_1^\kappa \right) = R_2^5 - R_1^2. \end{aligned} \tag{28}$$

Substituting our formulas (24) for the Maurer–Cartan invariants, we deduce that the commutator coefficients

$$Y_1 = -\,\frac{3\,Q_1 + Q_3}{12\,P}\,, \qquad Y_2 = \frac{3\,Q_2 + Q_4}{12\,P}\,, \tag{29}$$

are certain fourth order differential invariants. We now set

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = \mathcal{D}_1 P_2 - \mathcal{D}_2 P_1 = Y_1 P_1 + Y_2 P_2. \tag{30}$$

At this point we have constructed 3 independent fourth order differential invariants — namely P_1, P_2 and P_3 — by differentiation of the Pick invariant.

To obtain another fourth order invariant, we can differentiate any of the three:

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j, \qquad j = 1, 2, 3. \tag{31}$$

As long as at least one of the 2×2 determinants

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3, \tag{32}$$

† In more general contexts, the partial derivatives should be replaced by total derivatives with respect to x, y. Here, since we normalized both $I_{10} = \iota(u_x) = 0$ and $I_{01} = \iota(u_y) = 0$, the additional u derivative terms do not affrect the final formula.

we can solve (30–31) for the two fourth order differential invariants Y_1, Y_2. An explicit computation based on the recurrence relations (25) confirms that none of these determinants is identically zero, and so for generic non-singular surfaces, we can produce the invariants Y_1, Y_2 as certain rational combinations of the invariant derivatives of P up to order 3. The explicit formulas are rather complicated and so will not be written out here.

Note that if the Pick invariant is constant, the determinants (32) are all 0 and so the preceding argument breaks down. Indeed, it is possible that a surface with constant Pick invariant admit a non-constant fourth order differential invariant, [**5**]. An interesting challenge is to classify the degenerate equi-affine surfaces, for which all such determinants (32) are zero and so are characterized by the vanishing of certain fairly complicated polynomial combinations of the differential invariants. It is possible that, among the non-singular surfaces, only those with constant Pick invariant satisfy the degeneracy conditions, but so far I have no evidence that this is the case.

Summarizing and slightly simplifying, we have succeeded in expressing the following fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3, \qquad S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4, \tag{33}$$

as certain rational combinations of the invariant derivatives of the Pick invariant of order ≤ 3. The first two are multiples of P_1, P_2, whereas the latter two are simply related to Y_1, Y_2. Observe that we can express Q_1 and Q_3 in terms of S_2 and S_3.

To construct the final fourth order invariant, we return to the recurrence formulas (25) for the Q_j's. A direct computation using (24) shows that

$$\begin{aligned} 12\,P\big(\,\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3\,\big) &= 18\,P^2 (Q_0 - 2\,Q_2 + Q_4) - (18\,Q_1^2 + 36\,Q_1 Q_3 + 10\,Q_3^2) + \\ &\qquad + (9\,Q_0 Q_2 + 3\,Q_0 Q_4 + 36\,Q_2^2 + 15\,Q_2 Q_4 + Q_4^2) \\ &= 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2. \end{aligned} \tag{34}$$

Since all terms except the first depend on previously computed fourth order differential invariants, we are able to write the invariant Q_0 as an explicit (complicated) rational combination of the invariant derivatives, of orders ≤ 4, of the Pick invariant. Combining this with our previously constructed fourth order invariants, (33), we have indeed produced 5 functionally independent fourth order differential invariants by successively differentiating the Pick invariant. This completes the proof of the main result of this paper in the hyperbolic regime.

The Elliptic Case: The calculations are very similar, and only requires changing some of the signs. The Maurer–Cartan invariants are

$$\begin{aligned} R_1 &= \left(\tfrac{1}{2}\,P, -\tfrac{1}{2}\,P, \frac{3\,Q_1 - Q_3}{12\,P}\,, \tfrac{1}{4}\,Q_0 + \tfrac{1}{4}\,Q_2 - \tfrac{1}{2}\,P^2, \frac{-3\,Q_1 + Q_3}{12\,P}\,, -\tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, -1, 0 \right), \\ R_2 &= \left(0, 0, \frac{3\,Q_2 - Q_4}{12\,P} + \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, \frac{3\,Q_2 - Q_4}{12\,P} - \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_2 + \tfrac{1}{4}\,Q_4 - \tfrac{1}{2}\,P^2, 0, 1 \right). \end{aligned} \tag{35}$$

The first order derivatives of the Pick invariant $P = I_{30} = \iota(u_{xxx})$ are

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4} Q_0 - \tfrac{3}{4} Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4} Q_1 - \tfrac{3}{4} Q_3. \tag{36}$$

The commutation relation is

$$\mathcal{D}_3 = \left[\, \mathcal{D}_1, \mathcal{D}_2 \,\right] = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2, \tag{37}$$

where

$$Y_1 = -\,\frac{3\,Q_1 - Q_3}{12\,P}\,, \qquad Y_2 = -\,\frac{3\,Q_2 - Q_4}{12\,P}\,. \tag{38}$$

As before, we set $P_3 = \mathcal{D}_3 P = Y_1 P_1 + Y_2 P_2$, and can solve for Y_1, Y_2 provided one of the determinantal conditions (31) holds. At this stage we have produced the fourth order invariants

$$S_1 = Q_0 - 3\,Q_2, \qquad S_2 = Q_1 - 3\,Q_3, \qquad S_3 = 3\,Q_1 - Q_3, \qquad S_4 = 3\,Q_2 - Q_4. \tag{39}$$

Finally, the relation

$$\begin{aligned} 12\,P\big(\,\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3\,\big) &= -\,18\,P^2(Q_0 + 2\,Q_2 + Q_4) - (18\,Q_1^2 - 36\,Q_1 Q_3 + 10\,Q_3^2) + \\ &\qquad + (9\,Q_0 Q_2 - 3\,Q_0 Q_4 - 36\,Q_2^2 + 15\,Q_2 Q_4 - Q_4^2) \\ &= -\,48\,P^2 Q_0 + 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 - S_4^2 \end{aligned} \tag{40}$$

allows us to construct Q_0, and hence all of the fourth (and all higher) order differential invariants as rational invariant differential functions of the Pick invariant.

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