A BASIC INEQUALITY FOR THE STOKES OPERATOR RELATED TO THE NAVIER BOUNDARY CONDITION

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IMA Preprint Series # 2143

(November 2006)

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A basic inequality for the Stokes operator related to the Navier boundary condition

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Abstract. We show that $\|Au + \Delta u\|_{L^2(\Omega)} \le C_1 \|\nabla u\|_{L^2(\Omega)} + C_0 \|u\|_{L^2(\Omega)}$, where u belongs to the domain of A, the Stokes operator for divergence-free vector fields in the domain $\Omega \subset \mathbb{R}^3$ satisfying the Navier boundary condition. Moreover, in the case of thin domains, the constant C_1 is comparable with the small depth of the domains.

Keywords: Stokes operator, Navier boundary condition, thin domain, commutator estimate.

1. Introduction

In the study of the Navier–Stokes equations the Stokes operator $A = -P\Delta$, where P is the Leray projection, plays a crucial role. In the periodic domain Ω, we simply have

$$Au = -\Delta u, \text{ for } u \in D_A, \tag{1.1}$$

where D_A is the domain of A. However when Ω is a more general domain and u satisfies various boundary conditions rather than the periodicity one, relation (1.1) is, in general, no longer holds true. In those cases, the question is that: What is the difference between Au and $(-\Delta u)$? Clearly, one always has

$$\|Au + \Delta u\|_{L^2(\Omega)} \leq C\|u\|_{H^2(\Omega)}. \tag{1.2}$$

The interest now is the size of the constant C, or whether one can replace $\|u\|_{H^2(\Omega)}$ by a smaller norm. For example, it is shown in Proposition 3.9 of [3] that for the thin domain Ω_ε of the form (3.3) below with $h_0 = 0$, we have

$$\|Au + \Delta u\|_{L^2(\Omega_\varepsilon)} \leq C_2\varepsilon\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)} + C_1\|u\|_{L^2(\Omega_\varepsilon)}, \quad u \in D_A, \tag{1.3}$$

where ε is the small depth of the domain and the positive constants C_2 and C_1 are independent of ε. The domain D_A of the Stokes operator in this case consists of divergence-free vector fields in $H^2(\Omega_\varepsilon)$ that satisfy the Navier condition (2.1) on the top and bottom boundaries and satisfy the periodicity condition on the sides. (A related inequality for dilated two-layer thin domains appears in [2], Lemma 2.9.) Roughly speaking, (1.3) shows that Au is a small H^2-perturbation of $(-\Delta u)$ for $u \in D_A$. The current paper aims to improve (1.3) in several different contexts.

We will show that for a divergence-free vector field u satisfying the Navier boundary condition on the whole boundary $\partial\Omega$ of a more general domain Ω, the term Au is only a H^1-perturbation of $(-\Delta u)$. We have

$$\|Au + \Delta u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \quad u \in D_A. \tag{1.4}$$

Furthermore, in the context of thin domains Ω_ε as in (3.3) (including $h_0 \neq 0$), this estimate can be improved in terms of the small depth of the domains:

$$\|Au + \Delta u\|_{L^2(\Omega_\varepsilon)} \leq C(\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}), \quad u \in D_A. \tag{1.5}$$

For the similar result in spherical domains, see section 4. We will prove Ineq. (1.4) in section 2 and Ineq. (1.5) in section 3. For applications of the inequalities of this type, interested readers may look for our forthcoming papers on the Navier–Stokes equations.

2. General domains

We consider in this section an open, bounded, connected domain $\Omega \subset \mathbb{R}^3$ with C^3 boundary. A vector field $u = (u_1, u_2, u_3)$ in $\overline{\Omega}$ is said to satisfy the Navier boundary condition if

$$u \cdot N = 0 \quad \text{and} \quad [(Du)N]_{\tan} = 0, \tag{2.1}$$

on $\partial\Omega$, where $[\cdot]_{\tan}$ indicates the tangential part of the vector. Above, N is the unit outward normal vector on the boundary and Du is the symmetric part of the gradient matrix ∇u, that is, $Du = \frac{\nabla u + (\nabla u)^*}{2}$, where $(\nabla u)_{ij} = \partial_j u_i$, and $(\nabla u)^*$ is the transpose matrix of ∇u.

Let $H = \{u \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } u \cdot N = 0 \text{ on } \partial\Omega\}$. The Leray projection P is defined to be the orthogonal projection from $L^2(\Omega, \mathbb{R}^3)$ onto H. We have the Helmholtz-Leray decomposition

$$L^2(\Omega, \mathbb{R}^3) = H \oplus H^{\perp} \text{ where } H^{\perp} = \{\nabla\phi : \phi \in H^1(\Omega)\}. \tag{2.2}$$

There are geometric issues arising in the definition of the Stokes operator associated with the boundary condition (2.1), see e.g. [3]. What we need is that $A = P(-\Delta)$ on D_A where the domain D_A is contained in

$$\{u \in H^2(\Omega, \mathbb{R}^3),\ u \text{ satisfies } \nabla \cdot u = 0 \text{ in } \Omega \text{ and satisfies (2.1) on } \partial\Omega\}. \tag{2.3}$$

With a general domain Ω and a general element $u \in D_A$, the term Δu need not be tangential to the boundary $\partial\Omega$, and hence $Au \neq -\Delta u$.

Theorem 2.1. *Let $u \in D_A$, then*

$$\|Au + \Delta u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \tag{2.4}$$

where C is a positive constant depending on the domain.

Before proving Theorem 2.1, we recall the following lemma concerning $\nabla \times u$ on the boundary of the domain. While this result is proved in [1], we present the argument here for the convenience of the reader.

Lemma 2.2 ([1])**.** *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ such that $\Gamma_* = \partial\Omega \cap \mathcal{O} \neq \emptyset$. Let u belong to $C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ and satisfy (2.1) on Γ_*. Suppose $\check{N} \in C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ with the restriction $\check{N}\Big|_{\Gamma_*}$ being a unit normal vector field on Γ_*. Then*

$$\check{N} \times (\nabla \times u) = 2\check{N} \times (\check{N} \times ((\nabla\check{N})^* u)) \quad \text{on} \quad \Gamma_*. \tag{2.5}$$

Proof. Let $\omega = \nabla \times u$. From the identity $\check{N} \times \nabla(u \cdot \check{N}) = 0$ on Γ_*, we have

$$\begin{aligned} 0 &= \check{N} \times [(\nabla u)^* \check{N}] + \check{N} \times [(\nabla\check{N})^* u] \\ &= \check{N} \times [(Du)\check{N} - (Ku)\check{N}] + \check{N} \times [(\nabla\check{N})^* u], \end{aligned}$$

where $Ku = \frac{\nabla u - (\nabla u)^*}{2}$. Since $(Du)\check{N}$ is co-linear to $\check{N}$, we thus have

$$\check{N} \times [(\nabla\check{N})^* u] = \check{N} \times [(Ku)\check{N}] = \check{N} \times [(1/2)\omega \times \check{N}].$$

Therefore $\check{N} \times (\omega \times \check{N}) = 2\check{N} \times [(\nabla\check{N})^* u]$. Then use the identity

$$a \times (a \times (a \times b))) = -|a|^2(a \times b)$$

to obtain (2.5). □

Following is the basic lemma of this paper.

Lemma 2.3. *Let $u \in D_A$ and $\Phi \in H^{\perp}$. Then*

$$\left| \int_\Omega \Delta u \cdot \Phi dx \right| \leq C\|\Phi\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}, \tag{2.6}$$

where $C > 0$ depends on Ω.

Proof. Let $\omega = \nabla \times u$ and $\Phi = \nabla \phi$. By the density argument, we can assume u and Φ are smooth. We have $\nabla \times \omega = -\Delta u$ and $\nabla \times \Phi = 0$. Then

$$\begin{aligned}\int_\Omega \Delta u \cdot \Phi dx &= -\int_\Omega (\nabla \times \omega) \cdot \Phi dx \\ &= -\int_\Omega \omega \cdot (\nabla \times \Phi) dx - \int_{\partial\Omega} (\omega \times \Phi) \cdot N d\sigma \\ &= \int_{\partial\Omega} (\omega \times N) \cdot \Phi d\sigma.\end{aligned}$$

Let $N(x), x \in \Omega$, be a C^2-extension of the unit outward normal vector N from $\partial\Omega$ to the whole domain Ω. On $\overline{\Omega}$, we define $G(u) = N \times [(\nabla N)^* u]$ then by Lemma 2.2 we have

$$2N \times G(u)\Big|_{\partial\Omega} = N \times \omega. \tag{2.7}$$

We thus have

$$\begin{aligned}\int_\Omega \Delta u \cdot \Phi dx &= -\int_{\partial\Omega} 2(N \times G(u)) \cdot \Phi d\sigma = \int_{\partial\Omega} 2(\Phi \times G(u)) \cdot N d\sigma \\ &= 2\int_\Omega \nabla \cdot (\Phi \times G(u)) dx \\ &= 2\int_\Omega \Phi \cdot (\nabla \times G(u)) - (\nabla \times \Phi) \cdot G(u) dx \\ &= 2\int_\Omega \Phi \cdot (\nabla \times G(u)) dx.\end{aligned}$$

Since $|\nabla \times G(u)| \le C(|\nabla u| + |u|)$, we obtain

$$\left| \int_\Omega \Delta u \cdot \Phi dx \right| \le C \int_\Omega |\Phi|(|\nabla u| + |u|) dx,$$

and (2.6) follows. □

Proof of Theorem 2.1. Let $\Phi = Au + \Delta u = -P\Delta u + \Delta u$, then $\Phi \in H^\perp$. Since Au and Φ are orthogonal in $L^2(\Omega, \mathbb{R}^3)$, we have

$$\int_\Omega |\Phi|^2 dx = \int_\Omega (Au + \Delta u) \cdot \Phi dx = \int_\Omega \Delta u \cdot \Phi dx.$$

Applying Lemma 2.3, we obtain (2.4). □

Remark 2.4. The proof of (1.3) as presented in [3] involves the second order term $\Delta u \cdot N$ on $\partial\Omega$. Though having similar ideas, our proofs of Theorem 2.1 and Lemma 2.3 avoid using that higher order term, hence result in the improvement.

Remark 2.5. Concerning the size of constant C in Ineq. (1.2), it is proved in [4] that, in the context of Dirichlet boundary condition, one has

$$\|Au + \Delta u\|_{L^2(\Omega)} \le (\frac{1}{2} + \varepsilon)\|u\|_{H^2(\Omega)} + C_\varepsilon \|u\|_{H^1(\Omega)},$$

for $u \in D_A = H^2(\Omega, \mathbb{R}^3) \cap H_0^1(\Omega, \mathbb{R}^3) \cap H$, $\varepsilon > 0$.

3. Nearly flat domains

In this section, we consider three dimensional thin domains of the form

$$\Omega'_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, h_0(x_1, x_2) < x_3 < h_1(x_1, x_2)\}, \tag{3.1}$$

where $\varepsilon \in (0, 1]$, $h_0 = \varepsilon g_0$, $h_1 = \varepsilon g_1$, with g_0 and g_1 being given C^3 scalar functions in $\mathbb{R}^2$ satisfying the following periodicity condition

$$g_i(x' + \mathbf{e}_j) = g_i(x'), \quad x' = (x_1, x_2) \in \mathbb{R}^2,\ i = 0, 1,\ j = 1, 2,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. We assume that

$$g = g_1 - g_0 \geq \alpha > 0. \tag{3.2}$$

The boundary of Ω'_ε is $\Gamma' = \Gamma'_0 \cup \Gamma'_1$, where Γ'_0 and Γ'_1 are the bottom and the top of Ω'_ε:

$$\Gamma'_i = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, x_3 = h_i(x_1, x_2)\}, \quad i = 0, 1.$$

One of the representing domains of Ω'_ε is

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in (0, 1)^2, h_0(x_1, x_2) < x_3 < h_1(x_1, x_2)\}, \tag{3.3}$$

We study the divergence-free vector fields $u(x)$ in Ω'_ε that satisfy the periodicity condition

$$u(x + \mathbf{e}_j) = u(x) \quad \text{for all} \quad x \in \Omega'_\varepsilon, \quad j = 1, 2, \tag{3.4}$$

and the Navier boundary condition (2.1) on Γ'.

Let $L^2_{\rm per}(\Omega'_\varepsilon)$, resp. $H^k_{\rm per}(\Omega'_\varepsilon), k \geq 1$, be the closure with respect to the norm $\|\cdot\|_{L^2(\Omega_\varepsilon)}$, resp. $\|\cdot\|_{H^k(\Omega_\varepsilon)}$, of the set of all functions $\varphi \in C^\infty(\overline{\Omega'_\varepsilon})$ satisfying

$$\varphi(x + \mathbf{e}_j) = \varphi(x) \text{ for all } x \in \Omega'_\varepsilon,\ j = 1, 2.$$

We then have $L^2_{\rm per}(\Omega'_\varepsilon, \mathbb{R}^3) = H \oplus H^\perp$ where

$$H = \{u \in L^2_{\rm per}(\Omega'_\varepsilon, \mathbb{R}^3) : u \text{ satisfies } \nabla \cdot u = 0 \text{ in } \Omega'_\varepsilon \text{ and } u \cdot N = 0 \text{ on } \Gamma'\},$$
$$H^\perp = \{\nabla\phi : \phi \in H^1_{\rm per}(\Omega'_\varepsilon)\}.$$

In this case, P is the orthogonal projection from $L^2_{\rm per}(\Omega'_\varepsilon, \mathbb{R}^3)$ to H, the domain D_A is a subspace of $\{u \in H^2_{\rm per}(\Omega'_\varepsilon, \mathbb{R}^3) \cap H,\ u \text{ satisfies (2.1) on } \Gamma'\}$, and the Stokes operator $A = P(-\Delta)$ on D_A.

Theorem 3.1. *Let $u \in D_A$, then*

$$\|Au + \Delta u\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)} + C\|u\|_{L^2(\Omega_\varepsilon)}, \tag{3.5}$$

where the positive constant C is independent of ε.

The key point in the proof of Theorem 3.1 is to use a new $G(u)$ defined on $\overline{\Omega_\varepsilon}$ which satisfies (2.7) and gives a better estimate for $|\nabla \times G(u)|$. The argument used in [3] to find such $G(u)$ works for our general domains.

Note from (2.5) that if $\check{N}\Big|_{\Gamma_*} = \pm N$ then we have

$$\pm N \times (\nabla \times u) = \pm 2N \times (\check{N} \times ((\nabla\check{N})^* u)) \text{ on } \Gamma_*,$$

hence

$$N \times (\nabla \times u) = 2N \times (\check{N} \times ((\nabla\check{N})^* u)) \text{ on } \Gamma_*. \tag{3.6}$$

For $i = 0, 1$, let $\tilde{N}^i$ be the unit upward normal vectors on Γ'_i which can be extended to $\mathbb{R}^3$ by

$$\tilde{N}^i(x_1, x_2, x_3) = \frac{\big(-\partial_1 h_i(x_1, x_2), -\partial_2 h_i(x_1, x_2), 1\big)}{\sqrt{1 + |\partial_1 h_i(x_1, x_2)|^2 + |\partial_2 h_i(x_1, x_2)|^2}}.$$

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, let

$$\tilde{N}(x) = \frac{x_3 - h_0(x')}{\varepsilon g(x')} \tilde{N}^1(x) + \frac{h_1(x') - x_3}{\varepsilon g(x')} \tilde{N}^0(x). \tag{3.7}$$

We have $\tilde{N}^0\Big|_{\Gamma'_0} = -N$ and $\tilde{N}^1\Big|_{\Gamma'_1} = N$. We easily obtain the following estimates in Ω'_ε:

$$|\tilde{N}_j|, |\partial_j \tilde{N}| \le C\varepsilon \text{ for } j = 1, 2, \quad |\tilde{N}_3|, |\partial_3 \tilde{N}|, |\partial_k \partial_l \tilde{N}| \le C \text{ for } k, l = 1, 2, 3. \tag{3.8}$$

From (3.6) we have

$$N \times (\nabla \times u) = 2N \times G(u) \text{ on } \Gamma', \tag{3.9}$$

where $G(u)$ is defined on the closure of Ω'_ε by

$$G(u) = \tilde{N} \times [(\nabla \tilde{N})^* u] = \sum_{m=1}^{3} u_m G_m, \tag{3.10}$$

with $G_m = (\tilde{N} \times \nabla)\tilde{N}_m = \sum_{i,j,k=1}^3 \mathbf{e}_i \epsilon_{ijk} \tilde{N}_j \partial_k \tilde{N}_m$. As usual, ϵ_{ijk} is 1 if (i, j, k) is an even permutation of $(1, 2, 3)$, is (-1) if the permutation is odd, and is 0 otherwise. By virtue of (3.8), we have, in Ω'_ε, that $|\tilde{N}_j||\partial_k \tilde{N}_m| \le C\varepsilon$, for $j = 1, 2$, $k = 1, 2, 3$, or for $j = 3$, $k = 1, 2$; therefore $|\epsilon_{ijk} \tilde{N}_j \partial_k \tilde{N}_m| \le C\varepsilon$, and hence $|G_m| \le C\varepsilon$, for $m = 1, 2, 3$. It also follows from (3.8) that $|\nabla G_m| \le C$, for $m = 1, 2, 3$. Consequently,

$$|\nabla G(u)| \le C\varepsilon |\nabla u| + C|u| \text{ in } \Omega'_\varepsilon. \tag{3.11}$$

With this new $G(u)$ in Ω'_ε, the version of Lemma 2.3 for the thin domain is:

Lemma 3.2. *Let $u \in D_A$ and $\Phi \in H^\perp$. Then*

$$\left| \int_{\Omega_\varepsilon} \Delta u \cdot \Phi dx \right| \le C \|\Phi\|_{L^2(\Omega_\varepsilon)} (\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}), \tag{3.12}$$

where $C > 0$ is independent of ε.

Proof. The boundary of Ω_ε consists of four surfaces on the sides, the top Γ_1 and the bottom Γ_0, where $\Gamma_i = \Gamma'_i \cap \overline{\Omega_\varepsilon}$, $i = 0, 1$. Proceed as in Lemma 2.3 noticing that the surface integrals on the sides of Ω_ε vanish due to the periodicity of the integrands. Using (3.9), we have

$$\int_{\Omega_\varepsilon} \Delta u \cdot \Phi dx = - \int_{\Gamma_0 \cup \Gamma_1} 2(N \times G(u)) \cdot \Phi d\sigma = 2 \int_{\Omega_\varepsilon} \Phi \cdot (\nabla \times G(u)) dx,$$

where $G(u)$ is defined in (3.10). Thanks to (3.11),

$$\left| \int_{\Omega_\varepsilon} \Delta u \cdot \Phi dx \right| \le C \int_{\Omega_\varepsilon} |\Phi| (\varepsilon |\nabla u| + |u|) dx,$$

hence (3.12) follows. □

Proof of Theorem 3.1. Same as the proof of Theorem 2.1 with Lemma 3.2 being used instead of Lemma 2.3. □

4. Spherical domains

For the sake of simplicity, we consider the following simple spherical domains

$$\Omega_{R,R'} = \{x \in \mathbb{R}^3 : R < |x| < R'\},$$

where $R' > R > 0$. The functional spaces and operators are defined as in section 2. We obtain the following version of Theorem 2.1 with the constant C in (2.4) depending on R explicitly.

Theorem 4.1. *Let $R' > R > 0$ and $u \in D_A$, then*

$$\|Au + \Delta u\|_{L^2(\Omega_{R,R'})} \le C\left(\frac{1}{R^2}\|u\|_{L^2(\Omega_{R,R'})} + \frac{1}{R}\|\nabla u\|_{L^2(\Omega_{R,R'})}\right), \tag{4.1}$$

where $C > 0$ is independent of R and R'.

Proof. Let (θ, ϕ, r), $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $r \in [0, \infty)$, be the spherical coordinates and let $B = \{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r\}$ be the usual moving frame. In this case, $\tilde{N} = \mathbf{e}_r$, for every $r \in [R, R']$, plays the same role as the upward normal vectors $\tilde{N}$ defined in section 3. As in (3.10), let

$$G(u) = \tilde{N} \times [(\nabla\tilde{N})^* u] = \mathbf{e}_r \times [(\nabla \mathbf{e}_r)^* u].$$

We use the notation $[\cdot]_B$ to denote the presentation of a vector or a matrix with respect to the basis B. Let $u = U_\theta \mathbf{e}_\theta + U_\phi \mathbf{e}_\phi + U_r \mathbf{e}_r$, i.e., $[u]_B = U = (U_\theta, U_\phi, U_r)$. Calculations using in spherical coordinates yield

$$[\nabla \mathbf{e}_r]_B = \mathrm{diag}(r^{-1}, r^{-1}, 0) \text{ and } [G(u)]_B = r^{-1}(-U_\phi, U_\theta, 0).$$

It follows that

$$\begin{aligned}\nabla \times G(u) &= -\frac{1}{r}\partial_r U_\theta \mathbf{e}_\theta - \frac{1}{r}\partial_r U_\theta \mathbf{e}_\phi + \frac{1}{r^2 \sin\theta}\{\partial_\theta(\sin\theta U_\theta) + \partial_\phi U_\phi\}\mathbf{e}_r \\ &= -\frac{1}{r}Q_{13}\mathbf{e}_\theta - \frac{1}{r}Q_{23}\mathbf{e}_\phi + (\frac{1}{r}Q_{22} + \frac{U_\theta - U_r}{r^2})\mathbf{e}_r,\end{aligned}$$

where $Q = (Q_{ij})_{i,j=1,2,3}$ is the matrix $[\nabla u]_B$. Since $|Q| = |\nabla u|$ and $|U| = |u|$, we obtain

$$|\nabla \times G(u)| \le Cr^{-1}|\nabla u| + Cr^{-2}|u| \le CR^{-1}|\nabla u| + CR^{-2}|u|$$

(with possible $C = \sqrt{2}$). We then follow the proofs of Lemma 2.3 and Theorem 2.1. ∎

Remark 4.2. In studies of ocean flows, R is considered to be very large and $R' = (1+\varepsilon)R$ with small $\varepsilon \in (0, 1]$, then $\Omega_{R,R'}$ is a thin shell Ω_R^ε. The constant C in (4.1) is independent of ε, that is, independent of the depth of the domain. In this case, Ineq. (4.1) becomes

$$\|Au + \Delta u\|_{L^2(\Omega_R^\varepsilon)} \le \delta(R)\|u\|_{H^1(\Omega_R^\varepsilon)},$$

where $\lim_{R\to\infty} \delta(R) = 0$.

Acknowledgment. The author is indebted to the related work in [1, 3]. He would like to thank George Sell for his helpful discussions.

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