BINOMIAL D-MODULES

By

Alicia Dickenstein

Laura Felicia Matusevich

and

Ezra Miller

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

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ALICIA DICKENSTEIN, LAURA FELICIA MATUSEVICH, AND EZRA MILLER

The authors dedicate this work to the memory of Karin Gatermann, friend and colleague

ABSTRACT. We study quotients of the Weyl algebra by left ideals whose generators consist of an arbitrary $\mathbb{Z}^d$-graded binomial ideal I in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ along with Euler operators defined by the grading and a parameter $\beta \in \mathbb{C}^d$. We determine the parameters β for which these D-modules (i) are holonomic (equivalently, regular holonomic, when I is standard-graded); (ii) decompose as direct sums indexed by the primary components of I; and (iii) have holonomic rank greater than the rank for generic β. In each of these three cases, the parameters in question are precisely those outside of a certain explicitly described affine subspace arrangement in $\mathbb{C}^d$. In the special case of Horn hypergeometric D-modules, when I is a lattice basis ideal, we furthermore compute the generic holonomic rank combinatorially and write down a basis of solutions in terms of associated A-hypergeometric functions. Fundamental in this study is an explicit lattice point description of the primary components of an arbitrary binomial ideal in characteristic zero, which we derive from a characteristic-free combinatorial result on binomial ideals in affine semigroup rings.

CONTENTS

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1. INTRODUCTION

1.1. **Hypergeometric series.** A univariate power series is *hypergeometric* if the successive ratios of its coefficients are given by a fixed rational function. These functions, and the elegant differential equations they satisfy, have proven ubiquitous in mathematics. As a small example of this phenomenon, consider the Hermite polynomials. These hypergeometric functions naturally occur, for instance, in physics (energy levels of the harmonic oscillator) [CDL77], numerical analysis (Gaussian quadrature) [SB02], combinatorics (matching polynomials of complete graphs) [God81], and probability (iterated Itô integrals of standard Wiener processes) [Itô51].

Perhaps the most natural definition of hypergeometric power series in several variables is the following, whose bivariate specialization was studied by Jakob Horn as early as 1889 [Hor1889]. More references include [Hor31], the first of six articles, all in Mathematische Annalen between 1931 and 1940, and all containing "Hypergeometrische Funktionen zweier Verändlichen" (hypergeometric functions in two variables) in their titles.

Definition 1.1. A formal series $F(z) = \sum_{\alpha\in\mathbb{N}^m} a_\alpha z_1^{\alpha_1}\cdots z_m^{\alpha_m}$ in m variables with complex coefficients is *hypergeometric in the sense of Horn* if there exist rational functions $r_1, r_2, \ldots, r_m$ in m variables such that

$$\frac{a_{\alpha+e_k}}{a_\alpha} = r_k(\alpha) \quad \text{for all } \alpha\in\mathbb{N}^m \text{ and } k = 1,\ldots,m. \tag{1.1}$$

Here we denote by $e_1, \ldots, e_m$ the standard basis vectors of $\mathbb{N}^m$.

Write the rational functions of the previous definition as

$$r_k(\alpha) = p_k(\alpha)/q_k(\alpha + e_k) \quad k = 1,\ldots m,$$

where p_k and q_k are relatively prime polynomials and z_k divides q_k.

Since $g(z_1\partial_{z_1}, \ldots, z_m\partial_{z_m})z^\alpha = g(\alpha_1,\ldots,\alpha_m)z^\alpha$ for all monomials z^α, the series F satisfies the following *Horn hypergeometric system of differential equations*:

$$q_k(z_1\partial_{z_1},\ldots,z_m\partial_{z_m})F(z) = z_k p_k(z_1\partial_{z_1},\ldots,z_m\partial_{z_m})F(z) \quad k = 1,\ldots,m. \tag{1.2}$$

Of particular interest are the series where the numerators and denominators of the rational functions r_k factor into products of linear factors. (Contrast with the notion of "proper hypergeometric term" in [PWZ96].) Notice that by the fundamental theorem of algebra, this is not restrictive when the number of variables is $m = 1$.

1.2. **Binomial ideals and binomial D-modules.** The central objects of study in this article are the *binomial D-modules*, to be introduced in Definition 1.3, which reformulate and generalize the classical Horn hypergeometric systems, as we shall see in Section 1.4. Our definition is based on the point of view developed by Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89], and contains their hypergeometric systems as special cases; see Section 1.3.

To construct a binomial D-module, the starting point is an integer matrix A, about which we wish to be consistent throughout.

Convention 1.2. $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $a_1, \ldots, a_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of A is pointed (contains no lines), and all of the a_i are nonzero. We also assume that $\mathbb{Z}A = \mathbb{Z}^d$; that is, the columns of A span $\mathbb{Z}^d$ as a lattice.

The reformulation of Horn systems in Section 1.4 proceeds by a change of variables, so we will use $x = x_1, \ldots, x_n$ and $\partial = \partial_1, \ldots, \partial_n$ (where $\partial_i = \partial_{x_i}$), instead of $z_1, \ldots, z_m$ and $\partial_{z_1}, \ldots, \partial_{z_m}$, whenever we work in the binomial setting. The matrix A induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n] = \mathbb{C}[\partial]$, which we call the A-*grading*, by setting $\deg(\partial_i) = -a_i$. An ideal of $\mathbb{C}[\partial]$ is A-*graded* if it is generated by elements that are homogeneous for the A-grading. For example, a *binomial ideal* is generated by *binomials* $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$; such an ideal is A-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$ (in particular, monomials are allowed as generators of binomial ideals). The hypotheses on A mean that the A-grading is a *positive* $\mathbb{Z}^d$*-grading* [MS05, Chapter 8].

The Weyl algebra $D = D_n$ of linear partial differential operators, written with the variables x and ∂, is also naturally A-graded by additionally setting $\deg(x_i) = a_i$. Consequently, the *Euler operators* in our next definition are A-homogeneous of degree 0.

Definition 1.3. For each $i \in \{1, \ldots, d\}$, the i^{th} *Euler operator* is

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n.$$

Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$. (The dependence of the Euler operators E_i on the matrix A is suppressed from the notation.)

For an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta\rangle$ in the Weyl algebra D. The *binomial* D*-module* associated to I is $D/H_A(I, \beta)$.

We will explain in Section 1.4 how Horn systems correspond to the binomial D-modules arising from a very special class of binomial ideals called *lattice basis ideals*.

Our goal for the rest of this Introduction (and indeed, the rest of the paper) is to demonstrate not merely that the definition of binomial D-modules can be made in this generality—and that it leads to meaningful theorems—but that it *must* be made, even if one is interested only in classical questions concerning Horn hypergeometric systems, which arise from lattice basis ideals. Furthermore, once the definition has been made, most of what we wish to prove about Horn hypergeometric systems generalizes to all binomial D-modules.

1.3. **Toric ideals and A-hypergeometric systems.** The fundamental examples of binomial D-modules, and the ones which our definition most directly generalizes, are the A-*hypergeometric systems* (or *GKZ hypergeometric systems*) of Gelfand, Kapranov, and Zelevinsky [GKZ89]. Given A as in Convention 1.2, these are the left D-ideals $H_A(I_A, \beta)$, also denoted by $H_A(\beta)$, where

$$I_A = \langle \partial^u - \partial^v : Au = Av\rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n] \tag{1.3}$$

is the *toric ideal* for the matrix A. The systems $H_A(\beta)$ have many applications; for example, they arise naturally in the moduli theory of Calabi-Yau toric varieties, and (therefore) they play an important role in applications of mirror symmetry in mathematical physics [BvS95, Hor99, Hos04, HLY96].

The ideal I_A is a prime A-graded binomial ideal, and the quotient ring $\mathbb{C}[\partial]/I_A$ is the semigroup ring for the affine semigroup $\mathbb{N}A$ generated by the columns of A. There is a rich theory of toric ideals, toric varieties, and affine semigroup rings, whose core philosophy is to exploit the connection between the algebra of the semigroup ring $\mathbb{C}[\partial]/I_A = \mathbb{C}[\mathbb{N}A]$ and the combinatorics of the semigroup $\mathbb{N}A$. In this way, algebro-geometric results on toric varieties can be obtained by combinatorial means, and purely combinatorial facts about polyhedral geometry can be proved using algebraic techniques. We direct the reader to the texts [Ful93, GKZ94, MS05] for more information.

Much is known about A-hypergeometric D-modules. They are holonomic for all parameters [GKZ89, Ado94], and they are regular holonomic exactly when I_A is $\mathbb{Z}$-graded in the usual sense [Hot91, SW06]. In this case, (Gamma-)series expansions for the solutions of $H_A(\beta)$ centered at the origin and convergent in certain domains can be explicitly computed [GKZ89, SST00]. The generic (minimal) holonomic rank is known to be $\mathrm{vol}(A)$, the normalized volume of the convex hull of the columns of A and the origin [GKZ89, Ado94], and holonomic rank is independent of the parameter β if and only if the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay [GKZ89, Ado94, MMW05]. We will extend all of these results, suitably modified, to the general setting of binomial D-modules. The important caveat is that a general binomial D-module can exhibit behavior that is forbidden to GKZ systems (see Example 1.8, for instance), so it is impossible for the extension to be entirely straightforward.

1.4. **Binomial Horn systems.** Classical Horn systems, which we are about to define precisely, were first studied by Appell [App1880], Mellin [Mel21], and Horn [Hor1889]. They directly generalize the univariate hypergeometric equations for the functions ${}_pF_q$; see [SK85, Sla66] and the references therein. As we mentioned earlier, our motivation to consider binomial D-modules is that they contain as special cases these classical Horn systems. The definition of these systems involves a matrix B about which, like the matrix A from Convention 1.2, we wish to be consistent throughout.

Convention 1.4. Let $B = (b_{jk}) \in \mathbb{Z}^{n\times m}$ be an integer matrix of full rank $m \leq n$. Assume that every nonzero element of the column-span of B over the integers $\mathbb{Z}$ is *mixed*, meaning that it has at least one positive and one negative entry; in particular, the columns of B are mixed. We write $b_1, \dots, b_n$ for the rows of B. Having chosen B, we set $d = n - m$ and pick a matrix $A \in \mathbb{Z}^{d\times n}$ whose columns span $\mathbb{Z}^d$ as a lattice, such that $AB = 0$. In the case that $d \neq 0$, the mixedness hypothesis on B is equivalent to the pointedness assumption for A that appears in Convention 1.2. We do allow $d = 0$, in which case A is the empty matrix.

Definition 1.5. For a matrix $B \in \mathbb{Z}^{n\times m}$ as in Convention 1.4 and a vector $c = (c_1, \dots, c_n)$ in $\mathbb{C}^n$, the *classical Horn system with parameter* c is the left ideal $\mathrm{Horn}(B, c)$ in the Weyl algebra D_m generated by the m differential operators

$$q_k(\theta_z) - z_k p_k(\theta_z), \quad k = 1, \dots, m,$$

where $\theta_z = (\theta_{z_1}, \ldots, \theta_{z_m})$, $\theta_{z_k} = z_k \partial_{z_k}$ $(1 \leq k \leq m)$, and

$$q_k(\theta_z) = \prod_{b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (b_j \cdot \theta_z + c_j - \ell) \quad \text{and} \quad p_k(\theta_z) = \prod_{b_{jk}<0} \prod_{\ell=0}^{|b_{jk}|-1} (b_j \cdot \theta_z + c_j - \ell).$$

Using ideas of Gelfand, Kapranov, and Zelevinsky, the classical Horn systems can be reinterpreted as the following binomial D-modules, with $\beta = Ac$.

Definition 1.6. Fix integer matrices B and A as in Convention 1.4, and let $I(B)$ be the *lattice basis ideal* corresponding to this matrix, that is, the ideal in $\mathbb{C}[\partial]$ generated by the binomials

$$\prod_{b_{jk}>0} \partial_{x_j}^{b_{jk}} - \prod_{b_{jk}<0} \partial_{x_j}^{-b_{jk}} \quad \text{for} \quad 1 \leq k \leq m.$$

The *binomial Horn system with parameter* β is the left ideal $H(B, \beta) = H_A(I(B), \beta)$ in the Weyl algebra $D = D_n$.

The classical-to-binomial transformation proceeds via the surjection

$$\begin{array}{rcl} (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^m \\ (x_1, \ldots, x_n) & \mapsto & x^B = (\prod_{j=1}^n x_j^{b_{j1}}, \ldots, \prod_{j=1}^n x_j^{b_{jm}}), \end{array} \tag{1.4}$$

where $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ is the group of nonzero complex numbers. A solution $f(z_1, \ldots, z_m)$ of the classical Horn system $\operatorname{Horn}(B, c)$ gives rise to a solution $x^c f(x^B)$ of the binomial Horn system $H(B, Ac)$. That this indeed defines a vector space isomorphism between the (local) solution spaces was proved in [DMS05, Section 5] for $n > m$ in the homogeneous case, where the column sums of B are zero, but the proofs (which are elementary calculations taking only a page) go through verbatim for $n \geq m$ in the inhomogeneous case.

The transformation $f(z) \mapsto x^c f(x^B)$ takes classical series solutions supported on $\mathbb{N}^m$ to Puiseux series solutions supported on the translate $c + \ker(A) \subseteq \mathbb{C}^n$ of the kernel of A in $\mathbb{Z}^n$. (Note that $\ker(A)$ contains the lattice $\mathbb{Z}B$ spanned by the columns of B as a finite index subgroup.) More precisely, the differential equations $E - \beta$, which geometrically impose torus-equivariance infinitesimally under the action of (the Lie algebra of) $\ker((\mathbb{C}^*)^n \to (\mathbb{C}^*)^m)$, result in series supported on $c + \ker(A)$, while the binomials in the lattice basis ideal $I(B) \subseteq H(B, Ac)$ impose hypergeometric constraints on the coefficients.

Although the isomorphism $f(z) \mapsto x^c f(x^B)$ is only at the level of local holomorphic solutions, not D-modules, it preserves many of the pertinent features, including the dimensions of the spaces of local holomorphic solutions and the structure of their series expansions. Therefore, although the classical Horn systems are our motivation, we take the binomial formulation as our starting point: no result in this article depends logically on the classical-to-binomial equivalence.

1.5. **Holomorphic solutions to Horn systems.** The binomial rephrasing of Horn systems led to formulas in [GGR92] for Gamma-series solutions via A-hypergeometric theory. However, Gamma-series need not span the space of local holomorphic solutions of $H(B, \beta)$ at a point of $\mathbb{C}^n$ that is nonsingular for $H(B, \beta)$, even in the simplest cases. The reason is that Gamma-series are *fully supported*: there is a cone of dimension m (the maximum possible) whose lattice points correspond to monomials with nonzero coefficients. Generally speaking, Horn systems in dimension $m \geq 2$ tend to have many series solutions without full support.

Example 1.7. In the course of studying one of Appell's systems of two hypergeometric equations in $m = 2$ variables, Arthur Erdélyi [Erd50] mentions a modified form of the following example. Given any $\beta \in \mathbb{C}^2$ and the two matrices

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

satisfying Convention 1.4, the Puiseux monomial $x_1^{\beta_1/3} x_4^{\beta_2/3}$ is a solution of $H(B, \beta)$.

A key feature of the above example is that the solutions without full support persist for arbitrary choices of the parameter vector β. The fact that this phenomenon occurs in much more generality—for arbitrary dimension $m \geq 2$, in particular—was realized only recently [DMS05]. And it is not the sole peculiarity that arises in dimension $m \geq 2$: in view of the transformation to binomial Horn systems in Section 1.4, the following demonstrates that classical Horn systems can exhibit poor behavior for badly chosen parameters.

Example 1.8. Consider

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$H(B, \beta) = \langle \partial_1\partial_3 - \partial_2, \partial_1\partial_4 - \partial_2 \rangle + \langle x_1\partial_1 - x_2\partial_2 - \beta_1, x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_2 \rangle.$$

If $\beta_1 = 0$, then any (local holomorphic) bivariate function $f(x_3, x_4)$ annihilated by the operator $x_3\partial_3 + x_4\partial_4 - \beta_2$ is a solution of $H(B, \beta)$. The space of such functions is infinite-dimensional; in fact, it has uncountable dimension, as it contains all monomials $x_3^{w_3} x_4^{w_4}$ with $w_3, w_4 \in \mathbb{C}$ and $w_3 + w_4 = \beta_2$.

Erdélyi's goal for his study of the Appell system was to give bases of solutions that converged in different regions of $\mathbb{C}^2$, eventually covering the whole space, just as Kummer had done for the Gauss hypergeometric equation more than a century before [Kum1836]. There has been extensive work since then (see [SK85] and its references) on convergence of more general hypergeometric functions in two and three variables. But already for the classical case of Horn systems, where the phenomena in Examples 1.7 and 1.8 are commonplace, Erdélyi's work raises a number of fundamental questions that remain largely open (partial answers in dimension $m = 2$ being known [DMS05]; see Remark 1.16). The purpose of this article is to answer the following completely and precisely.

Questions 1.9. *Fix B as in Convention 1.4, and consider the Horn systems determined by B.*

1. *For which parameters does the space of local holomorphic solutions around a nonsingular point have finite dimension as a complex vector space?*

2. *What is a combinatorial formula for the minimum such dimension, over all possible choices of parameters?*
3. *Which parameters are generic, in the sense that the minimum dimension is attained?*
4. *How do (the supports of) series solutions centered at the origin look, combinatorially?*

These questions make sense simultaneously for classical Horn systems and binomial Horn systems, since the answers are invariant under the classical-to-binomial transformation. That the questions also make sense for binomial D-modules is our point of departure, for they can be addressed in this generality using answers to the following.

Questions 1.9 (continued)**.** *Consider the binomial D-modules $H_A(I, \beta)$ for varying $\beta \in \mathbb{C}^d$.*

5. *When is $D/H_A(I, \beta)$ a holonomic D-module?*
6. *When is $D/H_A(I, \beta)$ a* regular *holonomic D-module?*

The phenomena underlying all of the answers to Questions 1.9 can be described in terms of lattice point geometry, as one might hope, owing to the nature of hypergeometric recursions as relations between coefficients on monomials. The lattice point geometry is elementary, in the sense that it only requires constructions involving cosets and equivalence relations in lattices. However, modern techniques are required to make the descriptions quantitatively accurate and to prove them. In particular, our progress applies two distinct and substantial steps: precise advances in the combinatorial commutative algebra of binomial ideals in semigroup rings, and a functorial translation of those advances into D-module theory.

1.6. **Combinatorial answers to hypergeometric questions.** The supports of the various series solutions to $H(B, \beta)$ centered at the origin are controlled by how effectively the columns of B join the lattice points in the positive orthant $\mathbb{N}^n$. In essence, this is because the coefficients on a pair of Puiseux monomials are related by the binomial equations in $I(B)$ when their exponent vectors in $c + \ker(A)$ differ by a column of B. This observation prompts us to construct an undirected graph on the nodes $\mathbb{N}^n$ with an edge between pairs of points differing by a column of B. Each connected component, or *B-subgraph of* $\mathbb{N}^n$, is contained in a single fiber $(a + \mathbb{Z}B) \cap \mathbb{N}^n$ of the projection $\mathbb{N}^n \to \mathbb{Z}^n/\mathbb{Z}B$.

The geometry of B-subgraphs generalizes to an arbitrary binomial ideal I, which determines a *congruence* as follows: $u \sim v$ if $\partial^u - \lambda\partial^v \in I$ for some $\lambda \neq 0$. This generalization is key, as it allows us the flexibility to work with the congruences determined by various ideals related to $I(B)$, which might not themselves be lattice basis ideals. For example, when the binomial ideal I is the toric ideal

$$I_A = \langle \partial^u - \partial^v : u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle \subseteq \mathbb{C}[\partial]$$

for the matrix A, the congruence class of $\alpha = Aa \in \mathbb{Z}A = \mathbb{Z}^d$ for $a \in \mathbb{N}^n$ consists of the lattice points in the polyhedron

$$P_\alpha = \{u \in \mathbb{R}^n : Au = \alpha \text{ and } u \geq 0\} = (a + \ker(A)) \cap \mathbb{N}^n.$$

With this picture in mind, the B-subgraphs in P_α, or the congruence classes for any binomial ideal I (homogeneous for A as in Definition 1.3), typically consist of one big continent in the interior of P_α plus a number of surrounding islands.

The extent to which $I(B)$ differs from I_A is measured by which bridges must be built—and in which directions—to join various islands to the continent. To this end, let $J \subseteq \{1,\dots,n\}$, and write $\mathbb{Z}^J = \{v \in \mathbb{Z}^n : v_i = 0 \text{ for all } i \notin J\}$. For $\bar{J} = \{1,\dots,n\} \smallsetminus J$, $\mathbb{N}^{\bar{J}}$ is defined in a similar manner. Suppose that $L \subseteq \mathbb{Z}^J$ is a saturated sublattice, so $\mathbb{Z}^J/L$ is torsion-free. Just as $I(B)$ determines a congruence on $\mathbb{N}^n$, it determines one on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$. The sublattice L determines a coarsening of this congruence, by allowing bridges from u to v if $u - v \in L$. Certain choices of $L \subseteq \mathbb{Z}^J$ satisfying $L \subseteq \ker(A)$ are *associated* to $I(B)$ (see Section 1.7), and for these, there exist coarsened classes in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ whose images in $(\mathbb{Z}^J/L) \times \mathbb{N}^{\bar{J}}$ are finite; let us call these classes L-*bounded*. Each L-bounded class in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ lies in a single coset of $\ker(A)$, since $L \subseteq \ker(A)$, so its image in $\mathbb{Z}^d = \mathbb{Z}^n/\ker(A)$ is a well-defined point.

The D-module theoretic consequences of L-bounded classes depend on a crucial distinction; see Definition 3.6 and Remark 3.11 for more precision and an etymology.

Definition 1.10. An associated saturated sublattice $L \subseteq \mathbb{Z}^J \cap \ker(A)$ is called *toral* if $L = \mathbb{Z}^J \cap \ker(A)$; otherwise, $L \subsetneq \mathbb{Z}^J \cap \ker(A)$ is called *Andean*.

Example 1.11. [Example 1.8 continued] With A and B as in Example 1.8, there are two associated lattices, one with $J = \{1,2,3,4\}$, the other with $J = \{3,4\}$. The first one is toral, while the second is Andean.

In what follows, A_J denotes the submatrix of A whose columns are indexed by J. We write $\mathbb{Z}A_J \subseteq \mathbb{Z}^d = \mathbb{Z}A$ for the group generated by these columns, and $\mathbb{C}A_J \subseteq \mathbb{C}^d$ for the vector subspace they generate.

Observation 1.12 (cf. Theorem 3.2 and Lemma 4.5)**.** The images of the L-bounded classes for all of the Andean associated sublattices $L \subseteq \mathbb{Z}^J$ comprise a finite union of cosets of $\mathbb{Z}A_J$. The union over all J of the corresponding cosets of $\mathbb{C}A_J$ is an affine subspace arrangement in $\mathbb{C}^d$ called the *Andean arrangement* (Definition 5.1 and Lemma 5.2).

Example 1.13. [Example 1.11 continued] The Andean arrangement in this case is

$$\mathbb{C}\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] = \left\{ \left[\begin{smallmatrix} 0 \\ \beta_2 \end{smallmatrix}\right] : \beta_2 \in \mathbb{C} \right\}.$$

As we have already checked, the Horn system in Example 1.8 fails to be holonomic for this set of parameters.

Observation 1.14 (cf. Corollary 3.13 and its proof)**.** A class in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by a toral associated sublattice $L \subseteq \mathbb{Z}^J$ is L-bounded if and only if its image in $\mathbb{N}^{\bar{J}}$ is bounded. If $\mathbb{C}A_J = \mathbb{C}^d$, then the number $\mu(L,J)$ of such bounded images in $\mathbb{N}^{\bar{J}}$ is finite.

Answers 1.15. *The answers to Questions 1.9, phrased in the language of binomial Horn systems $H(B,\beta)$, are as follows.*

1. *(Theorem 5.3) The dimension is finite exactly for $-\beta$ not in the Andean arrangement.*
2. *(Theorem 5.10) The generic (minimum) rank is $\sum \mu(L,J) \cdot \mathrm{vol}(A_J)$, the sum being over all toral associated sublattices with $\mathbb{C}A_J = \mathbb{C}^d$, where $\mathrm{vol}(A_J)$ is the volume of the convex hull of A_J and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume* 1.

3. *(Definition 5.9 and Theorem 5.10) The minimum rank is attained precisely when $-\beta$ lies outside of an affine subspace arrangement determined by certain local cohomology modules, with the same flavor as (and containing) the Andean arrangement.*
4. *(Theorem 5.10 and Theorem 6.10) For general β, the $\mu(L,J)\cdot\mathrm{vol}(A_J)$ many solutions are supported on the L-bounded classes, with hypergeometric recursions determining the coefficients. In the regular holonomic case, only $g\cdot\mathrm{vol}(A)$ many Γ-series solutions have full support, where $g = |\ker(A)/\mathbb{Z}B|$ is the index of $\mathbb{Z}B$ in its saturation $\ker(A)$.*
5. *(Theorem 5.3) Holonomicity is equivalent to the finite dimension in Answer 1.15.1.*
6. *(Theorem 5.3) Holonomicity is equivalent to regular holonomicity when I is standard $\mathbb{Z}$-graded—i.e., the row-span of A contains the vector $[1\cdots 1]$. Conversely, if there exists a parameter β for which $D/H_A(I,\beta)$ is regular holonomic, then I is $\mathbb{Z}$-graded.*

In Answer 1.15.4, the solutions for toral sublattices $L = \ker(A)\cap\mathbb{Z}^J$ in which J is a proper subset of $\{1,\ldots,n\}$ give rise to solutions that are bounded in the $\mathbb{N}^{\bar{J}}$ directions, and hence supported on sets of dimension $\mathrm{rank}(L) = |J|-d < n-d = m$. Answer 1.15.6 is, given the other results in this paper, an (easy) consequence of the (hard) holonomic regularity results of Hotta [Hot91] and Schulze–Walther [SW06]. Finally, let us note again that most of the theorems quoted in Answers 1.15 are stated and proved in the context of arbitrary binomial D-modules, not just Horn systems.

Remark 1.16. We concentrate on the special case of Horn systems in Section 6. The systematic study of binomial Horn systems was started in [DMS05] under the hypothesis that m (the number of columns of B) is equal to 2. Our results here are more general than those found in [DMS05] (as we treat all binomial D-modules, not just those arising from lattice basis ideals of codimension 2), more refined (we have completely explicit control over the parameters) and stronger (for instance, our direct sum results hold at the level of D-modules and not just local solution spaces). On the other hand, the generic holonomicity of classical Horn D-modules (Definition 1.5) for $m > 2$ remains unproven, the bivariate case having been treated in [DMS05].

Example 1.17. [Example 1.7, continued] There are two associated sublattices $L\subseteq\mathbb{Z}^J$ here, both toral, and both satisfying $\mathbb{C}A_J = \mathbb{C}^2$: the sublattice $\ker(A)\subseteq\mathbb{Z}^4$, where $J = \{1,2,3,4\}$, and the sublattice $\mathbf{0}\subseteq\mathbb{Z}^J$ for $J=\{1,4\}$. Both of the multiplicities $\mu\big(\ker(A),\{1,2,3,4\}\big)$ and $\mu\big(\mathbf{0},\{1,4\}\big)$ equal 1, while $\mathrm{vol}(A)=3$ and $\mathrm{vol}(A_{\{1,4\}})=1$, the latter because $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$ form a basis for the lattice they generate. Hence there are four solutions in total, three of them with full support and one—namely the Puiseux monomial in Example 1.7—with support of dimension zero. See Example 1.18 for an (easy!) computation of these associated lattices and their multiplicities.

Throughout this article we will make repeated use of two quite different tools. The first is a description of the aforementioned associated lattices, which we develop in Sections 2 and 3. The second comes from A-hypergeometric theory [MMW05], and is homological in nature. We close this Introduction with a discussion of these tools.

1.7. **Binomial primary decomposition.** The geometry and combinatorics of lattice point congruences control the primary decomposition of arbitrary binomial ideals in characteristic zero; this is the content of Sections 2 and 3, particularly Theorem 2.14, Theorem 3.2, and Corollary 3.13. These sections are developed for all binomial ideals, instead of only for lattice basis ideals, because the binomial ideals arising naturally in the process of carrying out primary decompositions are sufficiently arbitrary that the general case contributes conceptual clarity without presenting additional obstacles. The developments here can be seen as a combinatorial refinement of the binomial primary decomposition theorem of Eisenbud and Sturmfels [ES96].

Our combinatorial study of binomial primary decomposition results in a natural language for quantifying which sublattices are associated, which cosets appear in Observation 1.12, and which bounded images appear in Observation 1.14. To be precise, a binomial prime ideal $I_{\rho,J}$ in $\mathbb{C}[\partial_1,\dots,\partial_n]$ is determined by a subset $J \subseteq \{1,\dots,n\}$ and a character $\rho : L \to \mathbb{C}^*$ for some sublattice $L \subseteq \mathbb{Z}^J$. The sublattice $L \subseteq \mathbb{Z}^J$ is associated, in the language of Section 1.6, when $I_{\rho,J}$ is associated to I in the usual commutative algebra sense, and the multiplicity $\mu(L,J)$ in Observation 1.14 is the commutative algebra multiplicity of $I_{\rho,J}$ in I.

Example 1.18. [Example 1.17, continued] The binomial Horn system is

$$H(B,\beta) = I(B) + \langle 3x_1\partial_1 + 2x_2\partial_2 + x_3\partial_3 - \beta_1, x_2\partial_2 + 2x_3\partial_3 + 3x_4\partial_4 - \beta_2\rangle \subseteq D_4.$$

The primary decomposition of the lattice basis ideal $I(B)$ in $\mathbb{C}[\partial_1,\partial_2,\partial_3,\partial_4]$ is

$$I(B) = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2\rangle = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle \cap \langle \partial_2, \partial_3\rangle.$$

The first of these components is the toric ideal $I_A = I_{\rho,J}$ of the twisted cubic curve, where $\rho : \ker(A) = \mathbb{Z}B \to \mathbb{C}^*$ is the trivial character and $J = \{1,2,3,4\}$. The ideal $\langle \partial_2, \partial_3\rangle$ is the binomial prime ideal $I_{\rho,J}$ for the (automatically) trivial character $\rho : \mathbf{0} \to \mathbb{C}^*$ and the subset $J = \{1,4\}$. Both of these ideals have multiplicity 1 in $I(B)$, which is a radical ideal. This explains the associated lattices and multiplicities in Example 1.17.

For a note on motivation, this project began with the conjectural statement of Theorem 6.10 (Answer 1.15.4), which we concluded must hold because of evidence derived from our knowledge of series solutions. Its proof reduced quickly to the statement of Example 3.14, which directed all of the developments in the rest of the paper. Our consequent use of B-subgraphs, and more generally the application of congruences toward the primary decomposition of binomial ideals, serves as an advertisement for hypergeometric intuition as inspiration for developments of independent interest in combinatorics and commutative algebra. That being said, the reader interested primarily (or solely) in combinatorics and commutative algebra should note that Sections 2 and 3, although inspired by hypergeometric ideas, are self-contained and do not involve D-modules.

1.8. **Euler-Koszul homology.** Binomial primary decomposition is not only the natural language for lattice point geometry, it is the reason why lattice point geometry governs the D-module theoretic properties of binomial D-modules. This we demonstrate by functorially translating the commutative algebra of A-graded primary decomposition directly into the D-module setting. The functor we employ is Euler-Koszul homology (see the opening of Section 4 for background and references), which allows us to pull apart the primary components of binomial ideals, thereby isolating the contribution of each to the solutions of the corresponding binomial D-module. Here we see again the need to work with general binomial D-modules: primary components of lattice basis ideals, and intersections of various collections of them, are more or less arbitrary A-homogeneous binomial ideals.

We stress at this point that the combinatorial geometric lattice-point description of binomial primary decomposition is a crucial prerequisite for the effective translation into the realm of D-modules. Indeed, semigroup gradings pervade the arguments demonstrating the fundamentally holonomic behavior of Euler-Koszul homology for toral modules (Theorem 4.12) and its resolutely non-holonomic behavior for Andean modules (Corollary 4.22). This is borne out in Examples 4.9 and 4.18, which say that quotients by primary ideals are either toral or Andean as $\mathbb{C}[\partial]$-modules, thus constituting the bridge from the commutative binomial theory in Sections 2 and 3 to the binomial D-module theory in Sections 4 and 5. Taming the homological (holonomic) and structural properties of binomial D-modules in Theorems 5.3, 5.8, and 5.10—which, together with Theorem 6.10 on series bases, form our core results—also rests squarely on having tight control over the interactions of primary decomposition with various semigroup gradings of the polynomial ring. The underlying phenomenon is thus:

Central principle. Just as toric ideals are the building blocks of binomial ideals, A-hypergeometric systems are the building blocks of binomial D-modules.

As a final indication of how structural results for binomial D-modules have concrete combinatorial implications for Horn hypergeometric systems, let us see how the primary decomposition in Example 1.18 results in the combinatorial multiplicity formula (Answer 1.15.2) for the holonomic rank at generic parameters β. The general result to which we appeal is Theorem 5.8: for generic parameters β, the binomial D-module $D/H_A(I,\beta)$ decomposes as a direct sum over the toral primary components of I.

Example 1.19. [Example 1.18, continued] The intersection in $\mathbb{C}^4 = \operatorname{Spec}(\mathbb{C}[\partial_1,\dots,\partial_4])$ of the two irreducible varieties in the zero set of $I(B)$ is the zero set of

$$\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle + \langle \partial_2,\partial_3\rangle = \langle \partial_1\partial_4, \partial_2,\partial_3\rangle.$$

The primary arrangement in Theorem 5.8 is, in this case, the line in $\mathbb{C}^2$ spanned by $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ union the line in $\mathbb{C}^2$ spanned by $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$. When β lies off the union of these two lines, Theorem 5.8 yields an isomorphism of D_4-modules:

$$\frac{D_4}{H(B,\beta)} \cong \frac{D_4}{\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle + \langle E-\beta\rangle} \oplus \frac{D_4}{\langle \partial_2,\partial_3\rangle + \langle E-\beta\rangle}.$$

The summands on the right-hand side are GKZ hypergeometric systems (up to extraneous vanishing variables in the $\langle \partial_2,\partial_3\rangle$ case) with holonomic ranks 3 and 1, respectively.

We conclude this introduction with some general background on D-modules. A left D-ideal $\mathcal{I}$ is *holonomic* if its characteristic variety has dimension n. Holonomicity has strong homological implications, making the class of holonomic D-modules a natural one to study. If $\mathcal{I}$ is holonomic, its *holonomic rank*, i.e. the dimension of the space of solutions of the D-ideal $\mathcal{I}$ that are holomorphic in a sufficiently small neighborhood of a point outside the singular locus, is finite (the converse of this result is not true). We refer to the texts [Bor87, Cou95, SST00] for introductory overviews of the theory of D-modules; we point out that the exposition in [SST00] is geared toward algorithms and computations. A treatment of D-modules with *regular singularities* can be found in [Bjö79, Bjö93].

2. Binomial ideals in affine semigroup rings

Our first eventual goal is to analyze the primary components of binomial ideals in polynomial rings over $\mathbb{C}$. As it turns out, our principal result along these lines (Theorem 3.2) is little more than a rephrasing of a statement (Theorem 2.14) about binomial ideals in arbitrary affine semigroup rings in which the associated prime comes from a face, combined with results of Eisenbud and Sturmfels [ES96]. The developments here stem from the observation that quotients by binomial ideals are naturally graded by noetherian commutative monoids. Our source for such monoids is the excellent book by Gilmer [Gil84]. For the special case of affine semigroups (finitely generated submonoids of free abelian groups), see [MS05, Chapter 7]. We work in this section over an arbitrary field $\Bbbk$.

Definition 2.1. A *congruence* on a commutative monoid Q is an equivalence relation $\sim$ with

$$u \sim v \implies u+w \sim v+w \quad \text{for all } w \in Q.$$

The *quotient monoid* $Q/{\sim}$ is the set of equivalence classes under addition.

Definition 2.2. The *semigroup algebra* $\Bbbk[Q]$ is the direct sum $\bigoplus_{u\in Q} \Bbbk \cdot \partial^u$, with multiplication $\partial^u\partial^v = \partial^{u+v}$. Any congruence $\sim$ on Q induces a $(Q/{\sim})$-grading on $\Bbbk[Q]$ in which the *monomial* ∂^u has degree $\Gamma \in Q/{\sim}$ whenever $u \in \Gamma$. A *binomial ideal* $I \subseteq \Bbbk[Q]$ is an ideal generated by *binomials* $\partial^u - \lambda\partial^v$, where $\lambda \in \Bbbk$ is a scalar, possibly equal to zero.

Example 2.3. A *pure difference* binomial ideal is generated by differences of monic monomials. If M is an integer matrix with q rows, for instance, set

$$\begin{aligned} I(M) &= \langle \partial^u - \partial^v : u - v \text{ is a column of } M,\ u, v \in \mathbb{N}^n\rangle \\ &= \langle \partial^{w_+} - \partial^{w_-} : w = w_+ - w_- \text{ is a column of } M\rangle \subseteq \Bbbk[\partial_1, \ldots, \partial_q] = \Bbbk[\mathbb{N}^q]. \end{aligned} \tag{2.1}$$

Here and in the remainder of this article we adopt the convention that, for an integer vector $w \in \mathbb{Z}^q$, w_+ is the vector whose i^{th} coordinate is w_i if $w_i \geq 0$ and 0 otherwise. The vector $w_- \in \mathbb{N}^q$ is defined via $w_- = w_+ - w$.

The first line (2.1) contains the second by definition; and the disjointness of the supports of w_+ and w_- implies that $\alpha = u - w_+ = v - w_-$ lies in $\mathbb{N}^q$ whenever $u - v = w$ is a column of M, so $\partial^u - \partial^v = \partial^\alpha(\partial^{w_+} - \partial^{w_-})$ lies in the second line whenever $\partial^u - \partial^v$ is a generator from the first line. Note that an ideal of $\Bbbk[\partial_1, \ldots, \partial_q]$ has the form described in (2.1) if and only if it is generated by differences of monomials with disjoint support. In particular, $I(B)$ is simply the lattice basis ideal for B if the matrix B has linearly independent columns.

Proposition 2.4. *A binomial ideal $I \subseteq \Bbbk[Q]$ determines a congruence $\sim_I$ under which*

$$u \sim_I v \text{ if } \partial^u - \lambda\partial^v \in I \text{ for some scalar } \lambda \neq 0.$$

The ideal I is graded by the quotient monoid $Q_I = Q/{\sim_I}$, and $\Bbbk[Q]/I$ has Q_I-graded Hilbert function 1 on every congruence class except the class $\{u \in Q : \partial^u \in I\}$ of monomials.

Proof. That $\sim_I$ is an equivalence relation is because $\partial^u - \lambda\partial^v \in I$ and $\partial^v - \lambda'\partial^w \in I$ implies $\partial^u - \lambda\lambda'\partial^w \in I$. It is a congruence because $\partial^u - \lambda\partial^v \in I$ implies that $\partial^{u+w} - \lambda\partial^{v+w} \in I$. The rest is similarly straightforward. □

Example 2.5. In the case of a pure difference binomial ideal $I(M)$ as in Example 2.3, the congruence classes under $\sim_{I(M)}$ from Proposition 2.4 are the M-subgraphs in the following definition, which will be useful later on (Examples 2.11 and 3.14, as well as much of Section 6).

Definition 2.6. Any integer matrix M with q rows defines an undirected graph $\Gamma(M)$ having vertex set $\mathbb{N}^q$ and an edge from u to v if $u - v$ or $v - u$ is a column of M. An M-*path* from u to v is a path in $\Gamma(M)$ from u to v. A subset of $\mathbb{N}^q$ is M-*connected* if every pair of vertices therein is joined by an M-path passing only through vertices in the subset. An M-*subgraph* of $\mathbb{N}^q$ is a maximal M-connected subset of $\mathbb{N}^q$ (a connected component of $\Gamma(M)$). An M-subgraph is *bounded* if it has finitely many vertices, and *unbounded* otherwise. (See Example 6.6 for a concrete computation.)

These M-subgraphs bear a marked resemblance to the concept of *fiber* in [Stu96, Chapter 4]. The interested reader will note, however, that even if these two notions have the same flavor, their definitions have mutually exclusive assumptions, since for a square matrix M, the corresponding matrix A is empty.

Given a face Φ of an affine semigroup $Q \subseteq \mathbb{Z}^\ell$, the *localization* of Q along Φ is the affine semigroup $Q + \mathbb{Z}\Phi$ obtained from Q by adjoining negatives of the elements in Φ. The algebraic version of this notion is a common tool for affine semigroup rings [MS05, Chapter 7]: for each $\Bbbk[Q]$-module V, let $V[\mathbb{Z}\Phi]$ denote its *homogeneous localization* along Φ, obtained by inverting ∂^ϕ for all $\phi \in \Phi$. For example, $\Bbbk[Q][\mathbb{Z}\Phi] \cong \Bbbk[Q + \mathbb{Z}\Phi]$. Writing

$$\mathfrak{p}_\Phi = \mathrm{span}_\Bbbk\{\partial^u : u \in Q \smallsetminus \Phi\} \subseteq \Bbbk[Q]$$

for the prime ideal of the face Φ, so that $\Bbbk[Q]/\mathfrak{p}_\Phi = \Bbbk[\Phi]$ is the affine semigroup ring for Φ, we find, as a consequence, that $\mathfrak{p}_\Phi[\mathbb{Z}\Phi] = \mathfrak{p}_{\mathbb{Z}\Phi} \subseteq \Bbbk[Q + \mathbb{Z}\Phi]$, because

$$\Bbbk[\mathbb{Z}\Phi] = \Bbbk[\Phi][\mathbb{Z}\Phi] = (\Bbbk[Q]/\mathfrak{p}_\Phi)[\mathbb{Z}\Phi] = \Bbbk[Q + \mathbb{Z}\Phi]/\mathfrak{p}_\Phi[\mathbb{Z}\Phi].$$

(We write equality signs to denote canonical isomorphisms.) For any ideal $I \subseteq \Bbbk[Q]$, the localization $I[\mathbb{Z}\Phi]$ equals the extension $I\,\Bbbk[Q + \mathbb{Z}\Phi]$ of I to $\Bbbk[Q + \mathbb{Z}\Phi]$, and we write

$$(I : \partial^\Phi) = I[\mathbb{Z}\Phi] \cap \Bbbk[Q], \tag{2.2}$$

the intersection taking place in $\Bbbk[Q + \mathbb{Z}\Phi]$. Equivalently, $(I : \partial^\Phi)$ is the usual colon ideal $(I : \partial^\phi)$ for any element ϕ sufficiently interior to Φ (for example, take ϕ to be a high multiple of the sum of the generators of Φ); in particular, $(I : \partial^\Phi)$ is a binomial ideal.

For the purpose of investigating $\mathfrak{p}_\Phi$-primary components, the ideal $(I : \partial^\Phi)$ is as good as I itself, since this colon operation does not affect such components, or better, since the natural map from $\Bbbk[Q]/(I : \partial^\Phi)$ to its homogeneous localization along Φ is injective. Combinatorially, what this means is the following.

Lemma 2.7. *A subset $\Gamma' \subseteq Q$ is a congruence class in $Q_{(I:\partial^\Phi)}$ determined by $(I : \partial^\Phi)$ if and only if $\Gamma' = \Gamma \cap Q$ for some class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under the congruence $\sim_{I[\mathbb{Z}\Phi]}$.* □

Lemma 2.8. *If a congruence class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under $\sim_{I[\mathbb{Z}\Phi]}$ has two distinct elements whose difference lies in $Q + \mathbb{Z}\Phi$, then for all $u \in \Gamma$ the monomial ∂^u maps to 0 in the (usual inhomogeneous) localization $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$ inverting all elements not in $\mathfrak{p}_\Phi$.*

Proof. Suppose $v \neq w \in \Gamma$ with $w - v \in Q + \mathbb{Z}\Phi$. The images in $\Bbbk[Q]/I$ of the monomials ∂^u for $u \in \Gamma$ are nonzero scalar multiples of each other, so it is enough to show that ∂^v maps to zero in $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. Since $w - v \in Q + \mathbb{Z}\Phi$, we have $\partial^{w-v} \in \Bbbk[Q + \mathbb{Z}\Phi]$. Therefore $1 - \lambda\partial^{w-v}$ lies outside of $\mathfrak{p}_{\mathbb{Z}\Phi}$ for all $\lambda \in \Bbbk$, because its image in $\Bbbk[\mathbb{Z}\Phi] = \Bbbk[Q + \mathbb{Z}\Phi]/\mathfrak{p}_{\mathbb{Z}\Phi}$ is either $1 - \lambda\partial^{w-v}$ or 1, according to whether or not $w - v \in \mathbb{Z}\Phi$. (The assumption $v \neq w$ was used here: if $v = w$, then for $\lambda = 1$, $1 - \lambda\partial^{w-v} = 0$.) Hence $1 - \lambda\partial^{w-v}$ maps to a unit in $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. It follows that ∂^v maps to 0, since $(1 - \lambda_{vw}\partial^{w-v})\partial^v = \partial^v - \lambda_{vw}\partial^w$ maps to 0 in $\Bbbk[Q]/I$ whenever $\partial^v - \lambda_{vw}\partial^w \in I$. □

Lemma 2.9. *A congruence class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under $\sim_{I[\mathbb{Z}\Phi]}$ is infinite if and only if it contains two distinct elements whose difference lies in $Q + \mathbb{Z}\Phi$.*

Proof. Let $\Gamma \subseteq Q + \mathbb{Z}\Phi$ be a congruence class. If $v, w \in \Gamma$ and $v - w \in Q + \mathbb{Z}\Phi$, then $w + \epsilon(v - w) \in \Gamma$ for all positive $\epsilon \in \mathbb{Z}$. On the other hand, assume Γ is infinite. There are two possibilities: either there are $v, w \in \Gamma$ with $v - w \in \mathbb{Z}\Phi$, or not. If so, then we are done, so assume not. Let $\mathbb{Z}^q$ be the quotient of $\mathbb{Z}^\ell/\mathbb{Z}\Phi$ modulo its torsion subgroup. (Here $\mathbb{Z}^\ell$ is the ambient lattice of Q.) The projection $\mathbb{Z}^\ell \to \mathbb{Z}^q$ induces a map from Γ to its image $\overline{\Gamma}$ that is finite-to-one. More precisely, if Γ' is the intersection of Γ with a coset of $\mathbb{Z}\Phi$ in $\ker(\mathbb{Z}^\ell \to \mathbb{Z}^q)$, then Γ' maps bijectively to its image $\overline{\Gamma}'$. There are only finitely many cosets, so some Γ' must be infinite, along with $\overline{\Gamma}'$. But $\overline{\Gamma}'$ is a subset of the affine semigroup $\overline{Q/\Phi}$, defined as the image of $Q + \mathbb{Z}\Phi$ in $\mathbb{Z}^q$. As $\overline{Q/\Phi}$ has unit group zero, every infinite subset contains two points whose difference lies in $\overline{Q/\Phi}$, and the corresponding lifts of these to Γ' have their difference in $Q + \mathbb{Z}\Phi$. □

Definition 2.10. Fix a face Φ of an affine semigroup Q. A subset $S \subseteq Q$ is an *ideal* if $Q + S \subseteq S$, and in that case we write $\Bbbk\{S\} = \langle \partial^u : u \in S\rangle = \mathrm{span}_\Bbbk\{\partial^u : u \in S\}$ for the monomial ideal in $\Bbbk[Q]$ having S as its $\Bbbk$-basis. An ideal S is $\mathbb{Z}\Phi$*-closed* if $S = Q \cap (S + \mathbb{Z}\Phi)$. If $\sim$ is a congruence on $Q + \mathbb{Z}\Phi$, then the *unbounded ideal* $U \subseteq Q$ is the ($\mathbb{Z}\Phi$-closed) ideal of elements $u \in Q$ whose congruence classes in $Q + \mathbb{Z}\Phi$ under $\sim$ are infinite. Finally, write $\mathcal{B}(Q + \mathbb{Z}\Phi)$ for the set of bounded (i.e., finite) congruence classes of $Q + \mathbb{Z}\Phi$ under $\sim$.

Example 2.11. Let M be as in Definition 2.6 and consider the congruence $\sim_{I(M)}$ on $Q = \mathbb{N}^q$. If $\Phi = \{0\}$, then the unbounded ideal $U \subseteq \mathbb{N}^q$ is the union of the unbounded M-subgraphs of $\mathbb{N}^q$, while $\mathcal{B}(\mathbb{N}^q)$ is the union of the bounded M-subgraphs.

Proposition 2.12. *Fix a face Φ of an affine semigroup Q, a binomial ideal $I \subseteq \Bbbk[Q]$, and a $\mathbb{Z}\Phi$-closed ideal $S \subseteq Q$ containing U under the congruence $\sim_{I[\mathbb{Z}\Phi]}$. Write $\mathcal{B} = \mathcal{B}(Q + \mathbb{Z}\Phi)$ for the bounded classes, J for the binomial ideal $(I : \partial^\Phi) + \Bbbk\{S\}$, and $\overline{Q} = (Q + \mathbb{Z}\Phi)_{I[\mathbb{Z}\Phi]}$.*

1. *$\Bbbk[Q]/J$ is graded by $\overline{Q}$, and its set of nonzero degrees is contained in $\mathcal{B}$.*
2. *The group $\mathbb{Z}\Phi \subseteq \overline{Q}$ acts freely on $\mathcal{B}$, and the $\Bbbk[\Phi]$-submodule $(\Bbbk[Q]/J)_T \subseteq \Bbbk[Q]/J$ in degrees from any orbit $T \subseteq \mathcal{B}$ is 0 or finitely generated and torsion-free of rank 1.*

3. *The quotient Q_J/Φ of the monoid $(Q+\mathbb{Z}\Phi)_{J[\mathbb{Z}\Phi]}$ by its subgroup $\mathbb{Z}\Phi$ is a partially ordered set if we define $\zeta \preceq \eta$ whenever $\zeta+\xi=\eta$ for some $\xi \in Q_J/\Phi$.*
4. *$\Bbbk[Q]/J$ is filtered by $\overline{Q}$-graded $\Bbbk[Q]$-submodules with associated graded module*
$$\mathrm{gr}(\Bbbk[Q]/J) = \bigoplus_{T\in\mathcal{B}/\Phi} (\Bbbk[Q]/J)_T, \quad \textit{where } \mathcal{B}/\Phi = \{\mathbb{Z}\Phi\text{-orbits } T \subseteq \mathcal{B}\},$$
the canonical isomorphism being as $\mathcal{B}$-graded $\Bbbk[\Phi]$-modules, although the left-hand side is naturally a $\Bbbk[Q]$-module annihilated by $\mathfrak{p}_\Phi$.
5. *If $(\Bbbk[Q]/J)_T \neq 0$ for only finitely many orbits $T\in\mathcal{B}/\Phi$, then J is a $\mathfrak{p}_\Phi$-primary ideal.*

Proof. The quotient $\Bbbk[Q]/(I : \partial^\Phi)$ is automatically $\overline{Q}$-graded by Proposition 2.4 applied to $Q+\mathbb{Z}\Phi$ and $I[\mathbb{Z}\Phi]$, given (2.2). The further quotient by $\Bbbk\{S\}$ is graded by $\mathcal{B}$ because $S \supseteq U$.

$\mathbb{Z}\Phi$ acts freely on $\mathcal{B}$ by Lemmas 2.8 and 2.9: if $\phi\in\mathbb{Z}\Phi$ and Γ is a bounded congruence class, then the translate $\phi+\Gamma$ is, as well; and if $\phi\neq 0$ then $\phi+\Gamma\neq\Gamma$, because each coset of $\mathbb{Z}\Phi$ intersects Γ at most once. Combined with the $\mathbb{Z}\Phi$-closedness of S, this shows that $\Bbbk[Q]/J$ is a $\Bbbk[\Phi]$-submodule of the free $\Bbbk[\mathbb{Z}\Phi]$-module whose basis consists of the $\mathbb{Z}\Phi$-orbits $T\subseteq\mathcal{B}$. Hence $(\Bbbk[Q]/J)_T$ is torsion-free (it might be zero, of course, if S happens to contain all of the monomials corresponding to congruence classes of Q arising from $\sim_{I[\mathbb{Z}\Phi]}$ classes in T). For item 2, it remains to show that $(\Bbbk[Q]/J)_T$ is finitely generated. Let $\mathcal{T}=\bigcup_{\Gamma\in T}\Gamma\cap Q$. By construction, $\mathcal{T}$ is the (finite) union of the intersections $Q\cap(\gamma+\mathbb{Z}\Phi)$ of Q with cosets of $\mathbb{Z}\Phi$ in $\mathbb{Z}^\ell$ for γ in any fixed $\Gamma\in T$. Such an intersection is a finitely generated Φ-set (a set closed under addition by Φ) by [Mil02a, Eq. (1) and Lemma 2.2] or [MS05, Theorem 11.13], where the $\Bbbk$-vector space it spans is identified as the set of monomials annihilated by $\Bbbk[\Phi]$ modulo an irreducible monomial ideal of $\Bbbk[Q]$. The images in $\Bbbk[Q]/J$ of the monomials corresponding to any generators for these Φ-sets generate $(\Bbbk[Q]/J)_T$.

The point of item 3 is that the monoid Q_J/Φ acts sufficiently like an affine semigroup whose only unit is the trivial one. To prove it, observe that Q_J/Φ consists, by item 1, of the (possibly empty set of) orbits $T\in\mathcal{B}$ such that $(\Bbbk[Q]/J)_T\neq 0$ plus one congruence class $\overline{S}$ for the monomials in J (if there are any). The proposed partial order has $T\prec\overline{S}$ for all orbits $T\in Q_J/\Phi$, and also $T\prec T+v$ if and only if $v\in(Q+\mathbb{Z}\Phi)\smallsetminus\mathbb{Z}\Phi$. This relation $\prec$ a priori defines a directed graph with vertex set Q_J/Φ, and we need it to have no directed cycles. The terminal nature of $\overline{S}$ implies that no cycle can contain $\overline{S}$, so suppose that $T=T+v$. For some $\phi\in\mathbb{Z}\Phi$, the translate $u+\phi$ lies in the same congruence class under $\sim_{I[\mathbb{Z}\Phi]}$ as $u+v$. Lemma 2.9 implies that $v-\phi$, and hence v itself, does not lie in $Q+\mathbb{Z}\Phi$.

For item 4, it suffices to find a total order $T_0, T_1, T_2, \ldots$ on $\mathcal{B}/\Phi$ such that $\bigoplus_{j\geq k}(\Bbbk[Q]/J)_{T_j}$ is a $\Bbbk[Q]$-submodule for all $k\in\mathbb{N}$. Use the partial order of $\mathcal{B}/\Phi$ via its inclusion in the monoid Q_J/Φ in item 3 for $S=U$. Any well-order refining this partial order will do.

Item 5 follows from items 2 and 4 because the associated primes of $\mathrm{gr}(\Bbbk[Q]/J)$ contain every associated prime of J for any finite filtration of $\Bbbk[Q]/J$ by $\Bbbk[Q]$-submodules. $\square$

For connections with toral and Andean modules (Definition 3.6), we record the following.

Corollary 2.13. *Fix notation as in Proposition 2.12. If I is homogeneous for a grading of $\Bbbk[Q]$ by a group $\mathcal{A}$ via a monoid morphism $Q \to \mathcal{A}$, then $\Bbbk[Q]/J$ and $\mathrm{gr}(\Bbbk[Q]/J)$ are $\mathcal{A}$-graded via a natural coarsening $\mathcal{B} \to \mathcal{A}$ that restricts to a group homomorphism $\mathbb{Z}\Phi \to \mathcal{A}$.*

Proof. The morphism $Q \to \mathcal{A}$ induces a morphism $\pi_{\mathcal{A}} : Q + \mathbb{Z}\Phi \to \mathcal{A}$ by the universal property of monoid localization. The morphism $\pi_{\mathcal{A}}$ is constant on the non-monomial congruence classes in Q_I precisely because I is $\mathcal{A}$-graded. It follows that $\pi_{\mathcal{A}}$ is constant on the non-monomial congruence classes in $(Q + \mathbb{Z}\Phi)_{I[\mathbb{Z}\Phi]}$. In particular, $\pi_{\mathcal{A}}$ is constant on the bounded classes $\mathcal{B}(Q + \mathbb{Z}\Phi)$, which therefore map to $\mathcal{A}$ to yield the natural coarsening. The group homomorphism $\mathbb{Z}\Phi \to \mathcal{A}$ is induced by the composite morphism $\mathbb{Z}\Phi \to (Q + \mathbb{Z}\Phi) \to \mathcal{A}$, which identifies the group $\mathbb{Z}\Phi$ with the $\mathbb{Z}\Phi$-orbit in $\mathcal{B}$ containing (the class of) 0. $\square$

Theorem 2.14. *Fix a face Φ of an affine semigroup Q and a binomial ideal $I \subseteq \Bbbk[Q]$. If $\mathfrak{p}_\Phi$ is minimal over I, then the $\mathfrak{p}_\Phi$-primary component of I is $(I : \partial^\Phi) + \Bbbk\{U\}$, where $(I : \partial^\Phi)$ is the binomial ideal (2.2) and $U \subseteq Q$ is the unbounded ideal (Definition 2.10) for $\sim_{I[\mathbb{Z}\Phi]}$. Furthermore, the only monomials in $(I : \partial^\Phi) + \Bbbk\{U\}$ are those of the form ∂^u for $u \in U$.*

Proof. The $\mathfrak{p}_\Phi$-primary component of I is the kernel of the localization homomorphism $\Bbbk[Q] \to (\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. As this factors through the homogeneous localization $\Bbbk[Q+\mathbb{Z}\Phi]/I[\mathbb{Z}\Phi]$, we find that $(I : \partial^\Phi)$ contains the kernel. Lemmas 2.8 and 2.9 imply that the kernel contains $\Bbbk\{U\}$. But already $(I : \partial^\Phi) + \Bbbk\{U\}$ is $\mathfrak{p}_\Phi$-primary by Proposition 2.12, so the quotient of $\Bbbk[Q]$ by it maps injectively to its localization at $\mathfrak{p}_\Phi$. To prove the last sentence of the theorem, observe that under the $\overline{Q}$-grading from Proposition 2.12.1, every monomial ∂^u outside of $\Bbbk\{U\}$ maps to a $\Bbbk$-vector space basis for the (1-dimensional) graded piece corresponding to the bounded congruence class containing u. $\square$

Example 2.15. One might hope that when $\mathfrak{p}_\Phi$ is an embedded prime of a binomial ideal I, the $\mathfrak{p}_\Phi$-primary components, or even perhaps the irreducible components, would be unique, if we require that they be finely graded (Hilbert function 0 or 1) as in Proposition 2.12. However, this fails even in simple examples, such as $\Bbbk[x,y]/\langle x^2 - xy, xy - y^2\rangle$. In this case, $I = \langle x^2 - xy, xy - y^2\rangle = \langle x^2, y\rangle \cap \langle x - y\rangle = \langle x, y^2\rangle \cap \langle x - y\rangle$ and Φ is the face $\{0\}$ of $Q = \mathbb{N}^2$, so that $I = (I : \partial^\Phi)$ by definition. The monoid Q_I, written multiplicatively, consists of 1, x, y, and a single element of degree i for each $i \geq 2$ representing the congruence class of the monomials of total degree i. Our two choices $\langle x^2, y\rangle$ and $\langle x, y^2\rangle$ for the irreducible component with associated prime $\langle x, y\rangle$ yield quotients of $\Bbbk[x,y]$ with different Q_I-graded Hilbert functions, the first nonzero in degree x and the second nonzero in degree y.

3. Primary components of binomial ideals

In this section, we express the primary components of binomial ideals in polynomial rings over the complex numbers as explicit sums of binomial and monomial ideals. We formulate our main result, Theorem 3.2, after recalling some essential results from [ES96]. In this section we work with the polynomial ring $\mathbb{C}[\partial]$ in (commuting) variables that we call $\partial = \partial_1, \ldots, \partial_n$ because of the transition to D-modules in later sections.

If $L \subseteq \mathbb{Z}^n$ is a sublattice, then the *lattice ideal* of L is

$$I_L = \langle \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \in L \rangle.$$

More generally, any *partial character* $\rho : L \to \mathbb{C}^*$ of $\mathbb{Z}^n$, which includes the data of both its domain lattice $L \subseteq \mathbb{Z}^n$ and the map to $\mathbb{C}^*$, determines a binomial ideal

$$I_\rho = \langle \partial^{u_+} - \rho(u)\partial^{u_-} : u = u_+ - u_- \in L \rangle.$$

(The ideal I_ρ is called $I_+(\rho)$ in [ES96].) The ideal I_ρ is prime if and only if L is a *saturated* sublattice of $\mathbb{Z}^n$, meaning that L equals its *saturation*, in general defined as

$$\mathrm{sat}(L) = (\mathbb{Q}L) \cap \mathbb{Z}^n,$$

where $\mathbb{Q}L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ is the rational vector space spanned by L in $\mathbb{Q}^n$. In fact, by [ES96, Corollary 2.6] every binomial prime ideal in $\mathbb{C}[\partial]$ has the form

$$I_{\rho,J} = I_\rho + \langle \partial_j : j \notin J \rangle \tag{3.1}$$

for some *saturated* partial character ρ (i.e., whose domain is a saturated sublattice) and subset $J \subseteq \{1, \ldots, n\}$ such that the binomial generators of I_ρ only involve variables ∂_j for $j \in J$ (some of which might actually be absent from the generators of I_ρ).

The characteristic zero part of the main result in [ES96], Theorem 7.1′, says that an irredundant primary decomposition of an arbitrary binomial ideal $I \subseteq \mathbb{C}[\partial]$ is given by

$$I = \bigcap_{I_{\rho,J} \in \mathrm{Ass}(I)} \mathrm{Hull}(I + I_\rho + \langle \partial_j : j \notin J \rangle^e) \tag{3.2}$$

for any large integer e, where Hull means to discard the primary components for embedded (i.e., nonminimal associated) primes. Our goal in this section is to be explicit about the Hull operation, first for the purpose of drawing conclusions about how the primary components interact with gradings of $\mathbb{C}[\partial]$, and later for the purpose of counting and writing down solutions to Horn systems and other hypergeometric systems. The salient feature of (3.2) is that $I + I_\rho + \langle \partial_j : j \notin J \rangle^e$ contains I_ρ. In contrast, this does not hold in positive characteristic, where the statement of [ES96, Theorem 7.1′] is the same except that $I_\rho + \langle \partial_j : j \notin J \rangle^e$ is replaced by a Frobenius power of $I_{\rho,J}$.

Example 3.1. Fix matrices A and B as in Convention 1.4. This identifies $\mathbb{Z}^d$ with the quotient of $\mathbb{Z}^n/\mathbb{Z}B$ modulo its torsion subgroup. Consider the *lattice basis ideal* corresponding to the lattice $\mathbb{Z}B = \{Bz : z \in \mathbb{Z}^m\}$, which is defined by

$$I(B) = \langle \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \text{ is a column of } B \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]. \tag{3.3}$$

The toric ideal I_A from (1.3) is an associated prime of $I(B)$, the primary component being I_A itself. More generally, all of the minimal primes of the lattice ideal $I_{\mathbb{Z}B}$, one of which is I_A, are minimal over $I(B)$ with multiplicity 1. These minimal primes are precisely the ideals I_ρ for partial characters $\rho : \mathrm{sat}(\mathbb{Z}B) \to \mathbb{C}^*$ of $\mathbb{Z}^n$ extending the trivial partial character on $\mathbb{Z}B$. The lattice ideal $I_{\mathbb{Z}B}$, which contains $I(B)$, is the intersection of these prime ideals. Hence $I_{\mathbb{Z}B}$ is a radical ideal, and every irreducible component of its zero set is isomorphic, as a subvariety of $\mathbb{C}^n$, to the variety of I_A.

In complete generality, each of the minimal primes of $I(B)$ arises, after row and column permutations, from a block decomposition of B of the form

$$\left[\begin{array}{c|c} N & B_J \\ \hline M & 0 \end{array}\right], \tag{3.4}$$

where M is a mixed submatrix of B of size $q \times p$ for some $0 \leq q \leq p \leq m$ [HS00]. (Matrices with $q = 0$ rows are automatically mixed; matrices with $q = 1$ row are never mixed.) We note that not all such decompositions correspond to minimal primes: the matrix M has to satisfy another condition which Hoşten and Shapiro call irreducibility [HS00, Definition 2.2 and Theorem 2.5]. If $I(B)$ is a complete intersection, then only square matrices M will appear in the block decompositions (3.4), by a result of Fischer and Shapiro [FS96].

For each partial character $\rho : \mathrm{sat}(\mathbb{Z}B_J) \to \mathbb{C}^*$ extending the trivial character on $\mathbb{Z}B_J$, the ideal $I_{\rho,J}$ is an associated prime of $I(B)$, where $J = J(M) = \{1,\ldots,n\} \smallsetminus \mathrm{rows}(M)$ indexes the $n-q$ rows not in M. We reiterate that the symbol ρ here includes the specification of the sublattice $\mathrm{sat}(\mathbb{Z}B_J) \subseteq \mathbb{Z}^n$. The corresponding primary component

$$\mathcal{C}_{\rho,J} = \mathrm{Hull}\big(I(B) + I_\rho + \langle \partial_j : j \notin J(M)\rangle^e\big)$$

of the lattice basis ideal $I(B)$ is simply I_ρ if $q = 0$, but will in general be non-radical when $q \geq 2$ (recall that $q = 1$ is impossible).

Our notation in the next theorem is as follows. If L is a saturated sublattice of $\mathbb{Z}^\ell$ for some ℓ, then we write $\mathbb{N}^\ell/L$ for the image of $\mathbb{N}^\ell$ in the torsion-free group $\mathbb{Z}^\ell/L$. Given a subset $J \subseteq \{1,\ldots,n\}$, let $\bar{J} = \{1,\ldots,n\} \smallsetminus J$ be its complement, and use these to index coordinate subspaces of $\mathbb{N}^n$ and $\mathbb{Z}^n$; in particular, $\mathbb{N}^n = \mathbb{N}^J \times \mathbb{N}^{\bar{J}}$. Adjoining additive inverses for the elements in $\mathbb{N}^J$ yields $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$, whose semigroup ring we denote by $\mathbb{C}[\partial][\partial_J^{-1}]$, so $\partial_J = \prod_{j\in J} \partial_j$. As in Definition 2.10, $\mathbb{C}\{S\}$ is the monomial ideal in $\mathbb{C}[\partial]$ having $\mathbb{C}$-basis S.

Consider a binomial ideal $I \subseteq \mathbb{C}[\partial]$, and let $I_{\rho,J}$ be a minimal prime of I. We may (by rescaling the variables ∂_j for $j \in J$) harmlessly assume that ρ is the trivial character on its lattice L, so that $I_\rho = I_L$ is the lattice ideal for L. The quotient $\mathbb{C}[\partial]/I_L$ is the affine semigroup ring $\mathbb{C}[Q]$ for $Q = \Phi \times \mathbb{N}^{\bar{J}}$. Now let us take the whole situation modulo I_L. The image of $I_{\rho,J} = I_L + \langle \partial_j : j \in \bar{J}\rangle$ is the prime ideal $\mathfrak{p}_\Phi \subseteq \Bbbk[Q]$ for the face Φ. The image in $\mathbb{C}[Q]$ of the binomial ideal I is a binomial ideal I', and $(I + I_L : \partial_J^\infty)$ has image $(I' : \partial^\Phi)$, as defined in (2.2). Finally, the image of $\widetilde{U}$ in Q is the unbounded ideal $U \subseteq Q$ (Definition 2.10) by construction. Now we can apply Theorem 2.14 to I' and obtain a combinatorial description of the component associated to $I_{\rho,J}$. These observations essentially prove the following result.

Theorem 3.2. *Fix a binomial ideal $I \subseteq \mathbb{C}[\partial]$ and an associated prime $I_{\rho,J}$ of I, where $\rho : L \to \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. Set $\Phi = \mathbb{N}^J/L$, and write $\sim$ for the congruence on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by the ideal $(I + I_\rho)[\mathbb{Z}^J] = (I + I_\rho)\mathbb{C}[\partial][\partial_J^{-1}]$.*

1. *If $I_{\rho,J}$ is a minimal prime of I and $\widetilde{U}$ is the set of $u \in \mathbb{N}^n$ whose congruence classes in $(\mathbb{Z}^J \times \mathbb{N}^{\bar{J}})/\sim$ have infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$, then the $I_{\rho,J}$-primary component of I is*

$$\mathcal{C}_{\rho,J} = \big(I + I_\rho : \partial_J^\infty\big) + \mathbb{C}\big\{\widetilde{U}\big\}.$$

Fix a monomial ideal $K \subseteq \mathbb{C}[\partial_j : j \in \bar{J}]$ containing a power of each available variable, and let $\approx$ be the congruence on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by $(I + I_\rho + K)\mathbb{C}[\partial][\partial_J^{-1}]$. Write $\widetilde{U}_K$ for the set of $u \in \mathbb{N}^n$ whose congruence classes in $(\mathbb{Z}^J \times \mathbb{N}^{\bar{J}})/{\approx}$ have infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$.

2. *The $I_{\rho,J}$-primary component of $\langle I + I_\rho + K\rangle \subseteq \mathbb{C}[\partial]$ is $(I + I_\rho + K : \partial_J^\infty) + \mathbb{C}\{\widetilde{U}_K\}$.*
3. *If K is contained in a sufficiently high power of $\langle \partial_j : j \in \bar{J}\rangle$, then*
$$\mathcal{C}_{\rho,J} = \big(I + I_\rho + K : \partial_J^\infty\big) + \mathbb{C}\big\{\widetilde{U}_K\big\}$$
is a valid choice of $I_{\rho,J}$-primary component for I.

The only monomials in the above primary components are those in $\mathbb{C}\{\widetilde{U}\}$ or $\mathbb{C}\{\widetilde{U}_K\}$.

Proof. We proved the first item in the paragraph preceding the statement of this theorem. From it, the second and third items follow easily by replacing I with $I + K$, given the primary decomposition in (3.2). □

Remark 3.3. One of the mysteries in [ES96] is why the primary components $\mathcal{C}$ of binomial ideals turn out to be generated by monomials and binomials. From the perspective of Theorem 3.2 and Proposition 2.12 together, this is because the primary components are *finely graded*: under some grading by a free abelian group, namely $\mathbb{Z}\Phi$, the vector space dimensions of the graded pieces of the quotient modulo the ideal $\mathcal{C}$ are all 0 or 1. In fact, via Lemma 2.8, the fine gradation is the root cause of primaryness.

Remark 3.4. Theorem 3.2 easily generalizes to arbitrary binomial ideals in arbitrary commutative noetherian semigroup rings over $\mathbb{C}$: simply choose a presentation as a quotient of a polynomial ring modulo a pure difference binomial ideal [Gil84, Theorem 7.11].

For the purpose of constructing D-modules from binomial ideals $I \subseteq \mathbb{C}[\partial]$, we need precise control over the interactions of primary components with various gradings on $\mathbb{C}[\partial]$. This is in fact one of our main reasons for including Theorem 3.2: the quotient of its statement by the toric ideal I_ρ puts us in the situation of Proposition 2.12 and Corollary 2.13, which provide excellent control over gradings. Therefore let us now fix an integer matrix A as in Convention 1.2, and recall that this datum determines a $\mathbb{Z}^d$-grading on $\mathbb{C}[\partial]$ in which $\deg(\partial_j) = -a_j$ is defined to be the negative of the j^{th} column of A (and recall $\mathbb{Z}A = \mathbb{Z}^d$).

Lemma 3.5. *Let $I \subseteq \mathbb{C}[\partial]$ be an A-graded binomial ideal and $\mathcal{C}_{\rho,J}$ a primary component, with $\rho : L \to \mathbb{C}^*$ for $L \subseteq \mathbb{Z}^J$. The image $\mathbb{Z}A_J$ of the homomorphism $\mathbb{Z}^J/L = \mathbb{Z}\Phi \to \mathbb{Z}^d$ induced by Corollary 2.13 (with $\mathcal{A} = \mathbb{Z}A = \mathbb{Z}^d$) is generated by the columns a_j of A indexed by $j \in J$, while their negatives generate the image of $\Phi = \mathbb{N}^J/L$, which we call $-\mathbb{N}A_J$.* □

To make things a little more concrete, let us give one more perspective on the homomorphism $\mathbb{Z}\Phi \to \mathbb{Z}^d$. Quite simply, the ideal $I_{\rho,J}$ is naturally graded by $\mathbb{Z}^J/L = \mathbb{Z}\Phi$, and the fact that it is also A-graded means that $L \subseteq \ker(\mathbb{Z}^n \to \mathbb{Z}^d)$, the map to $\mathbb{Z}^d$ being given by $-A$. (The real content of Corollary 2.13 lies with the action on the rest of B.)

The primary components of A-graded binomial ideals come in two flavors, as we are about to define. Identifying and exploiting the distinction between them is one of the primary goals of this paper. In particular, the distinction becomes amplified in the transition to binomial D-modules; see Theorem 4.12 and Corollary 4.22.

Definition 3.6. In the situation of Lemma 3.5, the quotients $\mathbb{C}[\partial]/I_{\rho,J}$ and $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ are

- *toral* if the homomorphism $\mathbb{Z}\Phi \to \mathbb{Z}^d$ is injective, and
- *Andean* if the homomorphism $\mathbb{Z}\Phi \to \mathbb{Z}^d$ is not injective.

We have already encountered toral and Andean lattices. We state for the record that Definitions 1.10 and 3.6 do indeed coincide.

Lemma 3.7. *Fix an associated prime $I_{\rho,J}$ of an A-graded binomial ideal I, where ρ is a partial character of a saturated lattice $L \subseteq \mathbb{Z}^J \cap \ker_{\mathbb{Z}}(A) = \ker_{\mathbb{Z}}(A_J)$. The quotient $\mathbb{C}[\partial]/I_{\rho,J}$ is toral if and only if $L = \ker_{\mathbb{Z}}(A_J)$; that is, if and only if the lattice L is toral.* $\square$

Here is another characterization of the notions of toral and Andean associated primes, this one involving the matrix A.

Lemma 3.8. *Given an A-graded binomial ideal I and an associated prime $I_{\rho,J}$, we have*

$$\dim(I_{\rho,J}) \geq \operatorname{rank}(A_J),$$

with equality if and only if $\mathbb{C}[\partial]/I_{\rho,J}$ is toral.

Proof. Rescale the variables and assume that $I_{\rho,J} = I_L$, the lattice ideal for a saturated lattice $L \subseteq \ker_{\mathbb{Z}}(A_J)$. The rank of L is at most $\#J - \operatorname{rank}(A_J)$; thus $\dim(I_L) = \#J - \operatorname{rank}(L) \geq \operatorname{rank}(A_J)$. Equality holds exactly when $L = \ker_{\mathbb{Z}}(A)$, i.e. when $\mathbb{C}[\partial]/I_{\rho,J}$ is toral. $\square$

Example 3.9. With notation as in Example 3.1, so $J = J(M)$, the quotient $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is toral if M is square and satisfies either $\det(M) \neq 0$ or $q = 0$. Otherwise, the quotient $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is Andean. (See Lemma 6.2 and Observation 6.3.)

Example 3.10. A binomial ideal $I \subseteq \mathbb{C}[\partial]$ maybe A-graded for different matrices A: in this case, which of the components of I are toral or Andean will change if we change the grading. For instance, the prime ideal $I = \langle \partial_1\partial_4 - \partial_2\partial_3 \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_4]$ is homogeneous for both the matrix $[1\,1\,1\,1]$ and the matrix $\left[\begin{smallmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1 \end{smallmatrix}\right]$. But $\mathbb{C}[\partial_1, \ldots, \partial_4]/I$ is Andean in the $[1\,1\,1\,1]$-grading, while it is toral in the $\left[\begin{smallmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1 \end{smallmatrix}\right]$-grading.

Remark 3.11. The notions of toral and Andean modules will be introduced generally in Definition 4.8 and 4.17. Although the ideals $I_{\rho,J}$ and $\mathcal{C}_{\rho,J}$ are rarely toral or Andean when considered as abstract modules, we nonetheless call them toral or Andean ideals. The latter adjective describes the geometry of the gradings on the $\mathbb{C}[\partial]$-modules $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$: collapsing (coarsening) the $\mathbb{Z}\Phi$-grading to the A-grading makes the $\mathbb{Z}\Phi$-graded degrees sit like a high thin mountain range over $\mathbb{Z}^d$, supported on finitely many translates of $\mathbb{Z}A_J$.

Example 3.12. Let $I = \langle bd - de, bc - ce, ab - ae, c^3 - ad^2, a^2d^2 - de^3, a^2cd - ce^3, a^3d - ae^3 \rangle$ be a binomial ideal in $\mathbb{C}[\partial]$, where we write $\partial = (\partial_1, \partial_2, \partial_3, \partial_4, \partial_5) = (a, b, c, d, e)$. Using

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{bmatrix}$$

one easily verifies that the binomial ideal I is graded by $\mathbb{Z}A = \mathbb{Z}^2$. If ω is a primitive cube root of unity ($\omega^3 = 1$), then I, which is a radical ideal, has the prime decomposition

$$\begin{aligned} I = \langle a, c, d\rangle &\cap \langle bc - ad, b^2 - ac, c^2 - bd, b - e\rangle \\ &\cap \langle \omega bc - ad, b^2 - \omega ac, \omega^2 c^2 - bd, b - e\rangle \\ &\cap \langle \omega^2 bc - ad, b^2 - \omega^2 ac, \omega c^2 - bd, b - e\rangle. \end{aligned}$$

The intersectand $\langle a, c, d\rangle$ equals the prime ideal $I_{\rho,J}$ for $J = \{2, 5\}$ and $L = \{0\} \subseteq \mathbb{Z}^J$. The homomorphism $\mathbb{Z}^J \to \mathbb{Z}^2$ is not injective since it maps both basis vectors to $\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$ (recall that $\deg(\partial_j) = -a_j$); therefore the prime ideal $\langle a, c, d\rangle$ is an Andean component of I. In contrast, the remaining three intersectands are the prime ideals $I_{\rho,J}$ for the three characters ρ that are defined on $\ker(A)$ but trivial on its index 3 sublattice $\mathbb{Z}B$ spanned by the columns of

$$B = \begin{bmatrix} -2 & -1 & 0 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $J = \{1, 2, 3, 4, 5\}$. These prime ideals are all toral by Lemma 3.7, with $\mathbb{Z}A_J = \mathbb{Z}A$.

The dichotomy between toral and Andean components of A-graded binomial ideals begins with the fact that toral ones admit an explicit description that is simpler than Theorem 3.2, although it follows without too much trouble from that result.

Corollary 3.13. *Fix an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$ and a toral associated prime $I_{\rho,J}$ of I. Define the binomial ideal $\bar{I} = I \cdot \mathbb{C}[\partial]/\langle \partial_j - 1 : j \in J\rangle$ by setting $\partial_j = 1$ for $j \in J$.*

1. *If $I_{\rho,J}$ is a minimal prime of I and $\overline{U} \subseteq \mathbb{N}^{\bar{J}}$ is the set of elements with infinite congruence class in $\mathbb{N}^{\bar{J}}_{\bar{I}}$ (Proposition 2.4), then the $I_{\rho,J}$-primary component of I is*
$$\mathcal{C}_{\rho,J} = \big(I + I_\rho : \partial_J^\infty\big) + \langle \partial^u : u \in \overline{U}\rangle.$$

Let $K \subseteq \mathbb{C}[\partial_j : j \in \bar{J}]$ be a monomial ideal containing a power of each available variable, and let $\overline{U}_K \subseteq \mathbb{N}^{\bar{J}}$ be the set of elements whose congruence classes in $\mathbb{N}^{\bar{J}}_{\bar{I}+K}$ are infinite.

2. *The $I_{\rho,J}$-primary component of $\langle I + I_\rho + K\rangle \subseteq \mathbb{C}[\partial]$ is $(I + I_\rho : \partial_J^\infty) + \langle \partial^u : u \in \overline{U}_K\rangle$.*
3. *If K is contained in a sufficiently high power of $\langle \partial_j : j \in \bar{J}\rangle$, then*
$$\mathcal{C}_{\rho,J} = \big(I + I_\rho : \partial_J^\infty\big) + \langle \partial^u : u \in \overline{U}_K\rangle$$
is a valid choice of $I_{\rho,J}$-primary component for I.

The only monomials in the above primary components are in $\langle \partial^u : u \in \overline{U}\rangle$ or $\langle \partial^u : u \in \overline{U}_K\rangle$.

Proof. Resume the notation from the statement and proof of Theorem 3.2. As in that proof, it suffices here to deal with the first item. In fact, the only thing to show is that $\widetilde{U}$ in Theorem 3.2 is the same as $\mathbb{N}^J \times \overline{U}$ here.

Recall that $I' \subseteq \mathbb{C}[Q]$ is the image of I modulo I_ρ. The congruence classes of $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ determined by $I'[\mathbb{Z}\Phi]$ are the projections under $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} \to \mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ of the $\sim$ congruence classes. Further projection of these classes to $\mathbb{N}^{\bar{J}}$ yields the congruence classes determined by the ideal $I'' \subseteq \mathbb{C}[\mathbb{N}^{\bar{J}}]$, where I'' is obtained from $I'[\mathbb{Z}\Phi]$ by setting $\partial^\phi = 1$ for all $\phi \in \mathbb{Z}\Phi$. This ideal I'' is just $\bar{I}$. Therefore we are reduced to showing that a congruence class in $\Phi \times \mathbb{N}^{\bar{J}}$ determined by $I'[\mathbb{Z}\Phi]$ is infinite if and only if its projection to $\mathbb{N}^{\bar{J}}$ is infinite. This is clearly true for the monomial congruence class in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$. For any other congruence class $\Gamma \subseteq \mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$, the homogeneity of I (and hence that of I') under the A-grading implies that Γ is contained within a coset of $\mathcal{K} = \ker(\mathbb{Z}\Phi \times \mathbb{Z}^{\bar{J}} \to \mathbb{Z}^d = \mathbb{Z}A)$. This kernel $\mathcal{K}$ intersects $\mathbb{Z}\Phi$ only at 0 because $I_{\rho,J}$ is toral. Therefore the projection of any coset of $\mathcal{K}$ to $\mathbb{Z}^{\bar{J}}$ is bijective onto its image. In particular, Γ is infinite if and only if its bijective image in $\mathbb{N}^{\bar{J}}$ is infinite. $\square$

Example 3.14. Resume the notation of Example 3.1. If $I_{\rho,J}$ is a toral minimal prime of $I(B)$ given by a decomposition as in (3.4), so $J = J(M)$, then

$$\mathcal{C}_{\rho,J} = I(B) + I_{\rho,J} + U_M,$$

where $U_M \subseteq \mathbb{C}[\partial_j : j \in \bar{J}]$ is the ideal $\mathbb{C}$-linearly spanned by by all monomials whose exponent vectors lie in the union of the unbounded M-subgraphs of $\mathbb{N}^{\bar{J}}$, as in Definition 2.6.

Remark 3.15. Corollary 3.13 need not always be false for an Andean component, but it can certainly fail: in the Andean case, there can be congruence classes in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ that are infinite only in the $\mathbb{Z}\Phi$ direction, so that their projections to $\mathbb{N}^{\bar{J}}$ are finite.

Remark 3.16. The methods of Section 2 work in arbitrary characteristic—and indeed, over a field $\Bbbk$ that can fail to be algebraically closed, and can even be finite—because we assumed that a prime ideal $\mathfrak{p}_\Phi$ for a face Φ is associated to our binomial ideal. In contrast, this section works only over an algebraically closed field of characteristic zero. However, it might be possible to produce similarly explicit binomial primary decompositions in positive characteristic by reducing to the situation in Section 2; this remains an open problem.

4. Euler-Koszul homology

The Euler operators in Definition 1.3 can be used to build a Koszul-like complex whose zeroth homology is the A-hypergeometric system in Definition 1.3. In its most basic form, this construction is due to Gelfand, Kapranov, and Zelevinsky [GKZ89], and was developed by Adolphson [Ado94, Ado99] and Okuyama [Oku06], among others. A functorial generalization was introduced in [MMW05], where it was proved to be homology-isomorphic to an ordinary Koszul complex detecting holonomic rank changes for varying parameters β. Here, we begin by reviewing the definitions from [MMW05, Section 4] (where more details can be found). We proceed to develop the foundations in the greater generality of toral and Andean modules (Definitions 4.8 and 4.17). Our main results for these are Theorems 4.12, 4.13, 4.15, and 4.16 for toral modules, and Corollary 4.22 for Andean modules.

Given a matrix A with columns $a_1, \ldots, a_n$ as in Convention 1.2, recall that the polynomial ring $\mathbb{C}[\partial]$ and the Weyl algebra $D = D_n$ are A-graded by $\deg(\partial_j) = -a_j$ and $\deg(x_j) = a_j$.

Under this A-grading, operators $E_1, \dots, E_d$, and in fact all of the products $x_j\partial_j \in D$, are homogeneous of degree 0.

Given an A-graded left D-module W, if $z \in W_\alpha$ is homogeneous of degree α then set $\deg_i(z) = \alpha_i$. The map $E_i - \beta_i : W \to W$ that sends each homogeneous element $z \in W$ to

$$(E_i - \beta_i) \circ z = (E_i - \beta_i - \deg_i(z))z, \tag{4.1}$$

and is extended $\mathbb{C}$-linearly to all of W, determines a D-linear endomorphism of W.

Definition 4.1. Fix $\beta \in \mathbb{C}^d$ and an A-graded $\mathbb{C}[\partial]$-module V. The *Euler-Koszul complex* $\mathcal{K}_\bullet(E - \beta; V)$ is the Koszul complex of left D-modules defined by the sequence $E - \beta$ of commuting endomorphisms on the left D-module $D \otimes_{\mathbb{C}[\partial]} V$ concentrated in homological degrees d to 0. The i^{th} *Euler-Koszul homology* of V is $\mathcal{H}_i(E - \beta; V) = H_i(\mathcal{K}_\bullet(E - \beta; V))$.

Example 4.2. Fix A and B as in Conventions 1.2 and 1.4.

1. The *binomial Horn D-module* with parameter β is $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I(B))$.
2. The *A-hypergeometric D-module* with parameter β is $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I_A)$; see (1.3).
3. If $I \subseteq \mathbb{C}[\partial]$ is any A-graded binomial ideal, then $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) = D/H_A(I, \beta)$.

Euler-Koszul homology behaves predictably with regard to A-graded translation.

Lemma 4.3. *Let V be an A-graded $\mathbb{C}[\partial]$-module and $\alpha \in \mathbb{Z}^d = \mathbb{Z}A$. If $V(\alpha)$ is the A-graded module with $V(\alpha)_{\alpha'} = V_{\alpha+\alpha'}$, then $\mathcal{H}_0(E - \beta; V(\alpha)) \cong \mathcal{H}_0(E - \beta + \alpha; V)(\alpha)$.* □

We shall see that Euler-Koszul homology has the useful property of detecting "where" a module is nonzero, the nonzeroness being measured in the following sense.

Definition 4.4. Let V be an A-graded $\mathbb{C}[\partial]$-module. The set of *true degrees* of V is

$$\operatorname{tdeg}(V) = \{\beta \in \mathbb{Z}^d : V_\beta \neq 0\}.$$

The set $\operatorname{qdeg}(V)$ of *quasidegrees* of V is the Zariski closure in $\mathbb{C}^d$ of its true degrees $\operatorname{tdeg}(V)$.

Because of the next lemma, we shall often refer to quasidegree sets as *arrangements*.

Lemma 4.5. *Let R be a noetherian A-graded ring that is finitely generated over its degree 0 piece. The quasidegree set of any finitely generated graded R-module is a finite union of affine subspaces of $\mathbb{C}^d$, each spanned by the degrees of some subset of the generators of R.*

Proof. Every A-graded module has an A-graded associated prime, and therefore a submodule isomorphic to an A-graded translate of a quotient by an A-graded prime. Now use Noetherian induction to conclude that every such module has a filtration whose successive quotients are A-translates of quotients of R modulo prime ideals. But being an integral domain, the true degree set of a quotient $R/\mathfrak{p}$ by a prime ideal $\mathfrak{p}$ is the affine semigroup generated by the degrees of the generators of R that remain nonzero in $R/\mathfrak{p}$. □

Example 4.6. If $V = \mathbb{C}[\partial]/\langle\partial_1, \partial_3, \partial_4\rangle$ from Example 3.12, then $\operatorname{qdeg}(V)$ is the diagonal line in $\mathbb{C}^2$. In contrast, the quotient by each one of the other three prime ideals there has quasidegree set equal to all of $\mathbb{C}^2$. It follows that $\operatorname{qdeg}(\mathbb{C}[\partial]/I) = \mathbb{C}^2$.

Let $\mathfrak{m}$ be the maximal ideal $\langle \partial_1, \ldots \partial_n \rangle$ of $\mathbb{C}[\partial]$. Since A is pointed with no nonzero columns, $\mathfrak{m}$ is the unique maximal A-graded ideal. Given an A-graded $\mathbb{C}[\partial]$-module V, its *local cohomology modules*

$$H^i_{\mathfrak{m}}(V) = \varinjlim_t \operatorname{Ext}^i_{\mathbb{C}[\partial]}(\mathbb{C}[\partial]/\mathfrak{m}^t, V)$$

supported at $\mathfrak{m}$ are A-graded; see [MS05, Chapter 13]. Even when V is finitely generated, its local cohomology modules $H^i_{\mathfrak{m}}(V)$ need not be; but their Matlis duals are, so their quasidegree sets are still arrangements.

Lemma 4.7. *If V is a finitely generated A-graded $\mathbb{C}[\partial]$-module, then the quasidegree set $\operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$ of the i^{th} local cohomology module of V is a union of finitely many integer translates of the complex subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$ spanned by $\{a_j : j \in J\}$ for various J.*

Proof. Let $\varepsilon_A = \sum_{j=1}^n a_j$. In the graded version [Mil02b, Theorem 6.3] of the Greenlees-May theorem [GM92], setting $\mathcal{E}$ equal to the injective hull of the residue field $\mathbb{C}[\partial]/\mathfrak{m}$ yields the natural A-graded local duality vector space isomorphism

$$\operatorname{Ext}^{n-i}_{\mathbb{C}[\partial]}(V, \mathbb{C}[\partial])_\alpha \cong \operatorname{Hom}_{\mathbb{C}}(H^i_{\mathfrak{m}}(V)_{-\alpha+\varepsilon_A}, \mathbb{C}).$$

(Use the case $\mathcal{G} = \mathbb{C}[\partial]$ to deduce that the right-hand side of [Mil02b, Theorem 6.3] is the derived Hom into the canonical module $\omega_{\mathbb{C}[\partial]}$, which is isomorphic as a graded module to the principal ideal $\langle \partial_1 \cdots \partial_n \rangle$; see also [BH93, Section 3.5] for $\mathbb{Z}$-graded local duality.) Hence $\varepsilon_A + \operatorname{qdeg}(H^i_{\mathfrak{m}}(V)) = -\operatorname{qdeg}(\operatorname{Ext}^{n-i}_{\mathbb{C}[\partial]}(V, \mathbb{C}[\partial]))$ is the negative of the quasidegree set of a finitely generated module. The result is now a consequence of Lemma 4.5. $\square$

4.1. **Toral modules.** Euler-Koszul homology was originally conceived of for *toric* modules [MMW05, Definition 4.5]. Here, we shall show that all of the main results in [MMW05] hold, with essentially the same proofs, for the following more general class of modules.

Definition 4.8. An A-graded $\mathbb{C}[\partial]$-module V is *natively toral* if there is a binomial prime ideal $I_{\rho,J}$ and a degree $\alpha \in \mathbb{Z}^d$ such that $V(\alpha) \cong \mathbb{C}[\partial]/I_{\rho,J}$ is a toral quotient (Definition 3.6). The module V is *toral* if it has a finite filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$ whose successive quotients V_k/V_{k-1} are all natively toral.

Example 4.9. $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is toral for any toral primary component $\mathcal{C}_{\rho,J}$ of any A-graded binomial ideal. This is a bit nontrivial, as $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ a priori only has—by Theorem 3.2 and Proposition 2.12—a finite filtration whose successive quotients are A-graded torsion-free modules of rank 1 over the affine semigroup ring $R = \mathbb{C}[\partial]/I_{\rho,J}$. That is, the successive quotients are not necessarily A-graded translates of R itself. However, every finitely generated A-graded R-module has a finite filtration whose successive quotients are A-graded translates of quotients $R/\mathfrak{p}_\Phi$ modulo prime ideals $\mathfrak{p}_\Phi$ for faces Φ, and these are natively toral.

The argument in Example 4.9 actually shows more.

Lemma 4.10. *If $W \subseteq V$ are A-graded modules with V toral, then W and V/W are toral.*

Proof. Intersecting any toral filtration of V with W yields a filtration of W whose successive quotients are toral because they are A-graded modules over natively toral quotients $\mathbb{C}[\partial]/I_{\rho,J}$. Hence W is toral. The same argument works for the image filtration in V/W. $\square$

We begin recounting the results of [MMW05] with an elementary observation about how Euler-Koszul homology works for modules killed by some of the variables; the proof is the same as [MMW05, Lemma 4.8]. For notation, let E_i^J be the operator obtained from E_i by setting the terms $x_j\partial_j$ to zero for $j \notin J$. This operator can be thought of as lying in the Weyl algebra D_J in the variables x_j and ∂_j for $j \in J$. Denote by $x_{\bar{J}}$ the x-variables for $j \notin J$.

Lemma 4.11. *If the variables ∂_j for $j \notin J$ annihilate an A-graded $\mathbb{C}[\partial]$-module V, then $D\otimes_{\mathbb{C}[\partial]}V \cong \mathbb{C}[x_{\bar{J}}]\otimes_{\mathbb{C}}(D_J\otimes_{\mathbb{C}[\partial_J]}V)$ as $D = D_{\bar{J}}\otimes_{\mathbb{C}}D_J$-modules. Acting by E_i on $D\otimes_{\mathbb{C}[\partial]}V$ as in (4.1) is the same as acting by E_i^J on the right-hand factor of $\mathbb{C}[x_{\bar{J}}]\otimes_{\mathbb{C}}(D_J\otimes_{\mathbb{C}[\partial_J]}V)$.* □

Many of the following results are stated in the context of *holonomic* D-modules, which by definition are the finitely generated left D-modules W with $\operatorname{Ext}^j_D(W, D) = 0$ for $j \neq n$. When W is holonomic, the vector space $\mathbb{C}(x)\otimes_{\mathbb{C}[x]} W$ over the field $\mathbb{C}(x)$ of rational functions in $x_1, \ldots, x_n$ has finite dimension equal to the *holonomic rank* $\operatorname{rank}(W)$ by a celebrated theorem of Kashiwara; see [SST00, Theorem 1.4.19 and Corollary 1.4.14].

We shall also be interested in whether our D-modules are *regular holonomic*, the definition of which can be found in [Bjö79]. For an A-hypergeometric D-module (Example 4.2), regular holonomicity is known [Hot91] to occur when A is *homogeneous*, meaning that there is a row vector $\psi \in \mathbb{Q}^d$ such that ψA equals the row vector $[1, \ldots, 1]$. In this case, the $\mathbb{Z}A = \mathbb{Z}^d$-grading on $\mathbb{C}[\partial]$ coarsens naturally to the *standard $\mathbb{Z}$-grading*, in which $\deg(\partial_j) = 1 \in \mathbb{Z}$ for all j.

Theorem 4.12. *If V is a toral $\mathbb{C}[\partial]$-module and $\beta \in \mathbb{C}^d$, then the Euler-Koszul homology $\mathcal{H}_i(E-\beta; V)$ is holonomic for all i. Moreover, the following are equivalent.*

1. *$\mathcal{H}_0(E-\beta; V)$ has holonomic rank 0.*
2. *$\mathcal{H}_0(E-\beta; V) = 0$.*
3. *$\mathcal{H}_i(E-\beta; V) = 0$ for all $i \geq 0$.*
4. *$-\beta \notin \operatorname{qdeg}(V)$.*

If, in addition, the matrix A is homogeneous, then $\mathcal{H}_i(E-\beta; V)$ is regular holonomic for all i.

Proof. This is the toral generalization of [MMW05, Proposition 5.1] and [MMW05, Proposition 5.3]. To see that it holds, start with [MMW05, Notation 4.4]: instead of only allowing submatrices of A corresponding to faces of the semigroup $\mathbb{N}A$, we allow submatrices A_J with arbitrary column sets $J \subseteq \{1, \ldots, n\}$. Then, in [MMW05, Definition 4.5], replace "toric" with "toral" and change S_{F_k} to $\mathbb{C}[\partial]/I_{\rho,J}$; that this defines toral modules is by Lemma 3.7.

The key is [MMW05, Lemma 4.9]. In the proof there, first replace $\mathcal{M}_\beta^F = D/H_A(I_A^F, \beta)$ by $D/H_A(I_{\rho,J}, \beta)$. Then observe that rescaling the variables via ρ induces an A-graded automorphism of D commuting with the construction of Euler-Koszul complexes (because $x_j\partial_j$ is invariant under the automorphism). Hence the theorem for natively toral modules need only be proved in the special case $\rho =$ identity. This allows us to use $I_{A_J} + \langle \partial_j : j \notin J\rangle$ instead of $I_{\rho,J}$. The rest of the proof of [MMW05, Lemma 4.9] goes through unchanged, and when A is homogeneous, provides regular holonomicity as a consequence of the analogous result for GKZ systems from [Hot91, SW06].

Now extend the proof of [MMW05, Proposition 5.1] to the toral setting. For the first paragraph of that proof, replace "toric" with "toral" and replace S_F by $\mathbb{C}[\partial]/I_{\rho,J}$. For the later paragraphs of the proof, begin by working with the module M there being native toral. This allows us to replace I_A, when it arises as an annihilator toward the end, with $I_{\rho,J}$, thereby proving the native toral case. For the arbitrary toral case, simply note that for any exact sequence $0 \to V' \to V \to V'' \to 0$ in which V' and V'' both have (regular) holonomic Euler-Koszul homology, each Euler-Koszul homology module of V is placed between two (regular) holonomic modules, and is hence (regular) holonomic.

Finally, to generalize [MMW05, Proposition 5.3], replace "toric" with "toral" in the statement and proof. Then, in the proof, replace I_A^F by $I_{\rho,J}$ and S_F by $\mathbb{C}[\partial]/I_{\rho,J}$. □

Next we record the toral generalization of [MMW05, Theorem 6.6].

Theorem 4.13. *The Euler-Koszul homology $\mathcal{H}_i(E-\beta;V)$ of a toral module V is nonzero for some $i \geq 1$ if and only if $-\beta \in \operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$ for some $i < d$. More precisely, if k equals the smallest homological degree i for which $-\beta \in \operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$, then $\mathcal{H}_{d-k}(E-\beta;V)$ is holonomic of nonzero rank while $\mathcal{H}_i(E-\beta;V) = 0$ for $i > d-k$.*

Proof. Begin by noting that $\operatorname{Ext}^i_{\mathbb{C}[\partial]}(V,\mathbb{C}[\partial])$ is toral whenever V is toral. This is the toral generalization of [MMW05, Lemma 6.1]; the same proof works, mutatis mutandis, replacing S_A in [MMW05] by $\mathbb{C}[\partial]/I_{\rho,J}$ here. Now extend [MMW05, Theorem 6.3] to the toral case: the only property of toric modules used in its proof is the holonomicity of Euler-Koszul homology, which we have shown is true for toral modules in Theorem 4.12. Finally, to torally extend the toric [MMW05, Theorem 6.6], start with the first sentence of the proof, which for toral modules is Lemma 4.14, below. After that, the proof goes through verbatim, given that we have shown the results it cites for toric modules to be true for toral modules. □

Lemma 4.14. *If V is toral, then its Krull dimension satisfies $\dim(V) = \dim(\operatorname{qdeg}(V)) \leq d$.*

Proof. For natively toral modules this follows from Lemma 3.7. For arbitrary toral modules, the Krull dimension and the dimension of the quasidegree set both equal the maximum of the corresponding dimensions for the composition factors in any toral filtration. □

One of the observations in [MMW05] is that hypergeometric systems $D/H_A(I,\beta)$ for varying β should be viewed as a family of D-modules fibered over $\mathbb{C}^d$. If (the holonomic rank function of the D-modules in) such a family is to behave well, it suffices to verify that it is a *holonomic family* [MMW05, Definition 2.1]. For families arising from toric modules this is done in [MMW05, Theorem 7.5], which we now generalize to the toral setting. As a matter of notation, let $b = b_1, \ldots, b_d$ be commuting variables of degree zero, so $D[b]$ is a polynomial algebra over the Weyl algebra D. For any A-graded $\mathbb{C}[\partial]$-module V, construct the *global Euler-Koszul complex* $\mathcal{K}_\bullet(E-b;V)$ of left $D[b]$-modules and *global Euler-Koszul homology* $\mathcal{H}_\bullet(E-b;V)$ by replacing D and β in Definition 4.1 with $D[b]$ and b here. Finally, if $\mathbb{C}(x)$ is the field of rational functions in $x_1,\ldots,x_n$, write $\mathcal{V}(x) = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} \mathcal{V}$ for any $\mathbb{C}[x]$-module $\mathcal{V}$, including $\mathcal{V} = \mathbb{C}[b][x]$, where we set $\mathcal{V}(x) = \mathbb{C}[b](x)$.

Theorem 4.15. *If V is toral, then the sheaf $\tilde{\mathcal{V}}$ on $\mathbb{C}^d$ whose global section module is $\mathcal{V} = \mathcal{H}_0(E-b;V)$ constitutes a holonomic family over $\mathbb{C}^d$; in other words, $\mathcal{V}_\beta = \mathcal{H}_0(E-\beta;V)$ is holonomic for all $\beta \in \mathbb{C}^d$, and $\mathcal{V}(x)$ is finitely generated as a module over $\mathbb{C}[b](x)$.*

Proof. [MMW05, Proposition 7.4] holds for $I_{\rho,J}$ in place of I_A^F after harmlessly rescaling the x and ξ variables inversely to each other, which affects neither $Ax\xi$ nor the initial ideal in question. Therefore we may, in the proof of [MMW05, Theorem 7.5], simply change "toric" to "toral" and base the induction again on $I_{\rho,J}$ and $\mathbb{C}[\partial]/I_{\rho,J}$ instead of I_A^F and S_A. □

Considering b_i and β_i as elements in the polynomial ring $\mathbb{C}[b]$, we can take ordinary Koszul homology $H_\bullet(b-\beta;W)$ for any $\mathbb{C}[b]$-module W. This gets used in the generalization of [MMW05, Theorem 8.2] to arbitrary A-graded $\mathbb{C}[\partial]$-modules, which we state along with the toral generalization of [MMW05, Theorem 9.1]. For the latter, we need also the *jump arrangement* $\mathcal{Z}_{\text{jump}}(V) = \bigcup_{i\le d-1} \operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$ of an A-graded module V over $\mathbb{C}[\partial]$.

Theorem 4.16. *If V is an A-graded $\mathbb{C}[\partial]$-module and $\mathcal{V} = \mathcal{H}_0(E-b;V)$, then*

$$\mathcal{H}_i(E-\beta;V) \cong H_i(b-\beta;\mathcal{V}),$$

the left and right sides being Euler-Koszul and ordinary Koszul homology, respectively. If, in addition, V is toral, then $-\beta \in \mathcal{Z}_{\text{jump}}(V)$ lies in the jump arrangement of V if and only if the holonomic rank of $\mathcal{H}_0(E-\beta;V)$ is not minimal (among all possible choices of β).

Proof. [MMW05, Theorem 8.2] and its proof both work verbatim for arbitrary A-graded $\mathbb{C}[\partial]$-modules. That being given, the proof of [MMW05, Theorem 9.1] works just as well for toral modules, since we have now seen that all of the earlier results in [MMW05] do. □

4.2. **Andean Modules.** The finiteness properties of toral modules encapsulated by Theorem 4.12 will be contrasted in Theorem 4.21 with the infiniteness that occurs for the modules in our next definition.

Definition 4.17. An A-graded $\mathbb{C}[\partial]$-module V is *natively Andean* if there is an $\alpha \in \mathbb{Z}^d$ and an Andean quotient ring $\mathbb{C}[\partial]/I_{\rho,J}$ (Definition 3.6) over which $V(\alpha)$ is torsion-free of rank 1 and admits a $\mathbb{Z}^J/L$-grading that refines the A-grading via $\mathbb{Z}^J/L \to \mathbb{Z}^d = \mathbb{Z}A$, where ρ is defined on $L \subseteq \mathbb{Z}^J$. V is *Andean* if it has a finite filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$ whose successive quotients V_k/V_{k-1} are all natively Andean.

Example 4.18. $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is Andean for any Andean primary component $\mathcal{C}_{\rho,J}$ of any A-graded binomial ideal. This follows immediately from Theorem 3.2 and Proposition 2.12.

Here is a weak form of Euler-Koszul rigidity for Andean modules (but see Corollary 4.22).

Lemma 4.19. *If V is an Andean module and $-\beta \notin \operatorname{qdeg}(V)$, then $\mathcal{H}_i(E-\beta;W) = 0 = \mathcal{H}_i(E-\beta;V/W)$ for all i and all A-graded submodules $W \subseteq V$.*

Proof. First assume that V is natively Andean. The torsion-freeness ensures that $\operatorname{qdeg}(V)$ is a $\mathbb{Z}^d$-translate of the complex span $\mathbb{C}A_J$ of the columns of A indexed by J, so let us also assume for the moment that $\operatorname{qdeg}(V) = \mathbb{C}A_J$. The result for this V and all of its A-graded submodules follows from Lemma 4.11, because the $\mathbb{C}$-linear span of $E_1^J - \beta_1, \ldots, E_d^J - \beta_d$

contains a nonzero scalar if $\beta \notin \mathbb{C}A_J$ (some linear combination of $E_1^J, \ldots, E_d^J$ is zero, while the corresponding linear combination of $\beta_1, \ldots, \beta_d$ is nonzero, and hence a unit).

The case where V is natively Andean (or a submodule thereof) and $\mathrm{qdeg}(V) = \alpha + \mathbb{C}A_J$ is proved by applying the above argument to $V(-\alpha)$, using Lemma 4.3. The case where V is a general Andean module is proved by induction on the length of an Andean filtration, using that $\mathrm{qdeg}(V) = \mathrm{qdeg}(V') \cup \mathrm{qdeg}(V'')$ whenever $0 \to V' \to V \to V'' \to 0$ is an exact sequence. Finally, for an A-graded submodule W of a general Andean module V, intersecting W with an Andean filtration of V yields a filtration of W whose successive quotients are submodules of native Andean modules. Hence the proof of vanishing of Euler-Koszul homology by induction on the length of the filtration still applies.

The vanishing of all $\mathcal{H}_i(E-\beta; V/W)$ follows easily from the vanishing for V and for W. $\square$

The following lemma will allow us to reduce to the case of Andean quotients $\mathbb{C}[\partial]/I_{\rho,J}$ whenever we need to work with natively Andean modules.

Lemma 4.20. *A natively Andean module V has a filtration whose successive quotients are A-graded translates of various quotients $\mathbb{C}[\partial]/I_{\rho,J}$, each being natively either toral or Andean. At least one of these quotients is natively Andean.*

Proof. By definition, V is torsion free of rank 1 over an Andean quotient $\mathbb{C}[\partial]/I_{\tau,S}$ where $S \subseteq \{1, \ldots, n\}$ and τ is a partial character (we use non-standard notation to avoid confusion with the statement we need to prove). Harmlessly rescaling the variables, we may assume that $I_{\tau,S} = I_L + \langle \partial_j : j \notin S \rangle$, so $\mathbb{C}[\partial]/I_{\tau,S}$ is a semigroup ring $\mathbb{C}[Q]$ for some $Q \subseteq \mathbb{Z}^S/L$. Replacing V with an A-graded translate, we may further assume that V is $\mathbb{Z}^S/L$-graded. Using Noetherian induction as in the proof of Lemma 4.5, we construct a filtration of V whose successive quotients are $\mathbb{Z}^S/L$-graded translates of quotients $\mathbb{C}[Q]/\mathfrak{p}_{Q'}$ modulo prime ideals $\mathfrak{p}_{Q'}$ for faces $Q' \subseteq Q$ (these are the $\mathbb{C}[\partial]/I_{\rho,J}$ of the statement). Each of these, being A-graded, is either natively toral or natively Andean. Moreover, if all of them were toral, then V would be toral as well, so the last assertion follows. $\square$

Now we combine the Euler-Koszul theory for Andean and toral modules to conclude that the hypergeometric D-modules associated to Andean modules, if nonzero, are very large.

Theorem 4.21. *If an A-graded $\mathbb{C}[\partial]$-module V possesses a surjection to an Andean module W, and if $-\beta \in \mathrm{qdeg}(W)$, then $\mathcal{H}_0(E-\beta; V)$ has uncountably many linearly independent solutions near any general point $x \in \mathbb{C}^n$; that is, $\mathrm{Hom}_D(\mathcal{H}_0(E-\beta; V), \mathcal{O}_x)$ is a vector space of uncountable dimension over $\mathbb{C}$, where $\mathcal{O}_x$ is the local ring of analytic germs at x.*

Proof. Since a surjection of $\mathbb{C}[\partial]$-modules induces a surjection of zeroth Euler-Koszul homology $\mathcal{H}_0(E-\beta; \cdot)$, we may assume that $V = W$ is Andean.

Consider an exact sequence $0 \to V' \to V \to V'' \to 0$ of Andean modules in which V'' is natively Andean. If $-\beta \notin \mathrm{qdeg}(V'')$, then $\mathcal{H}_0(E-\beta; V') \cong \mathcal{H}_0(E-\beta; V)$ by Lemma 4.19 for V'', so we may harmlessly replace V with V'. Continuing in this manner, using induction on the length of an Andean filtration of V, we may assume that $-\beta \in \mathrm{qdeg}(V'')$. But then, since $\mathcal{H}_0(E-\beta; V)$ always surjects onto $\mathcal{H}_0(E-\beta; V'')$, we may assume that $V = V''$ is natively Andean. By Lemma 4.3, we may further assume that V is torsion-free of rank 1 over some Andean quotient $R = \mathbb{C}[\partial]/I_{\rho,J}$, and that V contains R with no A-graded translation.

Using Lemma 4.20 and its notation, take a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$ in which each of the successive quotients V_k/V_{k-1} is a A-graded translate of some prime quotient that is natively either toral or Andean. We are free to choose $V_1 = R$, and we do so. Let k be the largest index such that V_k/V_{k-1} is Andean and $-\beta \in \mathrm{qdeg}(V_k/V_{k-1})$, noting that such an index exists because $V_1/V_0 = R$ satisfies the condition. Since V surjects onto V/V_{k-1}, we find that $\mathcal{H}_0(E-\beta; V)$ surjects onto $\mathcal{H}_0(E-\beta; V/V_{k-1})$. Therefore, replacing V by V/V_{k-1} and $\mathbb{Z}^d$-translating again via Lemma 4.3 if necessary, it is enough to prove the case $k = 1$, with $V_1 = R$.

If the above filtration has length $\ell > 1$, then the kernel and cokernel of the homomorphism $\mathcal{H}_0(E-\beta; V_{\ell-1}) \to \mathcal{H}_0(E-\beta; V)$ are holonomic, being $\mathcal{H}_i(E-\beta; V/V_{\ell-1})$ for $i \in \{0, 1\}$; this is by Theorem 4.12 if $V/V_{\ell-1}$ is toral, and by Lemma 4.19 if $V/V_{\ell-1}$ is Andean with $-\beta \notin \mathrm{qdeg}(V/V_{\ell-1})$. Therefore the desired result holds for V if and only if it holds for $V_{\ell-1}$. This argument reduces us to the case $\ell = 1$ by induction on ℓ, so we may assume that $V = R = \mathbb{C}[\partial]/I_{\rho,J}$.

The condition $-\beta \in \mathrm{qdeg}(R)$ means exactly that $-\beta$, or equivalently β, lies in the complex column span $\mathbb{C}A_J$. Let $\hat{A}$ be a matrix for the projection $\mathbb{Z}^J \to \mathbb{Z}^J/L$, and write $\mathbb{Z}\hat{A} = \mathbb{Z}^J/L$. If $\hat{\beta}$ is a vector in $\mathbb{C}\hat{A}$ mapping to β under the surjection to $\mathbb{C}A_J$ afforded by Lemma 3.5, then denote by $\hat{E} - \hat{\beta}$ the sequence of Euler operators associated to $\hat{A}$. Thought of as elements in the space of affine linear functions $\mathbb{Z}^J \to \mathbb{C}$, the Euler operators $E_1^J - \beta_1, \ldots, E_d^J - \beta_d$ truncated from $E - \beta$ generate a sublattice $\mathbb{Z}\{E^J - \beta\}$ properly contained in the sublattice $\mathbb{Z}\{\hat{E} - \hat{\beta}\}$ generated by $\hat{E} - \hat{\beta}$. The binomial hypergeometric system $D/H_{\hat{A}}(I_{\rho,J}, \hat{\beta})$ is holonomic of positive rank by Theorem 4.12 (for $\mathbb{Z}^J/L$-graded toral $\mathbb{C}[\partial_J]$-modules, via Lemma 4.11). Its solutions are also solutions of $D/H_A(I_{\rho,J}, \beta)$ because

$$H_A(I_{\rho,J}, \beta) = D \cdot \langle I_{\rho,J}, \mathbb{Z}\{E^J - \beta\}\rangle \subseteq D \cdot \langle I_{\rho,J}, \mathbb{Z}\{\hat{E} - \hat{\beta}\}\rangle = H_{\hat{A}}(I_{\rho,J}, \hat{\beta}).$$

On the other hand, for any pair of distinct lifts $\hat{\beta} \neq \hat{\beta}'$, the linear span of $\mathbb{Z}\{\hat{E} - \hat{\beta}\}$ together with $\mathbb{Z}\{\hat{E} - \hat{\beta}'\}$ contains a nonzero scalar. It follows that the solutions to $D/H_{\hat{A}}(I_{\rho,J}, \hat{\beta})$ for varying $\hat{\beta}$ are linearly independent. The direct sum of these (local) solution spaces is therefore an uncountable-dimensional subspace of the (local) solutions to $\mathcal{H}_0(E - \beta; R) = D/H_A(I_{\rho,J}, \beta)$. $\square$

Summarizing the above results, let us emphasize the dichotomy between toral and Andean modules by recording the Andean analogue of Theorem 4.12.

Corollary 4.22. *The following are equivalent for an Andean $\mathbb{C}[\partial]$-module V and $\beta \in \mathbb{C}^d$.*

0. $\mathcal{H}_0(E-\beta; V)$ has countable-dimensional local solution space.
1. $\mathcal{H}_0(E-\beta; V)$ has finite-dimensional local solution space.
2. $\mathcal{H}_0(E-\beta; V) = 0$.
3. $\mathcal{H}_i(E-\beta; V) = 0$ for all $i \geq 0$.
4. $-\beta \notin \mathrm{qdeg}(V)$.

5. BINOMIAL D-MODULES

Using the functoriality of Euler-Koszul homology, we now deduce the holonomicity, regularity, and other structural properties of arbitrary binomial D-modules, including the binomial Horn systems which motivated and presaged the developments here. Our first principal result is the specification, for any A-graded binomial ideal I, of an arrangement of finitely many affine subspaces of $\mathbb{C}^d$ such that the binomial D-module $D/H_A(I,\beta)$ is holonomic precisely when $-\beta$ lies outside of it (Theorem 5.3). Moreover, holonomicity occurs if and only if the vector space of local solutions to $H_A(I,\beta)$ has finite dimension. The subspace arrangement arises from the primary decomposition of I into its toral and Andean components. When $D/H_A(I,\beta)$ is holonomic, it is also regular holonomic if and only if I is $\mathbb{Z}$-graded in the standard sense. Finally, we construct another finite affine subspace arrangement in $\mathbb{C}^d$ such that for $-\beta$ outside of it, the binomial D-module splits as a direct sum of primary toral binomial D-modules (Theorem 5.8).

For the duration of this section, fix an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial_1,\ldots,\partial_n]$ and let

$$I = \bigcap_{I_{\rho,J}\in \operatorname{Ass}(I)} \mathcal{C}_{\rho,J} \tag{5.1}$$

be an irredundant primary decomposition as in Theorem 3.2. Thus, as in Examples 4.9 and 4.18, some of the quotients $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ are toral and some are Andean. Much of what we do is independent of the particular primary decomposition, since the data we typically need come from the quasidegrees of certain related modules. For example, the holonomicity in Theorem 5.3 is clearly independent of the primary decomposition.

Definition 5.1. The *Andean arrangement* $\mathcal{Z}_{\text{Andean}}(I)$ is the union of the quasidegree sets $\operatorname{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for the Andean primary components $\mathcal{C}_{\rho,J}$ of I.

Lemma 5.2. *The Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ is a union of finitely many integer translates of the subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$ for which there is an Andean associated prime $I_{\rho,J}$.*

Proof. Apply Lemma 4.5 to an Andean filtration of each Andean component $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. □

Theorem 5.3. *Given the A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, the following are equivalent.*

0. *The vector space of local solutions to $H_A(I,\beta)$ has countable dimension.*
1. *The vector space of local solutions to $H_A(I,\beta)$ has finite dimension.*
2. *The binomial D-module $D/H_A(I,\beta)$ is holonomic.*
3. *The Euler-Koszul homology $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I)$ is holonomic for all i.*
4. *$-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$.*

For I standard $\mathbb{Z}$-graded, these are equivalent to regular holonomicity of $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I)$. Moreover, the existence of a parameter β for which $\mathcal{H}_0(E-\beta;\mathbb{C}[\partial]/I)$ is regular holonomic implies that I is $\mathbb{Z}$-graded.

Proof. The last claim follows from the rest by Theorem 4.12 and results in [Hot91, SW06]. Item 1 trivially implies item 0. Item 2 implies item 1 because holonomic systems have finite rank. Item 3 implies item 2 by Definition 1.3 and Example 4.2. If $-\beta \in \mathcal{Z}_{\text{Andean}}(I)$, then $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for some Andean component $\mathcal{C}_{\rho,J}$, so item 0 implies item 4 by Theorem 4.21 for the surjection $\mathbb{C}[\partial]/I \twoheadrightarrow \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. Finally, item 4 implies item 3 by Theorem 4.12 and Proposition 5.4, below, given that $\mathbb{C}[\partial]/\bigcap_{I_{\rho,J}\text{ toral}}\mathcal{C}_{\rho,J}$ is a submodule of $\bigoplus_{I_{\rho,J}\text{ toral}}\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ and is hence toral. □

Proposition 5.4. *Let $I_{\text{toral}} = \bigcap_{I_{\rho,J}\text{ toral}}\mathcal{C}_{\rho,J}$ be the intersection of the toral primary components of I. If $-\beta$ lies outside of the Andean arrangement of I, then the natural surjection $\mathbb{C}[\partial]/I \twoheadrightarrow \mathbb{C}[\partial]/I_{\text{toral}}$ induces an isomorphism in Euler-Koszul homology:*

$$\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I_{\text{toral}}) \text{ for all } i \text{ when } -\beta \notin \mathcal{Z}_{\text{Andean}}(I).$$

Proof. If I_{Andean} is the intersection of the Andean primary components of I, then

$$\frac{I_{\text{toral}}}{I} = \frac{I_{\text{toral}}}{I_{\text{toral}} \cap I_{\text{Andean}}} \cong \frac{I_{\text{toral}} + I_{\text{Andean}}}{I_{\text{Andean}}}$$

is a submodule of $\mathbb{C}[\partial]/I_{\text{Andean}}$, which in turn is a submodule of $\bigoplus_{I_{\rho,J}\text{ Andean}}\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. Since $\mathcal{Z}_{\text{Andean}}(I)$ is the quasidegree set of this Andean direct sum, the exact sequence

$$0 \to \frac{I_{\text{toral}}}{I} \to \frac{\mathbb{C}[\partial]}{I} \to \frac{\mathbb{C}[\partial]}{I_{\text{toral}}} \to 0$$

yields isomorphisms $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I_{\text{toral}})$ of Euler-Koszul homology for all i, by Lemma 4.19 for I_{toral}/I. □

Now we move on to the question of when $D/H_A(I,\beta)$ splits into a direct sum.

Definition 5.5. The *primary cokernel module* P_I is defined by the exact sequence

$$0 \to \frac{\mathbb{C}[\partial]}{I} \to \bigoplus_{I_{\rho,J}\in\operatorname{Ass}(I)} \frac{\mathbb{C}[\partial]}{\mathcal{C}_{\rho,J}} \to P_I \to 0.$$

The *primary arrangement* is $\mathcal{Z}_{\text{primary}}(I) = \operatorname{qdeg}(P_I) \cup \mathcal{Z}_{\text{Andean}}(I)$.

Proposition 5.6. *The primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ is a union of finitely many integer translates of subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$. If there exists $\beta \in \mathbb{C}^d$ such that the local solution space of $H_A(I,\beta)$ has finite dimension, then $\mathcal{Z}_{\text{primary}}(I)$ is a proper Zariski-closed subset of $\mathbb{C}^d$.*

Proof. The first sentence is by Lemma 4.5. For the second sentence, let $(P_I)_{\text{toral}}$ be the image in P_I of the direct sum $\bigoplus_{\text{toral}}\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. A point in $\operatorname{qdeg}(P_I)$ that does not lie in $\mathcal{Z}_{\text{Andean}}(I)$ must necessarily be a quasidegree of $(P_I)_{\text{toral}}$; that is

$$\mathcal{Z}_{\text{primary}}(I) = \operatorname{qdeg}\big((P_I)_{\text{toral}}\big) \cup \mathcal{Z}_{\text{Andean}}(I). \tag{5.2}$$

The existence of our β immediately implies that $\mathcal{Z}_{\text{Andean}}(I)$ is a proper Zariski-closed subset of $\mathbb{C}^n$, so by (5.2) we need only prove the same thing for $\operatorname{qdeg}((P_I)_{\text{toral}})$. The module $(P_I)_{\text{toral}}$ is supported on the union of the toric subvarieties $T_{\rho,J} = \operatorname{Spec}(\mathbb{C}[\partial]/I_{\rho,J})$ for the toral associated primes of I; this much is by definition. However, the map $\mathbb{C}[\partial]/I \to \bigoplus_{\operatorname{Ass}(I)}\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is an isomorphism locally at a point x whenever x lies in only one of the associated varieties $T_{\rho,J}$ (toral or otherwise). Therefore $(P_I)_{\text{toral}}$ is supported on the union of the pairwise intersections of the toral toric varieties $T_{\rho,J}$ associated to I. Hence it is enough to show that if

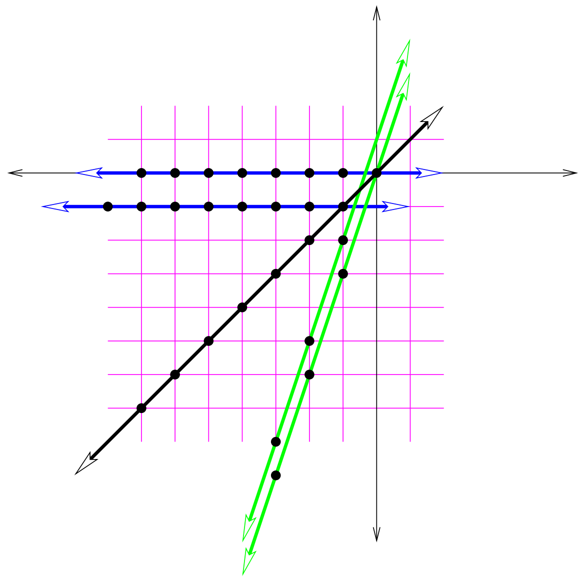

FIGURE 1. Primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ of the binomial ideal I in Example 5.7

R is the coordinate ring of the intersection $T_{\rho,J} \cap T_{\rho',J'}$ of any two distinct toral varieties, then $\mathrm{qdeg}(R)$ is a proper Zariski-closed subset of $\mathbb{C}^d$. This is a consequence of Lemma 4.14. □

Example 5.7. In the situation of Example 3.12, the primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ consists of the five bold lines in Figure 1. The diagonal line through $-\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ is the Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ by Example 4.6. On the other hand, the pairwise intersections of the toral components of I all equal $\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle$. This ideal has primary decomposition

$$\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle = \langle b^2, c, d, b-e\rangle \cap \langle a, b, c^2, b-e\rangle.$$

The set of true degrees of P_I that lie outside of $\mathcal{Z}_{\text{Andean}}(I)$ coincides with the true degree set $\mathrm{tdeg}(\mathbb{C}[a,b,c,d,e]/\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle)$, which consists simply of the A-degrees of the monomials in a, b, c, and d that are nonzero in this quotient. The exponent vectors of these monomials are those of the form

$$\begin{bmatrix}\alpha\\0\\0\\0\end{bmatrix}, \begin{bmatrix}\alpha\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\\delta\end{bmatrix}, \text{ or } \begin{bmatrix}0\\0\\1\\\delta\end{bmatrix}$$

for $\alpha \in \mathbb{N}$ and $\delta \in \mathbb{N}$, so $\mathrm{tdeg}(P_I) \smallsetminus \mathcal{Z}_{\text{Andean}}(I)$ consists of the lattice points having the form

$$\begin{bmatrix}-\alpha\\0\end{bmatrix}, \begin{bmatrix}-\alpha-1\\-1\end{bmatrix}, \begin{bmatrix}-\delta\\-3\delta\end{bmatrix}, \text{ or } \begin{bmatrix}-\delta-1\\-3\delta-2\end{bmatrix},$$

keeping in mind that the degrees of the variables are the negatives of the columns of A. These true degrees are plotted as black dots in Figure 1. The pair of horizontal lines comes from $\langle b^2, c, d, b-e\rangle$, while the pair of steep diagonal lines comes from $\langle b, c^2, d, b-e\rangle$.

Theorem 5.8. *Assume that $-\beta$ lies outside of the primary arrangement $\mathcal{Z}_{\text{primary}}(I)$. Then*

$$\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I) \cong \bigoplus_{I_{\rho,J}\text{ toral}} \mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$$

for all i, the sum being over all toral associated primes of I from (5.1). In particular,

$$D/H_A(I,\beta) \cong \bigoplus_{I_{\rho,J}\text{ toral}} D/H_A(\mathcal{C}_{\rho,J},\beta).$$

Proof. Assume that $-\beta \notin \mathcal{Z}_{\text{primary}}(I)$. Resuming the notation from the proof of Proposition 5.6, we have an exact sequence $0 \to (P_I)_{\text{toral}} \to P_I \to P_I/(P_I)_{\text{toral}} \to 0$. The direct sum $\bigoplus_{\text{Andean}} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ over the Andean components of I surjects onto $P_I/(P_I)_{\text{toral}}$. Hence, by Lemma 4.19, we deduce that $\mathcal{H}_i(E-\beta;P_I/(P_I)_{\text{toral}}) = 0$ for all i. Consequently, $\mathcal{H}_i(E-\beta;P_I) \cong \mathcal{H}_i(E-\beta;(P_I)_{\text{toral}})$ for all i. But the latter is zero for all i by Theorem 4.12 because $-\beta \notin \operatorname{qdeg}(P_I) \supseteq \operatorname{qdeg}((P_I)_{\text{toral}})$. Therefore, applying Euler-Koszul homology to the exact sequence in Definition 5.5, and using Lemma 4.19 to note that this kills the Andean summands, we have proved the first display. The second is simply the $i = 0$ case. □

Here is our final arrangement, outside of which the holonomic rank of $H_A(I,\beta)$ is minimal.

Definition 5.9. Given an A-graded binomial I, the *jump arrangement* of I is the union

$$\mathcal{Z}_{\text{jump}}(I) = \mathcal{Z}_{\text{Andean}}(I) \cup \bigcup_{i=0}^{d-1} \operatorname{qdeg}\big(H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_{\text{toral}})\big)$$

of the Andean arrangement of I with the quasidegrees of the local cohomology of $\mathbb{C}[\partial]/I_{\text{toral}}$ in cohomological degrees at most $d-1$.

Once the holonomic rank of a binomial D-module is minimal, we can quantify it exactly. Let $\mu_{\rho,J}$ be multiplicity of $I_{\rho,J}$ in I (or equivalently, in the primary component $\mathcal{C}_{\rho,J}$ of I). Denote by $\operatorname{vol}(A_J)$ the volume of the convex hull of A_J and the origin, normalized so that a lattice simplex in the group $\mathbb{Z}A_J$ generated by the columns of A_J has volume 1.

Theorem 5.10. *If $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$, then $H_A(I,\beta)$ has minimal rank at β if and only if $-\beta$ lies outside of the jump arrangement $\mathcal{Z}_{\text{jump}}(I)$, and this minimal rank is*

$$\operatorname{rank}\big(D/H_A(I,\beta)\big) = \sum_{I_{\rho,J}\text{ toral of dim. } d} \mu_{\rho,J}\cdot\operatorname{vol}(A_J).$$

Proof. Assume that $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$, and denote by X the complement of $-\mathcal{Z}_{\text{Andean}}(I)$ in $\mathbb{C}^d$. The global Euler-Koszul homology $\mathcal{H}_0(E-b;\mathbb{C}[\partial]/I)$ determines a sheaf on $\mathbb{C}^d$, and hence a sheaf $\mathcal{F}$ on X by restriction. We claim that $\mathcal{F}$ is a holonomic family [MMW05, Definition 2.1] over X. In fact, we claim that $\mathcal{F}$ is the restriction to X of the family determined by $\mathcal{H}_0(E-b;\mathbb{C}[\partial]/I_{\text{toral}})$, which is a holonomic family on all of $\mathbb{C}^d$ by Theorem 4.15. Our claim is immediate from the sheaf (i.e., global Euler-Koszul) version Proposition 5.4, which says that for all i, if $\beta \in X$ then $\mathcal{H}_i(E-b;\mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-b;\mathbb{C}[\partial]/I_{\text{toral}})$ in a neighborhood of β. This follows by the same proof as Proposition 5.4 itself, given the global version of Lemma 4.19. This global version, in turn, follows from the same proof as Lemma 4.19

itself with β_i replaced by b_i for all i, the point being that $b_i = (b_i - \beta_i) + \beta_i$ is a unit locally in $\mathbb{C}^d$ near β, since $b_i - \beta_i$ lies in the maximal ideal at β.

The statement about minimality of rank is now a consequence of Theorem 4.16 for $V = \mathbb{C}[\partial]/I_{\text{toral}}$, noting that the rank is infinite for $\beta \notin X$ by Theorem 5.3. To compute this minimal rank, we may assume that β is as generic as we like. In particular, we assume that $-\beta$ lies outside of the primary arrangement, and also (by Lemma 4.14) outside of $\operatorname{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for the components of dimension less than d. Using Theorem 5.8, we will be done once we show that $H_A(\mathcal{C}_{\rho,J}, \beta)$ has rank $\mu_{\rho,J} \cdot \operatorname{vol}(A_J)$ for generic β.

To do this, take a toral filtration of $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. We are guaranteed that the number of successive quotients of dimension d is precisely the multiplicity of $I_{\rho,J}$ in $\mathcal{C}_{\rho,J}$, and that all of the dimension d successive quotients are actually $\mathbb{Z}^d$-translates of $\mathbb{C}[\partial]/I_{\rho,J}$ itself. Therefore, choosing β to miss the quasidegree sets of the other successive quotients, we find that the rank of $H_A(\mathcal{C}_{\rho,J}, \beta)$ equals the multiplicity $\mu_{\rho,J}$ times the generic rank of $H_A(I_{\rho,J}, \beta) = H_{A_J}(I_{\rho,J}, \beta)$, which is $\operatorname{vol}(A_J)$ by [Ado94]. $\square$

Remark 5.11. The arrangement that we should require $-\beta$ to avoid for β to be called truly *generic* is the union of the jump arrangement $\mathcal{Z}_{\text{jump}}(I)$ and the *top arrangement* $\mathcal{Z}_{\text{top}}(I) = \operatorname{qdeg}\big(\bigoplus_{\text{toral}<d} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}\big)$, where the direct sum is over all toral components of I with $\dim(\mathbb{C}[\partial]/I_{\rho,J}) \leq d-1$. For $-\beta \notin \mathcal{Z}_{\text{jump}}(I) \cup \mathcal{Z}_{\text{top}}(I)$, the module $D/H_A(I, \beta)$ has minimal holonomic rank and decomposes as a direct sum over the dimension d toral components.

Corollary 5.12. *If I is standard $\mathbb{Z}$-graded without any Andean components, and $\mathbb{C}[\partial]/I$ has Krull dimension d, then the generic rank of $H_A(I, \beta)$ equals the $\mathbb{Z}$-graded degree of I.*

We close this section by illustrating a particular case of a Mellin system [Mel21, DS06]. Such systems arise when showing that algebraic functions satisfy hypergeometric equations. The goal of the example is to give an instance when the local solution space of the binomial D-module $D/H_A(I, \beta)$ for some nonzero parameter β fails to split as a direct sum of the local solution spaces to binomial D-modules arising from components. Note that $\beta = 0$ always lies in the primary arrangement: the residue field $\mathbb{C} = \mathbb{C}[\partial]/\mathfrak{m}$ is a quotient of every primary component $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ because the A-grading is positive (i.e., $\mathbb{N}A$ is a pointed semigroup).

Example 5.13. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 0 & 3 \\ -1 & -2 \end{bmatrix}.$$

In this case we have

$$I_{\mathbb{Z}B} = I(B) = \langle \partial_1^2\partial_4 - \partial_2^3, \partial_1\partial_4^2 - \partial_3^3 \rangle \subseteq \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4].$$

That is, the lattice basis ideal $I(B)$ coincides with the lattice ideal $I_{\mathbb{Z}B}$. The primary decomposition of $I_{\mathbb{Z}B}$ is obtained from that of the ideal I in Examples 3.12, 4.6, and 5.7 by omitting the Andean component $\langle a, c, d \rangle$ and erasing all occurrences of $b - e$. Thus the primary arrangement of $I_{\mathbb{Z}B}$ consists of the four lines in Figure 1 corresponding to toral components.

Let $\beta = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$. The solutions of the system $H_A\big(I_{\mathbb{Z}B}, \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]\big)$ are as follows. For $x = (x_1, x_2)$, let $z_1(t)$, $z_2(t)$ and $z_3(t)$ be the local roots in a neighborhood of $(0, 0)$ of

$$z^3 + x_1 z^2 + x_2 z + 1 = 0.$$

By [Stu00], a local basis of solutions of the A-hypergeometric system $H_A\big(\left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]\big) = H_A\big(I_A, \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]\big)$ for the toric ideal I_A (1.3) is given by the three roots of the homogeneous equation

$$x_0 z^3 + x_1 z^2 + x_2 z + x_3 = 0,$$

and the solutions for the other two components are the roots of

$$x_0 z^3 + x_1 z^2 + \omega x_2 z + x_3 = 0 \quad \text{and} \quad x_0 z^3 + x_1 z^2 + \omega^2 x_2 z + x_3 = 0,$$

where ω is a primitive cube root of 1. The system $H_A\big(I_{\mathbb{Z}B}, \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]\big)$ has nine algebraic solutions coming from the roots $z = z(x_0, x_1, x_2, x_3)$ of the above equations.

This looks good: the quotient $\mathbb{C}[\partial]/I_{\mathbb{Z}B}$ is Cohen-Macaulay, so $\mathcal{H}_0(E - \beta, \mathbb{C}[\partial]/I_{\mathbb{Z}B})$ has holonomic rank that is constant as a function of $\beta \in \mathbb{C}^2$, by the rank minimality in Theorem 5.10, and equal to 9 because $\operatorname{vol}(A) = 3$.

However, the nine algebraic solutions mentioned above only span a vector space of dimension 7, not 9. This means that there are two extra linearly independent local solutions, which are non-algebraic; see [DS06, Example 4.2, Theorem 4.3, Example 4.4].

The binomial D-module explanation for this collapsing from dimension 9 to dimension 7, and the concomitant extra two logarithmic solutions, is that $-\beta = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \mathcal{Z}_{\text{primary}}(I_{\mathbb{Z}B})$; again see Figure 1. Let us be more precise. The exact sequence in Definition 5.5 reads

$$0 \to \mathbb{C}[\partial]/I_{\mathbb{Z}B} \to R_0 \oplus R_1 \oplus R_2 \to P_{I_{\mathbb{Z}B}} \to 0,$$

where $R_i = \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4]/\langle \omega^i \partial_2\partial_3 - \partial_1\partial_4, \partial_2^2 - \omega^i \partial_1\partial_3, \omega^{2i}\partial_3^2 - \partial_2\partial_4 \rangle$. The surjection to $P_{I_{\mathbb{Z}B}}$ factors through the projection $R_0 \oplus R_1 \oplus R_2 \to \overline{R} \oplus \overline{R} \oplus \overline{R}$, where $\overline{R}$ is the monomial quotient $\mathbb{C}[\partial]/\langle \partial_2\partial_3, \partial_1\partial_4, \partial_2^2, \partial_1\partial_3, \partial_3^2, \partial_2\partial_4 \rangle$, the coordinate ring of the intersection scheme of any pair of irreducible components of the variety of $I_{\mathbb{Z}B}$. The image of $\mathbb{C}[\partial]/I_{\mathbb{Z}B}$ in this projection is the diagonal copy of $\overline{R}$, so $P_{I_{\mathbb{Z}B}}$ is a direct sum $\overline{R} \oplus \overline{R}$ of two copies of $\overline{R}$.

On the other hand, each of the rings R_i is also Cohen-Macaulay, so the only nonvanishing Euler-Koszul homology of $R_0 \oplus R_1 \oplus R_2$ is the zeroth. Thus we have an exact sequence

$$0 \to \mathcal{H}_1(E - \beta; P_{I_{\mathbb{Z}B}}) \to D/H_A(I_{\mathbb{Z}B}, \beta) \to \bigoplus_{i=0}^{2} \mathcal{H}_0(E - \beta; R_i) \to \mathcal{H}_0(E - \beta; P_{I_{\mathbb{Z}B}}) \to 0.$$

In general, for $-\beta$ lying on precisely one of the four lines in $\mathcal{Z}_{\text{primary}}(I_{\mathbb{Z}B}) = \operatorname{qdeg}(P_{I_{\mathbb{Z}B}})$, the leftmost and rightmost D-modules here have rank precisely 2, and this is the 2 that causes the dimension collapse and the pair of logarithmic solutions to appear.

Given our choice of parameter $\beta = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$, for instance, the $\mathcal{H}_1$ and the $\mathcal{H}_0$ in question are isomorphic to one another, since both are isomorphic to a direct sum of two copies of $\mathcal{H}_0(E - \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]; \partial_3\mathbb{C}[\partial]/\langle \partial_1, \partial_2, \partial_3 \rangle)$, where the ∂_3 in front of $\mathbb{C}[\partial]$ means to take an appropriate A-graded translate (namely by $\deg(\partial_2) = -\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$); this corresponds to the upper of the two steep diagonal lines in Figure 1.

6. HORN D-MODULES

We now return to the Horn hypergeometric D-modules—that is, binomial D-modules arising from lattice basis ideals—that motivated this work. Theorem 6.10, the main result of this section, provides a combinatorial formula for the generic rank of a binomial Horn system by explicitly describing a basis for its local solution space. The basis we construct involves GKZ hypergeometric functions.

Throughout this section, let B and A be integer matrices as in Convention 1.4. Since we have an explicit description for the toral components of a lattice basis ideal $I(B)$, namely Corollary 3.13, we make use of it to compute—just as explicitly—the local solutions for generic $\beta \in \mathbb{C}^d$ of the corresponding hypergeometric system.

Convention 6.1. Suppose that after permuting the rows and columns of B, there results a decomposition of B as in (3.4), where M is a $q \times p$ matrix of full rank q. Write $\bar{J} = \bar{J}(M)$ for the q rows occupied by M inside of B (before permuting), and let $J = \{1, \ldots, n\} \smallsetminus \bar{J}$ be the rows occupied by B_J. Split the variables $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$ into two blocks each:

$$x_J = \{x_j : j \in J\} \quad \text{and} \quad x_{\bar{J}} = \{x_j : j \notin J\}.$$
$$\partial_J = \{\partial_j : j \in J\} \quad \text{and} \quad \partial_{\bar{J}} = \{\partial_j : j \notin J\}.$$

As before, A_J is the submatrix of A with columns $\{a_j : j \in J\}$.

With the notation above, fix for the remainder of this article a toral prime $I_{\rho,J}$ of $I(B)$. Since $I(B)$ is generated by $m = n - d$ elements, $I_{\rho,J}$ has dimension at least d. On the other hand, toral primes can have dimension at most d, by Lemma 4.14. Thus we have:

Lemma 6.2. *All toral primes of a lattice basis ideal $I(B)$ have dimension exactly d and are minimal primes of $I(B)$.* □

Observation 6.3. Since the dimension of $I_{\rho,J}$ equals $n - p - (m - q) = d + q - p$, if $I_{\rho,J}$ is toral, the previous lemma implies that $q = p$. Thus, from now on, the matrix M is a $q \times q$ mixed invertible matrix (and q is allowed to be 0).

Recall that for a minimal prime $I_{\rho,J}$, the component $\mathcal{C}_{\rho,J}$ can be written as in Example 3.14:

$$\mathcal{C}_{\rho,J} = I(B) + I_\rho + U_M,$$

where $U_M \subseteq \mathbb{C}[\partial_{\bar{J}}]$ is the ideal $\mathbb{C}$-linearly spanned by all monomials whose exponent vectors lie in the union of the unbounded M-subgraphs of $\mathbb{N}^{\bar{J}}$.

In order to construct local solutions of $H_A(\mathcal{C}_{\rho,J}, \beta)$ we need two ingredients: local solutions of the GKZ-type system $H_{A_J}(I_\rho, \beta')$ and polynomial solutions of the constant coefficient system $I(M)$ from (2.1). As it turns out, solving the differential equations $I(M)$ is equivalent to finding the M-subgraphs of $\mathbb{N}^{\bar{J}}$.

Proposition 6.4. *Let M be a square mixed invertible integer matrix. For each M-subgraph Γ of $\mathbb{N}^{\bar{J}}$ fix some $\gamma \in \Gamma$.*

1. *The system $I(M)$ of differential equations has a unique formal power series solution of the form $G_{\Gamma,\gamma} = \sum_{u \in \Gamma} \lambda_u x^u$ in which $\lambda_\gamma = 1$.*

2. *The other coefficients λ_u of $G_{\Gamma,\gamma}$ for $u \in \Gamma$ are all nonzero.*
3. *The set $\{G_{\Gamma,\gamma} : \Gamma$ is an M-subgraph of $\mathbb{N}^{\bar{J}}\}$ is a basis for the space of all formal power series solutions of $I(M)$.*
4. *The set $\{G_{\Gamma,\gamma} : \Gamma$ is a bounded M-subgraph of $\mathbb{N}^{\bar{J}}\}$ is a basis for the space of polynomial solutions of $I(M)$.*

Proof. If $\Gamma = \{\gamma\}$ then $G_{\Gamma,\gamma} = x^\gamma$. We check that this is a solution of $I(M)$ working by contradiction. Let w be a column of M such that $\partial^{w_+}x^\gamma \neq \partial^{w_-}x^\gamma$. Then one of these terms is nonzero, say $\partial^{w_+}x^\gamma$, so that $\gamma - w_+ \in \mathbb{N}^{\bar{J}}$. But then $\gamma - w_+ + w_- = \gamma - w \in \mathbb{N}^{\bar{J}}$, and so $\gamma - w \in \Gamma$, a contradiction, because $\gamma - w \neq \gamma$ and Γ is a singleton.

Now assume that Γ is not a singleton, and fix $u \in \Gamma$ such that $u-\gamma = w$ is a column of M. We want to define the coefficients of $G_{\Gamma,\gamma}$, and we will start with λ_u. Since $u-\gamma = w = w_+-w_-$, we have $u-w_+ = \gamma-w_- \in \mathbb{N}^{\bar{J}}$, since u and γ both lie in $\mathbb{N}^{\bar{J}}$ and the supports of w_+ and w_- are disjoint. Set $\lambda_u = \partial^{w_-}(x^\gamma)/\partial^{w_+}(x^u)$, and observe that numerator and denominator are *nonzero* constant multiples of $x^{u-w_+} = x^{\gamma-w_-}$. Use this procedure to define the coefficients corresponding to the neighbors of γ. Now, if we know λ_u and we are given a neighbor $u' \in \Gamma$ of u, say $u' - u = w$, then set $\lambda_{u'} = \partial^{w_-}(x^u)/\lambda_u\partial^{w_+}(x^{u'})$. Propagating this procedure along Γ we obtain all of the coefficients λ_u. The formal power series $G_{\Gamma,\gamma}$ defined this way is tailor-made to be a solution of $I(M)$.

Since M-subgraphs are disjoint, it is clear that the series $G_{\Gamma,\gamma}$ are linearly independent. Now let $G = \sum_{u\in\mathbb{N}^q} \nu_u x^u$ be a formal power series solution of $I(M)$. We claim that $G - \nu_\gamma G_{\Gamma,\gamma}$ has coefficient zero on all monomials from Γ. This follows from the fact that $G-\nu_\gamma G_{\Gamma,\gamma}$ has coefficient zero on the monomial x^γ; indeed, if the difference contained a monomial from Γ, it would have to contain x^γ with a nonzero coefficient, as can be seen by the propagation argument from before. (The uniqueness of $G_{\Gamma,\gamma}$ also follows from this argument.) That our candidate for a basis of the power series solutions spans is now clear, and the statement for polynomial solutions has the same proof. $\square$

Remark 6.5. The system $I(M)$ is itself a Horn system; there are no Euler operators because M is invertible. We stress that it is a very special feature of hypergeometric differential equations that their irreducible (Puiseux) series solutions are determined (up to a constant multiple) by their supports. In general, this is far from being the case for systems of differential equations that are not hypergeometric.

We can use this correspondence between M-subgraphs and solutions of $I(M)$ to compute examples.

Example 6.6. Consider the 3×3 matrix

$$M = \begin{bmatrix} 1 & -5 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

A basis of solutions (with irreducible supports) of $I(M)$ is easily computed:

$$1, \quad x+y+z, \quad (x+y+z)^2, \quad (x+y+z)^3, \quad \text{and} \sum_{n\geq 4} \frac{(x+y+z)^n}{n!}.$$

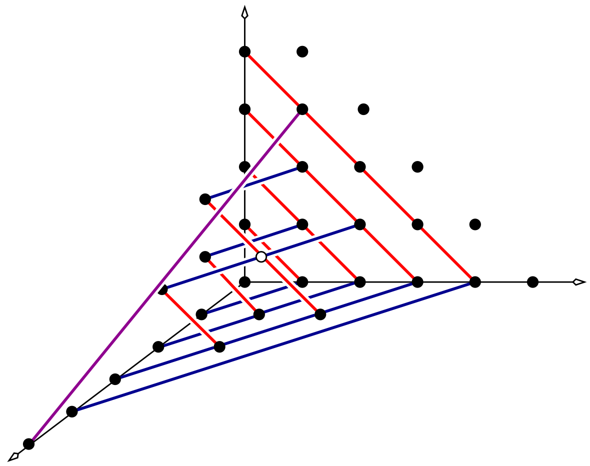

FIGURE 2. The M-subgraphs of $\mathbb{N}^3$

The M-subgraphs of $\mathbb{N}^3$ are the four slices $\{(a,b,c) \in \mathbb{N}^3 : a+b+c = n\}$ for $n \leq 3$; for $n \geq 4$, two consecutive slices are M-connected by $(5,-1,3)$, yielding one unbounded M-subgraph.

The following definition will allow us to determine a set of parameters β for which the system $H_A(\mathcal{C}_{\rho,J},\beta)$ has the explicit basis of solutions that we construct for Theorem 6.10.

Definition 6.7. A *facet* of A_J is a subset of its columns that is maximal among those minimizing nonzero linear functionals on $\mathbb{Z}^d$. For a facet σ of A_J let ν_σ be its *primitive support function*, the unique rational linear form satisfying

(1) $\nu_\sigma(\mathbb{Z}A_J) = \mathbb{Z}$,
(2) $\nu_\sigma(a_j) \geq 0$ for all $j \in J$,
(3) $\nu_\sigma(a_j) = 0$ for all $a_j \in \sigma$.

A parameter vector $\beta \in \mathbb{C}^d$ is A_J-*nonresonant* if $\nu_\sigma(\beta) \notin \mathbb{Z}$ for all facets σ of A_J. Note that if β is A_J-nonresonant, then so is $\beta + A_J(\gamma)$ for any $\gamma \in \mathbb{Z}^J$.

The reason nonresonant parameters are convenient to work with is the following.

Lemma 6.8. *If β is A_J-nonresonant, then for any $\gamma \in \mathbb{N}^J$, and for all torus translates I_ρ of the toric ideal I_{A_J}, right multiplication by ∂_J^γ induces a D_J-module isomorphism $D_J/H_{A_J}(I_\rho,\beta) \to D_J/H_{A_J}(I_\rho,\beta + A_J(\gamma))$, whose left inverse we denote by $\partial_J^{-\gamma}$.*

Proof. For $R = \mathbb{C}[\partial_J]/I_\rho$ there is an exact sequence $0 \to R \xrightarrow{\partial_J^\gamma} R \to R/\partial_J^\gamma R \to 0$. Since the multiplication by ∂_J^γ occurs in the right-hand factor of $D_J \otimes_{\mathbb{C}[\partial_J]} R$, the map on Euler-Koszul homology over D_J induced by ∂_J^γ corresponds to right multiplication. But $R/\partial_J^\gamma R$ is toral by Lemma 4.10, and its set of quasidegrees is the Zariski closure of $\{-A_J\vartheta :$

$\vartheta \in \mathbb{N}^J, \vartheta_i < \gamma_i$ for some $i \in J\}$, which is a finite subspace arrangement contained in the resonant parameters. Now apply Lemma 4.3 and Theorem 4.12 to complete the proof. $\square$

Remark 6.9. Denote by $\mathrm{Sol}(I_\rho, \beta)$ the space of local holomorphic solutions of $H_{A_J}(I_\rho, \beta)$ near a nonsingular point. Given $\gamma \in \mathbb{N}^J$, the D-module isomorphism in Lemma 6.8 induces a vector space isomorphism

$$\mathrm{Sol}(I_\rho, \beta) \longleftarrow \mathrm{Sol}(I_\rho, \beta + A_J\gamma)$$

given by differentiation by ∂_J^γ. If we denote the inverse of this map by $\partial_J^{-\gamma}$, a number of questions arise: for instance, given a local solution $f \in \mathrm{Sol}(I_\rho, \beta)$ where β is very generic, and taking for instance $J = \{1, 2\}$,

- is $\partial_{\{1,2\}}^{-(1,0)}\big(\partial_{\{1,2\}}^{-(0,1)} f\big)$ equal to $\partial_{\{1,2\}}^{-(1,1)} f$?
- is $\partial_{\{1,2\}}^{(1,1)}\big(\partial_{\{1,2\}}^{-(2,2)} f\big)$ equal to $\partial_{\{1,2\}}^{-(1,1)} f$?

Both questions have positive answers; their verification is based on the fact that the left and right inverses of a vector space isomorphism are the same. We conclude that $\partial_J^{-\gamma} f$ is well-defined for any $f \in \mathrm{Sol}(I_\rho, \beta + A_J\gamma)$ where β is very generic, and γ is an arbitrary *integer* vector.

At the level of D-modules $\partial_J^{-\gamma}$, $\gamma \in \mathbb{Z}^J$, is not well-defined, because the right and the left inverses of a D-isomorphism need not coincide.

Let φ be a polynomial solution of $I(M)$ having the form

$$\varphi = x_{\bar{J}}^{\alpha} \sum_{\alpha + Mv \in \Gamma_\alpha} c_v x_{\bar{J}}^{Mv},$$

where $\alpha \in \mathbb{N}^{\bar{J}}$ and Γ_α is the (bounded) M-subgraph containing α. We can choose a basis for the polynomial solution space for $I(M)$ consisting of such polynomials by Proposition 6.4. The cardinality of this basis is the number of bounded M-subgraphs, which we denote by μ_M. If $q = 0$, we assume that $\varphi = 1$ and set $\mu_M = 1$.

Given a local solution $f = f(x_J)$ of the system $H_{A_J}(\beta - A_{\bar{J}}(\alpha))$ for some $\alpha \in \mathbb{N}^{\bar{J}}$, define

$$F_{\varphi, f} = x_{\bar{J}}^{\alpha} \sum_{\alpha + Mv \in \Gamma_\alpha} c_v x_{\bar{J}}^{Mv} \partial_J^{-Nv}(f), \tag{6.1}$$

where $\partial_J^{-Nv} f$ is as in Remark 6.9. Note that if $q = 0$, we have $F_{1,f} = f$.

Consider a parameter vector $\beta \in \mathbb{C}^d$ such that $\beta - A_{\bar{J}}(\alpha)$ is A_J-nonresonant, for the finitely many $\alpha \in \mathbb{N}^{\bar{J}}$ that we fixed in our basis of the polynomial solutions of $I(M)$. We call such parameters β *very generic*. This condition is open and dense in the standard topology of $\mathbb{C}^d$, so that the rank of $H_A(\mathcal{C}_{\rho,J}, \beta)$ for such parameters equals the generic rank of this binomial D-module, in the sense of Theorem 5.10.

Theorem 6.10. *Fix a toral component $\mathcal{C}_{\rho,J}$ of $I(B)$ and a very generic parameter $\beta \in \mathbb{C}^d$. If φ runs over a basis for the polynomial solutions of $I(M)$ as in Proposition 6.4, and f runs over a local basis for the solutions of $H_{A_J}(I_\rho, \beta - A_{\bar{J}}(\alpha))$, then the $\mu_M \cdot \mathrm{vol}(A_J)$ many functions $F_{\varphi,f}$ form a local basis for the solution space of the binomial D-module $D/H_A(\mathcal{C}_{\rho,J}, \beta)$.*

Remark 6.11. Let us think about the regular ($\mathbb{Z}$-graded) case for a moment. We saw in the Introduction that a solution of $H_A(\beta)$ (or any of the binomial D-modules arising from a torus translate of I_A) is really a function in $m = n - d$ variables. In fact, for generic β, if we choose canonical series expansions as in [SST00], then their supports are translates cones of dimension m. This implies that the support of the series (6.1) has dimension $m - q$ (since the dimension of the support is given by f). In fact, this support is no longer the set of lattice points in a cone, but in a polyhedron whose recession cone has the correct dimension. Thus, the only fully supported solutions of $H_A(I(B),\beta) = H(B,\beta)$ arise from $H_A(I_{\mathbb{Z}B},\beta)$. In particular, there are no solutions with support of dimension $m-1$, because a matrix with $q = 1$ row is never mixed.

Before proving Theorem 6.10, let us see the construction (6.1) in some explicit examples.

Example 6.12. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 10 & 0 & 7 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \\ 4 & 5 & 0 \\ -3 & -5 & 0 \end{bmatrix}.$$

We concentrate on the decomposition

$$M = \begin{bmatrix} 4 & 5 \\ -3 & -5 \end{bmatrix}; \quad N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B_J = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

Note that $\mathbb{Z}B_J$ is saturated, so there is only one associated prime coming from this decomposition, namely $I_{\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]} + \langle \partial_4, \partial_5 \rangle$, and this is toral since $\det(M) \neq 0$.

The polynomial $\varphi = 5x_4^4x_5^2 + 2x_4^5 + 2x_5^5 + 40x_4x_5^3$ is a solution of the constant coefficient system $I(M)$. Let f be a local solution of the $\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]$-hypergeometric system that is homogeneous of degree $\beta - \left[\begin{smallmatrix} 6 \\ 40 \end{smallmatrix}\right]$. It can be verified that the following function is a solution of $H(B,\beta)$:

$$5x_4^4x_5^2 f + 2x_4^5\partial_1^1\partial_2^{-1}\partial_3^{-1} f + 2x_5^5\partial_2^{-1} f + 40x_4x_5^3\partial_1\partial_2^{-2}\partial_3^{-1} f.$$

In this example, the new solution we constructed has 1-dimensional support.

Example 6.13. Our procedure for constructing solutions works even when M is an $m \times m$ matrix, i.e., M is a maximal square submatrix of B. For instance, consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We concentrate on the component

$$M = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}; \quad N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad B_J = \varnothing.$$

Again, we only have one (toral) component, associated to $\langle \partial_3, \partial_4\rangle$. Let $p = x_1^2+2x_2$. This is a solution of $I(M) = \langle \partial_1^2 - \partial_2, \partial_1^3 - \partial_2^2\rangle$. Since B_J is empty, we need only consider solutions of the homogeneity equations that are functions of x_3 and x_4. Since $\det(M) = 1 \neq 0$, the complementary minor of A is also nonzero, and therefore there exists a unique monic monomial in x_3 and x_4 of each degree. To make a solution of $I + \langle E - \beta\rangle$, let $h = x_3^{w_3}x_4^{w_4}$ be the unique monic solution of the homogeneity equations with parameter $\beta - \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Then

$$w_4x_1^2x_3^{w_3}x_4^{w_4-1} + 2x_2x_3^{w_3}x_4^{w_4} = x_3^{w_3}x_4^{w_4-1}(w_4x_1^2 + 2x_2)$$

is the desired solution of $H(B, \beta)$.

Proof of Theorem 6.10. First note that if $q = 0$, all of the statements hold by construction. Therefore we assume that $q \geq 2$.

It is clear that all of the binomial generators of I_ρ annihilate $F_{\varphi,f}$. It is also easy to check that $F_{\varphi,f}$ satisfies the desired homogeneity equations. Let then $(\nu, \delta) \in \mathbb{Z}^J \times \mathbb{Z}^{\bar{J}}$ be one of the q columns of B involving N and M; i.e., $(\nu, \delta) = \left[\begin{smallmatrix}N\\M\end{smallmatrix}\right] e_k$ for some $k \in \bar{J}$. To prove that $(\partial_J^{\nu_+}\partial_{\bar{J}}^{\delta_+} - \partial_J^{\nu_-}\partial_{\bar{J}}^{\delta_-})(F_{\varphi,f}) = 0$, notice that $(\partial_{\bar{J}}^{\delta_+} - \partial_{\bar{J}}^{\delta_-})(\varphi) = 0$, which implies that for all v with $c_v \neq 0$, either $\partial_{\bar{J}}^{\delta_+}(x^{\alpha+Mv}) = 0$ or there exists another integer vector w with $c_w \neq 0$ such that

$$\partial_{\bar{J}}^{\delta_+}(c_v x_{\bar{J}}^{\alpha+Mv)}) = \partial_{\bar{J}}^{\delta_-}(c_w x_{\bar{J}}^{\alpha+Mw}).$$

In the first case, $\partial_J^{\nu_+}\partial_{\bar{J}}^{\delta_+}\left(c_v x_{\bar{J}}^{\alpha+Mv}\partial_J^{N(-v)}(f)\right) = 0$. In the second case, since the monomials on the left and right-hand sides of the above equation must have the same exponent vector, we see that $\delta = M(v - w) = M \cdot e_k$. But M is invertible by assumption, so that $v - w = e_k$. This implies that $\nu = Ne_k = N(v - w)$.

Consequently, $\partial_J^{N(-v)+\nu_+}(f) = \partial_J^{N(-v)+\nu_-}(f)$, and thus

$$\partial_{\bar{J}}^{\delta_+}\partial_J^{\nu_+}\left(c_v x_{\bar{J}}^{\alpha+Mv}\partial_J^{N(-v)}(f)\right) = \partial_{\bar{J}}^{\delta_-}\partial_J^{\nu_-}\left(c_w x_{\bar{J}}^{\alpha+Mw}\partial_J^{N(-w)}(f)\right).$$

Moreover, it is clear that the $F_{\varphi,f}$ are linearly independent.

Now we need to show that these functions span the local solution space of $H_A(\mathcal{C}_{\rho,J}, \beta)$. Let $F = F(x_1, \dots, x_n)$ be a local solution of $H_A(\mathcal{C}_{\rho,J}, \beta)$. Here we use the explicit description of $\mathcal{C}_{\rho,J}$ from Example 3.14. Since the monomials in U_M annihilate F, we can write

$$F = \sum_{\alpha:\Gamma_\alpha \text{ is bounded}} x_{\bar{J}}^\alpha h_\alpha(x_J),$$

where the sum runs over $\alpha \in \mathbb{N}^{\bar{J}}$ such that the M-subgraph Γ_α to which α belongs is bounded. That is, the sum runs over all α such that $\partial_{\bar{J}}^\alpha$ does not belong to U_M.

The functions h_α are solutions of $H_A(I_\rho, \beta - A_{\bar{J}}(\alpha))$, as is easy to check. Note that h_α may be zero.

Now it is time to use the equations $I(B)$. First, observe that we may assume that the $x_{\bar{J}}$-monomials in F belong to a single M-subgraph of $\mathbb{N}^{\bar{J}}$. This is because the only equations relating different summands from F are those from $I(B)$, which will relate a summand

$x_{\bar{J}}^{\alpha} h_\alpha(x_J)$ to a different summand $x_{\bar{J}}^{\gamma} h_\gamma(x_J)$ exactly when $\alpha - \gamma$ or $\gamma - \alpha$ is a column of M, thus staying within an M-subgraph.

So fix a bounded M-subgraph Γ and write

$$F = \sum_{\alpha \in \Gamma} x_{\bar{J}}^{\alpha} h_\alpha(x_J).$$

Fix $\alpha \in \Gamma$ such that $h_\alpha \neq 0$, write φ for the polynomial solution of $I(M)$ whose support is Γ, and let c be the (nonzero) coefficient of $x_{\bar{J}}^{\alpha}$ in φ. We want to show that $F = (1/c)F_{\varphi, h_\alpha}$. Since we know that $F - (1/c)F_{\varphi, h_\alpha}$ has support contained in Γ and has no summand with x^α, the desired equality will be a consequence of the following.

Claim. With the notation above, if $h_\alpha = 0$ then $F = 0$.

Proof of the Claim. Pick $\gamma \in \Gamma$ such that $\gamma - \alpha$ or $\alpha - \gamma$ is a column of M, say $\alpha - \gamma = Me_k$. The binomial from the corresponding column of B is $\partial_J^{Ne_{k+}} \partial_{\bar{J}}^{Me_{k+}} - \partial_J^{Ne_{k-}} \partial_{\bar{J}}^{Me_{k-}}$. Since this binomial annihilates F, and $\alpha - (Me_k)_+ = \gamma - (Me_k)_-$, we have

$$\partial_{\bar{J}}^{(Me_k)_+} x_{\bar{J}}^{\alpha} \partial_J^{(Ne_k)_+} h_\alpha = \partial_J^{(Me_k)_-} x_{\bar{J}}^{\gamma} \partial^{(Ne_k)_-} h_\gamma,$$

so that, as $h_\alpha = 0$,

$$\partial_{\bar{J}}^{(Me_k)_-} x_{\bar{J}}^{\gamma} \partial_J^{(Ne_k)_-} h_\gamma = 0.$$

Now, the first derivative in the previous expression is nonzero, so $\partial_J^{(Ne_k)_-} h_\gamma = 0$. But then $h_\gamma = 0$, since differentiation in any of the x_J variables is an isomorphism (which is why we need our parameter to be very generic).

Propagate the previous argument along Γ to finish the proof of the claim, and with it the proof of the theorem. □

Remark 6.14. When $\mathcal{C}_{\rho,J}$ is Andean (and β is generic), the above procedure produces no solutions, as expected, since in this case, $D/H_A(\mathcal{C}_{\rho,J}, \beta) = 0$ for generic β. The reason that the construction breaks in this situation is that there are no solutions for the "toric" part.

Corollary 6.15. *Let B as in Convention 1.4. If there exists a parameter β for which the binomial Horn system $H(B, \beta)$ has finite rank, then for generic parameters β, this rank is*

$$\mathrm{rank}(H(B,\beta)) = \sum_{\mathcal{C}_{\rho,J} \text{ toral}} \mu_M \cdot \mathrm{vol}(A_J) = \sum_{B = \left(\begin{smallmatrix} N & B_J \\ M & 0 \end{smallmatrix}\right)} \mu_M \cdot g(B_J) \cdot \mathrm{vol}(A_J),$$

the former sum being over all toral components $\mathcal{C}_{\rho,J}$ of the lattice basis ideal $I(B)$, and the latter sum being over all decompositions of B as in (3.4) with M invertible. Here, $g(B_J)$ is the cardinality of $\mathrm{sat}(\mathbb{Z}B_J)/\mathbb{Z}B_J$, and μ_M is the number of bounded M-subgraphs of $\mathbb{N}^{\bar{J}}$.

Proof. The first equality is a direct consequence of Theorem 6.10 and Theorem 5.8. Note that a comparison to Theorem 5.10 yields the fact that μ_M equals the multiplicity $\mu_{\rho,J}$ of $I_{\rho,J}$ as an associated prime of $I(B)$. For the second equality, the number of components arising from a decomposition (3.4) is $g(B_J)$ [ES96, Corollary 2.5]. □

Remark 6.16. The only sense in which our rank formula for Horn systems is not completely explicit is that it lacks an expression for the number μ_M of bounded M-subgraphs. In the case that $I(M)$ (or $I(B)$) is a complete intersection, Cattani and Dickenstein [CD05] can be applied to provide an explicit recursive formula for $\mu_M = \mu_{\rho,J}$. The general case—even just the toral case—of this computation is an open problem.

Example 6.17. Note in relation to Corollary 6.15, that there are examples of Horn systems $H(B,\beta)$ that fail to be holonomic *for all parameters*. Let

$$A = \begin{bmatrix} -3 & -1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\langle \partial_1, \partial_2 \rangle$ is an Andean prime of $I(B)$. The quasidegree set of the corresponding component is $\mathbb{C}\left[\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right] = \mathbb{C}^2$, which means that the Andean arrangement of $I(B)$ equals $\mathbb{C}^2$ and thus, $H(B,\beta)$ is non-holonomic for all parameters β.

A sufficient condition to guarantee holonomicity of $H(B,\beta)$ for generic parameters is to require that $I(B)$ be a complete intersection. Note that this is automatic for $m = 2$, so that the following result is a direct generalization of [DMS05, Theorem 8.1].

Proposition 6.18. *If $I(B)$ is a complete intersection, then the binomial Horn system $H(B,\beta)$ is holonomic for generic parameters β.*

Proof. If $I(B)$ is a complete intersection, its associated primes all have dimension d. In combinatorial terms, we encounter only square matrices M in the primary decomposition of $I(B)$. The component associated to a decomposition (3.4) is Andean exactly when $\det(M) = 0$, and in this case, the corresponding quasidegree set is $\mathbb{C}A_J \subsetneq \mathbb{C}^d$, as A_J does not have full rank. We conclude that the Andean arrangement of $I(B)$ is strictly contained in $\mathbb{C}^d$. $\square$

When $I(B)$ is standard $\mathbb{Z}$-graded and has no Andean components, we can obtain a cleaner rank formula, by noting that the sum in Corollary 6.15 equals the degree of $I(B)$.

Corollary 6.19. *Assume that $I(B)$ is standard $\mathbb{Z}$-graded and has no Andean components. Let $d_1, \ldots, d_m$ be the degrees of the generators of $I(B)$. Then*

$$\operatorname{rank}(H(B,\beta)) = d_1 \cdots d_m \quad \textit{for all } \beta \in \mathbb{C}^d.$$

Proof. Since $I(B)$ has no Andean components, $\mathbb{C}[\partial]/I(B)$ is toral. Moreover, $\mathbb{C}[\partial]/I(B)$ is Cohen-Macaulay, as $I(B)$ is a complete intersection by Lemma 6.2. Theorem 4.16 implies that the holonomic rank of $H(B,\beta)$ is constant. Now apply Corollary 5.12. $\square$

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(AD) Departamento de Matemática, FCEN, Universidad de Buenos Aires, (1428) Buenos Aires, Argentina.

E-mail address: alidick@dm.uba.ar

(LFM) Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104.

E-mail address: laura@math.tamu.edu

Current address: Department of Mathematics, Texas A&M University, College Station, TX 77843.

(EM) Department of Mathematics, University of Minnesota, Minneapolis, MN 55455.

E-mail address: ezra@math.umn.edu