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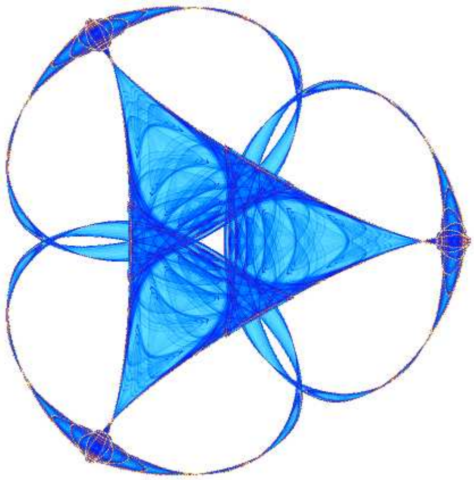

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SCALING LIMITS OF THE CHERN-SIMONS-HIGGS ENERGY

MATTHIAS KURZKE AND DANIEL SPIRN

Abstract. We continue our study in [16] of the Gamma limit of the Abelian Chern-Simons-Higgs energy

$$G_{csh} := \frac{1}{2}\int_U |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|\operatorname{curl} A_\varepsilon - h_{ex}|^2}{|u_\varepsilon|^2} + \frac{1}{\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 dx$$

on a bounded, simply connected, two dimensional domain where $\varepsilon \to 0$ and $\mu_\varepsilon \to \mu \in [0, +\infty]$. Under the critical scaling, $G_{csh} \approx |\log \varepsilon|^2$, we establish the Gamma limit when $\mu \in (0, +\infty]$, and as a consequence we are able to compute the first critical field $H_1 = H_1(U, \mu)$ for the nucleation of a vortex. Finally, we show failure of Gamma convergence when $\mu_\varepsilon \to 0$ (this includes the self-dual case). The method entails estimating in certain weak topologies the Jacobian $J(u_\varepsilon) = \det(\nabla u_\varepsilon)$ in terms of the Chern-Simons-Higgs energy E_{csh}.

1. Introduction

Abelian Chern-Simons-Higgs (CSH) theory serves as an anyon model [4, 9, 8, 24] and is a classical field theory defined on (2+1) dimensional Minkowski space. Such models have applications to the theory of high temperature superconductivity, quantum Hall effects and carry fractional charge values [4, 24]. This model has been the source of much interest in the physics community; the book of Yang [24] offers an excellent overview of Chern-Simons-Higgs and related theories. Letting $D_\alpha = (-\partial_\Phi, \nabla_A)$ then the CSH Lagrangian has the form

$$\mathcal{L}_{CSH} = \int D_\alpha \overline{D^\alpha u} + \frac{\mu^2}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - \frac{1}{\varepsilon^2}|u|^2\left(1-|u|^2\right)^2$$

where $\epsilon^{\alpha\beta\gamma}$ is the antisymmetric tensor. Here F_α is the Maxwell curvature tensor and $\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}$ is the Chern-Simons term. The associated Euler-Lagrange

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(CSH) equations are:

$$\partial_\Phi^2 u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right) \left(3\,|u|^2 - 1\right) \tag{1}$$

$$q = -\frac{\mu^2}{2} \operatorname{curl} A \tag{2}$$

$$j_A = \frac{\mu^2}{2} \left(E \times e_3\right) \tag{3}$$

where $q = (iu, \partial_\Phi u)$, $j_A = (iu, \nabla_A u)$, $E = \partial_t A - \nabla \Phi$, and $h = \operatorname{curl} A$.

Since $u : \mathbb{R}^2 \to \mathbb{C}$ we can induce the formation of topological vortices – regions where $|u| = 0$ and about which the winding number of the phase is nontrivial. Setting $u = \rho e^{i\varphi} \approx e^{i\varphi}$ over $\mathbb{R}^2$ and $\varphi = d\theta$ with $d \in \mathbb{Z}$, then $J_A \approx \operatorname{curl}(\nabla\varphi - A) = \operatorname{curl} \nabla \varphi - h$. Formally, if we integrate the curl of (3) over $\mathbb{R}^2$ then $2\pi d = \int_{\mathbb{R}^2} h dx$. Furthermore, integrating (2) over the plane yields

$$d = \frac{1}{2\pi} \int_{\mathbb{R}^2} h dx = -\frac{1}{\mu^2 \pi} \int_{\mathbb{R}^2} q dx.$$

As in Ginzburg-Landau theory, we see that the current and the magnetic field are quantized about a topological vortex; however, in CSH theory the magnetic field to induces a quantized electric charge, which can have arbitrary values, depending on μ. This quantized electric charge is a fundamental feature of Chern-Simons-Higgs theory.

Since the CSH equations serve as a model for high temperature superconductors, we include the possible presence of an applied magnetic field h_{ex}. If we look for solutions of (1)-(3) that are independent of time by setting $\partial_t u \equiv 0$, then we can remove the electric field potential Φ, and we are left with a set of coupled elliptic PDE's:

$$-\frac{\mu^2}{4} \frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^4} u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right) \left(3|u|^2 - 1\right) \tag{4}$$

$$0 = -\frac{\mu^2}{4} \operatorname{curl} \left(\frac{\operatorname{curl} A - h_{ex}}{|u|^2} \right) + j_A(u). \tag{5}$$

Equations (4)-(5) can be viewed as the Euler-Lagrange equations of the following Chern-Simons-Higgs energy

$$G_{csh}(u, A; h_{ex}) = \frac{1}{2} \int_U |\nabla_A u|^2 + \frac{\mu^2}{4} \frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2} |u|^2 \left(1 - |u|^2\right)^2 dx \tag{6}$$

for an applied magnetic field, h_{ex}, and a bounded, simply connected domain, $U \subset \mathbb{R}^2$. A discussion of the CSH theory on bounded domains can be found in [6].

We briefly describe some features of CSH topological vortices. First of all magnetic fields concentrate in an annular region about each topological vortex. This is in contrast to Ginzburg-Landau vortices, where the magnetic field concentrates at the site of the vortex. This concentration behavior is due to the $\frac{1}{|u|^2}$ part of the second term of (6). Second, in the $\varepsilon \to 0$ limit $|u_\varepsilon|$ relaxes to $\mathbb{S}^1 \cup \{0\}$, as opposed to $\mathbb{S}^1$ in the Ginzburg-Landau case. This implies that non-topological vortices (regions where $|u| = 0$ with trivial winding number about the region) are possible and potentially favorable. However, that such regions are of size $O(\varepsilon)$ if $|u_\varepsilon| \geq c_0 > 0$ on ∂U for some constant $c_0 = c_0(U)$, see [16].

1.1. **Prior results.** Up to now, most attention has focussed on the *self-dual* case where $\mu_\varepsilon \equiv \varepsilon$ and $h_{ex} \equiv 0$. In this case the CSH equations reduce, following Hong-Kim-Pac and Jackiw-Weinberg [8, 9], to a system of first order PDE's. Solutions can be recovered by solving (after a substitution) a Liouville-type elliptic equation, similar to the Jaffe-Taubes approach to solving self-dual solutions in Ginzburg-Landau theory [10]. It is impossible to give an adequate accounting of the extensive results on self-dual solutions to the Chern-Simons-Higgs equations, but we direct the reader to [4, 5, 6, 8, 9, 17, 23, 24] and the references therein.

We turn our attention to non self-dual Chern-Simons-Higgs theory. The first rigorous results to our knowledge for small ε and $\mu = O(1)$ for the CSH functional are those of Han-Kim [7], who studied sequential minimizers $\{u_\varepsilon, A_\varepsilon\}$ of (6) with $A_\varepsilon \equiv 0$ and Dirichlet boundary condition $u_\varepsilon = g$ on ∂U with $|g| = 1$. Their proofs are similar in spirit to Bethuel-Brezis-Helein [2] for the simplified Ginzburg-Landau energy

$$E_{gl}(u) = \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \tag{7}$$

and rely heavily on the maximum principle for $|u_\varepsilon|$. The maximum principle fails when gauge field $A_\varepsilon \not\equiv 0$, so another approach is needed.

In [16] the authors studied (6) with $A_\varepsilon \not\equiv 0$ in the Γ-convergence framework under various energy scalings for $\varepsilon \to 0$ and $\mu_\varepsilon \equiv \mu \in (0, +\infty)$. The techniques used are related to the approach of Jerrard-Soner [12, 13] combined with the Sandier [18] version of the vortex ball construction method of Jerrard [11] and Sandier [18]. Similar to their approach to studying (7), we first study the simplified functional

$$E_{csh}(u) = \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2)^2, \tag{8}$$

which helps us express the energetic limit of G_{csh}. Consider $v_\varepsilon = \frac{j(u_\varepsilon)}{r_\varepsilon}$, $a_\varepsilon = \frac{A_\varepsilon}{s_\varepsilon}$ and $h_{ex} = \frac{H}{t_\varepsilon}$ then

$$G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) \approx E_{csh}(|u_\varepsilon|) + \frac{r_\varepsilon^2}{2}\left[\left\|v_\varepsilon - \frac{s_\varepsilon}{r_\varepsilon}a_\varepsilon\right\|^2_{L^2(U\setminus\cup B_{r_j})}\right.$$
$$\left. + \frac{|\log\varepsilon|}{r_\varepsilon}\|\operatorname{curl} v_\varepsilon\|_{\mathcal{M}(\cup B_{r_j})} + \frac{\mu_\varepsilon^2}{4}\left\|\frac{\frac{s_\varepsilon}{r_\varepsilon}\operatorname{curl} a_\varepsilon - \frac{t_\varepsilon}{r_\varepsilon}H}{|u_\varepsilon|}\right\|^2_{L^2(U)}\right].$$

Such an energetic framework was proved in [16]. The critical scaling $G_{csh} \approx |\log\varepsilon|^2$ is the energy scaling where the order parameter interacts at the same order as the induced magnetic field, and we can see this from a simple scaling argument. Formally, for the induced magnetic field to interact with the order parameter we need both the $\operatorname{curl} v_\varepsilon$ and v_ε terms, so

$$r_\varepsilon \equiv |\log\varepsilon| \qquad s_\varepsilon \equiv r_\varepsilon,$$

and for the external magnetic field to interact with the induced magnetic field we need

$$t_\varepsilon \equiv r_\varepsilon$$

regardless of μ_ε. Thus,

$$G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) \approx \frac{|\log\varepsilon|^2}{2}\left[\|v_\varepsilon - a_\varepsilon\|^2_{L^2} + \|\operatorname{curl} v_\varepsilon\|_{\mathcal{M}} + \frac{\mu_\varepsilon^2}{4}\|\operatorname{curl} a_\varepsilon - H\|^2_{L^2}\right].$$

Therefore, we can describe the $G_{csh} \approx |\log\varepsilon|^2$ energy scaling as "critical".

In [16] we study $G_{csh} \approx |\log\varepsilon|^2$ for $\mu_\varepsilon \equiv \mu \in (0, +\infty)$, among other asymptotic limits. Our general assumptions are that $U \subset \mathbb{R}^2$ is a bounded, simply connected domain with smooth boundary. We take $\|u_\varepsilon\|_{L^\infty(U)} \leq C < +\infty$, so there are no nontopological vortices. For simplicity, we state the result in the Coulomb gauge, which amounts to considering only pairs (u, A) with $\nabla \cdot A = 0$ in U and $A \cdot \nu = 0$ on ∂U. These conditions can always be satisfied by an appropriate gauge transformation replacing (u, A) by $(ue^{i\chi}, A + \nabla\chi)$ without changing the energy. Finally, we assume that $\{u_\varepsilon\}$ is a sequence of functions in $H^1(U;\mathbb{C})$ whose traces on ∂U satisfy $|u_\varepsilon| \geq 1 - \frac{1}{|\log\varepsilon|}$, although we believe this assumption is technical and not crucial to the results. We recall the following results, stated here in the spirit of Γ-convergence; that is, separated into a compactness result combined with a lower bound for the energy and a construction that shows that the lower bound is essentially optimal.

Theorem 1 ([16]). *Assume that the external field satisfies $h_{ex} = H|\log\varepsilon|$ for some $H > 0$ and $\mu_\varepsilon \to \mu \in (0, +\infty)$. Consider a sequence $\{u_\varepsilon, A_\varepsilon\}$ with*

$$G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) \leq K|\log\varepsilon|^2,$$

$|u_\varepsilon| \geq 1 - \frac{1}{|\log\varepsilon|}$ on ∂U, and $\|u_\varepsilon\|_{L^\infty(U)} \leq C$. Set $a_\varepsilon = \frac{1}{|\log\varepsilon|}A_\varepsilon$, $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon)$, and $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|}$. Then $\{a_\varepsilon\}$ is weakly precompact in H^1, $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon)$ converges to v weakly in all L^p, $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2, and $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} \rightharpoonup w = \frac{1}{2}\operatorname{curl} v$ in Radon measure. Furthermore, the energy satisfies

$$\begin{aligned}(9)\quad &\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) \\ &\geq G(v,a;H) = \frac{1}{2}\left(\int_U |v-a|^2 + \frac{\mu^2}{4}|\operatorname{curl} a - H|^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right).\end{aligned}$$

Conversely, for any $a \in H^1(U;\mathbb{R}^2)$ and $v \in L^2(U;\mathbb{R}^2)$ such that $w = \frac{1}{2}\operatorname{curl} v$ is a Radon measure, there exists a sequence $\{u_\varepsilon\}$ in $H^1(U;\mathbb{C})$ with $|u_\varepsilon| = 1$ on ∂U and a sequence $\{A_\varepsilon\} \in H^1(U;\mathbb{C})$ in Coulomb gauge such that $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon) \rightharpoonup v$ in L^2, $w_\varepsilon = \frac{1}{|\log\varepsilon|}J(u_\varepsilon) \rightharpoonup w$ in $(C^{0,\beta})^*$, $a_\varepsilon = \frac{1}{|\log\varepsilon|}A_\varepsilon \rightharpoonup a$ in H^1, and such that (9) holds with equality.

Although Theorem 1 was established with $\mu_\varepsilon \equiv \mu \in (0,+\infty)$ a fixed constant, the $\mu_\varepsilon \to \mu \in (0,+\infty)$ case is a straightforward adaptation of the argument in Section 7 of [16]. As an application of the last theorem, we calculate the critical field h_{crit} for which vortices appear in nonzero minimizers of $G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex})$.

Corollary 2 ([16]). *As $\varepsilon \to 0$, the critical field h_{crit} is given asymptotically by $H_1(\mu)\,|\log\varepsilon|$, where*

$$H_1(\mu) = \frac{2}{\mu^2 \max_{\overline{U}} |z_\mu|}$$

and z_μ is the solution of

$$-\frac{\mu^2}{4}\Delta z_\mu + z_\mu + 1 = 0 \text{ in } U \qquad z_\mu = 0 \text{ on } \partial U.$$

Concerning the dependence on μ, we have that $\mu^2 H_1(\mu) \to 2$ as $\mu \to 0$. Furthermore, $H_1(\mu)$ is decreasing in μ and converges to a limit $\overline{H}(U) > 0$ as $\mu \to \infty$.

The corollary follows from Theorem 1 using some analysis of the limit functional.

1.2. **Results.** It is natural to ask whether the formal behavior of H_1 as $\mu \to 0$ or $\mu \to \infty$ in Corollary 2 hold in a rigorous sense. In particular we let $\mu_\varepsilon \to \mu \in \{0,+\infty\}$ and consider the Γ-limit of

$$G_{csh}(u,A;h_{ex}) = \frac{1}{2}\int_U |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|\operatorname{curl} A_\varepsilon - h_{ex}|^2}{|u_\varepsilon|^2} + \frac{1}{\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2$$

at the critical energy scaling $G_{csh} \approx |\log\varepsilon|^2$. In the $\mu_\varepsilon \to +\infty$ we have

Theorem 3 (Large μ_ε limit). *Assume $\mu_\varepsilon \to \infty$ and $G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) \leq K |\log\varepsilon|^2$. Furthermore, assume $\|u_\varepsilon\|_{L^\infty(U)} \leq C < +\infty$ and $\|u_\varepsilon\|_{L^\infty(\partial U)} \geq 1 - \frac{1}{|\log\varepsilon|}$. Assume that each $(u_\varepsilon, A_\varepsilon)$ is in Coulomb gauge, that is, $\nabla \cdot A_\varepsilon = 0$ in U and $A_\varepsilon \cdot \nu = 0$ on ∂U.*

Set $h_{ex} = H|\log\varepsilon|$, $a_\varepsilon = \frac{1}{|\log\varepsilon|}A_\varepsilon$, and $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon)$, then the following compactness and convergence statements hold:

(1) *$a_\varepsilon \to a = Ha_1$ strongly in H^1, where a_1 is the solution of the system*

$$\begin{aligned} \operatorname{curl} a_1 &= 0 \quad \text{in } U \\ \nabla \cdot a_1 &= 0 \quad \text{in } U \\ a_1 \cdot \nu &= 0 \quad \text{on } \partial U. \end{aligned}$$

(2) *For a subsequence, $v_\varepsilon \rightharpoonup v$ in all L^p with $p < 2$ and $\frac{v_\varepsilon}{|u_\varepsilon|} \to v$ in L^2. Furthermore, $\frac{J(u_\varepsilon)}{|\log\varepsilon|} \to \frac{1}{2}\operatorname{curl} v$ in $\left(C^{0,\beta}\right)^*$.*

The energy satisfies the Γ-$\liminf$ inequality

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} G_{csh}(u_e, A_\varepsilon; h_{ex}) \geq G(v) = \frac{1}{2}\left(\int_U |v-a|^2 + \|\operatorname{curl} v\|_{\mathscr{M}}\right). \tag{10}$$

A gauge-invariant form can be given as

$$G(v, a; H) = \begin{cases} G(v) & \text{if } \operatorname{curl} a = H \\ +\infty & \text{else.} \end{cases} \tag{11}$$

Conversely, for any $a \in H^1(U;\mathbb{R}^2)$ and $v \in L^2(U;\mathbb{R}^2)$ such that $w = \frac{1}{2}\operatorname{curl} v$ is a Radon measure, there exists a sequence $\{u_\varepsilon\}$ in $H^1(U;\mathbb{C})$ with $|u_\varepsilon| = 1$ on ∂U and a sequence $\{A_\varepsilon\} \in H^1(U;\mathbb{C})$ in Coulomb gauge such that $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon) \rightharpoonup v$ in L^2, $w_\varepsilon = \frac{1}{|\log\varepsilon|}J(u_\varepsilon) \rightharpoonup w$ in $(C^{0,\beta})^*$, $a_\varepsilon = \frac{1}{|\log\varepsilon|}A_\varepsilon \rightharpoonup a$ in H^1, and such that (10) holds with equality.

The proof of Theorem 3 relies on a Γ-convergence result for $E_{csh}(u_\varepsilon)$ established in [16], see the Appendix.

As an application of the last theorem, we calculate the critical field h_{crit} for which vortices appear in nonzero minimizers of $G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex})$.

Corollary 4 (Large μ_ε critical field). *As $\varepsilon \to 0$ and $\mu_\varepsilon \to +\infty$, the critical field h_{crit} is given asymptotically by $H_1|\log\varepsilon|$, where*

$$H_1 = \frac{2}{\max_{\overline{U}} |z_\infty|}$$

and z_∞ is the solution of

$$-\Delta z_\infty + 1 = 0 \text{ in } U \qquad z_\infty = 0 \text{ on } \partial U.$$

Hence the $\mu \to \infty$ behavior of H_1 in Corollary 2 holds rigorously. .

For the $\mu_\varepsilon \to 0$ case we find a **failure** of Γ-convergence in the critical $G_{csh} \approx |\log\varepsilon|^2$ scaling. In particular there can be a concentration phenomena of vortices at a single point, since a topological vortex costs very little energy when $\mu_\varepsilon \ll 1$. Therefore, we can pack vortices very close together and still make $G_{csh} \approx |\log\varepsilon|^2$.

Theorem 5 (Small μ_ε compactness failure)**.** *For any $\mu_\varepsilon \to 0$, there exists a sequence $\{u_\varepsilon, A_\varepsilon\}$ with $G_{csh}(u_\varepsilon, A_\varepsilon; 0) \leq K|\log\varepsilon|^2$ such that $\frac{1}{|\log\varepsilon|}j(u_\varepsilon)$ and $\frac{1}{|\log\varepsilon|}\frac{j(u_\varepsilon)}{|u_\varepsilon|}$ are not bounded in L^2.*

Remark 6. In particular Theorem 5 includes the **self-dual** case $\mu_\varepsilon = \varepsilon$ and $h_{ex} = 0$; therefore, the Γ-limit fails for large numbers of vortices.

2. Large μ_ε

The case of very large μ_ε turns out to be very similar to that of finite μ_ε, and we obtain a restricted limit functional by a similar Γ-convergence argument.

Proof of Theorem 3. After rescaling, the CSH energy satisfies

$$G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) = E_{csh}(|u_\varepsilon|) + \frac{|\log\varepsilon|^2}{2}\int_U \left|\frac{v_\varepsilon}{|u_\varepsilon|} - a_\varepsilon|u_\varepsilon|\right|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|\operatorname{curl} a_\varepsilon - H|^2}{|u_\varepsilon|^2}dx;$$

therefore,

$$\begin{aligned}\frac{\mu_\varepsilon^2}{8}\int_U |\operatorname{curl} a_\varepsilon - H|^2 dx &= \frac{\mu_\varepsilon^2}{8}\int_U \frac{|\operatorname{curl} a_\varepsilon - H|^2\,|u_\varepsilon|^2}{|u_\varepsilon|^2}dx \\ &\leq \frac{C^2\mu_\varepsilon^2}{8}\int_U \frac{|\operatorname{curl} a_\varepsilon - H|^2}{|u_\varepsilon|^2}dx \\ &\leq C^2\frac{G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex})}{|\log\varepsilon|^2} \\ &\leq C^2 K,\end{aligned}$$

where $C \geq \|u_\varepsilon\|_{L^\infty(U)}$ is one of our assumptions. From $\int_U |\operatorname{curl} a_\varepsilon - H|^2 \leq \frac{8C^2K}{\mu_\varepsilon^2} \to 0$ as $\varepsilon \to 0$ we obtain that $\operatorname{curl} a_\varepsilon \to H$ strongly in L^2, so $\operatorname{curl}(a_\varepsilon - Ha_1) \to 0$ in L^2. This, combined with the Coulomb gauge condition, implies elliptic regularity, and in turn the strong convergence of $(a_\varepsilon - Ha_1) \to 0$ in $H^1(U)$.

The compactness assertions for v_ε follow as in [16]: We decompose

$$\begin{aligned}G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) &= E_{csh}(u_\varepsilon) - \int_U j(u_\varepsilon)\cdot A_\varepsilon \\ &\quad + \int_U \frac{1}{2}|A_\varepsilon|^2|u_\varepsilon|^2 + \frac{\mu_\varepsilon^2}{8}\frac{|\operatorname{curl} A_\varepsilon - h_{ex}|^2}{|u_\varepsilon|^2}\end{aligned} \tag{12}$$

from which we see $E_{csh}(u_\varepsilon) \le G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) + \int_U j(u_\varepsilon)\cdot A_e$, and we can repeat our reasoning of [16] (which in turn is adapted from [13]) and estimate

$$\begin{aligned}|A_\varepsilon \cdot j(u_\varepsilon)| &\le \frac{1}{4}\frac{|j(u_\varepsilon)|^2}{|u_\varepsilon|^2} + |u_\varepsilon|^2|A_\varepsilon|^2\\ &\le \frac{1}{4}|\nabla u_\varepsilon|^2 + \left(|u_\varepsilon|\left|1-|u_\varepsilon|^2\right| + 1\right)|A_\varepsilon|^2\\ &\le \frac{1}{4}|\nabla u_\varepsilon|^2 + |A_\varepsilon|^2 + \frac{1}{4\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 + 2\varepsilon^2|A_\varepsilon|^4\\ &\le \frac{1}{2}E_{csh}(u_\varepsilon) + |A_\varepsilon|^2 + 2\varepsilon^2|A_\varepsilon|^4,\end{aligned}$$

where we used the simple estimate $x^2 \le x|1-x^2|+1$ for $x \ge 0$. By the uniform H^1 bound on a_ε, it follows via Sobolev embedding that

$$\frac{1}{2}E_{csh}(u_\varepsilon) \le K\left|\log\varepsilon\right|^2 + C\left|\log\varepsilon\right|^2 + C\varepsilon^2\left|\log\varepsilon\right|^4 \le C\left|\log\varepsilon\right|^2. \tag{13}$$

Inequality (13) lets us apply the compactness and lower bound results of Theorem 8 in the Appendix.

We decompose

$$G_{csh}(u_\varepsilon, A_\varepsilon; h_{ex}) = G^1_{csh}(u_\varepsilon, A_\varepsilon) + G^2_{csh}(u_\varepsilon, A_\varepsilon) + G^3_{csh}(u_\varepsilon, A_\varepsilon) + G^4_{csh}(u_\varepsilon, A_\varepsilon),$$

where

$$\begin{aligned}G^1_{csh}(u_\varepsilon, A_\varepsilon) &= E_{csh}(u_\varepsilon)\\ G^2_{csh}(u_\varepsilon, A_\varepsilon) &= \frac{\left|\log\varepsilon\right|^2}{2}\int_U |a_\varepsilon|^2 + \frac{\mu_\varepsilon^2\left|\log\varepsilon\right|^2}{8}\int_U |\operatorname{curl} a_\varepsilon - H|^2\\ G^3_{csh}(u_\varepsilon, A_\varepsilon) &= \frac{\left|\log\varepsilon\right|^2}{2}\int_U \left(|u_\varepsilon|^2 - 1\right)|a_\varepsilon|^2\\ G^4_{csh}(u_\varepsilon, A_\varepsilon) &= -\left|\log\varepsilon\right|^2\int_U a_\varepsilon\cdot v_\varepsilon.\end{aligned}$$

For G^1_{csh} we use the lower bound of Theorem 8 in the Appendix and obtain

$$\liminf_{\varepsilon\to 0}\frac{1}{\left|\log\varepsilon\right|^2}G^1_{csh}(u_\varepsilon, A_\varepsilon) \ge \frac{1}{2}\left(\|v\|^2_{L^2} + \|\operatorname{curl} v\|_{\mathcal{M}}\right).$$

Using the strong H^1 convergence we estimate

$$\liminf_{\varepsilon\to 0}\frac{1}{\left|\log\varepsilon\right|^2}G^2_{csh}(u_\varepsilon, A_\varepsilon) \ge \frac{1}{2}\left\|a\right\|^2_{L^2}.$$

For the third term, we use Corollary 2.4 of [16], which shows that $\|1-|u_\varepsilon|^2\|^2_{L^2} \le C\varepsilon^2\left|\log\varepsilon\right|^4$, hence

$$|G^3_{csh}(u_\varepsilon, A_\varepsilon)| \le \left(\int_U (|u_\varepsilon|^2-1)^2\right)^{1/2}\left(\int_U |A_\varepsilon|^4\right)^{1/2} \le C\varepsilon\left|\log\varepsilon\right|^4 \to 0.$$

The convergence

$$\frac{1}{|\log \varepsilon|^2} G^4_{csh}(u_\varepsilon, A_\varepsilon) \to -\int_U a \cdot v$$

follows from the weak convergence of v_ε in L^p with $p < 2$ and the strong convergence $a_\varepsilon \to a$ in L^q for all q that follows from Rellich-Kondrachov compactness. Summing up the terms, we obtain the lower bound as claimed.

The lower bounds can be complemented by matching upper bounds, yielding Γ-convergence. The construction is identical to the one used in the $\mu = O(1)$ case, see Sections 6 and 7 of [16]. □

As an application of the Γ-lim inf energy (10), we compute the asymptotic value of the critical field strength for vortex nucleation. This follows from the solution of a classical obstacle problem, commonly found in free boundary theory. The obstacle problem for Ginzburg-Landau can be found in [20].

Proof of Corollary 4. We compare $G(v, a; H)$ with $G(0, a; H)$; the difference is given by

$$\frac{1}{2}\int_U |v|^2 + \frac{1}{2}\|\operatorname{curl} v\|_{\mathscr{M}} - \int_U v \cdot a \geq \frac{1}{2}\|\operatorname{curl} v\|_{\mathscr{M}} - \int_U v \cdot a. \tag{14}$$

Define z_∞ as the solution of $\Delta z_\infty = -1$ in U, $z_\infty = 0$ on ∂U. Since $\operatorname{curl} a = H$ then $H\Delta z_\infty = -\operatorname{curl} a$ and $a = -H \operatorname{curl} z_\infty$. Integrating by parts, we note that $\int_U v \cdot a = H \int_U z_\infty \operatorname{curl} v$. It follows that

$$G(v, a; H) - G(0, a; H) \geq \int_U \left(\frac{1}{2} - \operatorname{sgn}(\operatorname{curl} v) z_\infty H\right) d\,\|\operatorname{curl} v\|,$$

which is positive for $H \max_{\overline{U}} |z_\infty| \leq \frac{1}{2}$. □

Remark 7. The critical field obtained in the proposition above is identical to the limit as $\mu \to \infty$ of the critical fields for $\mu = O(1)$ that were calculated in [16].

3. Small μ_ε

In this section we consider the situation where $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. We show that a simple application of the tools of the $\mu = O(1)$ case does not work here by means of a counterexample of a sequence $\{u_\varepsilon, A_\varepsilon\}$ where $G_{csh}(u_\varepsilon, A_\varepsilon) \leq K|\log \varepsilon|^2$ but $\frac{1}{|\log \varepsilon|}\frac{j(u_\varepsilon)}{|u_\varepsilon|}$ is not bounded in L^2. In particular we build a lattice of $|\log \varepsilon|$ vortices inside of a box of size μ_ε with energy of size $|\log \varepsilon|^2$. This is possible so long as μ_ε is not too small. (If $\mu_\varepsilon < C\varepsilon |\log \varepsilon|^{\frac{1}{2}}$ - as in the self-dual case - we use a modified construction.) Since $\mu_\varepsilon \ll 1$ then the scaled current blows up like $||v_\varepsilon||^2_{L^2} \gtrsim \log \frac{1}{\mu_\varepsilon}$.

Proof of Theorem 5. The idea of the construction is to use $v_\varepsilon = \frac{j(u_\varepsilon)}{|\log\varepsilon|}$ and $a_\varepsilon = \frac{A_\varepsilon}{|\log\varepsilon|}$ and to set $v_\varepsilon \approx a_\varepsilon$, to let $\frac{\mu_\varepsilon^2}{4}\int_U |\operatorname{curl} a_\varepsilon|^2 \approx C$ and $\|\operatorname{curl} v_\varepsilon\|_{\mathscr{M}} \approx C$ while $\int_U |v_\varepsilon|^2 \to \infty$. We do this by letting $\operatorname{curl} v_\varepsilon$ approach a delta function.

1. To achieve this, we define some auxiliary functions:

$$\Phi^s(r) := \begin{cases} \frac{4}{3}\log 2s - \frac{1}{3}\log s - \frac{1}{2} & \text{if } r < s \\ \frac{r^2}{6s^2} - \frac{1}{3}\log r + \frac{4}{3}\log 2s - \frac{2}{3} & \text{if } s < r < 2s \\ \log r & \text{else} \end{cases} \tag{15}$$

Note that $\Phi^s(r)$ is a smooth approximation to a solution of $\Delta v = \delta_0$ for $s \to 0$ and that $\int_{B_\ell(0)} \Delta\Phi^s = 2\pi\int_s^{2s} \frac{2r}{3s^2}dr = 2\pi$ for $\ell > 2s$. We also set $q_\varepsilon(r) = \min(1, \frac{r}{\varepsilon})$ Our example will use the domain $U = B_1(0)$. We choose N_ε points a_i in a square lattice of side length $\frac{\mu_\varepsilon}{|\log\varepsilon|^{1/2}} \ll \mu_\varepsilon$ inside $B_{\mu_\varepsilon}(0)$, so $N_\varepsilon = \pi\,|\log\varepsilon| + O(|\log\varepsilon|^{1/2})$.

We now set $\rho_\varepsilon(z) = \prod_{i=1}^{N_\varepsilon} q_\varepsilon(z - a_i)$, $\Phi_\varepsilon(z) = \sum_{i=1}^{N_\varepsilon} \Phi^s(z - a_i)$, and $\Psi_\varepsilon(z) = \sum_{i=1}^{N_\varepsilon} \log|z - a_i|$. We further set $V_\varepsilon = \operatorname{curl}\Psi_\varepsilon$ and $A_\varepsilon = \operatorname{curl}\Phi_\varepsilon$. To define u_ε, we choose a multi-valued function ϕ_ε with $\nabla\phi_\varepsilon = V_\varepsilon$ and set $u_\varepsilon = \rho_\varepsilon e^{i\phi_\varepsilon}$. In our following calculations, we assume $\varepsilon \le s \le \frac{\mu_\varepsilon}{4|\log\varepsilon|^{1/2}}$. We calculate the energy $G_{csh}(u_\varepsilon, A_\varepsilon)$ (the external field is zero in this example). In the case where $\mu_\varepsilon < 4\varepsilon\,|\log\varepsilon|^{1/2}$, we will use $\tilde\mu_\varepsilon = 4\varepsilon\,|\log\varepsilon|^{1/2}$ instead. Note that $|\nabla_{A_\varepsilon} u_\varepsilon|^2 = |\nabla\rho_\varepsilon|^2 + \rho_\varepsilon^2|V_\varepsilon - A_\varepsilon|^2$. From our construction, it follows that

$$\int_U |\nabla\rho_\varepsilon|^2 + \frac{1}{\varepsilon^2}\rho_\varepsilon^2(1-\rho_\varepsilon^2)^2 \le CN_\varepsilon.$$

For the curl term, we observe that $\operatorname{curl} A_\varepsilon = 0$ where $\rho_\varepsilon = 0$, hence

$$\frac{\mu_\varepsilon^2}{8}\int_U \frac{\operatorname{curl} A_\varepsilon^2}{\rho_\varepsilon^2} = N_\varepsilon \frac{\mu_\varepsilon^2}{8}\frac{4}{9s^4}\pi(4s^2 - s^2) = \frac{\mu_\varepsilon^2 \pi N_\varepsilon}{6s^2}.$$

For $V_\varepsilon - A_\varepsilon$, we find that

$$(-\partial_2, \partial_1)(\log|z| - \Phi^s(z)) = \begin{cases} \frac{1}{r^2}(-y, x) & r < s \\ \frac{2}{3r^2}(-y,x) + \frac{1}{3s^2}(-y,x) & s < r < 2s \\ 0 & r < 2s \end{cases}$$

and so since s is chosen such that the fields of different particles a_i are disjoint, we can calculate

(16)
$$\begin{aligned}\int_U \rho_\varepsilon^2|V_\varepsilon - A_\varepsilon|^2 &= 2\pi N_\varepsilon\left(\int_0^\varepsilon \frac{r^2}{\varepsilon^2 r^2} r dr + \int_\varepsilon^s \frac{1}{r^2} r dr + \int_s^{2s}\left(\frac{4}{9r^2}+\frac{4}{9s^2}+\frac{r^2}{9s^4}\right) r dr\right)\\ &= 2\pi N_\varepsilon(\frac{1}{2}+\log\frac{s}{\varepsilon}+\frac{4}{9}\log 2+\frac{13}{12})\\ &= N_\varepsilon(C+\pi\log\frac{s}{\varepsilon}).\end{aligned}$$

Summing up, we obtain a total energy of $(C+\pi\log\frac{s}{\varepsilon}+C\frac{\mu_\varepsilon^2}{s^2})N_\varepsilon$. Choosing $s=\frac{\mu_\varepsilon}{4|\log\varepsilon|^{1/2}}$, this is bounded by $C\left|\log\varepsilon\right|^2$ since $\mu_\varepsilon<1$.

2. If $\mu_\varepsilon < \tilde{\mu}_\varepsilon = 4\varepsilon\left|\log\varepsilon\right|^{1/2}$, the construction has to be done with $\tilde{\mu}_\varepsilon$ in place of μ_ε and $s=\varepsilon$, and the total energy is $(C+\pi\log\frac{2\varepsilon}{\varepsilon}+C\frac{\mu_\varepsilon^2}{\varepsilon^2})N_\varepsilon$, and now $\frac{\mu_\varepsilon^2}{\varepsilon^2}\le 16\left|\log\varepsilon\right|$, so the total energy is again bounded by $C\left|\log\varepsilon\right|^2$.

3. We still need to show the unboundedness of $\frac{1}{|\log\varepsilon|^2}\int_U\frac{|V_\varepsilon|^2}{|u_\varepsilon|^2}$. To do so, we estimate it from below by considering only the integral over $B_1\setminus B_{2\mu_\varepsilon}$. We note that

$$|V_\varepsilon|^2(z)=\sum_{i=1}^{N_\varepsilon}\sum_{j=1}^{N_\varepsilon}\left|\frac{z-a_i}{|z-a_i|^2}\cdot\frac{z-a_j}{|z-a_j|^2}\right|.$$

We estimate each of the terms from below, using that $|z|\ge 2\mu_\varepsilon$ and $|a_i|<\mu_\varepsilon$. It is clear that $\frac{1}{2}|z|\le|z-a_i|\le\frac{3}{2}|z|$.

Hence $(z-a_i)\cdot(z-a_j)=|z-a_i||z-a_j||\cos\alpha|\ge\frac{1}{4}|z|^2|\cos\alpha|$, where α is the angle between $(z-a_i)$ and $(z-a_j)$. Since $|a_i|\le\mu$ and $|z|\ge 2\mu$, this angle can be estimated from above, and a short argument in elementary geometry shows that $|\cos\alpha|\ge\frac{\sqrt{3}}{2}$. Hence $(z-a_i)\cdot(z-a_j)\ge\frac{\sqrt{3}}{8}|z|^2$, and the contribution of a single term in the sum can be estimated below by $\frac{c}{|z|^2}$ (with $c=\frac{2\sqrt{3}}{81}$), and we obtain

$$\int_U|V_\varepsilon|^2\ge CN_\varepsilon^2\int_{2\mu_\varepsilon}^1\frac{1}{r}dr=\tilde{C}\left|\log\varepsilon\right|^2\log\frac{1}{2\mu_\varepsilon}, \tag{17}$$

and this is $\gg\left|\log\varepsilon\right|^2$ for any $\mu_\varepsilon\to 0$.

4. In the case that $\mu_\varepsilon<\tilde{\mu}_\varepsilon$, we obtain the same bounds with $\tilde{\mu}_\varepsilon$ instead of μ_ε. However, $\log\frac{1}{2\tilde{\mu}_\varepsilon}=\log\frac{1}{8\varepsilon|\log\varepsilon|^{1/2}}\gg 1$ so our unboundedness statement for $\frac{1}{|\log\varepsilon|}V_\varepsilon$ still holds. □

4. APPENDIX

In order to establish Theorem 3 we use the following Γ-limit result for $E_{csh}\approx\left|\log\varepsilon\right|^2$ established in [16]:

Theorem 8 ([16])**.** *Assume $E_{csh}(u_\varepsilon) \leq K|\log\varepsilon|^2$, for some constant K and $\|u_\varepsilon\|_{L^\infty(\partial U)} \geq 1 - \frac{1}{|\log\varepsilon|}$.*

Set $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon)$ and $w_\varepsilon = \frac{1}{|\log\varepsilon|} J(u_\varepsilon) = \frac{1}{2}\operatorname{curl} v_\varepsilon$. Then $\{w_\varepsilon\}$ is precompact in the weak $(C^{0,\beta})^*$ topology and $\{v_\varepsilon\}$ is bounded in L^p for $1 \leq p < 2$. Furthermore, if $w_\varepsilon \rightharpoonup w = \frac{1}{2}\operatorname{curl} v$ and $v_\varepsilon \rightharpoonup v$, then also $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2, and the energy satisfies

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} E_{csh}(u_\varepsilon) \geq \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{18}$$

Conversely, for every $v \in L^2(U;\mathbb{R}^2)$ such that $w = \frac{1}{2}\operatorname{curl} v$ is a Radon measure, there exists a sequence $\{u_\varepsilon\}$ in $H^1(U;\mathbb{C})$ with $|u_\varepsilon| = 1$ on ∂U such that $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon) \rightharpoonup v$ in L^2 and $w_\varepsilon = \frac{1}{|\log\varepsilon|} J(u_\varepsilon) \rightharpoonup w$ in $(C^{0,\beta})^*$ such that the energy satisfies

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} E_{csh}(u_\varepsilon) = \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{19}$$

Proof. Jacobian estimates have been extremely useful in Ginzburg-Landau theory, [1, 3, 12, 13, 14, 19], and the proof of this theorem follows from the approach of Jerrard-Soner for Ginzburg-Landau, see [12]. In particular the theorem follows quickly from the Jacobian estimate

$$\left|\int \phi J(u_\varepsilon)dx\right| \leq \pi d \|\phi\|_{L^\infty} + C\varepsilon^\gamma \|\phi\|_{C^{0,1}} \tag{20}$$

where $d \approx \left\lfloor \frac{1}{\pi}\int \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|}dx \right\rfloor$, $\gamma \in (0,1)$, and C depends on $\int \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|}dx$. Estimate (20) follows from Proposition 9 and Proposition 10 below. □

We sketch the relationship between the Jacobian

$$J(u) = \det \nabla u = \frac{1}{2}\operatorname{curl} j(u)$$

and the energy density $e_{csh}(u)$ found in (20). Set $\phi \in C_c^{0,1}(U)$ a Lipschitz function vanishing on ∂U. We define $\Omega(t) = \{x \in U \text{ such that } \phi(x) > t\}$ then $\partial\Omega(t)$ is a level set ϕ. Let

$$\operatorname{Reg}(\phi) := \left\{ \begin{array}{l} t \in [0, \|\phi\|_{L^\infty}] \text{ such that } \partial\Omega(t) = \phi^{-1}(t), \\ \partial\Omega(t) \text{ rectifiable, and } \mathcal{H}^1(\partial\Omega(t)) < \infty \end{array} \right\}.$$

By the co-area formula $|\operatorname{Reg}(\phi)| = \|\phi\|_{L^\infty}$ and $t \in \operatorname{Reg}(\phi)$ implies $\partial\Omega(t)$ is a union of finite Jordan curves, $\Gamma_i(t)$. We set, as in [12],

$$\Gamma(t) := \cup\{\text{components of } \partial\Omega(t) \text{ such that } \min_{x\in\Gamma_i(t)} |u| > \frac{1}{2}\}.$$

We set $d \in \mathbb{Z}^+$ and define

$$D_d := \{t \in \operatorname{Reg}(\phi) : \Gamma(t) \text{ is nonempty and } |\deg(u;\Gamma(t))| \geq d+1\}.$$

We will choose d to be the least integer multiple of $\pi|\log\varepsilon|$ of the energy. Let us define

$$E_\phi(u) = \int_{\mathrm{spt}(\phi)} e_{csh}(u)dx$$

for short. The first estimate is rather direct and follows in spirit of [12]:

Proposition 9. *Suppose $U \subset \mathbb{R}^2$ and $u \in H^1(U;\mathbb{C})$ then*

$$\begin{aligned} \left|\int_U \phi J(u)dx\right| \leq \left(\pi d + \varepsilon^{\frac{1}{2}}\right) \|\phi\|_{L^\infty} \\ + \varepsilon^{\frac{1}{3}} \|\nabla\phi\|_{L^\infty} \left[2E_\phi^2(u) + 3E_\phi(u) + \frac{|\mathrm{spt}(\phi)|}{4}\right] \\ + \frac{|D_d|}{4} E_\phi(u) \end{aligned} \tag{21}$$

for any $\varepsilon \leq 1$.

In order to use (21) and close the estimate we need to control $|D_d|$. This is estimated by the following result.

Proposition 10. *Suppose $|u| \geq 1 - \frac{1}{|\log\varepsilon|}$ on ∂U and let $\varepsilon \in (0, e^{-2})$. Define*

$$\varepsilon^{\alpha_\varepsilon} = \varepsilon|\log\varepsilon|^2 \int_U e_{csh}(u)dx \qquad \eta_\varepsilon = \frac{1}{1 - \frac{2}{|\log\varepsilon|}} > 1,$$

then

$$|D_d| \leq 8\varepsilon^{\alpha_\varepsilon} \|\nabla\phi\|_{L^\infty} \left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\star} \int_V e_{csh}(u)dx\right)\right) \tag{22}$$

where $d_\star = d + 1$. Here $V \equiv \dot{\cup}_j B_{r_j}$ is the union of disjoint balls B_{r_j} with $B_{r_j} \subseteq \mathrm{spt}(\phi) \subseteq U$ and $\sum_j r_j = \frac{|D_d|}{2\|\nabla\phi\|_{L^\infty}}$.

Proposition 10 provides a good estimate for $|D_d|$ when $d \approx \left\lfloor \int_V \frac{e_{csh}(u_\varepsilon)}{\pi|\log\varepsilon|} \right\rfloor$, and this estimate follows from a variation of the Jerrard [11] and Sandier [18] vortex ball method. There is also a different approach [21] that can be taken to the estimate of $|D_d|$.

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