Acoustic Waves and Far-field Patterns in Two Dimensional Oceans with Porous-elastic Seabeds

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Abstract

In this paper we study the propagation of acoustic waves in two dimensional oceans with porous-elastic seabeds. First we outline our results developed for three dimensional ocean with rigid bottom and a depth dependent index of refraction. Then we extend this problem to an ocean bed consisting of porous, elastic sediment, lying over an elastic rock substrata.

1. Introduction

In a series of papers Gilbert and Xu [5]-[7] developed the theory for determining the shape of an unidentified target in a three-dimensional finite-depth ocean. The ocean, moreover, was assumed to have a depth dependent index of refraction, a pressure release surface, and a hard ocean bed. In the present work we begin our extension of this problem which will include an ocean bed consisting of porous, elastic sediment, lying over an elastic rock substrata. As the calculations corresponding to this theory promise to be more time consuming and delicate we shall develop first the computations for a two dimensional ocean, i.e. depth and range. The present paper develops the theoretical aspects of the problem. For purposes of exposition, we first rewrite our three-dimensional results into two-dimensional form. In the next two sections, we outline the scattering theory, in particular the properties of propagating far-field pattern in a two dimensional ocean with pressure free surface and a rigid bottom. Theorems which are more or less directly obtainable from our three-dimensional results are merely stated. Whenever formulas differ in the two-dimensional case they are explicitly indicated.

¿From section 4, we discuss the scattering of acoustic wave in an ocean with elastic-porous sediment. A Biot model presented in [1] and [2] is adopted. In section 4, we present the Green's function in the water column using the Fourier transform. In section 5, we find a normal mode expansion for the Green's function and show that only a finite number of modes propagate. Hence we can define the propagating far-field pattern vector as for the rigid bottom case in section 2 and 3.

2. The direct scattering problem for two-dimensional wave guides.

Let $I\!R_b^2 := \{(x,z) \in I\!R^2 : 0 \leq z \leq h\}$ where h is a positive constant. Let Ω be a bounded domain with a C^2 boundary having an outward unit normal ν , such that $\bar{\Omega} \subset I\!R_b^2$. The direct scattering problem in a stratified waveguide may be formulated as

$$u_{xx} + u_{zz} + k^2 n^2(z) u = 0 \quad \text{in} \quad I\!R_b^2 \setminus \bar{\Omega}, \tag{2.1}$$

$$u = 0 \quad \text{at} \quad z = 0 \tag{2.2}$$

$$u_z = 0 \quad \text{at} \quad z = h \tag{2.3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{2.4}$$

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(2.5) $$\lim_{|x| \to \infty} (\frac{\partial u_n^s}{\partial r} - ika_n u_n^s) = 0, \; r := |x|$$

where $u = u^i + u^s$, u^i is the incident wave and u^s is the scattered wave. The u_n^s is the n^{th} normal propagating mode of scattered wave u^s, i.e.

(2.6) $$u_n^s(x) = \int_0^h \phi_n(z) u^s(x,z) dz,$$

where $a_n, \phi_n(z)$ are the n^{th} eigenvalue and the n^{th} normalized eigenfunction respectively of the eigenvalue problem

(2.7) $$\phi_{zz} + k^2(n^2(z) - a^2)\phi = 0$$

(2.8) $$\phi(0) = 0$$

(2.9) $$\phi'(h) = 0.$$

For the well-posedness of the problem (2.1) - (2.5), we assume that $k \neq (2n+1)\pi/2h$ for $n = 0, 1, \cdots$.

A Green's function for the wave guide $\mathbb{R}_b^2$ is a function satisfying

(2.10) $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} + k^2 n^2(z) v = \delta(x - x_0)\delta(z - z_0)$$

(2.11) $$v(x, 0) = 0$$

(2.12) $$v_z(x, h) = 0$$

and

(2.13) $$\lim_{r \to \infty} (\frac{\partial v_n}{\partial r} - ika_n v_n) = 0, \; \; r = |x|$$

where $v_n(x) = \int_0^h \phi_n(z) v(x,z) dz$.

Using the Fourier transform we may construct the Green's function for the wave guide in the form

$$G(z, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} \hat{G}(z, \alpha) d\alpha.$$

Here

$$\hat{G}(z,\alpha) = \hat{p}_1(z_<,\alpha)\hat{p}_2(z_>,\alpha)/W(\alpha),$$

where $W(\alpha)$ is the Wronskian, $z_< := min\{z_0, z\}, z_> := max\{z_0, z\}$, and $\hat{p}_1$ is a modal solution satisfying the boundary condition at $z = 0$, where as $\hat{p}_2$ satisfies it at $z = h$.

The poles occur at the a_n, the zeros of $W(\alpha)$. Using the residue calculus, we obtain

$$G(z, x; z_0, x_0) = \sum_{n=0}^{\infty} \frac{e^{ika_n|x-x_0|}}{2ika_n} \phi_n(z)\phi_n(z_0). \tag{2.14}$$

If u is a solution of problem (2.1) - (2.5) in $C^2(I\!R_b^2 \setminus \bar{\Omega}) \bigcap C(I\!R_b^2 \setminus \Omega)$ such that the normal derivative on the boundary exists in the sense that the limit

$$\frac{\partial u}{\partial \nu}(x,z) = \lim_{h \to 0} (\nu(x,z),\ \nabla u((x,z) - h\nu(x,z))),\ (x,z) \in \partial\Omega$$

exists uniformly on $\partial\Omega$, then by Green's formula

$$\int_{\partial\Omega} \{u(\xi,\zeta)\frac{\partial G}{\partial \nu}(|\xi - x|, z, \zeta) - \frac{\partial u}{\partial \nu}(\xi,\zeta)G(|\xi - x|, z, \zeta)\} d\sigma_\xi$$

$$= \begin{cases} 0 & \text{if } (x,z) \in \Omega \\ u(x,z) & \text{if } (x,z) \in I\!R_b^2 \setminus \bar{\Omega} \end{cases} . \tag{2.15}$$

We now list several lemmas, the proof of which mirror the arguments used in the three dimensional wave guide. See, for example Xu [10].

Lemma 2.1 If $u \in C^2(I\!R_b^2 \setminus \bar{\Omega}) \bigcap C(I\!R_b^2 \setminus \Omega)$ is a solution of problem (2.1) - (2.5) with homogeneous boundary data ($u^i = 0$), then

$$u = 0(\frac{e^{-k|a_{N+1}|r}}{r^{1/2}}) \quad \text{as} \quad r \to \infty. \tag{2.16}$$

where N is the largest integer such that the eigenvalue $a_n^2 > 0$.

Lemma 2.2 Let $\nu = (\nu_x, \nu_z)$ be the inward normal vector of $\partial\Omega$ at (x,z). If $x\nu_x \leq 0$ holds for any $(x,z) \in \partial\Omega$, and u satisfies the assumption of Lemma 2.1 then

$$\frac{\partial u}{\partial x} = 0 \quad \text{for} (x,z) \in I\!R_b^2 \setminus \Omega \tag{2.17}$$

Theorem 2.3. Let $\Omega \subset I\!R_b^2$ be a bounded region with C^2 boundary, such that $\nu_x x \leq 0$ holds for $(x,z) \in \partial\Omega$. If $u \in C^2(I\!R_b^2 \setminus \bar{\Omega}) \bigcap C(I\!R_b^2 \setminus \Omega)$ is a solution of problem (2.1)-(2.5) with homogeneous boundary, then $u = 0$ in $I\!R_b^2 \setminus \Omega$.

We turn next to question of existence. To this end, we seek a solution in the form of a combined double-layer and single layer potential

$$u(x,z) = \int_{\partial\Omega} [\frac{\partial}{\partial \nu} G(|x - \xi|, z, \zeta) + \lambda G(|x - \xi|, z, \zeta)]\varphi(\xi,\zeta) d\sigma, \tag{2.18}$$

$$(x,z) \in I\!R_b^2 \setminus \bar{\Omega},$$

where $\mathrm{Im}\lambda > 0$.

We now need some information concerning the nature of the Green's function's singularity. Recalling that each Green's function is also a Levi function, it must satisfy the usual jump conditions (see for example Miranda [9] pg 35). i.e. if

$$v(x,z) := \int_{\partial\Omega} G(|x-\xi|, z, \zeta)\varphi(\xi,\zeta)d\sigma \tag{2.19}$$

$$w(x,z) := \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(|x-\xi|, z, \zeta)\varphi(\xi,\zeta)d\sigma \tag{2.20}$$

then $v(x,z)$ is continuous on $I\!R_b^2$ and

$$w^{\pm}(x,z) = \frac{1}{2}\varphi(x,z) \pm \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(|x-\xi|, z, \zeta)\varphi(\xi,\zeta)d\sigma, \ \ (x,z) \in \partial\Omega.$$

Let us denote by $\mathbf{S}$ and $\mathbf{K}$ the operators

$$\mathbf{S}\phi := 2\int_{\partial\Omega} G(|x-\xi|, z, \zeta)\varphi(\xi,\zeta)d\sigma, \tag{2.21}$$

$$\mathbf{K}\phi := 2\int_{\partial\Omega} \frac{\partial}{\partial\nu} G(|x-\xi|, z, \zeta)\varphi(\xi,\zeta)d\sigma. \tag{2.22}$$

In a manner similar to that used in [3] [10] we are able to prove.

Theorem 2.4: Let $\Omega \in I\!R_b^2$ be a bounded region with C^2 boundary such that $x\nu_x \leq 0$ holds for $(x,z) \in \partial\Omega$. Then the problem (2.1)-(2.5) is uniquely solvable. The solution can be written in the form of (2.18) where $\phi(\xi,\zeta)$ is the unique solution of the integral equation

$$\phi + \mathbf{K}\phi + \lambda\mathbf{S}\phi = 2\phi. \tag{2.23}$$

3. Far-field patterns and their properties in wave guides

As discussed in our earlier paper [6], the eigenvalue problem (2.7)-(2.9) has an infinite number of real eigenvalues and only a finite number of these are positive. Let N be the largest integer such that $a_N^2 > 0$. The solution of the problem (2.1)-(2.5) has the asymptotic expansion

$$u(x,z) = \sum_{n=0}^{N} c_n e^{ika_n|x|}\phi_n(z) + o(1), \ \ as\ |x| \to \infty. \tag{3.1}$$

We call the function

$$F(z) := \sum_{n=0}^{N} c_n \phi_n(z) \tag{3.2}$$

the "propagating far-field pattern".

We assume the incoming wave u^i is in the form

$$u^i(x,z;\alpha,\beta) = \sum_{n-0}^{N} \phi_n(\beta)\phi_n(z)\frac{e^{ika_n\alpha x}}{2ika_n}, \tag{3.3}$$

where $\alpha = \pm 1, \beta \in [0,h]$.

If the corresponding scattered wave is $u^s(x,z;\alpha,\beta)$, then

$$\begin{aligned} u^s(x,z;\alpha,\beta) &= \int_{\partial\Omega} (u^s \frac{\partial G}{\partial \nu} - G\frac{\partial u^s}{\partial \nu}) d\sigma \\ &= \sum_{n=0}^{N} \frac{e^{ika_n|x|}}{2ika_n}\phi_n(z)\int_{\partial\Omega} [u^s(\xi,\zeta;\alpha,\beta)\frac{\partial}{\partial \nu_\xi}(e^{-ika_n\frac{x}{|x|}\xi}\phi_n(\zeta)) \\ &- \frac{\partial}{\partial\nu_\xi}u^s(\xi,\zeta;\alpha,\beta)(e^{-ika_n\frac{x}{|x|}\xi}\phi_n(\zeta))]d\sigma_\xi + o(1), \end{aligned} \tag{3.4}$$

and the far-field pattern

$$\begin{aligned} F(x,z;\alpha,\beta) &= \sum_{n=0}^{N} \frac{\phi_n(z)}{2ika_n}\int_{\partial\Omega} [u^s(\xi,\zeta;\alpha,\beta)\frac{\partial}{\partial\nu_\xi}(e^{-ika_n\hat{x}\xi}\phi_n(\zeta)) \\ &- \frac{\partial}{\partial\nu_\xi}u^s(\xi,\zeta;\alpha,\beta)(e^{-ika_n\hat{x}\xi}\phi_n(\zeta))]d\sigma_\xi, \end{aligned} \tag{3.5}$$

where $\hat{x} = \frac{x}{|x|} = \pm 1$.

Similar to the three-dimensional case [7] we can prove

Theorem 3.1 Let $C_1 := \{-1,1\} \times [0,h]$. For any $(\hat{x},z),(\alpha,\beta) \in C_1$, we have

$$F(\hat{x},z,\alpha,\beta) = F(-\alpha,\beta,-\hat{x},z). \tag{3.6}$$

Now we introduce a class of solutions to the Helmholtz equation defined in all of $I\!R_b^2$. A similar definition has been given in $I\!R^2$ (cf [2]) and $I\!R_b^3$ (cf [5]-[7]).

Definition 3.1. A solution $v(x,z)$ of the Helmholtz equation in $I\!R_b^2$ satisfying

$$\lim_{r\to\infty} \frac{1}{r}\int_{\Omega_r} |v(x,z)|^2 dxdz < \infty, \tag{3.7}$$

where $\Omega_r = \{(x,z) \in I\!R_b^2,\ |x| \le r\}$ is called a generalized Herglotz wave function. From a straight forward calculation follows: (cf. [7]).

Theorem 3.2. Any generalized Herglotz wave function in $I\!R_b^2$ may be represented in the form of

$$v(x,z)=\sum_{|\xi|=1}\int_o^h g(\xi,\zeta)\sum_{n=0}^{N}\phi_n(\zeta)\phi_n(z)\frac{e^{ika_n\xi x}}{2ika_n}d\zeta \tag{3.8}$$

for some $g(-1,\zeta),g(1,\zeta)\in L^2[0,h]$. Conversely, any solution in the form of (3.8) for some $g(-1,\zeta),g(1,\zeta)\in L^2[0,h]$ is a generalized Herglotz wave function.

Define

$$V^N:=span\{\phi_0,\phi_1,\cdots,\phi_N\}, \tag{3.9}$$

$$\mathcal{S}:=\{F(x,z;1,\beta)-F(x,z;-1,\beta),\ \beta\in[0,h]\} \tag{3.10}$$

and

$$\mathcal{S}^\perp:=\{v\in V^N:\int_o^h uvdz=0\quad\text{for}\quad u\in\mathcal{S}\}. \tag{3.11}$$

Now we consider the properties of $\mathcal{S}$. If $g\in\mathcal{S}^\perp$, then

$$\sum_{|x|=1}\int_0^h[F(x,z;1,\beta)-F(x,z,-1,\beta)]\overline{g(x,z)}dz=0,\ \beta\in[0,h]. \tag{3.12}$$

It follows that

$$\sum_{|x|=1}\int_0^h F(x,z,\mp1,\beta)\overline{g(x,z)}dz=\sum_{n=0}^{N}c_n\phi_n(\beta),\ \beta\in[0,h]. \tag{3.13}$$

We can assume that $c_n=\phi_n(z_0)$ where $(0,z_0)\in\Omega$ and define

$$U^s(x,z)=\sum_{|x|=1}\int_o^h u^s(x,z;\alpha,\beta)\overline{g(-\alpha,\beta)}d\beta$$

then the far-field pattern of $u^s(x,z)$ is

$$\begin{aligned}&\sum_{|\alpha|=1}\int_0^h F(x,z;\alpha,\beta)\overline{g(-\alpha,\beta)}d\beta\\ &=\sum_{|\alpha|=1}\int_0^h F(-\alpha,\beta;-x,z)\overline{g(-\alpha,\beta)}d\beta\\ &=\sum_{|x|=1}\int_0^h F(x,z,\alpha,\beta)\overline{g(x,z)}dz\\ &=\sum_{n=0}^{N}\phi_n(z_0)\phi(\beta).\end{aligned}$$

It follows that for $|x| \geq R$, where R is a constant such that $\Omega_R \supset \bar{\Omega}$,

$$U^s(x,z) = \sum_{n=0}^{N} \phi_n(z_0)\phi_n(z)\frac{e^{ika_n|x|}}{2ika_n} + \sigma^N \tag{3.14}$$

where σ^N contains no propagating modes.

Theorem 3.3: Assume that k^2 is not an eigenvalue of the interior Dirichlet problem in Ω, and let v be the solution of the Dirichlet problem

$$\Delta v + k^2 n(z) v = 0 \quad \text{in} \quad \Omega, \tag{3.15}$$

such that

$$v(x,z) = \overline{U^s(x,z)} \quad \text{on} \quad \partial\Omega, \tag{3.16}$$

where $U^s(x,z)$ is given by (3.14). Then

(1) If v is an entire Herglotz wave function with Herglotz kernel $g \neq 0$, then $\mathcal{S}^\perp = \{g\}$.

(2) If v is not a generalized Herglotz wave function, $\mathcal{S}^\perp = \{0\}$.

Proof: This theorem may be proved indirectly, i.e. we prove (1) If v is an entire Herglotz wave function with Herglotz kernel g, and $\mathcal{S}^\perp = \{0\}$, then $g = 0$, (2) If $\mathcal{S}^\perp \neq \{0\}$, then v is an entire Herglotz wave function. If v is an entire Herglotz wave function, then

$$v(\xi,\zeta) = \sum_{|x|=1} \int_0^h g(x,z) \sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta)\frac{e^{ika_n x\xi}}{2ika_n} dz$$

for some $g(x) \in V_2^N$. Let $u^s(x,z;\alpha,\beta)$ be the scattered wave with respect to the incoming wave

$$u^i(x,z;\alpha,\beta) = \sum_{n=0}^{N} \phi_n(z)\phi_n(\beta)e^{ika_n\alpha x}, \alpha = \pm 1,\ \beta \in [0,h].$$

If $F(x,z;1,\beta)$ and $F(x,z;-1,\beta)$ are far-field patterns with respect to the incoming waves $u^i(x,z;1,\beta)$ and $u^i(x,z;-1,\beta)$ respectively, then we have

$$\begin{aligned}
&\sum_{|x|=1} \int_0^h [F(x,z;1,\beta) - F(x,z;-1,\beta)]\overline{g(x,z)}dz \\
&= \sum_{|x|=1} \int_0^h \Big\{ -\int_{\partial\Omega} (\sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta)\frac{e^{-ika_n x\xi}}{2ika_n})\frac{\partial u}{\partial \nu}(\xi,\zeta;1,\beta)d\sigma_\xi \\
&+ \int_{\partial\Omega} (\sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta)\frac{e^{-ika_n x\xi}}{2ika_n})\frac{\partial u}{\partial \nu}(\xi,\zeta;-1,\beta)d\sigma_\xi \Big\} \overline{g(x,z)}dz
\end{aligned}$$

where

$$u(x,z;\pm 1,\beta) = u^i(x,z;\pm 1,\beta) + u^s(x,z;\pm 1,\beta).$$

Since

$$\sum_{|x|=1}\int_0^h\Big[-\int_{\partial\Omega}(\sum_{n=0}^N\phi_n(z)\phi_n(\zeta)\frac{e^{ika_nx\xi}}{2ika_n})\frac{\partial u}{\partial\nu}(\xi,\zeta,\pm1,\beta)d\sigma_\xi\Big]\overline{g(x,z)}dz$$

$$=-\int_{\partial\Omega}\sum_{|x|=1}\int_0^h\overline{\Big[\sum_{n=0}^h\phi_n(z)\phi_n(\zeta)\frac{e^{ika_nx\xi}}{2ika_n}\Big]g(x,z)dz}\frac{\partial u}{\partial\nu}(\xi,\zeta;\pm1,\beta)d\sigma_\xi$$

$$=-\int_{\partial\Omega}\overline{U^s(\zeta,\xi)}\frac{\partial u}{\partial\nu}(\xi,\zeta;\pm1,\beta)d\sigma_\xi$$

$$=\int_{\partial\Omega}(\frac{\overline{\partial U^s}}{\partial\nu}u-\overline{U^s}\frac{\partial u}{\partial\nu})d\sigma_\xi$$

$$=\int_{\partial\Omega}(\frac{\overline{\partial U^s}}{\partial\nu}u^i-\overline{U^s}\frac{\partial u^i}{\partial\nu})d\sigma_\xi+\int_{\partial\Omega}(\frac{\overline{\partial U^s}}{\partial\nu}U^s-\overline{u^s}\frac{\partial u^s}{\partial\nu}d\sigma_\xi$$

$$=\sum_{|x|=R}\int_o^h(\frac{\overline{\partial U^s}}{\partial\nu}u^i-\overline{U^s}\frac{\partial u^i}{\partial\nu})d\sigma_\xi+\sum_{|x|=R}\int_o^h(\frac{\overline{\partial U^s}}{\partial\nu}u^s-\overline{U^s}\frac{\partial u^s}{\partial\nu})d\sigma_\xi$$

$$=\sum_{|x|=R}\int_o^h(\frac{\overline{\partial U^s}}{\partial\nu}u^i-\overline{U^s}\frac{\partial u^i}{\partial\nu})d\sigma_\xi,$$

where $R>0$ such that $\Omega_R\supset\Omega$.

In view of

$$\int_0^h\Big[\sum_{n=N+1}^\infty c_n\phi_n(z)\frac{e^{ika_n|x|}}{2ika_n}\Big]\frac{\partial}{\partial|x|}\Big[\sum_{n=0}^N\phi_n(\beta)\phi_n(z)\frac{e^{ika_n\xi}}{2ika_n}\Big]dz=0,$$

$$\int_0^h\frac{\partial}{\partial|x|}\Big[\sum_{n=N=1}^\infty c_n\phi_n(z)\frac{e^{ika_n|x|}}{2ika_n}\Big]\Big[\sum_{n=0}^N\phi_n(\beta)\phi^n(\xi)\frac{e^{ika_n\xi}}{2ika_n}\Big]dz=0,$$

and (2.14), (3.14), we obtain

$$\sum_{|x|=1}\int_0^h\Big[-\int_{\partial\Omega}(\sum_{n=0}^N\phi_n(z)\phi_n(\zeta)\frac{e^{ika_nx\xi}}{2ika_n}\frac{\partial u}{\partial\nu}(\zeta,\xi;\pm1,\beta))d\sigma_\xi\Big]\overline{g(x,z)}dz$$

$$=\sum_{|\zeta|=R}\int_0^h\Big[\frac{\partial G}{\partial\nu}(z_0,\xi;|\zeta|)u^i(\xi,\zeta,\pm1,\beta)-G(z_0,\zeta;|\xi|)\frac{\partial u^i}{\partial\nu}(\xi,\zeta;\pm1,\beta)\Big]dz$$

$$=u^i(0,z_0;\pm1,\beta)=\sum_{n=0}^N\phi_n(z_0)\phi_n(\beta).$$

Hence,

$$\sum_{|x|=1}\int_0^h\Big[F(x,z;1,\beta)-F(x,z;-1,\beta)\Big]\overline{g(x,z)}dz$$
$$=u^i(0,z_0;1,\beta)-u^i(0,z_0;-1,\beta)=0.$$

Therefore, $g\in\mathcal{S}^\perp$. But $\mathcal{S}^\perp=\{\phi\}$, hence $g=0$.

Now we prove (2). If $g\in\mathcal{S}^\perp$, then we have (3.13). If we define a function $w(x,z)$ by

$$w(x,z):=\sum_{|\alpha|=|}\int_0^h\overline{g(-\alpha,\beta)}\sum_{n=0}^N\phi_n(z)\phi_n(\beta)\frac{e^{ika_n\alpha x}}{2ika_n}d\beta,$$

then

$$w(x,z)=\sum_{|\alpha|=|}\int_0^h\overline{-g(\alpha,\beta)\sum_{n=0}^N\phi_n(z)\phi_n(\beta)\frac{e^{ika_n\alpha x}}{2ika_n}}d\beta=\overline{v(x,z)}.$$

Furthermore, for $(x,z)\in\partial\Omega$,

$$\begin{aligned}w(x,z)&=-\sum_{|\alpha|=1}\int_0^h\overline{g(-\alpha,\beta)}u^i(x,z;\alpha,\beta)d\beta\\&=\sum_{|\alpha|=1}\int_0^h\overline{g(-\alpha,\beta)}u^s(x,z;\alpha,\beta)d\beta\\&=u^s(x,z).\end{aligned}$$

Hence, v is a Herglotz function satisfynig (3.15) and (3.16).

4. Seismo-acoustic waves from a point source in an ocean with elasto-porous sediment

Following Yamamoto [11] and Badiey-Yamaoto [1] we use a two-dimensional Biot model of an elasto-porous sediment to model the ocean floor. In [1], this model was studied by single mode solution, and a numerical treatment was presented. In the conclusion of this paper, we treat this model by Fourior transform method and obtain a normal mode expansion of the Green's function in the water column.

The notations from this section are independent of that in previous sections. We introduce the following

$$I_w=[0,h],\ I_s=[h,d],$$
$$\mathbb{R}^2_w=\mathbb{R}^1\times I_w,\ \mathbb{R}^2_s=\mathbb{R}^1\times I_s$$

where $\mathbb{R}^2_w,\mathbb{R}^2_s$ represent the region occupied by the ocean and the porous sediment respectively. $S_0:=\mathbb{R}^1\times\{0\}$ forms the surface of the ocean, $S_h:=\mathbb{R}^1\times\{h\}$ the interface of the water and the sediment and $S_d:=\mathbb{R}^1\times\{d\}$ represents the bottom of the sediment.

In the porous sediment, the stresses are related to the strains, and hence, the x and z displacements u and w, by

$$\tau_{xx} = P\frac{\partial u}{\partial x} + F\frac{\partial w}{\partial z},$$
$$\tau_{zz} = F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z},$$
$$\tau_{xz} = L(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}),$$

where A, F, B and L are coefficients of the elastic moduli of transverse isotropy. We assume they are independent of the range x and continuously differentiable in the depth z.

The equation of equilibrium may be written as

$$\rho\ddot{u} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},$$
$$\rho\ddot{w} = \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x},$$

where ρ is the density of the porous matrix. This leads to a system of partial differential equations of the form

$$\rho\ddot{u} = \frac{\partial}{\partial x}(P\frac{\partial u}{\partial x}) + \frac{\partial}{\partial x}(F\frac{\partial w}{\partial z}) + \frac{\partial}{\partial z}\left(L[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}]\right)$$
$$\rho\ddot{w} = \frac{\partial}{\partial x}\left(L[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}]\right) + \frac{\partial}{\partial z}\left(F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z}\right)$$

Assuming there is a time harmonic motion we replace the displacements u, w by $u(x,z,t) = u(x,z)e^{i\omega t}$, $w(x,z,t) = w(x,z)e^{i\omega t}$, whereupon we obtain

$$(4.1)\qquad \begin{aligned} -u\rho\omega^2 &= \frac{\partial}{\partial x}\{P\frac{\partial u}{\partial x} + F\frac{\partial w}{\partial z}\} + \frac{\partial}{\partial z}\{L[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]\}, \\ -w\rho\omega^2 &= \frac{\partial}{\partial x}\{L[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]\} + \frac{\partial}{\partial z}\{F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z}\}. \end{aligned}$$

Let us now define the Fourier transform as

$$(4.2)\qquad \begin{aligned} \hat{u}(\alpha,z) &= \int_{-\infty}^{\infty} u(x,z)e^{i\alpha x}dx, \\ \hat{w}(\alpha,z) &= \int_{-\infty}^{\infty} w(x,z)e^{i\alpha x}dx; \end{aligned}$$

then the transformed equations, assuming the elastic coefficients are only z- dependent, become the system of ordinary equations

$$(4.3)\qquad \begin{aligned} \rho\omega^2\hat{u} &= \frac{d}{dz}[Lz)\frac{d\hat{u}}{dz}] + i\alpha F(z)\frac{d\hat{w}}{dz} + i\alpha\frac{d}{dz}[L(z)\hat{w}] - \alpha^2 P(z)\hat{u}, \\ \rho\omega^2\hat{w} &= \frac{d}{dz}[C(z)\frac{d\hat{w}}{dz} + i\alpha L(z)\frac{d\hat{u}}{dz} + i\alpha\frac{d}{dz}[F(z)\hat{u}] - \alpha^2 L(z)\hat{w}. \end{aligned}$$

We define functions r_1, r_2, r_3, r_4 as

$$
(4.4)\qquad
\begin{aligned}
r_1(z,\alpha) &= \hat{u}(z,\alpha), \quad ir_2(z,\alpha) = \hat{w}(z,\alpha),\\
r_3(z,\alpha) &= \int_{-\infty}^{\infty} \left[L\frac{\partial w}{\partial x} + L\frac{\partial u}{\partial z}\right] e^{i\alpha x}\,dx,\\
ir_4(z,\alpha) &= \int_{-\infty}^{\infty} \left[F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z}\right] e^{i\alpha x}\,dx.
\end{aligned}
$$

These functions satisfy equations

$$
(4.5)\qquad \frac{d}{dz}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha & \frac{-1}{L} & 0 \\ \frac{\alpha F}{B} & 0 & 0 & \frac{-1}{B} \\ \rho\omega^2 - \alpha^2(A - \frac{F^2}{B}) & 0 & 0 & \frac{-\alpha F}{B} \\ 0 & \rho\omega^2 & \alpha & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = 0,\ h < z < d.
$$

This system has the same form as equation (19) of [1].

We now consider the acoustic waves in the water column above the porous seabed. If $\phi(x,z,t) = \phi(x,z)e^{i\omega t}$ is the displacement potential in the water caused by time harmonic acoustic point source, then the acoustic pressure is given by

$$
p(x,z)e^{i\omega t} = -\rho_w \frac{\partial^2 \phi}{dt^2} = \rho_w \omega^2 \phi(x,z) e^{i\omega t},
$$

where ρ_w is the density of water. Since $\phi(x,z,t)$ satifies

$$
\triangle\phi - \frac{1}{c(z)^2}\phi_{tt} = Ae^{i\omega t}\delta(x - x_0)\delta(z - z_0),
$$

where A is a normalizing constant, $p(x,z)$ satisfies

$$
(4.6)\qquad \triangle p + \frac{\omega^2}{c(z)^2} p = \delta(x - x_0)\delta(z - z_0),
$$

and the Fourier transformation of $p(x,z)$,

$$
\hat{p}(z,\alpha) = \int_{-\infty}^{\infty} p(x,z)e^{i\alpha x}\,dx
$$

satisfies

$$
(4.7)\qquad \hat{p}'' + [\frac{\omega^2}{c(z)^2} - \alpha^2]\hat{p} = \delta(z - z_0), \text{ in } I_w.
$$

Now we consider the boundary conditions. We assume a pressure release condition on the ocean surface, i.e.,

$$
(4.8)\qquad p(x,z) = 0 \text{ on } S_0.
$$

The material supporting the porous sediment is assumed to be rigid and welded with the sediment, which implys that

$$u(x,z) = 0,\ w(x,z) = 0 \ \text{ on } S_d. \tag{4.9}$$

We need still to match acoustic waves in the porous seabed. We have

$$\begin{aligned} \tau_{xz} &= 0, \\ \tau_{zz} &= -p, \\ w &= -\frac{1}{\rho_w \omega^2} \frac{\partial p}{\partial z} \ \text{ on } S_h. \end{aligned} \tag{4.10}$$

The Fourier transforms for conditions (4.10) become

$$\begin{aligned} r_3(h,\alpha) &= 0, \\ ir_4(h,\alpha) &= -\rho_w \omega^2 \hat{p}(h,\alpha), \\ ir_2(h,\alpha) &= \frac{d}{dz}\hat{p}(h,\alpha). \end{aligned} \tag{4.11}$$

Moreover, we obtain from (4.8) and (4.9) that

$$\begin{aligned} \hat{p}(0,\alpha) &= 0, \\ r_1(d,\alpha) &= 0, \\ r_2(d,\alpha) &= 0. \end{aligned} \tag{4.12}$$

Collecting equations (4.5), (4.7) and boundary conditions (4.11) and (4.12), we obtain a boundary value problem (called problem G) which determines the Green's function. Hence, the Green's function, i.e., the seismo-acoustic wave caused by a time-harmonic acoustic point source in the water, can be obtained from the inverse Fourier transform.

5. Normal mode expansion of the Green's function in the water column

In this section we study the normal mode expansion of the Green's function from its inverse Fourior transform. Our interest is focused on the water column where the acoustic waves propagate. We present an algorithm to calculate the eigenvalues which correspond to normal modes and prove that there are only a finite number of them which propagate. Based on this analysis, we can define the *propagating far-field patterns* in the ocean with a porous elastic sediment. It has been noticed that in the porous sediment, there may be an additional sequence of modes besides that corresponding to the normal modes in the water column. The properties of these modes will be investigated in a further study.

We introduce the ratio $\zeta(\alpha)$ as

$$\zeta(\alpha) = \frac{r_2(h,\alpha)}{r_4(h,\alpha)}. \tag{5.1}$$

Let $\hat{p}(z,\alpha)$ be the solution of problem G, we can write $\hat{p}(z,\alpha)$ as

$$\hat{p}(z,\alpha) = \hat{p}_1(z_<,\alpha)\hat{p}_2(z_>,\alpha)/W(\alpha)$$

where $W(\alpha)$ is the Wronskian of $\hat{p}_1$ and $\hat{p}_2$, and $z_< := min\{z_0, z\}, z_> := max\{z_0, z\}$. $\hat{p}_1, \hat{p}_2$ satisfy

$$\hat{p}_1(0, \alpha) = 0,$$
$$\frac{d}{dz}\hat{p}_2(h, \alpha) + \rho_w \omega^2 \zeta(\alpha)\hat{p}_2(h, \alpha) = 0,$$

and the corresponding homogeneous equation of (4.7). Moreover,

$$\hat{p}_1(z_0^-, \alpha) = \hat{p}_2(z_0^+, \alpha),$$
$$\frac{d}{dz}\hat{p}_1(z_0^-, \alpha) - \frac{d}{dz}\hat{p}_2(z_0^+, \alpha) = -1.$$

Then, the Green's function in the water column can be obtained from

$$G(z, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} \hat{p}_1(z_<, \alpha)\hat{p}_2(z_>, \alpha)/W(\alpha)d\alpha.$$

We will prove that the zeros of the Wronskian $W(\alpha)$, $\alpha_n, n = 1, 2, \cdots, \infty$, do not have a finite accumulation point. Hence, if $\phi_n(z), n = 1, 2, \cdots, \infty$ are the solutions of the corresponding homogeneous equation of (4.7) for $\alpha = \alpha_n, n = 1, 2, \cdots, \infty$, then we can express the Green's function as

$$G(z, x; z_0, x_0) = \sum_{n=0}^{\infty} c_n e^{i\alpha_n |x - x_0|} \phi_n(z)\phi_n(z_0), \tag{5.2}$$

where $c_n = Res\{\frac{1}{W(\alpha)}, \alpha_n\}$.

We first define a complex number set σ_S whose elements may correspond to some modes which exist only in the sediment. Let $\mathbf{r}^{(i)} = (r_1^{(i)}, r_2^{(i)}, r_3^{(i)}, r_4^{(i)})^T$, $i = 1, 2$ be the solution of (4.5) with given data

$$(r_1^{(1)}, r_2^{(1)}, r_3^{(1)}, r_4^{(1)})(d, \alpha) = (0, 0, 0, 1),$$
$$(r_1^{(2)}, r_2^{(2)}, r_3^{(2)}, r_4^{(2)})(d, \alpha) = (0, 0, 1, 0).$$

Let

$$a_j^i(\alpha) = \int_h^d r_j^{(i)}(t, \alpha)dt, \;\; i = 1, 2, \; j = 1, 4.$$

We define the set σ_S now as

$$\sigma_S = \left\{ \alpha \in C \,\middle|\, \begin{pmatrix} [\rho\omega^2 - \alpha^2(A - \frac{F^2}{B})]a_1^{(1)} - \frac{\alpha F}{B} a_4^{(1)} = 0, \\ [\rho\omega^2 - \alpha^2(A - \frac{F^2}{B})]a_1^{(2)} - \frac{\alpha F}{B} a_4^{(2)} = 1 \end{pmatrix} \right\}.$$

Lemma 5.1 σ_S contains at most a countable number of points possessing no finite acumulation point.

Proof: Define

$$f(\alpha) = \rho\omega^2 - \alpha^2(A - \frac{F^2}{B})a_1^{(1)}(\alpha) - \frac{\alpha F}{B}a_4^{(1)}(\alpha)$$
$$+ \rho\omega^2 - \alpha^2(A - \frac{F^2}{B})a_1^{(2)}(\alpha) - \frac{\alpha F}{B}a_4^{(2)}(\alpha) - 1.$$

If $\alpha \in \sigma_S$, then $f(\alpha) = 0$. But $f(\alpha)$ is an analytic function of α and $f(\alpha)$ is not identical to 0, hence we have the Lemma.

Lemma 5.2 $\zeta(\alpha)$ is determined uniquely if $\alpha \in C \setminus \sigma_S$. Moreover, $\zeta(\alpha)$ is analytic in $C \setminus (\sigma_S \cup Z)$, where

$$Z = \left\{ \alpha \in C | - r_3^{(2)}(h,\alpha) r_4^{(1)}(z,\alpha) + r_3^{(1)}(h,\alpha) r_4^{(2)}(z,\alpha) = 0 \right\}.$$

Proof: Any solution satisfying (4.5), (4.11) and (4.12) can be written as

$$\mathbf{r}(z,\alpha) = c_1 \mathbf{r}^{(1)}(z,\alpha) + c_2 \mathbf{r}^{(2)}(z,\alpha),$$

where the c_1, c_2 are complex numbers. If $\alpha \in C \setminus \sigma_S$, then $r_3^{(1)}(h,\alpha)$ and $r_3^{(2)}(h,\alpha)$ are not both zero. Assuming $r_3^{(1)}(h,\alpha) \neq 0$, we have from from (4.11)

$$c_1 = -\frac{r_3^{(2)}(h,\alpha)}{r_3^{(1)}(h,\alpha)} c_2.$$

$$\mathbf{r}(z,\alpha) = c_2 \left[-\frac{r_3^{(2)}(h,\alpha)}{r_3^{(1)}(h,\alpha)} \mathbf{r}^{(1)}(z,\alpha) + \mathbf{r}^{(2)}(z,\alpha) \right].$$

Hence, $\zeta(\alpha) = r_2(h,\alpha)/r_4(h,\alpha)$ is determined and analytic too, provided $\alpha \notin Z$. If $r_3^1(h,\alpha) = 0$, then $c_2 = 0$ and we have the same conclusion. The last statement follows from the fact that $r_2(h,\alpha)$ and $r_4(h,\alpha)$ are both analytic functions of α.

Lemma 5.3 $W(\alpha)$ has at most a countable number of zeros.

Proof: If $\alpha \in C \setminus (\sigma_S \cup Z)$, then

$$W(\alpha) = \hat{p}_1(h,\alpha)\frac{d}{dz}\hat{p}_2(h,\alpha) - \hat{p}_2(h,\alpha)\frac{d}{dz}\hat{p}_1(h,\alpha)$$
$$= \hat{p}_2(h,\alpha)\left[-\rho_w \omega^2 \zeta(\alpha)\hat{p}_1(h,\alpha) - \frac{d}{dz}\hat{p}_1(h,\alpha) \right]$$

is analytic. So for $\alpha \in C \setminus (\sigma_S \cup Z)$, $W(\alpha)$ has at most countable number of zeros having ∞ as the only possible acumulation point. In view of Lemma 5.1 and that if $\alpha \in Z \setminus \sigma_S$, then $\hat{p}(h,\alpha) = 0$, we have proved the Lemma.

Lemma 5.4 $W(\alpha)$ has at most a finite number of real zero points.

Proof: If there were infinite number of real α_n which are zero points of $W(\alpha)$, then $|\alpha_n| \to \infty,\ as\ n \to \infty$. The corresponding eigenfunctions $\phi_n(z)$ can be written as

$$\phi_n(z) = sinh\left(\sqrt{\alpha_n^2 - \frac{\omega^2}{c_0^2}} z\right) + \int_0^z K(z,s) sinh\left(\sqrt{\alpha_n^2 - \frac{\omega^2}{c_0^2}} s\right) ds,$$

where c_0 is a chosen constant and $K(z,s)$ is a transmutation kernel which does not depend on α. (For the transmutation representation see [8]). For some positive number δ, we have

$$|\zeta(\alpha_n)| = \left| \frac{\frac{d}{dz}\phi(h,\alpha_n)}{-\rho_w \omega^2 \phi(h,\alpha_n)} \right| > \delta |\alpha_n| \to \infty, \ as \ |\alpha_n| \to \infty. \tag{5.3}$$

However, from (4.5) r_2, r_4 satisfy

$$\mathbf{D}\begin{pmatrix} r_2 \\ r_4 \end{pmatrix} + \alpha^2 \begin{pmatrix} r_2 \\ r_4 \end{pmatrix} = -\alpha^3 \left(A - \frac{F^2}{B} \right) \begin{pmatrix} 0 \\ r_1 \end{pmatrix}, \tag{5.4}$$

where

$$\mathbf{D}\begin{pmatrix} r_2 \\ r_4 \end{pmatrix} = \begin{pmatrix} (\frac{B}{F} r_2')' - (\frac{1}{F} r_4)' - \frac{1}{L} r_4' - \frac{1}{B} r_4 \\ \frac{B}{F} r_4'' + \frac{B}{F}(\rho\omega^2 r_2)' + \frac{B^2}{F^2}\rho\omega^2 r_2' + \frac{B}{F}\rho\omega^2 r_4 \end{pmatrix}.$$

The fundamental matrix of operator $\mathbf{D} + \alpha^2$ is in the order $O(1)$ as $\alpha \to \infty$. Expanding r_1 in terms of α^{-n} we can see that $r_2 = O(r_4)$, hence $\zeta(\alpha) = O(1)$ as $\alpha \to \infty$. This contradicts (5.3).

In view of Lemma 5.1-5.4, we have

Theorem Let $\alpha_n, \phi_n(z), n = 1, 2, \cdots, \infty$ be the eigensolution to the eigenvalue problem (4.5), (4.11), (4.12) and

$$\phi'' + [\frac{\omega^2}{c(z)^2} - \alpha^2]\phi = 0. \text{ in } I_w. \tag{5.4}$$

The outgoing Green's function in the water column has a normal mode expansion (5.2) which has an asymptotic representation

$$G(z, x; z_0, x_0) = \sum_{n=0}^{N} c_n e^{i\alpha_n |x - x_0|} \phi_n(z) \phi_n(z_0) + o(1), \tag{5.5}$$

where N is the number of real eigenvalues.

Using (5.2) and (5.5), we can study the properties of the far-field pattern of the scattered wave as we did for the rigid bottom case in sections 2 and 3.

In conclusion, we would like to propose some questions which desire further consideration. A better understanding of these problems will give useful information in posing the inverse scattering problems in the ocean with porous-elastic sediment.

(1) The eigenvalue problem (4.5), (4.11), (4.12) and (5.4) is a nonselfadjoint eigenvalue problem. For $\alpha \in \sigma_S$, the first condition of (4.11) is not independent of the last two conditions of (4.12). This may result in some extra eigenvalues in the sediment column. What properties does σ_S have?

(2) How does the coefficients of the porous elastic modeli affect the eigenvalues and modes? How much information is lost in the sediment?

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Eingegangen am 12, Juli 1991.