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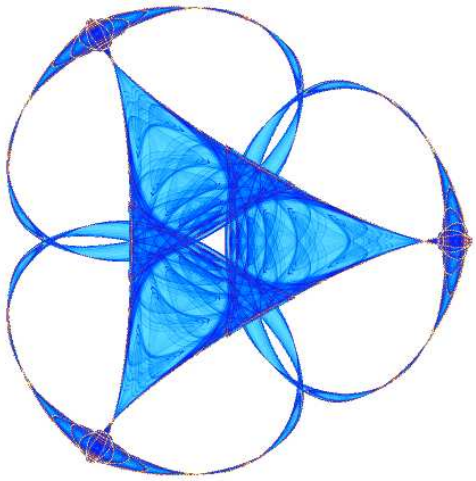

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EXPLICIT SOLUTIONS TO THE KORTEWEG-DE VRIES EQUATION ON THE HALF LINE

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Abstract: Certain explicit solutions to the Korteweg-de Vries equation on the half line are presented. Such solutions consist of algebraic combinations of real and complex exponential functions, and their initial values correspond to rational reflection coefficients in the associated Schrödinger equation. In the reflectionless case such solutions reduce to pure N-soliton solutions. An illustrative example is provided.

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Short title: Explicit solutions to the KdV equation

Consider the celebrated Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \eta \frac{\partial u}{\partial x} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where x and t denote the spatial and temporal variables, respectively, and η is a nonnegative constant [1,2] that can be chosen as 0 or 1. The KdV equation is used to model [3-5] the propagation of water waves in long, narrow, shallow channels; it also arises in other areas such as hydromagnetic waves in a cold plasma, ion-acoustic waves, and acoustic waves in harmonic crystals.

The KdV equation is one of the most well-known and most widely analyzed nonlinear partial differential equations. It has many remarkable aspects [4,5]. For example, it possesses traveling wave solutions known as solitons. The numerical studies on the KdV equation by Zabusky and Kruskal [3] led to the discovery of multi-soliton solutions, where the individual solitons interact nonlinearly at close distance and then move away from each other without changing their shapes. In their celebrated paper [6] Gardner, Greene, Kruskal, and Miura showed that the initial-value problem for the KdV equation can be solved via the "inverse scattering transform" associated with the Schrödinger equation. This led to the discovery that certain nonlinear partial differential equations are "completely integrable:" They can be solved via an inverse scattering transform and they have some interesting common properties such as possessing soliton solutions, Lax pairs, and infinitely many conserved quantities.

Despite all the advances in the field of integrable nonlinear partial differential equations, explicit solutions (closed-form solutions expressed in terms of elementary functions) are mainly known in the pure soliton cases, and there only seem to be a few results [7] otherwise. For example, a pure N-soliton solution to the KdV equation can be written explicitly as [4,5]

$$u(x,t) = -2 \frac{\partial^2 \log\left(\det \Gamma(x;t)\right)}{\partial x^2}, \tag{2}$$

where $\Gamma(x;t)$ is the $N \times N$ matrix whose (j,l) entry is given by [8]

$$\Gamma_{jl} = \delta_{jl} + \frac{c_j\, e^{-2\kappa_j x + 8\kappa_j^3 t + 2\eta\kappa_j t}}{\kappa_j + \kappa_l}, \qquad 1 \leq j,l \leq N,$$

with δ_{jl} denoting the Kronecker delta, the κ_j are N distinct positive constants corresponding to the bound states of $u(x,0)$, and the c_j are N positive constants known as the bound-state norming constants. Pure soliton solutions are trivial in the sense that the potential $u(x,0)$ corresponds [4,5] to a zero reflection coefficient in the Schrödinger equation. There are many important ways explicit solutions to the KdV equation may help us to understand nonlinearity better. For example, it is of great importance [1,9] to determine function spaces containing the initial data $u(x,0)$ so that $u(x,t)$ is globally well behaved (i.e. does not blow up during $t \in [0,+\infty)$) or only locally well behaved (i.e. remains finite during $t \in [0,\tau)$ and blows up at some finite time τ), and explicit solutions may help us to understand the global or local well posedness of initial-value problems for the KdV equation. The lack of explicit solutions to the KdV equation is partially responsible for the importance of its numerical analysis. Explicit solutions to the KdV equation may help us to understand local, global, and asymptotic behavior of its solutions without worrying about numerical errors, and such solutions may also be useful to check the accuracy in numerical methods.

In this letter we present a method leading to certain explicit solutions to the KdV equation on the half line. In our explicit solutions, the corresponding potential $u(x,0)$ is real valued and integrable, has a finite first moment, and corresponds to a rational reflection coefficient. In case the corresponding reflection coefficient is analytic on the upper half complex plane $\mathbf{C}^+$, our solution $u(x,t)$ reduces to the pure N-soliton solution given in (2). Our method can also be extended to obtain explicit solutions to the nonlinear Schrödinger equation (NLS) on the half line when the reflection coefficient for the initial potential is a rational function, but we will not deal with the NLS here. We will present the mathematical details of our method elsewhere. Here, we only outline our method and

present an explicit example of a global solution to (1) in the form of (2) by explicitly displaying $\det \Gamma(x,t)$ with a nontrivial reflection coefficient.

Our method works as follows. First, given $u(x,0)$ with $x > 0$, we extend it to the whole line by choosing $u(x,0) \equiv 0$ for $x < 0$ and we uniquely determine the corresponding scattering data $\{R, \{\kappa_j\}, \{c_j\}\}$. Here, $R(k)$ is the corresponding right reflection coefficient [10], the set of constants κ_j with $0 < \kappa_1 < \cdots < \kappa_N$ corresponds to the bound-states associated with the full-line potential $u(x,0)$, and the set of constants c_j corresponds to the associated bound-state norming constants. The construction of $\{R, \{\kappa_j\}, \{c_j\}\}$ can be accomplished through the following steps:

(a) Given $u(x,0)$ for $x > 0$, uniquely determine the corresponding Jost solution $f_{\rm r}(k,x)$ from the right by solving the initial-value problem for the half-line Schrödinger equation

$$\frac{d^2 f_{\rm r}}{dx^2} + k^2 f_{\rm r} = u(x,0)\, f_{\rm r}; \qquad f_{\rm r}(k,0) = 1, \quad \frac{df_{\rm r}(k,0)}{dx} = -ik.$$

(b) Recover the corresponding right reflection coefficient R and the transmission coefficient T with the help of the asymptotics [10-14] of $f_{\rm r}$ as $x \to +\infty$, namely by using

$$f_{\rm r}(k,x) = \frac{1}{T(k)}\, e^{-ikx} + \frac{R(k)}{T(k)}\, e^{ikx} + o(1), \qquad x \to +\infty.$$

It is known [10-14] that T is related to R via

$$T(k) = \prod_{j=1}^{N} \left(\frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds\, \frac{\log(1 - |R(s)|^2)}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+}, \quad (3)$$

where $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$ and the 0^+ indicates that the limit from $\mathbf{C}^+$ should be used to evaluate $T(k)$ for real k values.

(c) Construct the set $\{\kappa_j\}_{j=1}^N$ by using (3).

(d) Construct the set of positive constants $\{c_j\}_{j=1}^N$ by using [10]

$$c_j = -[\mathrm{Res}(T, i\kappa_j)]^2 \left[\frac{1}{2\kappa_j} + \int_0^{\infty} dx\, f_{\rm r}(i\kappa_j, x)^2 \right],$$

where the purely imaginary constant $\mathrm{Res}(T, i\kappa_j)$ denotes the residue of T at $k = i\kappa_j$.

Having constructed R which is a rational function of k, we determine all its poles in $\mathbf{C}^+$ and the coefficients in the partial fraction expansion of R at such poles. It is known [10-14] that $R(-k^*) = R(k)^*$ with the asterisk denoting complex conjugation, and hence such poles are either located on the positive imaginary axis $\mathbf{I}^+$ or they occur in pairs symmetrically located with respect to $\mathbf{I}^+$. Let us use M to denote the number of poles in $\mathbf{C}^+$, and let us order them in such a way that the first n pairs are located off $\mathbf{I}^+$ at $k = \pm\alpha_j + i\beta_j$ with $\alpha_j > 0$ and $0 < \beta_1 \leq \cdots \leq \beta_n$; in case several distinct α_j values correspond to the same β_j, we can further arrange α_j in increasing order. We choose our notation so that the remaining $M - n$ poles occur at $k = i\omega_j$ on $\mathbf{I}^+$ with $0 < \omega_{n+1} < \cdots < \omega_M$. We let m_j indicate the multiplicity of the jth pole for $j = 1, \ldots, M$.

Let ΠR denote the part of the partial fraction expansion of R containing only the poles in $\mathbf{C}^+$. We have

$$\Pi R(k) = \sum_{j=1}^{n} \sum_{s=1}^{m_j} \left[\frac{(-i)^s(\epsilon_{js} + i\gamma_{js})}{(k - i\beta_j - \alpha_j)^s} + \frac{(-i)^s(\epsilon_{js} - i\gamma_{js})}{(k - i\beta_j + \alpha_j)^s} \right] + \sum_{j=n+1}^{M} \sum_{s=1}^{m_j} \frac{(-i)^s r_{js}}{(k - i\omega_j)^s}. \qquad (4)$$

As a result of $R(-k^*) = R(k)^*$, the constants ϵ_{js}, γ_{js}, and r_{js} appearing in (4) are all real; in fact, we have

$$\epsilon_{js} + i\gamma_{js} = \frac{i^s}{(m_j - s)!} \frac{d^{m_j - s}}{dk^{m_j - s}} \left[R(k) \, (k - \alpha_j - i\beta_j)^{m_j} \right] \Big|_{k = \alpha_j + i\beta_j}, \qquad j = 1, \ldots, n,$$

$$r_{js} = \frac{i^s}{(m_j - s)!} \frac{d^{m_j - s}}{dk^{m_j - s}} \left[R(k) \, (k - i\omega_j)^{m_j} \right] \Big|_{k = i\omega_j}, \qquad j = n + 1, \ldots, M.$$

For $j = 1, \ldots, n$, let us define $C_j := 2 \left[\begin{array}{cccc} \gamma_{jm_j} & \epsilon_{jm_j} & \ldots & \gamma_{j1} \quad \epsilon_{j1} \end{array} \right]$ and

$$A_j := \begin{bmatrix} \Lambda_j & -I_2 & 0 & \ldots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \ldots & 0 & \Lambda_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where I_2 denotes the 2×2 unit matrix, each column vector B_j has $2m_j$ components, each A_j has size $2m_j \times 2m_j$, and each 2×2 matrix Λ_j is defined as

$$\Lambda_j := \begin{bmatrix} \beta_j & \alpha_j \\ -\alpha_j & \beta_j \end{bmatrix}.$$

Similarly, for $j = n+1, \ldots, M$, let

$$A_j := \begin{bmatrix} \omega_j & -1 & 0 & \ldots & 0 & 0 \\ 0 & \omega_j & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_j & -1 \\ 0 & 0 & 0 & \ldots & 0 & \omega_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := [\, r_{jm_j} \quad \ldots \quad r_{j1} \,],$$

where each column vector B_j has m_j components and each A_j has size $m_j \times m_j$. Note that we can write (4) as

$$\Pi R(k) = -i\, [\, C_1 \quad \ldots \quad C_M \,] \begin{bmatrix} (k - iA_1)^{-1} & 0 & \ldots & 0 \\ 0 & (k - iA_2)^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (k - iA_M)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_M \end{bmatrix}.$$

The above expression corresponds to a minimal realization [15] of ΠR. Associated with the bound-state data $\{\kappa_j, c_j\}_{j=1}^N$, we let

$$A_{M+j} := \kappa_j, \quad C_{M+j} := c_j, \quad B_{M+j} := 1, \qquad j = 1, \ldots, N.$$

Let us also define

$$A := \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{M+N} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ \vdots \\ B_{M+N} \end{bmatrix}, \quad C := [\, C_1 \quad \ldots \quad C_{M+N} \,]. \qquad (5)$$

Note that A is a $P \times P$ block square matrix, B is a column P-vector, and C is a row P-vector, where P is the constant given by

$$P := N + 2\sum_{j=1}^{n} m_j + \sum_{j=n+1}^{M} m_j.$$

Having constructed the constant matrices A, B, C as in (5), let us define

$$\Gamma(x;t) := I_P + \int_x^\infty dz\, e^{-zA} BC e^{-zA} e^{8tA^3+2\eta At}, \tag{6}$$

where I_P is the $P \times P$ unit matrix. Our main result is that the quantity $u(x,t)$ given as

$$u(x,t) = -2\frac{\partial}{\partial x}\left[\frac{\frac{\partial}{\partial x}\det\Gamma(x;t)}{\det\Gamma(x;t)}\right], \tag{7}$$

is a solution to (1) as long as $\det\Gamma(x;t) > 0$ or, equivalently, as long as the matrix $\Gamma(x;t)$ is invertible. It is known [16] that $\det\Gamma(x;0) > 0$ for $x \geq 0$. As seen from (6), the matrix $\Gamma(x;t)$ can be explicitly constructed from A, B, and C and also $\det\Gamma(x;t)$ is simply an algebraic combination of (real and complex) exponential functions, each of which is linear in x and t. Moreover, $\Gamma(x,0) \to I_P$ as $x \to +\infty$. As a result, $\det\Gamma(x;t) > 0$ for all $x \geq 0$ and $t \in [0,\tau)$ for some $\tau > 0$. There are two possibilities: If $\tau = +\infty$ then the solution $u(x,t)$ given in (7) is a global-in-time solution to (1); otherwise, it is a local-in-time solution.

The proof that (7) satisfies (1) when $\Gamma(x;t)$ is invertible can be outlined as follows. The solution to (1) via the inverse scattering transform is obtained as in the diagram

$$\begin{array}{ccc} \{R(k),\{\kappa_j\},\{c_j\}\} & \xleftarrow[\text{direct scattering}]{} & u(x,0) \\ \Big\downarrow{\scriptstyle \text{time evolution}} & & \Big\downarrow{\scriptstyle \text{solution to KdV}} \\ \{R(k)\,e^{8ik^3t-2i\eta kt},\{\kappa_j\},\{c_j\,e^{8\kappa_j^3t+2\eta\kappa_j t}\}\} & \xrightarrow{\text{inverse scattering}} & u(x,t) \end{array} \tag{8}$$

The inverse scattering step in (8) for $x > 0$ can be accomplished by solving the time-evolved Marchenko equation [4,5,8]

$$K(x,y;t) + \Omega(x+y;t) + \int_x^\infty dz\, K(x,z;t)\,\Omega(y+z;t) = 0, \qquad y > x > 0, \tag{9}$$

where the Marchenko kernel $\Omega(y;t)$ is given by

$$\Omega(y;t) := \frac{1}{2\pi}\int_{-\infty}^\infty dk\, R(k)\, e^{8ik^3t-2i\eta kt+iky} + \sum_{j=1}^N c_j\, e^{8\kappa_j^3t+2\eta\kappa_j t-\kappa_j y}. \tag{10}$$

If $t = 0$ in (10) then we can explicitly evaluate $\Omega(y;0)$ in terms of A, B, C given in (5), and this can be accomplished with the help of the generalized Cauchy integral formula by using a contour integration along the boundary of $\mathbf{C}^+$. In general, we cannot evaluate $\Omega(y;t)$ the same way for all $t > 0$, although there are cases when we can do this; for example, if all the eigenvalues of $8A^3 + 2\eta A$ have nonpositive real parts, then we can explicitly evaluate $\Omega(y;t)$ and obtain

$$\Omega(y;t) = Ce^{8tA^3+2\eta At-yA}B. \tag{11}$$

It turns out that the evaluation of (10) as (11) yields (13), which is a solution to (1) as long as $\Gamma(x;t)$ is invertible. As discussed earlier, this invertibility holds for all $x \geq 0$ and $t \in [0,\tau)$ for some $\tau > 0$, where the value of τ depends on the value of η and the entries of the constant matrices A and C given in (5). We can write $\Omega(x+y;t)$ as a dot product of a P-vector not containing x and a P-vector not containing y. This separability is easily seen from (11) by writing

$$\Omega(x+y;t) = Ce^{8tA^3+2\eta At-xA}e^{-yA}B,$$

where $Ce^{8tA^3+2\eta At-xA}$ is a row P-vector and $e^{-yA}B$ is a column P-vector. The degeneracy of the kernel $\Omega(y;t)$ allows us to solve (9) explicitly by algebraic means. In fact, its explicit solution is given by

$$K(x,y;t) = -Ce^{8tA^3+2\eta At-Ax}\Gamma(x;t)^{-1}e^{-yA}B, \tag{12}$$

where $\Gamma(x;t)$ is the matrix in (6). Finally, the time-evolved potential $u(x,t)$, which is also a solution to (1), is obtained from (12) via [4,5,8]

$$u(x,t) = -2\frac{\partial K(x,x;t)}{\partial x},$$

leading to

$$u(x,t) = 2\frac{\partial}{\partial x}\left[Ce^{8tA^3+2\eta At-Ax}\Gamma(x;t)^{-1}e^{-xA}B\right]. \tag{13}$$

From (6) and (13) we obtain

$$u(x,t) = -2\frac{\partial}{\partial x}\,\mathrm{tr}\left[\Gamma(x;t)^{-1}\frac{\partial}{\partial x}\Gamma(x;t)\right], \tag{14}$$

where we have used the fact that in evaluating the trace of a product of two matrices, we can change the order in the product. Using Theorem 7.3 on p. 38 of [17], we can write (14) also as (2) or (7). We note that it can independently and directly be verified that $u(x,t)$ given in (13) is a solution to (1) as long as $\Gamma(x;t)$ is invertible.

We will now illustrate our method by an explicit example. Consider the scattering data with no bound states and

$$\Pi R(k) = \frac{-2i\epsilon(k-i/2)-\sqrt{3}\gamma}{(k-i/2)^2-3/4}, \tag{15}$$

where ϵ and γ are some positive constants. Using (15) in (5) we obtain

$$A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = 2\begin{bmatrix} \gamma & \epsilon \end{bmatrix}. \tag{16}$$

The use of (16) in (6) results in

$$\begin{aligned}\det\Gamma(x;t) =& 1 - \frac{3}{4}\left(\epsilon^2+\gamma^2\right)e^{2(\eta-8)t-2x} \\ &+ \frac{1}{2}\,e^{(\eta-8)t-x}\left[(\sqrt{3}\epsilon-\gamma)\,\sin(\sqrt{3}\eta t-\sqrt{3}x) + (\epsilon+\sqrt{3}\gamma)\,\cos(\sqrt{3}\eta t-\sqrt{3}x)\right].\end{aligned} \tag{17}$$

Note that $\det\Gamma(x;t) > 0$ for all $x,t \geq 0$ if $(\epsilon^2+\gamma^2) < 4/9$ and $0 \leq \eta \leq 8$. It can directly be verified that $u(x,t)$ obtained as in (7) with $\det\Gamma(x;t)$ given in (17) solves (1) and hence it is a global-in-time solution. A Mathematica animation of that solution is available [18]. Adding bound states to (15) in our example results in global-in-time solutions containing solitons, an example and its animation of which are available [18].

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