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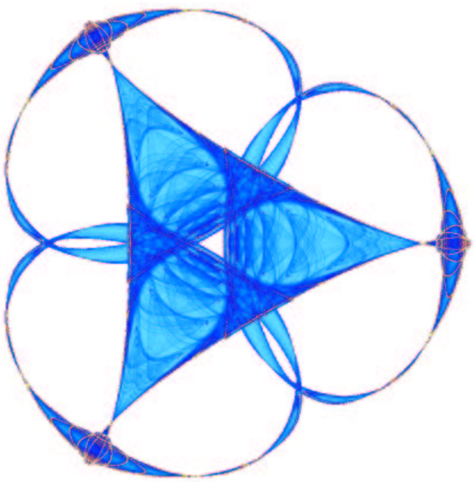

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YOUNG MEASURES AND ORDER-DISORDER TRANSITION IN STATIONARY FLOW OF LIQUID CRYSTALS

M. CARME CALDERER* AND ALEXANDER PANCHENKO†

Abstract. We study a system of nonlinear second order ordinary differential equations modeling Poiseuille flow of liquid crystals with variable degree of orientation, at the limit of large Ericksen number. The system is singularly perturbed and degenerate, and as a result the solutions are highly oscillatory. We obtain the relations satisfied by the Young measures generated by sequences of weak solutions, and show that the persistent oscillations are encoded in the Young measure generated by the molecular alignment variable. The effective equations correspond to the macroscopic isotropic Newtonian flow with a liquid crystalline microstructure indicating a remnant alignment.

Key words. Nematic liquid crystals, Young measures, non-Newtonian flows, singular perturbations, effective viscosity.

AMS subject classifications. 34D15, 34E10, 76A05, 76A15.

1. Introduction. In this article we study stationary flow of nematic liquid crystals with large Ericksen number, $\mathcal{E}$, in terms of the Young measures generated by sequences of weak solutions of the governing equations. It is experimentally well known that liquid crystal flows with large Ericksen number present a high density of defects and texture which increases with increasing values of $\mathcal{E}$ (cf. [19], [11], [16], [13] and [18]).

The system that we analyze consists of ordinary differential equations for the variable fields $s(x)$, $\phi(x)$ and $v(x)$, with $x \in [-1,1]$ with $\mathbf{R}^2 \times (-1,1)$, representing the domain of the flow. The governing system is highly nonlinear, non-autonomous and singularly perturbed with respect to the small parameter $\mu = \mathcal{E}^{-1}$. Its principal part as well as the boundary conditions become degenerate at $s = 0$. These combined features result in a highly oscillatory behavior of weak solutions. The goal of the present analysis is to encode oscillations persistent at the limit $\mu \to 0$ into Young measures.

We study a plane Poiseuille flow, which is driven by a prescribed pressure gradient, with vanishing velocity field at the boundary. The variable ϕ corresponds to the angle between the unit molecular director, $\mathbf{n} = (\sin\phi, 0, \cos\phi)$, and the velocity, $\mathbf{v} = (0, 0, v(x))$, of the flow. The variable degree of orientation, $s \in (-\frac{1}{2}, 1)$, gives the quality of alignment of the molecules with the director field, with $s = 1$ corresponding to perfect alignment, and $s = -\frac{1}{2}$ describing the case with molecules placed on a plane perpendicular to $\mathbf{n}$. Especially relevant to the present study is the isotropic case, $s = 0$, with randomly oriented molecules. Points, lines or planes in the flow region with $s = 0$ correspond to nematic liquid crystal defects, with undefined ϕ. Moreover, ϕ becomes discontinuous across defect lines and planes. The variables s and $\mathbf{n}$ correspond to an eigenvalue and eigenvector, respectively, of the optically uniaxial and traceless order tensor Q. The latter represents a second order moment of the molecular orientation field of a rigid polymer.

The equations that we analyze follow from those derived by Ericksen to model flow of liquid crystals with variable degree of orientation [8]. They yield the Ericksen-Leslie

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equations when the order parameter s is taken to be a nonzero constant. Relevant features of the model include the Helmholtz free energy, and the viscous, anisotropic, stress tensor. The latter is characterized by a set of anisotropic viscosity functions, $\alpha_i(s)$, $1 \leq i \leq 6$, known as Leslie coefficients (in particular, $\frac{1}{2}\alpha_4(0)$ represents the Newtonian viscosity). The free energy density is of the form $a_1|\nabla s|^2 + a_2 s^2|\nabla \mathbf{n}|^2 + \mathcal{J}^{-1}f(s)$, with a_1, a_2 and $\mathcal{J}$ denoting positive, dimensionless material parameters. The scalar function $f(s)$ represents a multi-well potential, favoring special directions of alignment at equilibrium. As a result of the elastic and viscous contributions to the model, the nature of the flow is fully non-Newtonian.

The flow behavior is determined mostly by three nondimensional parameter groups, the *Reynolds* number $\mathcal{R}$, the *Interface* number $\mathcal{I}$, and the *Ericksen* number $\mathcal{E}$. The latter measures the ratio of the viscous torque of the flow with respect to the elastic one. The condition of $\mathcal{E}$ being large corresponds to flow with large pressure gradient, and also to the case of viscous torque dominating the elastic one. The parameter $\mathcal{I}$ is associated with the free energy required to maintain defects in the flow, and it corresponds to the quotient of the bulk elastic energy and the gradient part of the Helmholtz free energy. The quantity $\mathcal{J}^{-1} = \mathcal{I}\mathcal{E}^{-1}$ appears as a coefficient in the free energy, as previously indicated. The derivation of the model studied in this article, the physical and non-dimensional parameter groups can be found in [3], [4] and [5].

We observe that s identically zero is not a solution of the problem. For an arbitrary flow domain, the bulk isotropic state, $s \equiv 0$, can only be realized at equilibrium, if permitted by the boundary conditions. In the case of Poiseuille flow, prescribing a non-zero pressure gradient excludes $s = 0$ from being an equilibrium solution. Consequently, one of the main outcomes of our study is to show that the isotropic state can be nearly realized in an effective sense.

Intuitively, one expects that for large Ericksen number, viscosity effects are dominant, and therefore the molecular alignment (associated with $s \neq 0$) is destroyed. The absence of alignment is indicated by $s = 0$. In this sense, the limit of large Ericksen number should represent the transition from order to disorder. In section 5, we show (Theorem 5.1) that there is indeed a sequence of generalized solutions such that in the limit $\mu \to 0$, s tends to zero uniformly, and v becomes the Newtonian velocity field of Poiseuille flow. This alone would indicate a perfect isotropic limit. However, the Young measure generated by ϕ satisfies the additional moment relations, indicating a residual molecular alignment. Although at a macroscopic scale the flow is isotropic, a liquid crystalline microstructure is present. The oscillatory behavior of solutions at the limit of large Ericksen number was numerically detected by the simulations performed in [14]. This provided the motivation for the present study.

In addition to the governing system being singularly perturbed as $\mu \to 0$, it becomes degenerate at $s = 0$. One mathematical difficulty is that standard methods of analysis of singular perturbations [7] cannot be readily applied here. On the other hand, owing to the singularly perturbed nature of the system, a priori bounds for the derivatives are not uniform in μ, and therefore, embedding theorems cannot be applied to obtain compactness of s. We overcome these difficulties by constructing tight bounds in terms of super-solutions and sub-solutions. They are both solutions of a classical variational problem for particle motion in a central force field [1]. The upper and lower bounds on s so obtained tend to 0 as $\mu \to 0$, which entails compactness. This leads to partial stability in the sense that s converges to zero uniformly as $\mu \to 0$, while ϕ oscillates with increasing frequency.

Another interesting aspect of the system being driven to degeneracy at the limit

$\mu \to 0$ manifests itself in the prescription of boundary conditions. Since the system is of second order with respect to s and ϕ, one expects that, both, $s(\pm 1)$ and $\phi'(\pm 1)$ (or $\phi(\pm 1)$) should be prescribed. We choose $s(\pm 1) = \mu$. However, in passing to the limit $\mu \to 0$, we find that the boundary conditions $\phi'(\pm 1)$ cannot be freely chosen, but depend on $s(\pm 1)$. In addition, $|\phi'(\pm 1)| \to \infty$ as $\mu \to 0$, in agreement with the fact that our limiting process amounts to creating a boundary defect. The unboundedness of ϕ' causes ϕ to be undefined at the defect location as expected. So, boundary values for the angular variable become redundant at the limit. The choice of $s(\pm 1) = \mu$ does not detract from generality. Our purpose is to study the oscillatory phenomenon on the bulk, and avoid boundary layer contributions that would appear in the case that s is not driven to zero at the boundary. These contributions could be incorporated in the current analysis using the techniques previously developed in [3], [4] and [5], but we do not attempt such an analysis here.

The article is organized as follows. Following the introductory section 1, the statement of the problem is presented in section 2. We developed a priori bounds and necessary conditions for Young measures in section 3. The main technical result of the paper is stated in section 4, with Theorem 4.3 being the focus of the work. A discussion of the effective equations is presented in section 5, with Theorem 5.1 stating the physical conclusions.

2. Formulation of the Problem. We study the following system of differential equations on the interval $I = (-1, 1)$:

$$\mu \left(a_1 s'' - a_2 s (\phi')^2 \right) = G_1(s, \phi, x), \tag{2.1}$$

$$\mu a_2 (s^2 \phi')' = G_2(s, \phi, x), \tag{2.2}$$

where

$$G_1(s, \phi, x) = \frac{1}{2} \beta_1(s) g^{-1}(s, \phi) x \sin 2\phi + \frac{1}{\mathcal{J}} \frac{df}{ds}(s), \tag{2.3}$$

$$G_2(s, \phi, x) = \frac{1}{2} (\gamma_1(s) + \gamma_2(s) \cos 2\phi) g^{-1}(s, \phi) x. \tag{2.4}$$

Equations (2.1)–(2.4) model plane Poiseuille flow of nematic liquid crystals with variable degree of orientation. The previously introduced functions $s(x)$ and $\phi(x)$ are the unknown fields of the problem. In (2.1)–(2.4), $'$ denotes derivative with respect to the independent variable $x \in I$. The boundary conditions are prescribed as follows,

$$s(-1) = \mu, \quad s(1) = \mu, \tag{2.5}$$

$$\phi'(-1) = A(\mu), \quad \phi'(1) = B(\mu). \tag{2.6}$$

As mentioned in the introduction, $A(\mu)$ and $B(\mu)$ cannot be arbitrarily prescribed for sufficiently small $\mu > 0$. The system (2.1)–(2.4) contains positive scalar parameters μ, $\mathcal{J}$, a_1 and a_2 discussed in the introduction. For more details on these quantities see [3], [4] and [5].

Let $\tau > 0$ be a given material parameter. We suppose that β_1, γ_1 and γ_2 are smooth functions of s on the interval $(-\frac{1}{2}, 1)$, and that $g(s, \phi)$ is also smooth. We assume that the following hypotheses hold:

$$\beta_1 < 0, \quad \gamma_1 > 0, \tag{2.7}$$

$$\gamma_1(s) = O(s^2), \quad \gamma_2(s) = O(s), \quad \text{for } s \text{ close to } 0, \tag{2.8}$$

$$g(s, \phi) \geq \tau > 0. \tag{2.9}$$

It follows from assumptions (2.8) and (2.9), that $\frac{1}{s}G_2(s)$ is a smooth function of s, and satisfies

$$\frac{1}{s}G_2(s) = O(1), \tag{2.10}$$

for $|s|$ small. Equations (2.1)–(2.9) completely describe the problem studied in the paper. The key observation that enables us to overcome the lack of uniform bounds mentioned in the introduction is as follows. If the right hand sides of equations (2.1) and (2.2) are replaced with $\frac{1}{\mathcal{J}}df/ds$ and zero, respectively, and, in addition, $a_1 = a_2$, one obtains the classical system modeling motion of a particle in a central force field with potential $-\frac{1}{\mathcal{J}}f(s)$. (see, e.g. [1]). In that case the variable x is time, and $s(x), \phi(x)$ are the radial and angular particle coordinates, respectively. This system has two first integrals: the angular momentum and the total mechanical energy, which make the problem completely integrable. In this paper, this classical system (with different potentials) is used to construct sub- and super-solutions of (2.1), (2.2) (see the proof of the theorem 4.3).

In the remainder of this section we outline the derivation of the governing system from the Leslie-Ericksen equations studied previously in [3], [4], [5]). The equations considered in these papers are as follows,

$$\mu\left(a_1 s'' - a_2 s(\phi')^2\right) = \beta_1(s)v'\sin\phi\cos\phi + \frac{1}{\mathcal{J}}\frac{df}{ds}(s) \tag{2.11}$$

$$\mu a_2 (s^2\phi')' = \left(\gamma_1(s) + \gamma_2(s)\cos 2\phi\right)v' \tag{2.12}$$

$$\frac{1}{\mathcal{R}}(g(s,\phi)v')' = 1, \tag{2.13}$$

Equation (2.13) corresponds to the balance of linear momentum. Its right hand side represents the prescribed pressure gradient, and $\mathcal{R}$ is the Reynolds number. Since we are interested in the behavior of solutions when $\mathcal{E}$ is large and the Reynolds number $\mathcal{R}$ is moderate, there is no loss of generality in assuming $\mathcal{R} = 1$. The function $\beta_1(s)$ is a Leslie coefficient, and $\gamma_1(s), \gamma_2(s)$ and $g(s,\phi)$ are given in terms of $\alpha_i(s)$, $i = 1, \ldots, 5$, as follows,

$$\gamma_1 := \alpha_3 - \alpha_2, \quad \gamma_2 := \alpha_2 + \alpha_3, \tag{2.14}$$

$$\begin{aligned} g(s,\phi) &= \frac{1}{2}\alpha_4(s) + \alpha_1(s)\sin^2\phi\cos^2\phi \\ &+ \frac{1}{2}(\alpha_5 - \alpha_2)(s)\sin^2\phi + \frac{1}{2}\alpha_3(s)\cos^2\phi. \end{aligned} \tag{2.15}$$

In the sequel (Section 5), we shall make use of of the fact (see e.g. [3], [4], [5]) that $\alpha_j(0) = 0$ for $j \neq 4$. This implies

$$g(0,\phi) = \frac{1}{2}\alpha_4(0) > 0, \tag{2.16}$$

where $\frac{1}{2}\alpha_4(0)$ is the Newtonian viscosity. The second law of thermodynamics requires that the system be dissipative. This imposes inequality restrictions on the Leslie coefficients $\beta_1(s)$ and $\alpha_i(s)$ (see [12]). As a consequence, we have that $\beta_1 < 0, \gamma_1 > 0$ and

$$g(s,\phi) \geq 0, \quad s \in [-\frac{1}{2}, 1], \quad \phi \in \mathbf{R}. \tag{2.17}$$

For the forthcoming analysis, we will impose a stronger assumption (2.9), where $\tau > 0$ is a material parameter related to the Newtonian viscosity. Such a strict inequality is an immediate consequence of $\mathcal{E} \neq 0$ [3]. Additional assumptions on Leslie coefficients consistent with kinetic theory of polymers are discussed in [3] and [4].

Next, we solve equation (2.13) for the velocity v, obtaining

$$v'(x) = (x + C)g^{-1}(s(x), \phi(x)), \tag{2.18}$$

where C is a constant of integration. The boundary conditions $v(\pm 1) = 0$ imply,

$$v(x) = \int_{-1}^{x} tg^{-1}(s(t), \phi(t))\, dt + C \int_{-1}^{x} g^{-1}(s(t), \phi(t))\, dt,$$

and

$$C = -\left(\int_{-1}^{1} g^{-1}(s(t), \phi(t))\, dt\right)^{-1} \int_{-1}^{1} tg^{-1}(s(t), \phi(t))\, dt. \tag{2.19}$$

It should be noted that C is a functional of s, ϕ taking constant values for a given flow. A calculation using the positivity of g yields $-1 < C \leq 1$. Postulating translational invariance of the equations, it is not difficult to show that there exists an interval J (which may be different for different μ) such that $C = 0$ when the equations are considered on J. Calculations involving C are analogous to those with the remaining terms. So, without loss of generality, we will set $C = 0$ in the equations. We also point out that this does not change the qualitative behavior of solutions because of the translational invariance. Combining (2.11), (2.12) and (2.18), we obtain the reduced system (2.1)–(2.4).

3. A Priori Bounds and Necessary Conditions for Young Measures. In this section we introduce the concept of Young measures generated by sequences of weak solutions of the problem. We derive integral identities and a priori estimates satisfied by weak solutions. Passing to the limit $\mu \to 0$ in the weak formulation of the equations yields algebraic momentum relations satisfied by the Young measures. Such relations can be appropriately interpreted as the effective equations of the system.

A weak solution of the system (2.1)-(2.2) is a pair of functions (s, ϕ) such that $s \in W^{1,2}(I)$, $s\phi \in W^{1,2}(I)$, and for all test functions $h \in C_0^1(I)$, the integral identities

$$-\mu a_1 \int_I s'h'dx - \mu a_2 \int_I s(\phi')^2 h dx = \int_I G_1 h dx, \tag{3.1}$$

$$-\mu a_2 \int_I s^2 \phi' h' dx = \int_I G_2 h dx, \tag{3.2}$$

hold, where the right hand sides G_1 and G_2 are given by (2.3) and (2.4), respectively.

For each $\mu > 0$, existence of weak solutions satisfying $s \in (-\frac{1}{2}, 1), |\phi| \leq \frac{\pi}{2}$ together with the boundary conditions (2.5), (2.6) can obtained using the existence theorem in [2].

3.1. L^2**-bounds.** The L^2-estimates will be obtained from the following

Proposition 3.1.

Let s, ϕ be a sufficiently smooth solution of (2.1) and (2.2). Then the following identities hold:

$$-\mu a_1 \int_I |s'|^2\, dx + \mu a_1 (s's(1) - s's(-1)) - \mu a_2 \int_I s^2 (\phi')^2\, dx = \int_I G_1(x, s, \phi) s\, dx, \tag{3.3}$$

$$-\mu a_2 \int_I s^2(\phi')^2 dx + \mu a_2((s^2\phi'\phi)(1) - (s^2\phi'\phi)(-1)) = \int_I G_2(x,s,\phi)\phi dx. \tag{3.4}$$

Proof. Multiplying equation (2.1) by s and integrating by parts yields,

$$\mu a_1 \int_I (s')^2 dx + \mu a_1(s's(1) - s's(-1)) + \mu a_2 \int_I s^2|\phi'|^2 dx = \int_I G_1(x,s,\phi) s dx.$$

Likewise, multiplication of equation (2.2) by ϕ and integration by parts yields (3.4). ☐ The uniform boundedness of G_1 and G_2 with respect to μ follows from the analogous property of s and ϕ. If, in addition, the boundary terms $b_1(\mu) \equiv \mu a_1(s(1)s'(1) - s(-1)s'(-1))$ and $b_2(\mu) \equiv \mu a_2((s^2\phi'\phi)(1) - (s^2\phi'\phi)(-1))$ are also uniformly bounded, then the identities (3.3) and (3.4) yield the following uniform apriori bounds:

$$\begin{aligned} \mu^{\frac{1}{2}} \|s'\|_{L^2(I)} &\leq C \\ \mu^{\frac{1}{2}} \|s\phi'\|_{L^2(I)} &\leq C, \end{aligned} \tag{3.5}$$

where $C > 0$ is independent of μ.

Note that $b_1(\mu)$ and $b_2(\mu)$ vanish when $s(\pm 1) = 0$. Otherwise, if $s'(\pm 1)$ and $\phi'(\pm 1)$ do not grow too fast as $\mu \to 0$, and s and ϕ are uniformly bounded, then $b_1(\mu)$ and $b_2(\mu)$ are also uniformly bounded. Such statement follows from the estimates on the boundary layer terms valid for a large class of boundary conditions. Indeed, near the boundary $x = \pm 1$, s can be well approximated by a boundary layer term S, satisfying $S(x) = O(e^{\frac{-|x-1|}{\mu^{1/2}}})$, for μ close to 0 (cf. [3], and [4]). Moreover,

$$|s'(\pm 1)| \leq \frac{C}{\mu^{1/2}}, \tag{3.6}$$

where C is independent of μ. A related property of the solutions of the governing equations for small μ is the oscillatory behavior of s about $s = 0$ on the interval I; an estimate on the number of oscillations gives $N = O(\mu^{-\frac{1}{2}})$. Moreover, the first and the last zeroes of s in I approach the boundary as $\mu \to 0$ [2]. Numerical simulations of such a behavior are presented in [14].

3.2. Momentum relations for Young Measures. The estimates (3.5) allow for some control of the oscillations and yield existence of the microscopic length scale $l \sim \mu^{1/2}|I|$. Unfortunately, the bounds on the derivatives are not uniform in μ, so it is not possible to extract subsequences convergent weakly in $W^{1,2}(I)$ as $\mu \to 0$. Since s, ϕ are bounded pointwise, we can extract subsequences weak-*convergent in $L^\infty(I)$, and hence weakly convergent in $L^2(I)$. However, this convergence cannot be improved to strong due to highly oscillatory behavior of s and ϕ. Such a behavior is appropriately encoded in the Young measure generated by sequences of weak solutions [20]. In [17], this measure is defined as follows.

Definition 1. *A Young (parameterized) measure is a family of probability measures $\lambda = \{\lambda_x\}_{x\in\Omega}$ associated with a sequence of functions $f_j : \Omega \subset \mathbf{R}^N \to \mathbf{R}^m$ such that*
(i) $\mathrm{supp}(\lambda_x) \subset \mathbf{R}^m$,

(ii) λ_x *depend measurably on* $x \in \Omega$, *which means that for any continuous function* $\phi : \mathbf{R}^N \to \mathbf{R}$, *the function*

$$\overline{\phi}(x) = \int \phi(y) d\lambda_x(y) = \langle \phi(y), \lambda_x(y) \rangle$$

is (Lebesgue) measurable,
(iii) If the sequence $\phi(f_j)$ *converges weakly in* $L^p(\Omega)$, $1 \le p < \infty$ *(weak-*$\star$ *in* $L^\infty(\Omega)$*), then the weak limit is the function*

$$\overline{\phi}(x) = \int \phi(\xi) d\lambda_x(\xi).$$

In the sequel, we use the following facts about Young measures. The first ([17], th. 6.2) is the existence theorem.

THEOREM 3.2. *Let* $\Omega \subset \mathbf{R}^N$ *be a (Lebesgue) measurable set and let* $z_j : \Omega \to \mathbf{R}^m$ *be measurable functions such that*

$$\sup_j \int_\Omega g(|z_j|) dx < \infty,$$

where $g : [0, \infty]$ *is a continuous, nondecreasing function such that* $\lim_{t\to\infty} g(t) = \infty$.

Then there exists a subsequence, not relabeled, and a family of probability measures, $\lambda = \{\lambda_x\}_{x\in\Omega}$ *(the associated Young measure) with the property that whenever the sequence* $\{\psi(x, z_j(x))\}$ *is weakly convergent in* $L^1(\Omega)$ *for any Caratheodory function* $\psi(x, \xi) : \Omega \times \mathbf{R}^m \to [-\infty, \infty]$, *the weak limit is the function*

$$\overline{\psi}(x) = \int_{\mathbf{R}^m} \psi(x, \xi) d\lambda_x(\xi).$$

The second fact concerns Young measures generated by sequences of vector functions, for which strong convergence holds for some of the components, but not for all of them ([17], Proposition 6.13).

THEOREM 3.3. *Let* $z_j = (u_j, v_j) : \Omega \to \mathbf{R}^d \times \mathbf{R}^m$ *be a bounded sequence in* $L^p(\Omega)$ *such that* $\{u_j\}$ *converges strongly to* u *in* $L^p(\Omega)$. *If* $\lambda = \{\lambda_x\}_{x\in\Omega}$ *is the Young measure associated with* z_j, *then* $\lambda_x = \delta_{u(x)} \otimes \nu_x$ *for (Lebesgue) almost all* $x \in \Omega$, *where* $\{\nu_x\}_{x\in\Omega}$ *is the Young measure corresponding to* $\{v_j\}$.

Young measures provide a description of the weak limits of the non-weakly continuous functions G_1 and G_2 in (2.1) and (2.2). We denote by λ_x the Young measure associated with the sequences of weak solutions (s^μ, ϕ^μ). By definition, for any continuous function H on $[-1/2, 1] \times [-\pi/2, \pi/2]$, the sequence $H(s^\mu, \phi^\mu)$ converges weakly-* to

$$\bar{H}(x) = \int_{-1/2}^{1} \int_{-\pi/2}^{\pi/2} H(s, \phi) d\lambda_x(s, \phi),$$

It is a well known fact ([17]) that Young measures are in general difficult to compute. In particular, there is no general method for derivation of conditions imposed on Young measures by the non-linear differential constraints on generating sequences. Available results concern sequences generated by gradients [10]. A generalization of [10] to sequences solving certain constant-coefficient linear partial differential equations is obtained in [9].

In the present case it is possible to derive some necessary conditions for Young measures passing to the limit in the integral identities (3.1) and (3.2). In the next proposition we use notation $a = (a_2/a_1)^{1/2}$.

PROPOSITION 3.4. *Let λ_x be the Young measure generated by a sequence of weak solutions (s^μ, ϕ^μ) satisfying the estimates (3.5). Then for any test function $h \in C_0^1(I)$, the following relations hold:*

$$\int \big(G_1(x,y,z)\sin az + \frac{1}{ay}G_2(x,y,z)\cos az\big)d\lambda_x(y,z)h(x)dx = 0, \tag{3.7}$$

$$\int \big(G_1(x,y,z)\cos az - \frac{1}{ay}G_2(x,y,z)\sin az\big)d\lambda_x(y,z)h(x)dx = 0. \tag{3.8}$$

Proof. Let $w(x)$ be a test function. In the integral identities (3.1), we first replace h with the test function $\sin(a\phi)w$. Next, we formally substitute $\frac{1}{s}a\cos(a\phi)w$ for h in (3.2), and add the resulting identities. This yields,

$$-\mu a_1 \int (s\sin a\phi)'w'dx = \int (G_1 \sin a\phi + \frac{1}{as}G_2\cos a\phi)wdx. \tag{3.9}$$

To justify the use of $\frac{1}{s}a\cos(a\phi)w$ as a test function, we point out that $\frac{1}{s}G_2$ is a smooth function of s and satisfies (2.10). Thus, the right hand side of the previous equation is well defined. This allows us to approximate $\frac{1}{s}$ by a sequence of test functions and then pass to the limit in the resulting integral identities.

The estimates (3.5) on derivatives yield

$$\| \mu^{1/2}(s\sin a\phi)' \|_{L^2(I)} \leq C$$

with C independent of μ. Hence the integral on the left tends to zero as $\mu \to 0$. Using the definition of Young measures to pass to the limit in the right hand side we obtain the first equation in (3.7).

Next, using $\cos a\phi w$ as a test function in (3.1) and $-\frac{A}{s}\sin a\phi w$ in (3.2) and summing up, we have

$$-\mu a_1 \int (s\cos a\phi)'w'\,dx = \int (G_1\cos a\phi - \frac{1}{as}G_2\sin a\phi)w\,dx. \tag{3.10}$$

Again, the limit of the integral on the left is zero. Passing to the limit in (3.10) yields the second equation in (3.7). □

Remark 3.1. The method of proof is based on the following formal procedure. Consider the original system (2.1)-(2.2):

$$\mu(a_1 s'' - a_2 s(\phi')^2) = G_1(x,s,\phi),$$

$$\mu a_2 (s^2\phi')' = G_2(x,s,\phi).$$

If we multiply the first equation of the system by $\sin a\phi$, the second equation by $1/s\cos a\phi$ and add the resulting equations, we obtain

$$\mu a_1(s\sin a\phi)'' = G_1\sin a\phi + \frac{G_2}{s}\cos a\phi. \tag{3.11}$$

Similarly, multiplying the first equation by $\cos a\phi$, the second one by $-1/s\sin a\phi$ and adding the results, we get

$$\mu a_1(s\cos a\phi)'' = G_1\cos a\phi - \frac{G_2}{s}\sin a\phi. \tag{3.12}$$

The momentum relations from the proposition are obtained by passing to the limit (in the sense of distributions) in the integral identities corresponding to the system (3.11), (3.12), and thus they are the effective equations for this new system. Moreover relations (3.7) (3.8) are also effective equations for the original system, since their solutions satisfy the differential equations almost everywhere, and, for each $\mu > 0$, the zero sets of the functions $\cos a\phi, \sin a\phi$ and s can be shown to be countable by standard Sturm-Liouville results as in [2].

Remark 3.2. It is interesting to ask to what extent the relations from the proposition characterize Young measures generated by sequences of weak solutions of (2.1) and (2.2). The measures in question satisfy (3.7) and (3.8), but these conditions are far from being sufficient, since they can be obtained for Young measures generated by different equations. These equations may contain, for instance, terms with linear combinations of higher-order derivatives of s and $s\phi'$ multiplied by sufficiently large powers of μ.

4. Generalized Solutions and Isotropic Defects. In this section, we construct generalized solutions s^μ, ϕ^μ from a special class of weak solutions. Generalized solutions are such that s^μ approaches 0, uniformly on I, and ϕ^μ is bounded and presents multiple jump discontinuities. Such jump discontinuities may correspond to isotropic (plane) defects in stationary three dimensional flow.

The numerical experiments carried out in [14] indicate partial stability, which means that $s^\mu \to 0$ as $\mu \to 0$, and ϕ^μ oscillates with a frequency of the order of $\mu^{-1/2}$. In this subsection, we obtain sufficient conditions for such type of behavior. For this, we appeal to results from the theory of ordinary differential equations; the main technical tool is Nagumo's theorem [15], (stated as theorem 4.1 below). This theorem provides sufficient conditions for existence of solutions satisfying pointwise upper and lower bounds, constructed from sub– and super solutions. This result allows us to prove theorem 4.3 on existence of partially stable solutions.

The strategy of the proof of theorem 4.3 is as follows. First, we consider arbitrary solutions satisfying boundary conditions that depend on μ (see theorem 4.3 for the precise formulation). Then, we show that among these, there is a large class of boundary conditions for which s^μ is bounded by a multiple of μ. Nagumo's theorem guarantees existence of such solutions. To apply the theorem, we first show that sub– and super solutions of the system can be constructed from solutions of a classical variational problem that models the motion of a particle in a central force field. The solutions of the variational systems depend on the first integral parameters. By appropriate choice of the parameters, we find sub- and super solutions satisfying the desired bounds. An additional difficulty that we encounter is that Nagumo's theorem concerns classical solutions, while the present system may have singularities near the points where $s = 0$. We deal with this problem by examining solutions only on the set where the absolute value of s^μ is larger than μ, and then use upper and lower bounds to demonstrate that, on this set, $|s^\mu| \le 2\mu$.

Nagumo's theorem concerns solutions of the Dirichlet problem on the interval $[a, b]$ for a single second order equation

$$u'' = F(x, u, u'), \tag{4.1}$$

$$u(a) = u_0; \qquad u(b) = u_1.$$

Assume that F is a continuous function of its arguments satisfying the condition

$$F(x, u, z) = O(|z|^2) \text{ as } \quad |z| \to \infty, \tag{4.2}$$

for all (x, u) in a rectangle $[a, b] \times [\alpha, \beta]$. The following theorem is due to Nagumo [15].

THEOREM 4.1. *Suppose that F satisfies (4.2), and there exist functions $\alpha(x), \beta(x)$ with the properties*
i) $\alpha, \beta \in C^2([a, b])$;
ii) $\alpha(x) \leq \beta(x)$;
iii) $\alpha'' \geq F(x, \alpha, \alpha')$, $\beta'' \leq F(x, \beta, \beta')$;
iv) $\alpha(a) \leq u_0 \leq \beta(a)$, $\alpha(b) \leq u_1 \leq \beta(b)$. Then the problem (4.1) has a solution $u(x) \in C^2([a, b])$ such that

$$\alpha(x) \leq u(x) \leq \beta(x) \tag{4.3}$$

on $[a, b]$.

This theorem has been used by Howes and Chang [7] to study stability of singularly perturbed ODE.We note that condition (4.2) is not the most general one (see [7] for other types of conditions), but it is sufficient for our purpose. We point out that stability conditions for systems studied in [7] do not apply to the present case, since the vector solution (s, ϕ) is not expected to be stable.

According to this theorem, to show that $s^\mu \to 0$, it is enough to construct sequences of bounds $\alpha^\mu(x), \beta^\mu(x)$ for solutions s^μ of the first equation (2.1) which converge to zero uniformly on I. Since we are dealing with a system, the bounds for s must be uniform in ϕ. First we consider the case when $\frac{1}{\mathcal{J}}$ tends to infinity as μ approaches zero.

THEOREM 4.2. *We assume that hypotheses (2.7)–(2.9) are satisfied. Let*

$$\frac{1}{\mathcal{J}(\mu)} \to \infty \quad \text{as} \quad \mu \to 0.$$

Suppose that $f \in C^1(-\frac{1}{2}, 1)$ has an isolated local minimum at $s = 0$, and ϕ is an arbitrary differentiable function on I.

Then, there exist a decreasing sequence $\{\mu_n\} \subset (0, 1]$, $\mu_n \to 0$ as $n \to \infty$, and sequences of real numbers $\mathcal{L}_n$ and $\mathcal{M}_n$, such that for every n, there exists a solution s_n to the scalar boundary value problem,

$$\mu_n a_1 s'' = \mu_n a_2 s(\phi')^2 - \frac{1}{2}\beta_1(s) x \frac{\sin 2\phi}{g(s, \cos 2\phi)} + \frac{1}{\mathcal{J}(\mu_n)} \frac{df}{ds}(s), \tag{4.4}$$

$$s(-1) = \mathcal{L}_n, \qquad s(1) = \mathcal{M}_n,$$

with the property that $s_n \to 0$, as $n \to \infty$, uniformly on I. Moreover, the rate of convergence is independent of ϕ.

Proof. By the assumptions on f, there exists a fixed interval E, containing zero, such that $\frac{df}{ds} > 0$ for $s > 0$, and $\frac{df}{ds} < 0$ for $s < 0$. Let us choose a strictly decreasing sequence $\{\sigma_n\} \subset E$, $\sigma_n > 0$, and such that, $\lim_{n \to \infty} \sigma_n = 0$. Likewise, we select an increasing sequence $\{\alpha_n\} \subset E$, $\alpha_n < 0$, and such that $\lim_{n\to\infty} \alpha_n = 0$. For each positive integer n, we can find a real number $\mu_n > 0$ such that

$$\mu a_2 \sigma_n (\phi')^2 + \frac{1}{\mathcal{J}}(\mu)\frac{df}{ds}(\sigma_n) - \frac{1}{2}\beta_1(\sigma_n) x \frac{\sin 2\phi}{g(s,\cos 2\phi)} \geq 0, \quad \text{and}$$
$$\mu a_2 \alpha_n (\phi')^2 + \frac{1}{\mathcal{J}}(\mu)\frac{df}{ds}(\alpha_n) - \frac{1}{2}\beta_1(\alpha_n) x \frac{\sin 2\phi}{g(s,\cos 2\phi)} \leq 0$$

for all $0 < \mu \leq \mu_n$. Indeed, by assumptions (2.7)–(2.9), the last term on the left hand side of the previous expressions is bounded by a constant $\frac{1}{2}\frac{m}{\tau}$, where $m > 0$ is a bound on $x\beta_1$. For each $n > 0$, we now choose $\alpha_n < \mathcal{L}_n, \mathcal{M}_n < \sigma_n$. The conclusion follows from Theorem 4.1, if we use the constant functions σ_n and α_n as, respectively, upper and lower bounds. ∎

Remark 4.1. We point out that the condition $\mathcal{J} \to 0$ (i.e., $\frac{1}{\mathcal{J}} \to \infty$) as $\mu \to 0$ represents the Leslie-Ericksen limit of the theory, that is the case when the liquid crystal is described by the director $\mathbf{n}$, with the order parameter taking a constant value corresponding to a critical point of the bulk energy $f(s)$.

Next, we consider the more subtle case of $\frac{1}{\mathcal{J}}$ bounded. We assume that $\frac{1}{\mathcal{J}}$ and f satisfy the following conditions:
i) $\frac{1}{\mathcal{J}}(\mu)$ is bounded by a constant M, for all $\mu \leq 1$,
ii) $f(s) \in C^2(-\frac{1}{2}, 1)$,
iii) there exist $a, b \in (-\frac{1}{2}, 1)$ such that $|\frac{df}{ds}(s)| \leq M$ for $s \in [a,b]$, $\frac{df}{ds}(s) > 0$ on $(b,1)$ and $\frac{df}{ds}(s) < 0$ on $(-\frac{1}{2}, a)$.

We observe that the last condition on f may allow several potential wells between $s = -\frac{1}{2}$ and $s = 1$. This may include pure nematic liquid crystals as well as compounds ([6], chapter 1).

THEOREM 4.3. *We assume that hypotheses (2.7)–(2.9) are satisfied. Suppose that conditions i),ii) and iii) are fulfilled. Consider sequences $\{s^\mu, \phi^\mu\}$ of weak solutions of (2.1-2.2) such that*

$$s(-1) = s(1) = \mu, \tag{4.5}$$

hold. Then there exist $A(\mu), B(\mu) \in \mathbf{R}$ such that sequences $\{s^\mu, \phi^\mu\}$ additionally satisfying

$$(\phi^\mu)'(-1) = A(\mu), \quad (\phi^\mu)'(1) = B(\mu), \tag{4.6}$$

have the property that $s^\mu \to 0$ as $\mu \to 0$, uniformly on I.

Proof. Fix $\mu > 0$, and consider solutions (s,ϕ) of the problem

$$\mu a_1 s'' = \mu a_2 s (\phi')^2 - \frac{\beta_1 x \sin 2\phi}{2g(s,\phi)} + \frac{1}{\mathcal{J}}\frac{df}{ds}(s), \tag{4.7}$$

$$\mu a_2 (s^2\phi')' = \frac{x}{g}(\gamma_1(s) + \gamma_2(s)\cos 2\phi), \tag{4.8}$$

satisfying boundary conditions (4.5) and (4.6), with the values $A(\mu)$ and $B(\mu)$ to be specified later. Although this solution is weak, the estimates (3.5) imply continuity of s.

First, we integrate (4.8) to obtain

$$s^2(x)\phi'(x) = \frac{1}{\mu a_2}(\mu^2 A + p(x)), \tag{4.9}$$

where

$$p(x) = \int_{-1}^{x} \frac{y}{g(s(y), \phi(y))} (\gamma_1(s(y)) + \gamma_2(s(y)) \cos 2\phi(y))\, dy. \tag{4.10}$$

Next, we represent I as the union of the sets $E^+ = \{x \in I : s > \mu\}$, $E^- = \{x \in I : s < \mu\}$ and $E_0 = \{x \in I : |s| \le \mu\}$, and define s^+ and s^- to be restrictions of s to E^+, E^-, respectively. Since E^+ is open, it can be represented as $E^+ = \cup_{j=1}^{\infty} I_j$, where $I_j = (a_j, b_j)$ are open intervals, and $s(a_j) = s(b_j) = \mu$. Let us consider the system (4.7), (4.8) on some interval I_j. Since $s \ge \mu$ on I_j, we can obtain s from (4.9) and substitute it into (4.7):

$$\mu a_1 s'' = \frac{1}{\mu a_2 s^3} \left(\mu^2 A + p(x)\right)^2 + \frac{\beta_1(s) x \sin 2\phi}{2g(s, \phi)} + \frac{1}{\mathcal{J}} \frac{df}{ds}(s), \tag{4.11}$$

From now on, the strategy of the proof is as follows.
1. Use Nagumo's theorem to construct, for each choice of ϕ, a solution u of (4.11) on I_j, satisfying the boundary conditions (4.5), and such that $-2\mu \le u \le 2\mu$ on I_j. To obtain existence of u, we shall choose A, B sufficiently large. The same choice of A, B works for all I_j, which provides u on E^+. The analogous argument yields existence of u on E^-.
2. Once the choice of A, B is made, the boundary conditions (4.5), (4.6) are specified, and we obtain a particular solution (S, Φ) of (4.7) and (4.8).
3. We prove that u, corresponding to the choice $\phi = \Phi$ in (4.11), coincides with S on the intervals where $|S| > \mu$. This gives an upper bound on S.
4. For each μ, define $s^\mu = S$, $\phi^\mu = \Phi$.

Now we begin implementing the previously described strategy. First, note that the right hand side of (4.11) is a continuous function of s when $x \in I_j$, and the Nagumo condition (4.2) is satisfied, since the right hand side is independent of s'. The bounds on u must be independent of the choice of ϕ, which would allow us to adjust ϕ later on when $A = \phi'(-1)$ is chosen. We begin with the construction of the upper bound. To construct the bound, we use the comparison variational system,

$$\mu a_1 q'' = \mu a_2 q (\psi')^2 - \frac{dh}{dq}(q), \tag{4.12}$$

$$\mu a_2 (q^2 \psi')' = 0. \tag{4.13}$$

This system is classical (see [1]), at least when $a_1 = a_2$. It describes the motion of a particle in a central force field with the potential h. In that case, x denotes time, and q, ψ represent the radial and angular particle coordinates, respectively. The following properties of (4.12) and (4.13) are well known. The system has two first integrals, the angular momentum and the total energy, respectively:

$$M = q^2 \psi',$$
$$E = \frac{1}{2}\mu a_1 (q')^2 + \frac{1}{2}\mu a_2 q^2 (\psi')^2 + h(q).$$

Combining these two equations allows us to rewrite,

$$E = \frac{1}{2}\mu a_1 (q')^2 + V(q), \quad \text{where}$$
$$V(q) = \frac{1}{2}\mu a_2 \frac{M^2}{q^2} + h(q),$$

is the effective potential energy. The evolution of q is described by the autonomous differential equation

$$\mu a_1 q'' = \mu a_2 \frac{M^2}{q^3} - \frac{dh}{dq}(q). \tag{4.14}$$

We point out that it is possible to choose E, M and h (non-uniquely) so that $V(q)$ has a minimum at $q_0 = 3/2\mu$, and such that the solution, $q(x)$ remains close to q_0 for all $x \in I_j$; in such a case, there are positive constants c_1, c_2, with the property that $c_1\mu \le q \le c_2\mu$, and $c_1 > 1, c_2 < 2$. The function q is an upper bound for u provided the inequality,

$$\mu a_2 \frac{M^2}{q^3} - h'(q) \le \frac{1}{\mu a_2 q^3}\left(\mu^2 A - C_1\right)^2 - \frac{C_2}{\tau} + \frac{1}{\mathcal{J}}\frac{df}{ds}(q), \tag{4.15}$$

holds. Here the constant C_1 denotes an upper bound of $|p(x)|$, and C_2 an upper bound of $|\beta_1(q)x|$. Since $q > 0$, the right hand side of (4.15) is dominated by that of (4.11). We point out that C_1 and C_2 are independent of s, ϕ and μ. We also note that, since $q \in (c_1\mu, c_2\mu)$, the second and third terms in the right hand side of (4.15) are bounded independent of μ. So, in order to satisfy inequality (4.15), it is sufficient to choose $A(\mu)$ large enough so that

$$(\mu^2 A - C_1)^2 - (\mu a_2)^2 M^2 \ge K, \tag{4.16}$$

where $K > 0$ is independent of μ. Then, for $\mu \in (0, 1]$, the first term in the right hand side is dominant. We observe that the condition (4.16) is independent of the choice of the interval I_j, so it will remain the same when we consider different intervals. This completes the construction of the upper bound.

The construction of the lower bound is similar. We only need to choose q to be negative with the absolute value on the order of μ. Since the power of q in the dominant term is odd, we obtain a lower bound $q \ge -2\mu$. Now by Nagumo's theorem, for each choice of ϕ, there exists u such that the pair u, ϕ solves the (4.11) and such that $|u| \le 2\mu$ holds on each I_j, which means that we have u defined on E^+.

Next, a slight modification of the preceding argument provides existence of the solution u on E^-. Since A has already been determined, we can now determine B using equations (4.5) and (4.9). Specifically, set

$$B = \frac{1}{\mu^3 a_2}(\mu^2 A + C_1). \tag{4.17}$$

So, the boundary conditions, A and B of ϕ' are now specified. Thus, solving the system (4.7), (4.8) subject to boundary conditions (4.5) and (4.6), yields a solution (S, Φ).

We observe from relations (4.16) and (4.17) that the quantities A and B are independent of the particular choice of s used in the definition of the level sets $E^{\pm}$.

We now consider the sets $E^{\pm}$ corresponding to the fields (S, Φ). Arguing as above, we construct the solution u, on the new sets $E^{\pm}$, and denote the restrictions of S to E^+ (E^-) by S^+ (S^-). Next we show that u and S^+ are equal on I_j. Suppose, otherwise, that $S^+ \ge u$ on some interval $L \subset I_j$ and $S^+ = u$ at the endpoints of L. Consider the function $v = S^+ - u$. Substituting u, S^+ into (4.7) and subtracting the resulting equations we find that

$$\mu a_1 v'' = \mu a_2 v(\Phi')^2 - \frac{1}{2}\Big(\frac{\beta_1(S^+)}{g(S^+, \Phi)} - \frac{\beta_1(u)}{g(u, \Phi)}\Big) x \sin 2\Phi + \tag{4.18}$$

$$\frac{1}{\mathcal{J}}(\frac{df}{ds}(S^+) - \frac{df}{ds}(u)).$$

Since $\frac{\beta_1}{g(s,\Phi)}$ is Lipschitz in s, we note that $|\big(\frac{\beta_1(S^+)}{g(S^+,\Phi)} - \frac{\beta_1(u)}{g(u,\Phi)}\big)x\sin 2\Phi| \leq C_g|v|$, holds, where C_g is independent of the choice of I_j. Furthermore, for μ sufficiently close to zero, because of condition iii), $\frac{df}{ds}(u)$ belongs to the interval on which $\frac{df}{ds}$ and $\frac{d^2f}{ds^2}$ are bounded. Now if S^+ is also in that interval, then using condition ii), we can write $|\frac{df}{ds}(S^+) - \frac{df}{ds}(u)| \leq C_f v$, where C_f is independent of I_j. Otherwise, if S^+ is large, $\frac{df}{ds}(S^+) > |\frac{df}{ds}(u)|$, by condition iii). Hence, the right hand side of (4.18) is positive, for all nonnegative v. Then (4.18) implies that $v'' \geq 0$ on L. Hence, v is a convex function on L which is nonnegative and satisfies zero boundary conditions. Thus, v must be identically zero on L.

Similarly, we prove that if $u \geq S^+$ on some interval, then in fact $u = S^+$. From this and continuity of S^+, we conclude that $u = S^+$ on I_j and the original solution S satisfies,

$$\mu \leq S \leq 2\mu \tag{4.19}$$

on I_j. Since the analogous arguments apply to all intervals in E^+, we obtain that S satisfies the inequality (4.19) in E^+. Moreover, $|S| \leq \mu$ in E_0. It remains to prove that $-2\mu \leq S \leq -\mu$ on E^-. The proof is analogous to that in the case of E^+.

Finally, for each $\mu \in (0,1]$, define $s^\mu = S$, $\phi^\mu = \Phi$. Since $|s^\mu| \leq 2\mu$ on I, and the sequence s^μ converges to zero uniformly as $\mu \to 0$. □

Remark 4.2. Equations (4.16) and (4.17) suggest that $A = O(\mu^{-2}), B = O(\mu^{-3})$ as $\mu \to 0$.

Remark 4.3. The second part of the proof provides a partial uniqueness argument for solutions of the system (2.1)–(2.4), with the boundary conditions satisfying (4.16), (4.17). It seems that such an argument can be used to prove uniqueness for each fixed $\mu > 0$. This will be the subject of the future research.

The proof of the previous theorem brings out some relevant physical aspects of the problem as well as related mathematical issues. The former arguments strongly reflect the interplay between the mechanisms responsible for the oscillatory behavior of solutions and the degenerate nature of the boundary conditions for s close to 0. (Let us recall that ϕ is undefined when $s = 0$.) In order to illustrate such features, let us consider the boundary value problem with prescribed nonzero boundary conditions on s, and also prescribed values of ϕ'. If we now allow the boundary values of s approach 0, it is natural to expect that the boundary values of ϕ' cannot be independently chosen. This is indeed the nature of Theorem 4.3. Specifically, the restriction on $\phi'(\pm 1)$ imposed by $s(\pm 1) = \mu$, with μ small is contained in inequality (4.16).

A consequence of the fact that $\phi'(\pm 1)$ is large, as indicated by (4.16), is that ϕ' is positive on I, so ϕ is increasing on the intervals of continuity. Consequently, ϕ may be large in such intervals. In order to ensure boundedness of ϕ, we will make use of the shift invariance of the system (2.1)-(2.2). Indeed, if s, ϕ is a solution, then $s, \phi + k\pi$ is also a solution for any integer k. Starting with an increasing $\tilde{\phi}$, we can split the interval I into subintervals on which $k\pi \leq \tilde{\phi} \leq (k+1)\pi$ and, then define ϕ by shifting appropriately on each subinterval. The function ϕ obtained in such a fashion will be bounded, rapidly oscillating and discontinuous. We will refer to solutions with rapidly oscillating discontinuous ϕ as generalized to distinguish them from the weak solutions. The discontinuities of ϕ are associated with liquid crystal defects.

Corollary 4.4. *Let $\tilde{s}^\mu, \tilde{\phi}^\mu$ be a sequence of weak solutions constructed in the Theorem 4.3. Then there exists a sequence of generalized solutions s^μ, ϕ^μ such that*

i) $s^\mu \to 0$ *uniformly on* I*;*
ii) $|\phi^\mu| \leq \frac{\pi}{2}$*;*
iii) for each $\mu > 0$*, the support of the distribution* $(\tilde{\phi}^\mu)' - (\phi^\mu)'$ *is a finite set of points;*
iv) for any $h \in C_0^1(I)$*,* s^μ, ϕ^μ *satisfy the integral identities*

$$\mu a_1 \int_I (s^\mu)' h' \, dx - \mu a_2 \int_I s^\mu ((\phi^\mu)')^2 h \, dx = \int_I G_1(s^\mu, \phi^\mu) h \, dx, \tag{4.20}$$
$$\mu a_2 \int_I (s^\mu)^2 (\tilde{\phi}^\mu)' h' \, dx = \int_I G_2(s^\mu, \phi^\mu) h \, dx.$$

Proof. Let J be the largest integer such that $\phi(-1) \geq J\pi - \frac{\pi}{2}$. The interval I can be represented as a union of N_μ disjoint intervals I_k, $k = 0, 1, 2, ...N_\mu$, such that $x \in I_k$ when

$$\pi(J + k) - \frac{\pi}{2} \leq \phi(x) < \pi(J + k) + \frac{\pi}{2},$$

holds. Starting with $\tilde{s}^\mu, \tilde{\phi}^\mu$, we define s^μ, ϕ^μ as follows:

$$s^\mu = \tilde{s}^\mu,$$
$$\phi^\mu = \tilde{\phi}^\mu - \pi(J + k),$$

for $x \in I_k$. Note that the distributional derivative of ϕ^μ is not locally integrable. We have, however,

$$(\phi^\mu)' = (\tilde{\phi}^\mu)',$$

when both are restricted to the complement of the set of the endpoints of I_k.

Next we observe that $G_l(s, \phi) = G_l(s, \phi + m\pi)$, where $l = 1, 2$ and m is an integer. Hence, replacing $\tilde{\phi}^\mu$ by ϕ^μ in the right hand sides of the integral identities for the weak solutions which yields (4.20). □

Remark 4.4. Note that, in general, it is not possible to replace $(\tilde{\phi}^\mu)'$ by $(\phi^\mu)'$ in the left hand side of the second identity.

5. Effective Configurations. The limiting process yields effective governing equations and configurations of Newtonian Poiseuille flow, with constant viscosity. However additional equations associated to microstructural phenomena also arise at the limit. They may be related to the occurrence of remnant ordered states on a microscopic scale.

Let us first recall the equations (2.11)–(2.13) and the definition of g:

$$g(s, \phi) = \frac{1}{2}\alpha_4(s) + \alpha_1(s) \sin^2\phi \cos^2\phi + \frac{1}{2}(\alpha_5 - \alpha_2)(s) \sin^2\phi + \frac{1}{2}\alpha_3(s) \cos^2\phi. \tag{5.1}$$

If $\{s^\mu, \phi^\mu\}$ is a sequence of functions such that $s^\mu \to 0$ uniformly, then by (2.16)

$$g(s^\mu, \phi^\mu) \to \frac{1}{2}\alpha_4(0),$$

uniformly on I, as μ approaches zero. Up to a subsequence, the solution v_μ of (2.13) tends to a limit v_0 strongly in $L^2(I)$, and weakly in $W^{1,2}(I)$. Hence, the product

$g(s^\mu, \phi^\mu)v'_\mu$ converges to $\frac{1}{2}\alpha_4(0)v'_0$, weakly in $L^2(I)$. This implies that v_0 satisfies the effective equation

$$\eta_{\text{eff}} v''_0 = 1, \tag{5.2}$$

which is the Newtonian Poiseuille flow with constant effective viscosity $\eta_{\text{eff}} = \frac{1}{2}\alpha_4(0)$. When the Ericksen number is large and the Reynolds number is of order 1, the typical viscosity is much larger than the typical elasticity. It is natural to expect that alignment of the molecules will be destroyed by the diffusion, so that liquid crystal flow is that of an isotropic liquid with a constant viscosity. If that were the case, equation (5.2) would be the only effective equation of the limiting flow. Rigorous analysis suggests, however, that one should also consider the Young measure ν_x generated by the sequence ϕ^μ. In the Remark 5, following the proof of the theorem 5.1, we explain that ν_x is non-trivial, which means that it cannot have the form $\delta(z - \overline{\phi}(x))$ for any function $\overline{\phi}(x)$. Since localization of ν_x at $\overline{\phi}$ signals strong convergence, this implies that the sequence ϕ^μ cannot converge strongly. Therefore, ν_x must describe "possible disordered states" compatible with the boundary conditions and macroscopic flow.

Combining Theorem 3.3 and Proposition 3.4 with the partial stability Theorem 4.3, we obtain

THEOREM 5.1. *Let $\tilde{s}^\mu, \tilde{\phi}^\mu$ be a sequence of weak solutions from the Theorem 4.3 satisfying the a priori estimates*

$$\| \tilde{s}' \|_{L^2(I)} \leq C,$$

$$\| \tilde{s}\tilde{\phi}' \|_{L^2(I)} \leq C,$$

with C independent of μ. Let (s^μ, ϕ^μ) be a corresponding sequence of generalized solutions. Then, up to a subsequence,
i) $s^\mu \to 0$ uniformly on I;
ii) The sequence ϕ^μ generates a Young measure ν_x satisfying moment relations

$$\int_{-1}^{1} \int_{-\pi/2}^{\pi/2} \Big(G_1(0,z,x)\sin az - \frac{1}{a}\frac{G_2}{z}(0,z,x)\cos az\Big) d\nu_x(z) h(x)\, dx = 0, \tag{5.3}$$

$$\int_{-1}^{1} \int_{-\pi/2}^{\pi/2} \Big(G_1(0,z,x)\cos az + \frac{1}{a}\frac{G_2}{z}(0,z,x)\sin az\Big) d\nu_x(z) h(x)\, dx = 0,$$

for each $h \in C_0^1(I)$. In (5.3), $a = (a_2/a_1)^{1/2}$;
iii) The sequence $s((\tilde{\phi}^\mu)')^2$ converges to a measure ρ in the sense of distributions. Moreover,

$$\rho = \int_{-\pi/2}^{\pi/2} G_1(0,z,x) d\nu_x(z). \tag{5.4}$$

Proof. Part i) follows directly from Theorem 4.3. Since s^μ converges to zero in L^2, the corresponding Young measure is $\delta(y)$. Hence, part ii) follows from the Theorem 3.3 and momentum relations in Proposition 3.4.

Next, consider the integral identity

$$-\mu a_1 \int_I s'h'dx + \mu a_2 \int_I s(\phi')^2 h\, dx = \int_I G_1(x,s,\phi)h\, dx.$$

Since $\| s' \|_{L^2(I)} \leq C\mu^{-1/2}$ with C independent of μ, the first integral on the left converges to zero as $\mu \to 0$. The integral in the right hand side converges to

$$\int_I \int_{-\pi/2}^{\pi/2} G_1(0,z,x) d\nu_x(z) h(x) dx.$$

This yields,

$$\lim_{\mu\to 0} \int_I s(\phi')^2 h\, dx = \int_I \int_{-\pi/2}^{\pi/2} G_1(0,z,x) d\nu_x(z) h(x)\, dx,$$

for all $h \in C_0^1(I)$. Since this space is dense in $C_0(I)$, the equality above holds for all $h \in C_0(I)$. Hence, the distributional limit of $s(\phi')^2$ is a Radon measure ρ such that

$$\int_I h d\rho = \int_I h \int_{-\pi/2}^{\pi/2} G_1(0,z,x) d\nu_x(z) dx$$

for all $h \in C_0(I)$. □

Remark 5.1. Strong convergence of ϕ^μ is incompatible with the moment relations (5.3). If strong convergence takes place, then $\nu_x(z) = \delta(x - \overline{\phi}(x))$ for some function $\overline{\phi}$ (the strong limit). If that were the case, then from (5.3) we would obtain

$$G_1(0,\overline{\phi},x) \sin a\overline{\phi} - \frac{1}{a}\left(\frac{G_2}{s}\right)(0,\overline{\phi},x) \cos a\overline{\phi} = 0, \tag{5.5}$$

$$G_1(0,\overline{\phi},x) \cos a\overline{\phi} + \frac{1}{a}\left(\frac{G_2}{s}\right)(0,\overline{\phi},x) \sin a\overline{\phi} = 0, \tag{5.6}$$

and thus

$$G_1(0,\overline{\phi},x) = 0, \tag{5.7}$$

$$\left(\frac{G_2}{s}\right)(0,\overline{\phi},x) = 0. \tag{5.8}$$

This means that $s = 0, \phi = \overline{\phi}$ is an equilibrium solution of (2.1, 2.2). Direct computation shows that $s = 0$ cannot be an equilibrium solution for any $\overline{\phi}$.

6. Conclusions. We study the oscillatory behavior of the solutions of the governing equations modeling Poiseuille flow of liquid crystals with variable degree of orientation, at the limit of large Ericksen number $\mathcal{E}$. The governing equations are singularly perturbed and highly degenerate. The oscillations of s occur about the isotropic value $s = 0$. At points where s vanishes the angle of alignment ϕ is discontinuous and ϕ' becomes unbounded. This situation corresponds to the presence of defects in the flow. We obtain necessary condition for the Young measures generated by sequences of solutions, and show that the persistent oscillatory behavior is encoded in the Young measure generated by ϕ. We prove a partial stability result establishing uniform convergence of s to 0 as $\mu = \mathcal{E}^{-1} \to 0$, and the increasingly oscillatory behavior of ϕ. Compactness of s allows us to pass to the limit in the governing system and obtain the effective equations. The latter consist of the Newtonian flow equation together with the algebraic relations for the Young measure generated by ϕ. This suggest that macroscopically the flow is isotropic and Newtonian with a remaining liquid crystalline microstructure.

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