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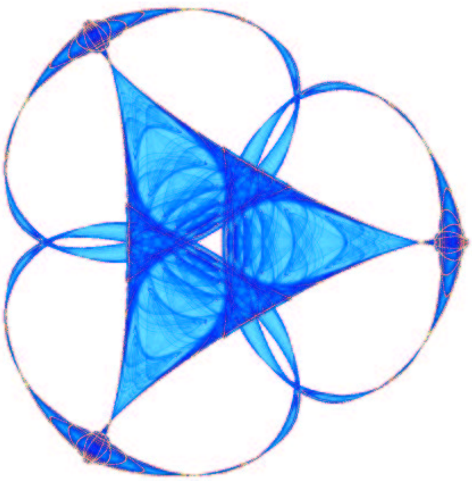

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Geodesic Flow and Two (Super) Component Analog of the Camassa–Holm Equation

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Abstract

We derive the 2-component Camassa–Holm equation and corresponding $N = 1$ super generalization as geodesic flows with respect to the H^1 metric on the extended Bott-Virasoro and superconformal groups, respectively.

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1 Introduction

About ten years ago, Rosenau, [19], introduced a class of solitary waves with compact support as solutions of certain wave equations with nonlinear dispersion. It was found that the solutions of such systems unchanged from collision and were thus called *compactons*. The discovery that solitons may compactify under nonlinear dispersion inspired further investigation of the role of nonlinear dispersion. It has been known for quite sometime that nonlinear dispersion causes wave breaking or lead to the formation of corners or cusps, but not the above novel feature within the framework of integrable systems. In fact, apart from solutions without decaying tails, that is compactons, other forms of exotic solutions with discontinuous derivatives in the form of peaks and cusps have been found in various models with nonlinear dispersion, and these solutions are named peakons and cuspon respectively.

In an earlier work, the second author showed with Rosenau [17] that a simple scaling argument shows that most integrable bihamiltonian systems are governed by tri-Hamiltonian structures. They formulated a method of 'tri- Hamiltonian duality" in which a recombination of the Hamiltonian operators leads to integrable hierarchies endowed with nonlinear dispersion that supports compactons or peakons.

Let us consider a bihamiltonian system appearing as the n-th member of a hierarchy of an integrable evolution equations

$$u_t = F_n[u] = J_1 \frac{\delta H_{n+1}}{\delta u} = J_2 \frac{\delta H_n}{\delta u} \qquad n = 0, 1, 2, \cdots \tag{1}$$

where $H_n[u]$ are the Hamiltonians. The members of the hierarchy are successively generated by the recursion operator

$$\mathcal{R} = J_2 J_1^{-1}.$$

The formalism of [17] can be best described through example. The Korteweg–deVries equation

$$u_t = u_{xxx} + 3uu_x, \tag{2}$$

can be written in bihamiltonian form (1) using the two compatible Hamiltonian operators

$$J_1 = D, \qquad J_2 = D^3 + uD + Du \qquad \text{where} \;\; D \equiv \frac{d}{dx}$$

and

$$H_1 = \frac{1}{2}\int u^2 \, dx, \qquad H_2 = \frac{1}{2}\int (-u_x^2 + u^3) \, dx.$$

The tri- Hamiltonian duality procedure is implemented as follows:

- A simple scaling argument shows that J_2 is in fact the sum of two compatible Hamiltonian operators, namely $K_2 = D^3$ and $K_3 = uD + Du$, so that $K_1 = J_1, K_2, K_3$ form a triple of mutually compatible Hamiltonian operators.
- Thus, when we can recombine the Hamiltonian triple as transfer the leading term D^3 from J_2 to J_1, thereby constructing the Hamiltonian pairs $\hat{J}_1 = K_2 \pm K_1 = D^3 \pm D$. The resulting self-adjoint operator $S = 1 \pm D^2$ is used to define the new field variable $\rho = Su = u \pm u_{xx}$.
- Finally, the second Hamiltonian structure is constructed by replacing u by ρ in the remaining part of the original Hamiltonian operator K_3, so that $\hat{J}_2 = \rho D + D\rho$. Note that this change of variables does not affect $\hat{J}_1$.

As a result of this procedure, we recover the compacton analogue of KdV as

$$\rho_t = \hat{J}_1 \frac{\delta \hat{H}_2}{\delta \rho} = \hat{J}_2 \frac{\delta \hat{H}_1}{\delta \rho}, \tag{3}$$

where

$$\hat{H}_1 = \frac{1}{2}\int u\rho\, dx = \frac{1}{2}\int (u^2 \mp u_x^2)\, dx, \qquad \hat{H}_2 = \frac{1}{2}\int (u^3 \mp u u_x^2)\, dx.$$

Therefore, (3) reduces to the celebrated Camassa–Holm equation [2, 3]:

$$u_t \pm u_{xxt} = 3uu_x \pm \left(uu_{xx} + \tfrac{1}{2}u_x^2\right)_x . \tag{4}$$

Remark: The choice of plus sign leads to an integrable equation which yields compacton, whereas the minus sign gives the peakon equation derived by Camassa–Holm, whose solitary wave solutions have a sharp corner at the crest.

The Ito equation Let us study the Ito equation [12],

$$\begin{aligned} u_t &= u_{xxx} + 3uu_x + vv_x , \\ v_t &= (uv)_x . \end{aligned} \tag{5}$$

which is a protypical example of a two component KdV equation. Once again, this can be expressed in bihamiltonian form using the following two Hamiltonian operators

$$J_1 = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \qquad J_2 = \begin{pmatrix} D^3 + uD + Du & vD \\ Dv & 0 \end{pmatrix},$$

with Hamiltonians

$$H_1 = \frac{1}{2}\int (u^2 + v^2)\, dx, \qquad H_2 = \frac{1}{2}\int (u^3 + uv^2 - u_x^2)\, dx.$$

Let us transfer again the leading term D^3 from the first Hamiltonian operator to the second. We obtain the first Hamiltonian operator for the new equation

$$\hat{J}_1 = \begin{pmatrix} D^3 \pm D & 0 \\ 0 & D \end{pmatrix} \equiv D \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the new variables become

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The second Hamiltonian structure follows from the truncated part of the original Hamiltonian operator J_2, so that

$$\hat{J}_2 = \begin{pmatrix} \rho D & vD \\ Dv & 0 \end{pmatrix}$$

with

$$\hat{H}_1 = \frac{1}{2}\int (u\rho + v^2)\, dx \qquad \hat{H}_2 = \frac{1}{2}\int (u^3 + uv^2 \mp uu_x^2)\, dx$$

Therefore (3) take the explicit form

$$\begin{aligned} u_t \pm u_{xxt} &= 3uu_x + vv_x \pm uu_{xx} + \tfrac{1}{2}\left(uu_{xx} + \tfrac{1}{2}u_x^2\right)_x , \\ v_t &= (uv)_x . \end{aligned} \tag{6}$$

Motivation The Camassa–Holm equation was derived physically as a shallow water wave equation by Camassa and Holm. Later, Misiolek [16] showed that, like the KdV equation [18], it can also be characterized as a geodesic flow on the Bott–Virasoro group.

Recently a 2-component generalization of the Camassa–Holm equation has drawn lot of interest among scientists. Last year, in Bologna[1] Dubrovin, Grava, Zhang and Falqui (the group at SISSA) said they have been working on multi-component analogues of Camassa–Holm, using reciprocal transformations and studying how the Hamiltonian structures change under such non-local transformations [4]. Falqui [7] showed that the 2-component system as above does admit analogues of peakons, but they have a different shape from the Camassa–Holm peakons owing to the difference in the corresponding Green's functions. In the work of the SISSA group, they write the system in different variables which make the properties more transparent.

Following Ebin-Marsden [6], we enlarge $\mathit{Diff}(S^1)$ to a Hilbert manifold $\mathit{Diff}^s(S^1)$, consisting of the diffeomorphisms of the Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local charts, one can check whether the derivations of order $\leq s$ are square integrable. The Lie algebra of $\mathit{Diff}^s(S^1)$ is denoted by $Vect^s(S^1)$.

In this paper we show that a 2-component generalization of the Camassa–Holm equation and its super analog also follow from the geodesic with respect to the H^1 metric on the semidirect product space $\mathit{Diff}^s(S^1) \ltimes S^1$ and its supergroup respectively. In fact, it is known that numerous coupled KdV equations [9, 10, 11] follow from geodesic flows of the right invariant L^2 metric on the semidirect product group $\widehat{\mathit{Diff}(S^1) \ltimes C^\infty(S^1)}$, [1, 15].

2 Preliminaries

The Lie algebra of $\mathit{Diff}^s(S^1) \ltimes C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathfrak{g} = Vect^s(S^1) \ltimes C^\infty(S^1).$$

An element of $\mathfrak{g}$ is a pair

$$\left(f(x)\frac{d}{dx}, a(x)\right), \qquad \text{where} \qquad f(x)\frac{d}{dx} \in Vect(S^1), \quad \text{and} \quad a(x) \in C^\infty(S^1).$$

[1]We like to thank Andy Hone for this information.

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$
\begin{aligned}
\omega_1\left(\left(f(x)\frac{d}{dx},a(x)\right),\left(g\frac{d}{dx},b\right)\right) &= \int_{S^1} f'(x)g''(x)dx \\
\omega_2\left(\left(f(x)\frac{d}{dx},a(x)\right),\left(g\frac{d}{dx},b\right)\right) &= \int_{S^1} [\, f''(x)b(x) - g''(x)a(x)\,]dx \\
\omega_3\left(\left(f(x)\frac{d}{dx},a(x)\right),\left(g\frac{d}{dx},b\right)\right) &= 2\int_{S^1} a(x)b'(x)dx.
\end{aligned} \tag{7}
$$

The first cocycle ω_1 is the well-known Gelfand–Fuchs cocycle. The Virasoro algebra

$$Vir = Vect^s(S^1) \oplus \mathbb{R}$$

is the unique non-trivial central extension of $Vect^s(S^1)$ based on the Gelfand–Fuchs cocycle. The space $C^\infty(S^1) \oplus \mathbb{R}$ is identified as *regular part* of the dual space to the Virasoro algebra. The pairing between this space and the Virasoro algebra is given by:

$$\left\langle\, (u(x),a)\,,\, \left(f(x)\frac{d}{dx},a(x)\right)\right\rangle = \int_{S^1} u(x)f(x)dx + a\alpha.$$

Similarly we consider the following extension of $\mathfrak{g}$,

$$\widehat{\mathfrak{g}} = Vect^s(S^1) \ltimes C^\infty(S^1) \oplus \mathbb{R}^3. \tag{8}$$

The commutation relation in $\widehat{\mathfrak{g}}$ is given by

$$\left[\left(f\frac{d}{dx},a,\alpha\right),\left(g\frac{d}{dx},b,\beta\right)\right] := \left((fg'-f'g)\frac{d}{dx}, fb'-ga', \omega\right) \tag{9}$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{R}^3$, and where $\omega = (\omega_1,\omega_2,\omega_3)$ are the cocycles. However, we will not require cocycle term to derive the 2-component generalization of the Camassa–Holm equation, and so they are omitted.

Let

$$\widehat{\mathfrak{g}}^*_{reg} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$$

denote the *regular part* of the dual space $\widehat{\mathfrak{g}}^*$ to the Lie algebra $\widehat{\mathfrak{g}}$, under the following pairing:

$$\langle\, \hat{u}\,,\, \hat{f}\,\rangle = \int_{S^1} [\, f(x)u(x) + a(x)v(x)\,]\,dx + \alpha\cdot\gamma, \tag{10}$$

where $\hat{u} = (u(x),v,\gamma) \in \widehat{\mathfrak{g}}^*_{reg}$, $\hat{f} = \left(f\frac{d}{dx},a,\alpha\right) \in \widehat{\mathfrak{g}}$. Of particular interest are the coadjoint orbits in $\widehat{\mathfrak{g}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev, [8], have constructed some of the corresponding representations.

Let us introduce H^1 inner product on the algebra $\widehat{\mathfrak{g}}$

$$\langle\, \hat{f}\,,\, \hat{g}\,\rangle_{H^1} = \int_{S^1} [\, f(x)g(x) + a(x)b(x) + \partial_x f(x)\partial_x g(x)\,]\,dx + \alpha\cdot\beta, \tag{11}$$

where

$$\hat{f} = \left(f\frac{d}{dx}, a, \alpha \right), \qquad \hat{g} = \left(g\frac{d}{dx}, b, \beta \right).$$

Now we compute :

Lemma 2.1 *The coadjoint operator with respect to the H^1 inner product is given by*

$$ad^*_{\hat{f}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (1-\partial^2)^{-1}[2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'v(x)] \\ f'v(x) + f(x)v'(x) \end{pmatrix}. \tag{12}$$

Proof: From the definition it follows that

$$\begin{aligned}\langle\, ad^*_{\hat{f}}\hat{u}\,,\, \hat{g}\,\rangle_{H^1} &= \langle\, \hat{u}\,,\, [\hat{f}, \hat{g}]\,\rangle_{H^1} \\ &= -\int_{S^1} [\,(fg' - f'g)u - (fb' - ga')v - \partial_x(fg' - f'g)u\,]dx.\end{aligned}$$

After computing all the terms by integrating by parts and using the fact that the functions $f(x), g(x), u(x)$ and $a(x), b(x), v(x)$ are periodic, the right hand side can be expressed as above.

Let us compute now the left hand side:

$$\begin{aligned} ad^*_{\hat{f}} \begin{pmatrix} u \\ v \end{pmatrix} &= \int_{S^1} [\,(ad^*_{\hat{f}}u)g + (ad^*_{\hat{f}}u)'g' + (ad^*_{\hat{f}}v)b\,]\; dx \\ &= \int_{S^1} [\,[(1-\partial^2)ad^*_{\hat{f}}u]g + (ad^*_{\hat{f}}v)b\,]\; dx = \Big\langle\, ((1-\partial^2)ad^*_{\hat{f}}u\,,\, (ad^*_{\hat{f}}v)), (g, b)\,\Big\rangle \end{aligned}$$

Thus by equating the the right and left hand sides, we obtain the desired formula. □

We conclude that the Hamiltonian operator arising from the induced Lie–Poisson structure is

$$\begin{pmatrix} D\rho + \rho D & vD \\ Dv & 0 \end{pmatrix}, \tag{13}$$

where $\rho = (1-\partial_x^2)u$. We conclude that

Theorem 2.2 *A curve*

$$\hat{c}(t) = \left(u(x,t)\frac{d}{dx}, v(x,t), \gamma \right) \subset \mathfrak{g}$$

defines a geodesic in the H^1 metric if and only if

$$\begin{aligned} u_t - u_{xxt} &= u_{xxx} + 3uu_x + vv_x - \left(uu_{xx} + \tfrac{1}{2}u_x^2 \right)_x \\ v_t &= 2(uv)_x. \end{aligned} \tag{14}$$

3 Geodesic flow and superintegrable systems

The first and foremost characteristic property of a superalgebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$ graded. A vector superspace is a $\mathbb{Z}_2$ graded vector space $V = V_B \oplus V_F$. An element v of V_B (resp. V_F) is said to be even or bonsonic (resp. odd or fermionic). The super-commutator of a pair of elements $v, w \in V$ is defined to be the element

$$[\, v, w\,] = vw - (-1)^{\bar{v}\bar{w}} wv.$$

The generalized Neveu-Schwartz superalgebra has two parts, bosonic (even) and fermionic (odd). These are given by

$$S\mathfrak{g}_B = Vect^s(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3, \qquad\qquad S\mathfrak{g}_F = C^\infty(S^1) \oplus C^\infty(S^1). \tag{15}$$

There are *three* different actions:
(A) action of the bosonic part on the bosonic part, discussed earlier.
(B) action of the bosonic part on the fermionic part, given by

$$\begin{gathered}[(f(x)\frac{d}{dx}, a(x)), (\phi(x), \alpha(x))] := \\ \begin{pmatrix} f(x)\phi' - \frac{1}{2}f'(x)\phi(x) \\ f(x)\alpha'(x) + \frac{1}{2}f'(x)\alpha(x) - \frac{1}{2}a'(x)\phi(x) \end{pmatrix}\end{gathered} \tag{16}$$

(C) action of the fermionic part on the fermionic part, given by

$$[\ ,\ ]_+ : S\mathfrak{g}_F \otimes S\mathfrak{g}_F \longrightarrow S\mathfrak{g}_B$$

$$[(\phi(x), \alpha(x)), (\psi(x), \beta(x))]_+ = (\phi\psi\frac{d}{dx}, \phi\beta + \alpha\psi, \omega_F), \tag{17}$$

where ω_F is a fermionic cocycle, which is given by

$$\begin{aligned} \omega_{F1}((\phi, \alpha), (\psi, \beta)) &= 2\int_{S^1} \phi'(x)\psi'(x)dx, \\ \omega_{F2}((\phi, \alpha), (\psi, \beta)) &= -2\int_{S^1} (\phi'(x)\beta(x) + \psi'\alpha(x))dx, \\ \omega_{F3}((\phi, \alpha), (\psi, \beta)) &= 4\int_{S^1} \alpha(x)\beta(x)dx. \end{aligned} \tag{18}$$

The supercocycle ω_S has two parts, the bosonic and the fermionic:

$$\omega_S = \omega_B \oplus \omega_F.$$

The bosonic part ω_B is identical to ω, given by the equation (7).

Once again we do not require a cocycle term to establish the supersymmetric 2-component generalization of the Camassa–Holm equation.

Definition 3.1 *The H^1 pairing between the regular part of the dual space $S\widehat{\mathfrak{g}}^*$ and $S\mathfrak{g}$ is given by*

$$\left\langle (u(x), v(x), \psi(x), \beta)\,,\ (f(x)\frac{d}{dx}, a(x), \phi(x), \alpha) \right\rangle_{H^1} =$$

$$\int_{S^1} f(x)u(x)dx + \int_{S^1} f_x u_x\, dx + \int_{S^1} a(x)v(x)\, dx \tag{19}$$

$$+ \int_{S^1} \phi(x)\psi(x)dx + \int_{S^1} \phi_x\psi_x\, dx + \int_{S^1} \alpha(x)\beta(x)dx$$

Let us compute the coadjoint action with respect to the H^1 norm.

Lemma 3.2

$$ad^*_{\widehat{f}} \begin{pmatrix} u(x) \\ v(x) \\ \psi(x) \\ \beta(x) \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} (1-\partial^2)^{-1}[2f'(1-\partial^2)u(x) + (1-\partial^2)u'f + a'v + \frac{1}{2}(1-\partial^2)\psi'\phi + \frac{3}{2}(1-\partial^2)\psi\phi'] \\ f'v + fv' + \frac{1}{2}(\beta'\phi + \beta\phi') \\ (1-\partial^2)^{-1}[f(1-\partial^2)\psi' + \frac{3}{2}f'(1-\partial^2)\psi + \frac{1}{2}a'\beta + (1-\partial^2)u\phi + v\alpha] \\ f\beta' + \frac{1}{2}f'\beta + v\phi \end{pmatrix}$$

Sketch of the Proof: Using the definition of the coadjoint action

$$\langle\, ad^*_{\widehat{f}}\widehat{u}\,,\, \widehat{g}\,\rangle_{H^1} = \langle\, \widehat{f}\,,\, [\widehat{u}, \widehat{g}]\,\rangle_{H^1}$$

with

$$\widehat{f} = \begin{pmatrix} f(x) \\ a(x) \\ \phi(x) \\ \alpha(x) \end{pmatrix}, \qquad \widehat{u} = \begin{pmatrix} u(x) \\ v(x) \\ \psi(x) \\ \beta(x) \end{pmatrix}, \qquad \widehat{g} = \begin{pmatrix} g(x) \\ b(x) \\ \chi(x) \\ \gamma(x) \end{pmatrix},$$

we obtain

$$\left\langle (u, v, \psi, \beta)]\,,\ \begin{pmatrix} (fg' - f'g)\frac{d}{dx} + \phi\chi\frac{d}{dx} \\ fb' - ga' + \phi\gamma + \alpha\chi \\ f\chi' - \frac{1}{2}f'\chi + g\phi' - \frac{1}{2}g'\phi \\ f\gamma' + \frac{1}{2}f'\gamma - \frac{1}{2}a'\gamma + g\alpha' + \frac{1}{2}g'\alpha - \frac{1}{2}b'\phi \end{pmatrix} \right\rangle_{H^1}.$$

This would give us the right hand side without the $(1-\partial^2)^{-1}$ term, which appears on the left hand side:

$$
\begin{aligned}
L.H.S. = & \int_{S^1} (ad^*_{\hat{f}} u) g dx + \int_{S^1} (ad^*_{\hat{f}} u)' g' dx \int_{S^1} (ad^*_{\hat{f}} v) b \, dx \\
& + \int_{S^1} (ad^*_{\hat{f}} \psi) \phi \, dx + \int_{S^1} (ad^*_{\hat{f}} \psi') \phi' \, dx + \int_{S^1} (ad^*_{\hat{f}} \beta) \alpha \, dx \\
= & \int_{S^1} [(1-\partial^2) ad^*_{\hat{f}} u] g dx + \int_{S^1} (ad^*_{\hat{f}} v) b \, dx \\
& + \int_{S^1} [(1-\partial^2) ad^*_{\hat{f}} \psi] \phi dx + \int_{S^1} (ad^*_{\hat{f}} \beta) \alpha \, dx.
\end{aligned}
$$

Equating the right and left hand sides, we obtain the desired formula. □

Therefore, if we use the Euler-Poincaré equation and the computational trick used in [5], we obtain the supersymmetric version of the two component Camassa–Holm equation:

$$
\begin{aligned}
m_t &= 2mu_x + m_x u + (vv_x) + 3\xi\xi'', \\
v_t &= 2(uv)_x + \beta'\xi' + \beta\xi'', \\
(1-\partial^2)\xi_t &= 4m\xi' + 3m'\xi + 2\xi''', \\
\beta_t &= 2u\beta' + u'\beta + 2v\xi',
\end{aligned}
\qquad \text{where} \qquad m = u - u_{xx}.
$$

Corollary 3.3

$$
ad^*_{\hat{f}} \widehat{u} = \begin{pmatrix} 2uf'(x) + u'f + a'v + f''' + \frac{1}{2}\psi'\phi + \frac{3}{2}\psi\phi' \\ f'v + fv' + \frac{1}{2}(\beta'\phi + \beta\phi' \\ f\psi' + \frac{3}{2}f'\psi + \frac{1}{2}a'\beta + u\phi + v\alpha + 2\phi'' \\ f\beta' + \frac{1}{2}f'\beta + v\phi \end{pmatrix} \tag{21}
$$

Thus, we recover a supersymmetric version of Ito equation [12] given by

$$
\begin{aligned}
u_t &= 6uu_x + 2(vv_x) + u_{xxx} + 3\xi\xi'', \\
v_t &= 2(uv)_x + \beta'\xi' + \beta\xi'', \\
\xi_t &= 4u\xi' + 3u'\xi + 2\xi''', \\
\beta_t &= 2u\beta' + u'\beta + 2v\xi'.
\end{aligned} \tag{22}
$$

Corollary 3.4 *In the supersymmetric Ito equation* (22):
(A) if we set the super variables $\xi = \beta = 0$, we recover the Ito equation.
(B) If we set $v = \beta = 0$, we obtain

$$
\begin{aligned}
u_t &= 6uu_x + u_{xxx} + 3\xi\xi'' \\
\xi_t &= 4u\xi' + 3u'\xi + 2\xi''',
\end{aligned} \tag{23}
$$

which is a fermionic extension of KdV equation and, modulo rescalings, is the super KdV equation of Mathieu and Manin–Radul, [13, 14].

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