CW Mode Structure and Constraint Beamforming in a Waveguide with Unknown Large Inclusions

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Abstract

In this paper, we continue our study in combining the matched-field method with the boundary integral equation method from inverse scattering theory to study a sound source localization problem in a shallow ocean with an unknown large inclusion. We assume that there is an unknown inclusion embedded in a shallow water waveguide. The existence of the unknown inclusion changes the propagating field greatly. Therefore, neglecting the existence of the unknown inclusion will lead to substantial mismatching in the matched-field signal processing. To compensate for environmental unknown, we send in a number of acoustic waves emitted from known locations, which scatter off the unidentified inclusion and are received by a hydrophone array. We discuss the CW mode structure in such a waveguide, then present a method to generate approximately the replica field from an acoustic source using the previously recorded information. Combining the information of these scattered waves and the signal from the target, we present an optimum beamforming algorithm to estimate the location of the CW source. A numerical simulation using this method is presented.

1 Introduction

Matched-field processing has been studied extensively in recent years as a method for localization of underwater sound sources [1] [2] [9] [11] [12]. In matched-field processing, a model is used to simulate the acoustic field propagating from an object to a receiver array. Using this model, one generate replica fields corresponding to object locations, and estimate the location of the object by observing which of these replica fields best matches the received data. This requires that we have a good model which truly reflects the ocean environment. However, there are unknown structures and uncertainties in the medium which affect the accuracy of the model. In fact, as an example, if we consider a shallow ocean with an unknown large inclusion, we find that the existence of the

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unknown inclusion changes the propagating field greatly. (The propagating fields from a point source in a waveguide with and without the inclusion are plotted in figures 4 and 5 respectively.) Therefore, neglecting the existence of the unknown inclusion will lead to substantial mismatching in the matched-field signal processing.

In our recent papers, we combine the matched-field method with the boundary integral equation method from inverse scattering theory to study a sound source localization problem in a shallow ocean with an unknown large inclusion [14], [17]. To compensate for environmental unknown, one straightforward method of avoiding mismatch is to use the inverse scattering method (ref. [6] [7] [15]) to reconstruct the shape of the unknown inclusion and then use the BIEM method in [16] to estimate the location of the source. However, sometimes one is only interested in locating the sound source. In that case, the method above is not a wise choice because the reconstruction of the unknown inclusion requires a large amount of information and very heavy computation. Moreover, the ill-posedness of the reconstruction problem will cause unnecessary error. In [14], [17], we present a method which uses the idea of inverse scattering without actually computing the shape of the unknown inclusion. To localize a continuous wave source we send in a number of "mode waves" which scatter off the unidentified inclusion and are received by a hydrophone array. By combining the information from these scattered waves with the signal from the point source we present an algorithm to estimate the location of the CW source. A numerical simulation shows that this algorithm works well when the signal-to-noise ratio is not too small. However, there are some concerns to this method. First, it is not easy to generate the single mode wave in practice. Secondly, the estimator used there is sensitive to noise.

In this paper, we extend our study in combining the matched-field method with the inverse scattering method to study a sound source localization problem in a shallow ocean with an unknown large inclusion from three aspects. First, we present the method for more general type of acoustic sources. Second, instead of using "mode waves" as incident wave to compensate for environmental unknown, we send in a number of **point source** acoustic waves emitted from known locations, which scatter off the unidentified inclusion and are received by a hydrophone array. Third, we use a constraint beamformer as estimator instead of the simple least squares estimator in [16]. In section 2, we discuss the CW mode structure in such a waveguide, which provides a theoretical foundation for our method. In section 3, we present a method to generate approximately the replica field from an acoustic source using the previously recorded information. In section 4, combining the information of these scattered waves and the signal from the target, we present an **constraint beamforming** algorithm to estimate the location of the CW source. In section 5, a numerical simulation using this method is presented.

2 CW mode structure in a waveguide with inclusions

We denote the waveguide with depth d as $\mathbf{R}_d^2 = \{(x_1, x_2)| -\infty < x_1 < \infty, 0 \leq x_2 \leq d\}$. An inclusion which is a bounded region located in the waveguide is denoted as Ω. This inclusion can be a part of the ocean bottom (see figure 1 and figure 2). Based on the linear theory of acoustics [13], the acoustic field generated by a transmitter is characterized by a real-values acoustic potential function

$$u = u(t, \mathbf{x}),\ t \in R,\ \mathbf{x} = (x_1, x_2) \in \mathbf{R}_d^2 \setminus \Omega, \tag{2.1}$$

which satisfies the inhomogeneous wave equation

$$\frac{\partial^2}{\partial t^2} u - c^2 \triangle u = f(t, \mathbf{x})\ for\ t \in R,\ \mathbf{x} \in \mathbf{R}_d^2 \setminus \Omega. \tag{2.2}$$

Here c is the sound velocity, $\triangle$ is the Laplacian and $f(t, \mathbf{x})$ is a known function characteristic of the transmitter. CW mode fields are generated by source functions of the form

$$f(t, \mathbf{x}) = g_1(\mathbf{x}) \cos \omega t + g_2(\mathbf{x}) \sin \omega t = Re\{g(\mathbf{x}) e^{-i\omega t}\}, \tag{2.3}$$

where $\omega > 0$ is a fixed frequency and $g = g_1 + i g_2$. If the wave is emitted from a point source located at $\mathbf{x} = \mathbf{x}^s = (x_1^s, x_2^s)$,

$$g(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}^s). \tag{2.4}$$

The corresponding CW mode field has the same time dependence:

$$u(t, \mathbf{x}) = p_1(\mathbf{x}) \cos \omega t + p_2(\mathbf{x}) \sin \omega t = Re\{p(\mathbf{x}) e^{-i\omega t}\}, \tag{2.5}$$

where $p = p_1 + i p_2$. $u(t, \mathbf{x})$ satisfies the wave equation with $f(t, \mathbf{x})$ defined by (2.3). It follows that the complex-valued wave function $p(\mathbf{x})$ satisfies

$$\triangle p(\mathbf{x}) + k^2 p(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} = (x_1, x_2) \in \mathbf{R}_d^2 \setminus \overline{\Omega}, \tag{2.6}$$

where $k = \omega/c$ is called the wave number.

For the sake of exposition, we assume that the water waveguide has an acoustic free surface at $x_2 = 0$ and the flat part of the bottom is acoustic rigid. Then, $p(\mathbf{x})$ satisfies also the following condition:

$$p(x_1, 0) = 0, \quad \frac{\partial p}{\partial x_2}(\mathbf{x}) = 0, \; for \; \mathbf{x} \in \Gamma_d \setminus \Omega, \tag{2.7}$$

$$\mathbf{B}p(\mathbf{x}) = 0 \;\text{ for }\; \mathbf{x} \in \partial\Omega \cap \mathbf{R}^2_d, \tag{2.8}$$

Here $\Gamma_d = \{\mathbf{x} | x_2 = d\}$ and $\mathbf{B}p$ is a symbolic notation corresponding to the property of the boundary of the inclusion. For example, $\mathbf{B}p = p$ for sound soft object; $\mathbf{B}p = \frac{\partial p}{\partial n}$ for rigid object, and so on. Moreover, $p(\mathbf{x})$ satisfies an outgoing radiating condition, i.e., for $|x_1| \to \infty$, $p(\mathbf{x})$ has an expansion

$$p(x_1, x_2) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x_2) e^{ik_n |x_1|}, \tag{2.9}$$

where $k_n = [k^2 - (n - \frac{1}{2})^2 \frac{\pi^2}{d^2}]^{1/2}$ is the horizontal wavenumber, and the coefficients α_n depend on $g(\mathbf{x})$ and the sign of x_1, and

$$\phi_n(x_2) = \sin[(n - \frac{1}{2})\frac{\pi}{d} x_2]. \tag{2.10}$$

The primary CW mode field is the CW mode field $p_0(\mathbf{x})$ generated by $g(\mathbf{x})$ when no scattering object is present. It is given by

$$p_0(\mathbf{x}; g) = \int_D G_0(\mathbf{x}; \mathbf{y}) g(\mathbf{y}) d\mathbf{y}, \;\; for \;\; \mathbf{x} \in \mathbf{R}^2_d, \tag{2.11}$$

where $D = supp(g) = \{\mathbf{x} | g(\mathbf{x}) \neq 0\}$ and

$$G_0(\mathbf{x}; \mathbf{x}^s) = \sum_{n=1}^{\infty} \left\{ \frac{1}{2k_n i} \right\} \phi_n(x_2) \phi_n(x_2^s) e^{ik_n |x_1 - x_1^s|}. \tag{2.12}$$

Here we denote $p_0 = p_0(\mathbf{x}; g)$ to emphasize the dependence of p_0 upon the source g. We may omit g if there is no confuse caused. If it is a point source at $\mathbf{x}^s$, then $g(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}^s)$ and

$$p_0(\mathbf{x}; g) = G_0(\mathbf{x}; \mathbf{x}^s), \;\; for \;\; \mathbf{x} \in \mathbf{R}^2_d. \tag{2.13}$$

The CW mode field which is produced when p_0 is scattered by the inclusion Ω is denoted by p_{sc}. The total CW mode field

$$p(\mathbf{x}) = p_0(\mathbf{x}) + p_{sc}(\mathbf{x}) \tag{2.14}$$

satisfies (2.6)-(2.9) implies that $p_{sc} = p - p_0$ is a solution of the problem

$$\triangle\, p_{sc}(\mathbf{x}) + k^2 p_{sc}(\mathbf{x}) = 0,\ \mathbf{x} \in \mathbf{R}^2_d \setminus \overline{\Omega}, \tag{2.15}$$

$$p_{sc}(x_1, 0) = 0, \quad \frac{\partial p_{sc}}{\partial x_2}(\mathbf{x}) = 0,\ for\ \mathbf{x} \in \Gamma_d \setminus \Omega, \tag{2.16}$$

$$\mathbf{B} p_{sc}(\mathbf{x}) = -\mathbf{B} p_0(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \partial\Omega \cap \mathbf{R}^2_d, \tag{2.17}$$

and $p_{sc}(\mathbf{x})$ is out-going as $|x_1| \to \infty$.

By Green's formula, the CW scattered field p_{sc} has an integral representation

$$p_{sc}(\mathbf{x}) = -\int_{\partial\Omega} \left\{ p_{sc}(y) \frac{\partial G_0(\mathbf{x};\mathbf{y})}{\partial \nu_y} - G_0(\mathbf{x};\mathbf{y}) \frac{\partial p_{sc}(y)}{\partial \nu_y} \right\} d\sigma_y, \text{ for } \mathbf{x} \in \mathbf{R}^2_d \setminus \overline{\Omega}. \tag{2.18}$$

Using the boundary condition (2.17), we can obtain a representation

$$p_{sc}(\mathbf{x}; g) = \mathbf{T_B} \{p_0\} (\mathbf{x}; g), \tag{2.19}$$

where $\mathbf{T_B}$ is a linear integral operator depending on boundary condition $\mathbf{B}p_0$. Readers who are interested in the forms of $\mathbf{T_B}$ could refer to the Appendix of this paper. Again we use $p_{sc}(\mathbf{x}; g)$ to emphasize that p_{sc} depends on source g. We will use the same notation for total field p when this dependence is to be emphasized.

If $\mathbf{x}$ is in the far-field where $|x_1|$ is large, say, $|x_1| > \sup_{\mathbf{y} \in D\cup\Omega}\{|y_1|\}$, then $p_0(\mathbf{x}; g)$ represented by (2.11) has asymptotic representation

$$\begin{aligned} p_0(\mathbf{x}; g) &= \sum_{n=1}^{\infty} \left\{ \frac{1}{2k_n i} \right\} \int_D \phi_n(y_2) e^{-sign\{x_1\} i k_n y_1} g(y) dy\, \phi_n(x_2) e^{i k_n |x_1|} \\ &= \sum_{n=1}^{\infty} V_n(g) \phi_n(x_2) e^{i k_n |x_1|}, \end{aligned} \tag{2.20}$$

where

$$V_n(g) = \left\{ \frac{1}{2k_n i} \right\} \int_D \phi_n(y_2) e^{-sign\{x_1\} i k_n y_1} g(y) dy,\ (n = 1, 2, \cdots, \infty) \tag{2.21}$$

depend on the source function g.

Similarly, p_{sc} represented by (2.19) has asymptotic representation

$$p_{sc}(\mathbf{x}; g) = \sum_{n=1}^{\infty} V_n(g) \mathbf{T_B} \left\{ \phi_n(\cdot) e^{i k_n |\cdot|} \right\} (\mathbf{x}), \tag{2.22}$$

where $\mathbf{T_B}$ depends on the boundary condition on $\partial\Omega$.

Hence, we can represent the CW mode fields in the far-field by

$$p(\mathbf{x};g) = \sum_{n=1}^{\infty} V_n(g) \left[\phi_n(x_2)e^{ik_n|x_1|} + \mathbf{T_B}\left\{\phi_n(\cdot)e^{ik_n|\cdot|}\right\}(\mathbf{x})\right]. \tag{2.23}$$

For point source CW field, (2,23) becomes

$$p(\mathbf{x};\mathbf{x}^s) = \sum_{n=1}^{\infty} \left\{\frac{1}{2k_n i}\right\} \phi_n(x_2^s)e^{-sign\{x_1\}ik_n x_1^s} \left[\phi_n(x_2)e^{ik_n|x_1|} + \mathbf{T_B}\left\{\phi_n(\cdot)e^{ik_n|\cdot|}\right\}(\mathbf{x})\right]. \tag{2.24}$$

Formula (2.23) (or (2.24) for point source) is a basic formula for constructing replica fields for the purpose of matched-field processing.

3 Construction of replica fields

To simplify our discussion, from now on we assume that the location of a receiver, $\mathbf{x}$, is fixed and far away from the inclusion and the source. More specifically, we assume that $x_1 > \max\{0, \sup_{\mathbf{y}\in D\cup\Omega}\{y_1\}\}$. If we truncate (2.23) at the order of N, then the CW mode field is approximated by the sum of a finite number of modes. Each mode is consist of two factors $V_n(g)$ and $p_n(\mathbf{x})$, where

$$V_n(g) = \left\{\frac{1}{2k_n i}\right\} \int_D \phi_n(y_2)e^{-ik_n y_1} g(y)dy \tag{3.1}$$

and

$$p_n(\mathbf{x}) = \phi_n(x_2)e^{ik_n x_1} + \mathbf{T}_B\left\{\phi_n(\cdot)e^{ik_n|\cdot|}\right\}(\mathbf{x}). \tag{3.2}$$

Physically, $V_n(g)$ stands for the amplitude of the n^{th} mode generated by the source g. $p_n(\mathbf{x})$ stands for the CW mode total field generated by an incident "mode wave"

$$u_n^i(\mathbf{x}) = \phi_n(x_2)e^{ik_n x_1}.$$

That is, the incident "mode wave" $u_n^i(\mathbf{x})$ scatters off the inclusion and produces the scattered wave

$$u_n^s = \mathbf{T}_B\left\{u_n^i\right\}(\mathbf{x}).$$

The sum of the incident and scattered waves is the total CW field $p_n = u_n^i + u_n^s$. Clearly, $V_n(g)$ depends on the sound source and does not depend on the inclusion and the locations of receivers. On the other hand, $p_n(\mathbf{x})$ depends on the inclusion and the locations of the receivers, but does not depend on the sound source. The above observation implies that we can construct the replica field by constructing $V_n(g)$ and $p_n(\mathbf{x})$ separately.

3.1 Construction of $p_n(\mathbf{x})$

If we know the environment perfectly, that is, we know the inclusion Ω, then we can compute $p_n(\mathbf{x})$ by solving an integral equation to obtain $\mathbf{T_B}$ (ref. Appendix and [15]). Hence, $p_n(\mathbf{x})$ is given by (3.2).

The more interesting case is that Ω is unknown. From the physical sense of $p_n(\mathbf{x})$, we send in a number of incident waves

$$u_n^i(\mathbf{x}) = \phi_n(x_2)e^{ik_n x_1},\ n = 1, 2, \cdots, N,$$

and detect the produced field. These fields are approximations to $p_n(\mathbf{x}), (n = 1, 2, \cdots, N)$.

However, it is not easy to produce a single mode wave $u_n^i(\mathbf{x})$ in practise. Therefore, it will be helpful if we can approximate $p_n(\mathbf{x})$ using point source wave as incident wave. We can reach this goal in the following way. ¿From (2.24), we know that the CW fields produced by acoustic point sources at $\mathbf{x}_j^s, (j = 1, 2, 3, \cdots, N)$ are approximately

$$p(\mathbf{x}; \mathbf{x}_j^s) = \sum_{n=1}^{N} \left\{ \frac{1}{2k_n i} \right\} \phi_n(x_{2,j}^s) e^{-ik_n x_{1,j}^s} p_n(\mathbf{x}),\ (j = 1, 2, 3, \cdots, N). \tag{3.3}$$

Let

$$\mathbf{P}^*(\mathbf{x}) = [p(\mathbf{x}; \mathbf{x}_1^s), p(\mathbf{x}; \mathbf{x}_2^s), \cdots, p(\mathbf{x}; \mathbf{x}_N^s)]^T, \tag{3.4}$$

$$\mathbf{P}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_N(\mathbf{x})]^T, \tag{3.5}$$

$$\mathbf{A} = \frac{1}{2k_n i} \begin{bmatrix} \phi_1(x_{2,1}^s)e^{-ik_1 x_{1,1}^s} & \phi_1(x_{2,2}^s)e^{-ik_1 x_{1,2}^s} & \cdots & \phi_1(x_{2,N}^s)e^{-ik_1 x_{1,N}^s} \\ \phi_2(x_{2,1}^s)e^{-ik_1 x_{1,1}^s} & \phi_2(x_{2,2}^s)e^{-ik_1 x_{1,2}^s} & \cdots & \phi_2(x_{2,N}^s)e^{-ik_1 x_{1,N}^s} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N(x_{2,1}^s)e^{-ik_N x_{1,1}^s} & \phi_N(x_{2,2}^s)e^{-ik_N x_{1,2}^s} & \cdots & \phi_N(x_{2,N}^s)e^{-ik_N x_{1,N}^s} \end{bmatrix}. \tag{3.6}$$

We have

$$\mathbf{A}\mathbf{P}(\mathbf{x}) = \mathbf{P}^*(\mathbf{x}). \tag{3.7}$$

We can choose $\mathbf{x}_j^s,\ j = 1, 2, 3, \cdots, N$ properly so that $\mathbf{A}$ is regular, hence

$$\mathbf{P}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{P}^*(\mathbf{x}). \tag{3.8}$$

3.2 Construction of $V_n(g)$

We discuss two simple types of acoustic sources here as examples. However, the method used here can be applied to more general source determination problems.

1. Point acoustic source:

$$V_n(\mathbf{x}^s) = \phi_n(x_2^s)e^{-ik_n x_1^s}. \tag{3.9}$$

There are two real parameters x_1^s, x_2^s to be determined.

2. Uniformly distributed circular source: Let $\epsilon > 0$ be a constant,

$$g(\mathbf{x}) = \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if} |\mathbf{x} - \mathbf{x}^s| < \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then

$$V_n(\mathbf{x}^s, \epsilon) = \left\{ \frac{1}{2k_n \pi i} \right\} \int_0^1 \int_0^{2\pi} \phi_n(x_2^s + \epsilon r \sin(\theta)) e^{-ik_n(x_1^s + \epsilon r \cos(\theta))} r d\theta dr \tag{3.11}$$

There are three parameters x_1^s, x_2^s and ϵ to be determined.

4 A linear constraint beamformer

In the literature of matched-field processing, many methods have been discussed. Here a linear constraint beamformer is adapted for our processing. In this section we only apply the method to our application. For more discussion, the reader could refer [1] [8] and the papers referred there.

Let $\{p_l^*\}$ be the detected data set consisting of the acoustic pressure field p_l^* sampled at the hydrophones located at $\mathbf{x}^l, l = 1, 2, \cdots, L$. For statistic purpose, each data is sampled m times (so p_l^* may be thought of a m dimensional vector). We use superscripts T, H to indicates the matrix transposition and matrix Hermitian transposition. Denote

$$\mathbf{X} = [p_1^*, p_2^*, \cdots, p_L^*]^T. \tag{4.1}$$

An L-element filter vector to be applied to the beam steered array data is defined as

$$\mathbf{W} = [W_1, W_2, \cdots, W_L]^T. \tag{4.2}$$

The beamformer output for this vector filter is

$$\mathbf{Y} = \mathbf{W}^H \mathbf{X}. \tag{4.3}$$

The power output of the beamformer is

$$E[|\mathbf{Y}|^2] = \mathbf{W}^H E[\mathbf{X}\mathbf{X}^H]\mathbf{W} =: \mathbf{W}^H\mathbf{R}\mathbf{W}, \tag{4.4}$$

where $E[\cdot]$ denotes the expectation operator and $\mathbf{R} = E[\mathbf{X}\mathbf{X}^H]$.

Let $\mathbf{E}$ be the "probe" replica. If the receivers are located in the far-field at $\mathbf{x}_l, l = 1, 2, \cdots, L$, then

$$\mathbf{E} = [\mathbf{P}(\mathbf{x}_1), \mathbf{P}(\mathbf{x}_2), \cdots, \mathbf{P}(\mathbf{x}_L)]^T\mathbf{V}(g), \tag{4.5}$$

where

$$\mathbf{V}(g) = [V_1(g), V_2(g), \cdots, V_N(g)]^T. \tag{4.6}$$

We wish to minimize the beamformer output power $E[|\mathbf{Y}|^2]$ under the constraint

$$\mathbf{E}^H\mathbf{W} = 1. \tag{4.7}$$

The solution to this problem is

$$\mathbf{W}_{opt} = \mathbf{R}^{-1}\mathbf{E}[\mathbf{E}^H\mathbf{R}^{-1}\mathbf{E}]^{-1}, \tag{4.8}$$

and the power output of the optimal beamformer is

$$U_{opt} := \mathbf{W}_{opt}^H\mathbf{R}\mathbf{W}_{opt} = [\mathbf{E}^H\mathbf{R}^{-1}\mathbf{E}]^{-1}. \tag{4.9}$$

Clearly U_{opt} depends on the source g. If g can be characterized by a few parameters, we can represent U_{opt} as a function of a few veriables. For example, for the case of acoustic point source, $U_{opt} = U_{opt}(\mathbf{x}^s)$. For the case of distributed source in section 3.2.2, $U_{opt} = U_{opt}(\mathbf{x}^s, \epsilon)$. Scanning the power output (4.9) in a searching area, we obtain a peak output when the replica field matchs the detected field.

Two factors may affect the accuracy of the resolution; the noise of detected field, which is contained in $\mathbf{R}$, and the unknown of environment, which is contained in the replica $\mathbf{E}$. Our method compensates for the noise by large number of sampling and for the environment unknown by inverse scattering of incident waves from given point sources.

5 Numerical simulations

Computer simulations using the method above were carried out on the Cray2 at the Minnesota Supercomputer Center. We used our approximate boundary integral equation method described in [15] to compute the propagating fields. A subroutine using transformation method presented in [10] (p. 203) was modified to produce random numbers as Gaussian noise.

The configuration for the computer simulations is depicted in figure 3.

We assume the waveguide has a depth of 100 meters. The sound speed is assumed to be $1500m/s$. An acoustic point source S located at $(-350/\pi, 100/\pi)$ emits a CW field at the frequency $f = 30Hz$. The hydrophone array is arranged vertically at $(700/\pi, 2.5j), j = 0, 1, \cdots, 40$. There is an inclusion Ω with a rigid surface which occupies the region $\{(x_1, x_2) \,|\, x_1^2 + 4(x_2 - 50)^2 \le (120/\pi)^2\}$. If the waveguide is normalized to a depth of π, then the normalized wave number is $k = 4$, which means there are four propagating modes for the acoustic wave at the given frequency.

5.1 Generating the CW mode propagating field from the source

Using the boundary integral equation method, we compute the propagating field $p(\mathbf{x}; \mathbf{x}^s)$ where $\mathbf{x}^s = (-350/\pi, 100/\pi)$. The primary field is given by (2.12) with truncation at $n = 30$. A contour plot of the propagating wave with source at $\mathbf{x}^s = (-350/\pi, 100/\pi)$ is plotted in figure 4. For comparison, a contour plot of the propagating wave with a source at $\mathbf{x}^s = (-350/\pi, 100/\pi)$ in an unperturbed waveguide is plotted in figure 5. The scattered field p_{sc} is plotted in figure 6.

In particular, we obtain $p_m^* = p(700/\pi, 2.5m; \mathbf{x}^s), m = 0, 1, \cdots, 40$. To make the data more realistic, for each receiver we generated 200 random numbers ϵ_j^l $(j = 1, 2, \cdots, 200; l = 0, 1, \cdots, 40)$ as Gaussian noise which is added to the computed field. That is,

$$p_l^* = [p_l^* + \epsilon_1^l, p_l^* + \epsilon_2^l, \cdots, p_l^* + \epsilon_{200}^l]^T, \; l = 0, 1, \cdots, 40.$$

Using these data as the repeatly detected data, we calculate

$$\mathbf{R} = E[\mathbf{X}\mathbf{X}^H] = \frac{1}{200} \begin{bmatrix} p_0^{*T} \\ p_1^{*T} \\ \cdot \\ \cdot \\ \cdot \\ p_{40}^{*T} \end{bmatrix} [\overline{p_0^*}, \overline{p_1^*}, \cdots, \overline{p_{40}^*}], \tag{5.1}$$

where $\overline{p_l^*}$ is the complex conjugate of p_l^*, $l = 0, 1, \cdots, 40$.

5.2 Construct the 'probe' replica E

We construct the 'probe' replica $\mathbf{E}$ using (4.5). For each given search location $\mathbf{x}^s$, $\mathbf{V}(\mathbf{x}^s)$ is calculate by (4.6) and (3.9). However, since we assume the inclusion is unknown, the data $[\mathbf{P}(\mathbf{x}_0), \mathbf{P}(\mathbf{x}_1), \cdots, \mathbf{P}(\mathbf{x}_{40})]^T$ cannot be calculated directly from integral equation (3.2). They are indeed computed by (3.8) using detected data $[\mathbf{P}^*(\mathbf{x}_0), \mathbf{P}^*(\mathbf{x}_1), \cdots, \mathbf{P}^*(\mathbf{x}_{40})]^T$. Therefore, $\mathbf{E}$ must contain some noise. In our numerical experiment, we generate the detected data $[\mathbf{P}^*(\mathbf{x}_0), \mathbf{P}^*(\mathbf{x}_1), \cdots, \mathbf{P}^*(\mathbf{x}_{40})]^T$ by adding Gaussian noise in the same way as we do to p_l^*. Then we compute their mathematical expectations $E[\mathbf{P}^*(\mathbf{x}_l)], (l = 0, 1, \cdots, 40)$ and $\mathbf{P}(\mathbf{x}_l) = \mathbf{A}^{-1} E[\mathbf{P}^*(\mathbf{x}_l)], (l = 0, 1, \cdots, 40)$. The replica $\mathbf{E}$ is calculated by (4.5):

$$\mathbf{E} = [E[\mathbf{P}^*(\mathbf{x}_0)], E[\mathbf{P}^*(\mathbf{x}_1)], \cdots, E[\mathbf{P}^*(\mathbf{x}_{40})]^T \mathbf{A}^{-T} \mathbf{V}(\mathbf{x}^s).$$

5.3 Numerical examples

Obtaining $\mathbf{R}$ and $\mathbf{E}$, we search the area of $[-650/\pi, -50/\pi] \times [0, 100]$, and plot the optimal beamformer power output (4.9). Following are three sets of graphs from our numerical experiment.

Remark: In the graphs, the waveguide is scaled by a factor $100/\pi$. For example, the search area is scaled as $[-6.5, -0.5] \times [0, \pi]$. The source is at the coordinate $(-3.5, 1)$. We label the receivers from 0 to 40 in the context. However, they are labeled from 1 to 41 in some graphs.

Example 1. Mismatching caused by unknown inclusion.

The purpose of this group of graphs is to show that the existence of unknown inclusion changes the propagating wave and causes mismatching. We can compensate for this unknown by our inverse scattering method. These simple results show the necessary and advantage of the method in the discussed situation.

Figure 7. Detected CW field $|p(700/\pi, x_2^l)|$, $(l = 0, 1, \cdots, 40)$ in a waveguide with inclusion, simulated by BIEM.

Figure 8. Detected CW field $|p(700/\pi, x_2^l)|$, $(l = 0, 1, \cdots, 40)$ in a waveguide without inclusion, simulated by the method of separation of variables.

Figure 9. Power output of constraint beamformer, $20log|U_{opt}(\mathbf{x})|$, $\mathbf{x} \in [-6.5, -0.5] \times [0, \pi]$, with compensation for the unknown inclusion.

Figure 10. Power output of constraint beamformer, $20log|U_{opt}(\mathbf{x})|$, $\mathbf{x} \in [-6.5, -0.5] \times [0, \pi]$, without compensation for the unknown inclusion. Mismatching is observed.

Example 2. This group of graphs compares the constraint beamformer method and the simple least squares method in high signal-to-noise ratio situation. The signal-to-noise ratio is $10.05dB$. The unknown inclusion is compensated by inverse scattering. The results show that both estimators give good resolution in this high signal-to-noise ratio case.

Figure 11. CW field $Re\{p(700/\pi, x_2^l)\}, (l = 0, 1, \cdots, 40)$, calculated by BIEM without adding Gaussian noise.

Figure 12. A sample of Gaussian noise $Re\{\epsilon_l\}, (l = 0, 1, \cdots, 40)$.

Figure 13. A sample of detected CW field, the sum of the CW signal in Figure 11 and Gaussian noise in Figure 12. $S/N = 10.05dB$.

Figure 14. Power output of constraint beamformer, $20log|U_{opt}(\mathbf{x}|$, $\mathbf{x} \in [-6.5, -0.5] \times [0, \pi]$.

Figure 15. Power output of constraint beamformer, filter threshold set at $E[p] + 0.9(p_{max} - E[p])$, where $E[p]$ is the average of the power output and p_{max} is the maximum power output.

For comparison, we plot the estimator output using a simple least squares estimator (see [16])

$$F_p(\mathbf{x}^s) = \left[\sum_{l=1}^{L} |p(\mathbf{x}_l; \mathbf{x}^s) - p_l^*|^2 \right]^{-1}. \tag{5.2}$$

Figure 16. Power output of the estimator (5.2), $|F_p(\mathbf{x}^s)|$, $\mathbf{x}^s \in [-6.5, -0.5] \times [0, \pi]$.

Figure 17. Power output of the estimator (5.2) with filter threshold set at $E[p] + 0.9(p_{max} - E[p])$.

Figure 14 and figure 17 show that for large signal-to-noise ratio, both estimators give good resolutions.

Example 3. This group of graphs is to serve the same purpose as that in example 2 for a lower signal-to-noise ratio case. The graphs in this group are parallel to that in example 2. The signal-to-noise ratio is $-6.85dB$. In this case, the combination of inverse scattering and constraint beamformer can still give a reasonable estimate for the source location. However, the LS estimator is no longer giving a reliable estimate.

Figure 18. CW field calculated by BIEM, without adding Gaussian noise.

Figure 19. A sample of Gaussian noise.

Figure 20. A sample of detected CW field, $S/N = -6.85dB$.

Figure 21. Power output of constraint beamformer.

Figure 22. Power output of constraint beamformer, filter threshold set at $E[p] + 0.9(p_{max} - E[p])$.

Figure 23. Power output of the estimator (5.2).

Figure 24. Power output of the estimator (5.2) with filter threshold set at $E[p] + 0.9(p_{max} - E[p])$.

It is also interesting to compare the corresponding graphs in example 2 and 3. ¿From figure 21 and figure 14, we see that the basic structure of the power output from the constraint beamformer is similar. However, the outputs from the LS-estimator (5.2) (figure 16 and figure 23) are completely different. These may be seen more clearly in the filtered output. While figure 17 shows that the resolution is close to the true location, figure 24 gives a complete wrong resolution.

6 Conclusions

1. A technique for compensating for environmental unknowns in matched field processing has been described. The method combines inverse scattering method with matched field processing. By illuminating the search space with incident waves from known locations, the effect of unknown inhomogeneities in the environment on matched field processing can be compensated. The advantages of this compensation on matched field processing gain and localization can be clearly seen by the numerical simulations.

2. A constraint beamforming method is used to compensate the random noise. The method makes use of the BIEM to separate the unknown source location and the known locations of receivers. This reduces greatly the computation of replica in a perturbed waveguide. From our numerical experiment, we found that the constraint beamformer method has better robust property than the Least squares estimator method. We anticipate that any good processing methods such as Multiple Constraint Method (MCM), Maximum Likelihood Method (MLM) can be used for combining with inverse scattering method and further improve the results.

3. BIEM is essential in the numerical simulation of the method presented in this paper. Our numerical results provide a very positive conclusion to the method. It is interesting to evaluate the method in laboratory experiment or sea experiment.

7 Appendix: Construction of scattered fields by BIEM

In this section, we give an outline of the BIEM for constructing scattered field in a waveguide with large inclusions. As described in Section 2, the scattered field p_{sc} satisfies a boundary value problem

$$\triangle p_{sc}(\mathbf{x}) + k^2 p_{sc}(\mathbf{x}) = 0,\ \mathbf{x} \in \mathbf{R}_d^2 \setminus \overline{\Omega}, \tag{7.1}$$

$$p_{sc}(x_1, 0) = 0, \quad \frac{\partial p_{sc}}{\partial x_2}(\mathbf{x}) = 0,\ for\ \mathbf{x} \in \Gamma_d \setminus \Omega, \tag{7.2}$$

$$\mathbf{B} p_{sc}(\mathbf{x}) = -\mathbf{B} p_0(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \partial\Omega \cap \mathbf{R}_d^2, \tag{7.3}$$

and $p_{sc}(\mathbf{x})$ is out-going as $|x_1| \to \infty$.

Here the condition on the boundary of the inclusion $\partial\Omega \cap \mathbf{R}_d^2$ may be different depending upon the acoustic property of the boundary. In general, three kinds of conditions are often used. That is, Dirichlet condition, Neumann condition and Robin condition (the impedance condition). In either case, we can construct the operator $\mathbf{T_B}$ by BIEM.

We will need the following integral operators.

$$\mathbf{K}\psi(\mathbf{x}) := 2\int_{\partial\Omega} \frac{\partial G_0}{\partial \nu_{\mathbf{y}}}(\mathbf{x};\mathbf{y})\psi(\mathbf{y}) d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x} \in \partial\Omega. \tag{7.4}$$

$$\mathbf{K}'\psi(\mathbf{x}) := 2\int_{\partial\Omega} \frac{\partial G_0}{\partial \nu_{\mathbf{x}}}(\mathbf{x};\mathbf{y})\psi(\mathbf{y}) d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x} \in \partial\Omega. \tag{7.5}$$

$$\mathbf{S}\psi(\mathbf{x}) := 2\int_{\partial\Omega} G_0(\mathbf{x};\mathbf{y})\psi(\mathbf{y}) d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x} \in \partial\Omega. \tag{7.6}$$

7.1 Dirichlet problem

If the inclusion has an acoustic soft boundary, the boundary condition (7.3) becomes (see also [16])

$$p_{sc}(\mathbf{x}; g) = -p_0(\mathbf{x}; g) \ \text{ for } \ \mathbf{x} \in \partial\Omega \cap \mathbf{R}_d^2. \tag{7.7}$$

We represent p_{sc} by a double layer potential

$$p_{sc}(\mathbf{x}; g) = -\int_{\partial\Omega} \frac{\partial G_0(\mathbf{x};\mathbf{y})}{\partial \nu_{\mathbf{y}}} \psi(\mathbf{x}; g) d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x} \in \mathbf{R}_d^2 \setminus \overline{\Omega}, \tag{7.8}$$

where ψ is the solution of the boundary integral equation

$$\psi(\mathbf{x};g)+2\int_{\partial\Omega}\frac{\partial G_0}{\partial\nu_{\mathbf{y}}}(\mathbf{x};\mathbf{y})\psi(\mathbf{x};g)d\sigma_y=-2p_0(\mathbf{x};g),\ \text{for } \mathbf{x}\in\partial\Omega. \tag{7.9}$$

If k is not an eigenvalue of the interior Neumann problem in Ω, then (7.9) has a unique solution.

Using the notation in (7.4), we denote the boundary integral equation (7.9) as

$$\psi+\mathbf{K}\psi=-2p_0. \tag{7.10}$$

If k is not an eigenvalue of the interior Neumann problem in Ω, then $\mathbf{I}+\mathbf{K}$ is invertible. We can write

$$\psi(\mathbf{x};g)=-2(\mathbf{I}+\mathbf{K})^{-1}\left\{p_0\right\}(\mathbf{x};g), \tag{7.11}$$

and

$$\mathbf{T_B}\{p_0\}(\mathbf{x};g)=2\int_{\partial\Omega}\frac{\partial G_0(\mathbf{x};\mathbf{y})}{\partial\nu_{\mathbf{y}}}(\mathbf{I}+\mathbf{K})^{-1}\left\{p_0\right\}(\mathbf{y};g)d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x}\in\mathbf{R}_d^2\setminus\overline{\Omega}. \tag{7.12}$$

7.2 Neumann problem

If the inclusion has an acoustic rigid boundary, the boundary condition (7.3) becomes

$$\frac{\partial p_{sc}}{\partial\nu_x}(\mathbf{x};g)=-\frac{\partial p_0(\mathbf{x};g)}{\partial\nu_x}\quad\text{for}\quad \mathbf{x}\in\partial\Omega\cap\mathbf{R}_d^2. \tag{7.13}$$

We represent p_{sc} by a single layer potential

$$p_{sc}(\mathbf{x};g)=-\int_{\partial\Omega}G_0(\mathbf{x};\mathbf{y})\psi(\mathbf{y};g)d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x}\in\mathbf{R}_d^2\setminus\overline{\Omega}, \tag{7.14}$$

where ψ is the solution of the boundary integral equation

$$\psi-\mathbf{K}'\psi=2\frac{\partial p_0}{\partial\nu}, \tag{7.15}$$

where $\mathbf{K}'$ is the integral operator defined by (7.5). By the theory of Fredholm integral equations, if k is not an eigenvalue of the interior Dirichlet problem in Ω , then $\mathbf{I}-\mathbf{K}'$ is invertible. We can write

$$\psi(\mathbf{x};g)=2(\mathbf{I}-\mathbf{K}')^{-1}\left\{\frac{\partial p_0}{\partial\nu_{\mathbf{x}}}\right\}(\mathbf{x};g), \tag{7.16}$$

and

$$\mathbf{T_B}\{p_0\}(\mathbf{x};g)=-2\int_{\partial\Omega}G_0(\mathbf{x};\mathbf{y})(\mathbf{I}-\mathbf{K}')^{-1}\left\{\frac{\partial p_0}{\partial\nu_{\mathbf{y}}}\right\}(\mathbf{y};g)d\sigma_{\mathbf{y}},\ \text{for } \mathbf{x}\in\mathbf{R}_d^2\setminus\overline{\Omega}. \tag{7.17}$$

7.3 Robin problem

If the inclusion has an acoustic impedance boundary, the boundary condition (7.3) becomes

$$\frac{\partial p_{sc}}{\partial \nu_x}(\mathbf{x};g) + \lambda p_{sc}(\mathbf{x};g) = -\frac{\partial p_0(\mathbf{x};g)}{\partial \nu_x} - \lambda p_0(\mathbf{x};g) \ \text{ for } \ \mathbf{x} \in \partial\Omega \cap \mathbf{R}^2_d, \tag{7.18}$$

where λ is a given positive function. We represent p_{sc} by a single layer potential

$$p_{sc}(\mathbf{x};g) = -\int_{\partial\Omega} G_0(\mathbf{x};\mathbf{y})\psi(\mathbf{y};g)d\sigma_{\mathbf{y}}, \ \text{for } \mathbf{x} \in \mathbf{R}^2_d \setminus \overline{\Omega}, \tag{7.19}$$

where ψ is the solution of the boundary integral equation

$$\psi - \mathbf{K}'\psi - \lambda\mathbf{S}\psi = 2\frac{\partial p_0}{\partial \nu_{\mathbf{y}}} + 2\lambda p_0, \tag{7.20}$$

By the theory of Fredholm integral equations, if k is not an eigenvalue of the interior Dirichlet problem in Ω , then $\mathbf{I} - \mathbf{K}' - \lambda\mathbf{S}$ is invertible. We can write

$$\psi(\mathbf{x};g) = 2(\mathbf{I} - \mathbf{K}' - \lambda\mathbf{S})^{-1}\left\{\frac{\partial p_0}{\partial \nu_{\mathbf{x}}} + \lambda p_0\right\}(\mathbf{x};g), \tag{7.21}$$

and

$$\mathbf{T_B}\{p_0\}(\mathbf{x};g) = -2\int_{\partial\Omega} G_0(\mathbf{x};\mathbf{y})(\mathbf{I} - \mathbf{K}' - \lambda\mathbf{S})^{-1}\left\{\frac{\partial p_0}{\partial \nu_{\mathbf{y}}} + \lambda p_0\right\}(\mathbf{x};g)d\sigma_{\mathbf{y}}, \ \text{for } \mathbf{x} \in \mathbf{R}^2_d \setminus \overline{\Omega}. \tag{7.22}$$

The essential part in computation of the wave field involves the numerical solution of the boundary integral equation (7.10), (7.15) or (7.20). For the closed curve case, a numerical method is presented in [15] [16] for Dirichlet problem, and in [17] for Neumann problem. The method can be also used for other cases with some modification.

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