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By

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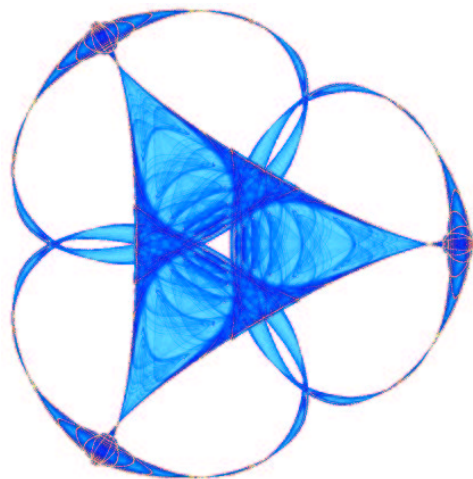
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MATHEMATICAL FRAMEWORK FOR CURRENT DENSITY IMAGING DUE TO DISCHARGE OF ELECTRO-MUSCULAR DISRUPTION DEVICES

JEEHYUN LEE[†], JIN KEUN SEO[†], AND EUNG JE WOO[‡]

ABSTRACT. Electro-muscular disruption (EMD) devices such as TASER M26 and X26 have been used as a less-than-lethal weapon. Such EMD devices shoot a pair of darts toward an intended target to generate an incapacitating electrical shock. In the use of the EMD device, there have been controversial questions about its safety and effectiveness. To address these questions, we need to investigate the distribution of the current density \mathbf{J} inside the target produced by the EMD device. One approach is to develop a computational model providing a quantitative and reliable analysis about the distribution of \mathbf{J} . In this paper, we set up a mathematical model of a typical EMD shock, bearing in mind that we are aiming to compute the current density distribution inside the human body with a pair of inserted darts. The safety issue of TASER is directly related to the magnitude of $|\mathbf{J}|$ at the region of the darts where the current density \mathbf{J} is highly concentrated. Hence, fine computation of \mathbf{J} near the dart is essential. For such numerical simulations, serious computational difficulties are encountered in dealing with the darts having two different very sharp corners, tip of needle and tip of barb. The boundary of a small fishhook-shaped dart inside a large computational domain and the presence of corner singularities require a very fine mesh leading to a formidable amount of numerical computations. To circumvent these difficulties, we developed a new method of computing \mathbf{J} , called the multiple point source method (MPSM). It has a potential to provide effective analysis and more accurate estimate of \mathbf{J} near fishhook-shaped darts. Numerical experiments show that the MPSM is just fit for the study of EMD shocks.

Key words: electro-muscular disruption (EMD) device, electrical current density, Maxwell equations, non-smooth boundary, elliptic partial differential equations, corner singularity

AMS Subject Classifications: 93A30, 32S05, 92C55, 33K28, 35M10, 35R30

1. INTRODUCTION

Electro-muscular disruption (EMD) devices such as TASER M26 and X26 are being used as a less-than-lethal weapon [10]. These devices generate a powerful electrical shock to completely override the central nervous system of a victim and directly control a large amount of skeletal muscles. It shoots a pair of fishhook-shaped darts attached to 15-foot wires toward an intended target. When the darts are inserted in the victims's skin or clothing, the device generates a debilitating electrical shock. Although these devices are intended to reduce the probability of fatality or permanent injury compared with other lethal weapons, there are ongoing controversial arguments over its safety as the number of death increases with more frequent uses of such EMD devices. See [5, 4, 8, 9, 7]. Numerous investigations on the implications of EMD shocks are, therefore, requested to create a guidance that helps device users to minimize the risk of injury or fatality of victims.

When the pair of darts strike a victim and an electrical circuit is formed, a current density $\mathbf{J} = (J_x, J_y, J_z)$ is established inside the body. The distribution of \mathbf{J} is influenced by several factors such as the shape and location of darts, distance between darts, geometry of the body, and its conductivity distribution. The knowledge of \mathbf{J} is necessary for the investigation of the risk of ventricular fibrillation (VF), seizure, skin burn, and so on.



FIGURE 1. Fishhook-shaped dart used in typical EMD devices.

In this paper, we suggest a basic study to undertake a three-dimensional numerical modelling of an EMD shock as a way to assess its biophysical implications. We setup a mathematical model of the EMD shock that can correctly simulate the real situation. Accurate computation of \mathbf{J} near the dart is required to properly assess the risk of skin burn injury and VF in the worst case. In numerical computations of \mathbf{J} using such a model, however, there are serious difficulties in dealing with the region of the fishhook-shaped dart since \mathbf{J} is highly concentrated there. As shown in figure 1, the dart is not axially symmetric and the inserted dart creates a three-dimensional computational domain with two different corners, tip of needle and tip of barb, where singularities of $|\mathbf{J}|$ occur. Therefore, dealing with these corners requires a very fine mesh to get enough numerical accuracy, and it may lead to a formidable amount of numerical computations and memory space. To circumvent these difficulties, we develop a new method of computing \mathbf{J} , called the multiple point source method (MPSM), that provides an analytical representation formula for the potential distribution near the fishhook-shaped dart. Eliminating the requirement of a very fine mesh, the MPSM seems to be an ideal method for us to model EMD shocks using a conventional PC.

In this paper, we try to explain the whole process of the current density imaging during EMD shocks; motivation, mathematical modelling, computational method, and a future study of experimental validation using animal subjects. We hope that the proposed computer model to simulate the effect of shocks help examine competing claims about the benefits and risks of TASERs [5, 9, 10].

2. MATHEMATICAL MODELLING OF EMD SHOCK

2.1. Problem definition. In this section, we formulate a mathematical model of EMD shocks, bearing in mind that we are aiming to compute the current density distribution inside the body generated by a pair of inserted darts. We denote the three-dimensional domain of the subject as Ω with its boundary $\partial\Omega$. Let $\mathbf{r} = (x, y, z)$ denote a position vector. Since an EMD shock is induced by very short pulses of stimulating current, the subject Ω can be assumed to have an isotropic conductivity σ that is a positive function defined in Ω . The darts, denoted by \mathcal{E}_1 and \mathcal{E}_2 , are fired from the device and have a fishhook-shaped geometry with 9mm length and 0.8mm diameter as shown in figure 1.

An EMD device transmits electricity to the body Ω through the pair of darts, and it stimulates the subject with about 2 to 15A body current and 50,000V peak voltage. We assume the injection current through the pair of darts is I A. The current I gives rise to a current density $\mathbf{J} = (J_x, J_y, J_z)$ inside the subject $\Omega_s := \Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}$ depicted in figure 2. For a given amount of injection current I , we have the following data along the darts:

$$-\int_{\partial\mathcal{E}_1} \mathbf{J} \cdot \mathbf{n} \, ds = I = \int_{\partial\mathcal{E}_2} \mathbf{J} \cdot \mathbf{n} \, ds$$

where $\partial\mathcal{E}_j$ is the boundary of the dart \mathcal{E}_j intersecting Ω , \mathbf{n} the unit outward normal vector to the boundary $\partial\Omega_s$, and ds the surface area element. The current density \mathbf{J} can be expressed as

$$\mathbf{J} = (J_x, J_y, J_z) = -\sigma \nabla V \tag{1}$$

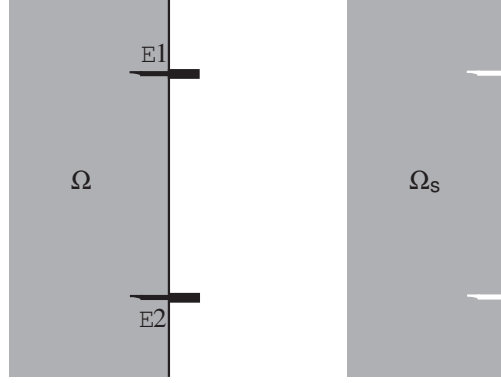


FIGURE 2. Simplified model of a computational domain with two darts.

where V is the voltage inside the body Ω_s . Since the dart is a very good conductor, V can be assumed to be constant on each dart. Setting the voltage on the dart \mathcal{E}_2 to be zero, the voltage V satisfies the following mixed boundary value problem with the elliptic PDE:

$$\begin{cases} \nabla \cdot (\sigma(\mathbf{r})\nabla V(\mathbf{r})) = 0 & \text{in } \Omega_s := \Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2} \\ V = \text{a constant on } \partial\mathcal{E}_1, \quad V = 0 & \text{on } \partial\mathcal{E}_2 \\ \sigma \frac{\partial V}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2} \\ \int_{\partial\mathcal{E}_1} \sigma \frac{\partial V}{\partial \mathbf{n}} ds = I = - \int_{\partial\mathcal{E}_2} \sigma \frac{\partial V}{\partial \mathbf{n}} ds \end{cases} \quad (2)$$

where $\frac{\partial V}{\partial \mathbf{n}}$ is the outward normal derivative of V . Here, $\partial\mathcal{E}_j$ means the boundary of \mathcal{E}_j inside Ω .

The model (2) is inconvenient as a computational model, so we need to modify it maintaining its structure. For a moment, we assume the voltage V at the dart \mathcal{E}_1 is α and let $u = \frac{2V}{\alpha} - 1$. Then u satisfies the following PDE:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega_s \\ u = 1 & \text{on } \mathcal{E}_1 \quad \text{and} \quad u = -1 & \text{on } \mathcal{E}_2 \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}. \end{cases} \quad (3)$$

We need to determine α to compute V directly from u using the relation $V = \frac{\alpha}{2}(u + 1)$. Taking normal derivative to $V = \frac{\alpha}{2}(u + 1)$ and integrating it over $\partial\mathcal{E}_1$, we have $\int_{\partial\mathcal{E}_1} \sigma \frac{\partial V}{\partial \mathbf{n}} ds = \frac{\alpha}{2} \int_{\partial\mathcal{E}_1} \sigma \frac{\partial u}{\partial \mathbf{n}} ds$ and obtain the following linear relation:

$$V(\mathbf{r}) = \frac{\int_{\partial\mathcal{E}_1} \sigma \frac{\partial u}{\partial \mathbf{n}} ds}{I} (u(\mathbf{r}) + 1), \quad \mathbf{r} \in \Omega_s.$$

Hence, it suffices to compute u in the simplified model (3) for the computation of $\mathbf{J} = -\sigma \nabla V$ in the model (2).

For an accurate analysis of the distribution of \mathbf{J} near the darts, we should carefully model the exact shape of the dart depicted in figure 1. Assuming that the dart \mathcal{E}_1 is lying along the z -axis, we may express its geometry as follows:

$$\begin{aligned} \mathcal{E}_1 := & \left\{ \mathbf{r} \in \mathbb{R}^3 : \sqrt{x^2 + y^2} < 0.4, 0 < z \leq 6 \right\} \\ & \cup \left\{ \mathbf{r} \in \mathbb{R}^3 : \sqrt{x^2 + \left(y + \frac{2}{15}(z - 6)\right)^2} \leq \frac{2(9-z)}{15}, 6 \leq z < 9 \right\} \\ & \cup \left\{ \mathbf{r} \in \mathbb{R}^3 : \sqrt{x^2 + \left(y + \frac{6}{5}(z - 6.4)\right)^2} \leq 0.2(z - 5.4), 5.4 \leq z < 6.4 \right\}. \end{aligned} \quad (4)$$

2.2. Concentration of \mathbf{J} near fishhook-shaped dart. In the region where darts are inserted, very high concentration of \mathbf{J} is established. To see this in a quantitative way, let us compare $|\mathbf{J}|$ on \mathcal{E}_1 with $|\mathbf{J}|$ over the zero equipotential surface $\Gamma_0 := \{\mathbf{r} \in \Omega : u(\mathbf{r}) = 0\}$. We can view Γ_0 roughly as a cross-sectional surface lying somewhere between two darts since Γ_0 separates the body Ω_s into two parts $\Omega_s^+ := \{\mathbf{r} \in \Omega_s : u(\mathbf{r}) > 0\}$ and $\Omega_s^- = \Omega_s \setminus \Omega_s^+$. According to the divergence theorem, we have

$$0 = \int_{\Omega_s^+} \nabla \cdot (\sigma \nabla u) d\mathbf{r} = \int_{\partial\Omega_s^+} \sigma \frac{\partial u}{\partial \mathbf{n}} dS = \int_{\partial\mathcal{E}_1} \sigma \frac{\partial u}{\partial \mathbf{n}} dS + \int_{\Gamma_0} \sigma \frac{\partial u}{\partial \mathbf{n}} dS$$

where $\Omega_s^+ := \{\mathbf{r} \in \Omega_s : u(\mathbf{r}) > 0\}$. Since the normal vector on the equipotential surface is $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ on $\partial\mathcal{E}_1$ and $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ on Γ_0 , the above identity becomes

$$\int_{\partial\mathcal{E}_1} |\mathbf{J}| dS = c \int_{\partial\mathcal{E}_1} |\sigma \nabla u| dS = c \int_{\Gamma_0} |\sigma \nabla u| dS = \int_{\Gamma_0} |\mathbf{J}| dS$$

where $c = \frac{1}{I} \int_{\partial\mathcal{E}_1} \sigma \frac{\partial u}{\partial \mathbf{n}} ds$ is the scaling constant. Note that the surface area of the dart is approximately $0.8\pi \times 9\text{mm}^2$ and the surface area of a cross-section of a normal adult body would be bigger than $200 \times 300\text{mm}^2$. Hence, we may conclude that

$$[\text{average } |\mathbf{J}| \text{ on } \partial\mathcal{E}_1] \geq 2000 \times [\text{average } |\mathbf{J}| \text{ on the cross-section } \Gamma_0].$$

For the risk characterization, therefore, we must focus on \mathbf{J} near the darts, while we may disregard \mathbf{J} at some distance from the darts. In particular, we should carefully investigate the magnitude of \mathbf{J} near the dart \mathcal{E}_1 where it is placed with the shortest distance to the heart as the worst case.

3. MULTIPLE POINT SOURCE METHOD (MPSM)

3.1. Homogeneous half-space model. In the model (3), we need to develop a computational method providing reliable estimates of \mathbf{J} near darts with various positions. The computational domain has two corners at the dart, tip of needle and tip of barb, where singularities of \mathbf{J} occur. Conventional FEM requires a very fine mesh near the fishhook-shaped corners to get enough numerical accuracy and it leads to a large-scale computation. To circumvent this problem, we develop a new method of computing \mathbf{J} near the dart, called the multiple point source method (MPSM). The MPSM seems to be ideal for models that have complicated singularities at very sharp small corners in a large three-dimensional domain.

To explain the MPSM, we begin with considering the case where the subject is the upper half-space $\mathbb{R}_+^3 = \{\mathbf{r} = (x, y, z) : z > 0\}$ and has a homogeneous conductivity. Let u_1 be the solution of (3) with Ω replaced by \mathbb{R}_+^3 and $\lim_{\mathbf{r} \rightarrow \infty} u_1(\mathbf{r}) = 0$. Then we have

$$\begin{cases} \nabla^2 u_1 = 0 & \text{in } \mathbb{R}_+^3 \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2} \\ u_1 = 1 & \text{on } \partial\mathcal{E}_1 \quad \text{and} \quad u = -1 \quad \text{on } \partial\mathcal{E}_2 \\ \frac{\partial u_1}{\partial \mathbf{n}} = 0 & \text{on } \partial\mathbb{R}_+^3 \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2} \\ \lim_{\mathbf{r} \rightarrow \infty} u_1(\mathbf{r}) = 0 \end{cases} \quad (5)$$

where $\nabla^2 = \nabla \cdot \nabla$ denotes the Laplacian.

The basic idea of the MPSM is to view the dart as a sum of a sequence of point sources lying inside the dart and their image sources with respect to the z -plane. Let $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{M_1}$ be a sequence of points lying inside the dart \mathcal{E}_1 and let \mathbf{a}_j^- be the reflected point of \mathbf{a}_j to the z -plane. Similarly, let $\mathbf{b}_0, \dots, \mathbf{b}_{M_2} \in \mathcal{E}_2$ with their reflected points \mathbf{b}_j^- . Now we consider the multiple-source function

$$W(\mathbf{r}) = \sum_{j=0}^{M_1} q_j^1 \left(\frac{1}{|\mathbf{r} - \mathbf{a}_j|} + \frac{1}{|\mathbf{r} - \mathbf{a}_j^-|} \right) - \sum_{j=0}^{M_2} q_j^2 \left(\frac{1}{|\mathbf{r} - \mathbf{b}_j|} + \frac{1}{|\mathbf{r} - \mathbf{b}_j^-|} \right) \quad (6)$$

where q_j^1 and q_j^2 are positive point charges. This W satisfies

$$\nabla^2 W(\mathbf{r}) = 0 \quad \text{in } \Omega_s = R_+^3 \setminus (\mathcal{E}_1 \cup \mathcal{E}_2) \quad \text{and} \quad \lim_{|\mathbf{r}| \rightarrow \infty} W(\mathbf{r}) = 0.$$

Due to the presence of the image sources at \mathbf{a}_j^- and \mathbf{b}_j^- , we have

$$\frac{\partial}{\partial z} W(\mathbf{r}) = 0 \quad \text{on the } z\text{-plane.}$$

Hence, W satisfies all the conditions of the model (3) but the boundary conditions on the darts. If we could approximate $W \approx 1$ on \mathcal{E}_1 and $W \approx -1$ on \mathcal{E}_2 , then $W \approx u_1$ in the entire domain Ω_s according to the standard maximum principle.

In order for W to be a good approximation of u_1 , we need to find appropriate source points and point charges such that $W \approx 1$ on \mathcal{E}_1 and $W \approx -1$ on \mathcal{E}_2 . To be precise, we need to find the vectors $\mathbf{a}_0, \dots, \mathbf{a}_{M_1} \in \mathcal{E}_1$, $\mathbf{b}_0, \dots, \mathbf{b}_{M_2} \in \mathcal{E}_2$, and positive quantities $q_0^1, \dots, q_{M_1}^1, q_0^2, \dots, q_{M_2}^2$ which minimize the following functional

$$\begin{aligned} & \int_{\partial\mathcal{E}_1} |1 - W(\mathbf{r})|^2 ds + \int_{\partial\mathcal{E}_2} |1 + W(\mathbf{r})|^2 ds \\ &= \int_{\partial\mathcal{E}_1} \left| 1 - \sum_{j=0}^{M_1} q_j^1 F_j^1(\mathbf{r}) + \sum_{j=0}^{M_2} q_j^2 F_j^2(\mathbf{r}) \right|^2 ds \\ &+ \int_{\partial\mathcal{E}_2} \left| 1 + \sum_{j=0}^{M_1} q_j^1 F_j^1(\mathbf{r}) - \sum_{j=0}^{M_2} q_j^2 F_j^2(\mathbf{r}) \right|^2 ds \end{aligned} \quad (7)$$

where

$$F_j^1(\mathbf{r}) := \frac{1}{|\mathbf{r} - \mathbf{a}_j|} + \frac{1}{|\mathbf{r} - \mathbf{a}_j^+|} \quad \text{and} \quad F_j^2(\mathbf{r}) := \frac{1}{|\mathbf{r} - \mathbf{b}_j|} + \frac{1}{|\mathbf{r} - \mathbf{b}_j^+|}.$$

To simplify the minimization problem, we may choose \mathbf{a}_j and \mathbf{b}_j by considering the distribution of the electric potential due to the special geometry of the fishhook-shaped dart \mathcal{E}_1 . For example, one can choose

$$\mathbf{a}_j = \begin{cases} (0, -0.4 + \frac{2}{5}(1.4)^{j-8}, 9 - 3(1.4)^{j-8}), & 0 \leq j \leq 7 \\ (0, 0, \frac{5(j-8)}{8}), & 8 \leq j \leq 15 \\ (0, 0, 5 + \frac{j-16}{8}), & 16 \leq j \leq 23 \\ (0, \frac{6(j-24)}{40}, 6.4 - \frac{j-24}{8}), & 24 \leq j \leq 33. \end{cases}$$

Some theoretical insight about corner singularity is generally required for an appropriate choice of such source points.

With the given $\mathbf{a}_0, \dots, \mathbf{a}_{M_1}, \mathbf{b}_0, \dots, \mathbf{b}_{M_2}$, the minimization problem in (7) can be reduced to find a vector $\mathbf{q} = (q_0^1, \dots, q_{M_1}^1, q_0^2, \dots, q_{M_2}^2)$ minimizing the following functional:

$$\Phi(\mathbf{q}) = \int_{\partial\mathcal{E}_1} |1 - \mathbf{q} \cdot \mathbb{F}(\mathbf{r})|^2 ds + \int_{\partial\mathcal{E}_2} |1 + \mathbf{q} \cdot \mathbb{F}(\mathbf{r})|^2 ds \quad (8)$$

where

$$\mathbb{F}(\mathbf{r}) = (F_0^1(\mathbf{r}), \dots, F_{M_1}^1(\mathbf{r}), -F_0^2(\mathbf{r}), \dots, -F_{M_2}^2(\mathbf{r})).$$

To discretize (8), we choose a sequence of points on the boundary of darts such as $\mathbf{r}_1^1, \dots, \mathbf{r}_{N_1}^1 \in \partial\mathcal{E}_1$ and $\mathbf{r}_1^2, \dots, \mathbf{r}_{N_2}^2 \in \partial\mathcal{E}_2$. Then, this minimization problem can be simplified to find \mathbf{q} that minimizes the following sum:

$$\tilde{\Phi}(\mathbf{q}) = \sum_{k=0}^{N_1} |1 - \mathbf{q} \cdot \mathbb{F}(\mathbf{r}_k^1)|^2 + \sum_{k=0}^{N_2} |1 + \mathbf{q} \cdot \mathbb{F}(\mathbf{r}_k^2)|^2. \quad (9)$$

For readers who try to test the MPSM, we provide an example of \mathbf{r}_j^1 :

$$\mathbf{r}_j^1 = \begin{cases} \left(0, -0.4, 9 - 3(1.4)^{j-N} \right), & j = 0, \dots, N-1 \\ \left(0, 0.4, \frac{5(j-N)}{N} \right), & N \leq j \leq 2N-1 \\ \left(0, 0.4, 5 + \frac{(j-2N)}{N} \right), & 2N \leq j \leq 3N-1 \\ \left(0, 0.4 + \frac{6(j-3N)}{5N}, 6.4 - \frac{j-3N}{N} \right), & 3N \leq j \leq 4N-1 \\ \left(0, 0.4, \frac{6(j-4N)}{N} \right), & 4N \leq j \leq 5N-1 \end{cases} \quad \text{and } N = 8.$$

If \mathbf{q}^* is a minimizer of the functional $\tilde{\Phi}(\mathbf{q})$ in (9), then the solution u_1 can be approximated as

$$u_1(\mathbf{r}) \approx \mathbf{q}^* \cdot \mathbb{F}(\mathbf{r}), \quad \mathbf{r} \in \mathbb{R}_+^3 \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}. \quad (10)$$

3.1.1. Remark on MPSM. The MPSM is very flexible and it can be viewed as a dart-drawing process by appropriately arranging point sources at \mathbf{a}_j . Some knowledge about corner singularities in elliptic PDE and layer potential theory may help determine a desirable arrangement of \mathbf{a}_j by looking at the geometry of the dart, or one can learn it heuristically by using a computer program. Let us state a simple theoretical lemma that is related to the MPSM.

Lemma 3.1. *Let u_1 be the solution of (5). There exist $q^1 \in L^2(\partial\mathcal{E}_1^*)$ and $q^2 \in L^2(\partial\mathcal{E}_2^*)$ such that q^1 and q^2 are symmetric with respect to z -axis and*

$$u_1(\mathbf{r}) = \mathcal{S}_{\partial\mathcal{E}_1^*} q^1(\mathbf{r}) - \mathcal{S}_{\partial\mathcal{E}_2^*} q^2(\mathbf{r}) \quad \text{in } \mathbb{R}_+^3 \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}$$

where \mathcal{E}_j^* is the interior domain of $\overline{\mathcal{E}_j \cup \{\mathbf{r} \in \mathbb{R}^3 : (x, y, -z) \in \mathcal{E}_j\}}$, $j = 1, 2$ and $\mathcal{S}_{\partial\mathcal{E}_j^*} q^j$ is the single layer potential $\mathcal{S}_{\partial\mathcal{E}_j^*} q^j(\mathbf{r}) := \int_{\partial\mathcal{E}_j^*} \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} q^j(\mathbf{r}') ds_{\mathbf{r}'}$.

We will give a brief proof. We extend u_1 into the lower half-space by even reflection, so the extended function u_1^* is defined by $u_1^*(x, y, -z) = u_1(x, y, z)$ for $(x, y, z) \in \mathbb{R}_+^3 \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}$. Then u_1^* satisfies the Dirichlet problem

$$\nabla^2 u_1^* = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{E}_1^* \cup \mathcal{E}_2^*}, \quad u_1^* = 1 \quad \text{on } \partial\mathcal{E}_1^*, \quad u_1^* = -1 \quad \text{on } \partial\mathcal{E}_2^*.$$

Then the proof follows from the invertibility of single layer potential on Lipschitz domain [24]. \square

Since the dart \mathcal{E}_j is very thin and sharp, it is like a curve. Hence, the above lemma can be modified as follows. For appropriately given curves γ_j contained in \mathcal{E}_j^* , there exist charges $q^j \in L^2(\gamma_j)$ such that

$$\int_{\gamma_1} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} q^1(\mathbf{r}') dl_{\mathbf{r}'} - \int_{\gamma_2} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} q^2(\mathbf{r}') dl_{\mathbf{r}'} \approx \begin{cases} 1 & \text{on } \partial\mathcal{E}_1 \\ -1 & \text{on } \partial\mathcal{E}_2 \end{cases}$$

where dl is the length element. The MPSM can be viewed as a discretized version of the above approximation.

3.2. Computation of \mathbf{J} in bounded domain and finite geometry effect.

3.2.1. Hexahedral domain with homogeneous conductivity. Now, we consider the solution u of (3) in a bounded domain Ω_s with a homogeneous conductivity. We begin with the case where Ω is a hexahedral domain with $\sigma = 1$ as shown in figure 3(a). To be precise, let

$$\Omega := \{\mathbf{r} = (x, y, z) : -100\text{mm} < x < 400\text{mm}, |y| < 100\text{mm}, 0 < z < 150\text{mm}\}. \quad (11)$$

Assume that the dart \mathcal{E}_1 strikes at the origin. The other dart \mathcal{E}_2 strikes L mm below \mathcal{E}_1 and $100\text{mm} < L < 300\text{mm}$, that is, $\mathcal{E}_2 = \{\mathbf{r} + (L, 0, 0) : \mathbf{r} \in \mathcal{E}_1\}$.

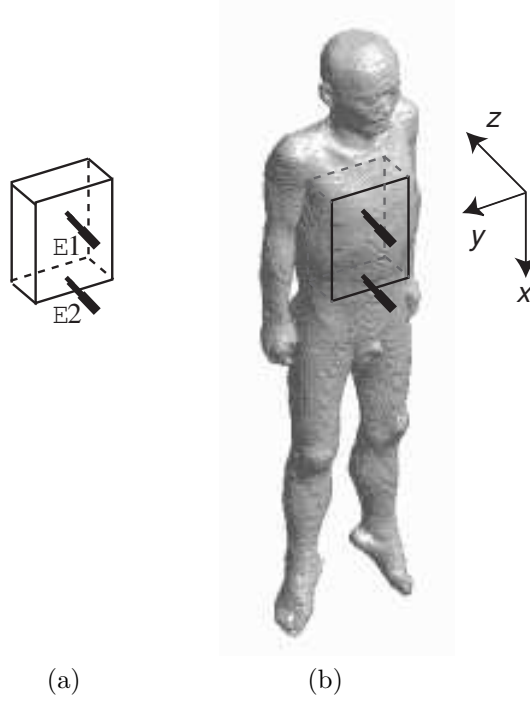


FIGURE 3. (a) Hexahedral domain and (b) human body model.

For the computation of u , we make a good use of the approximation $u_1(\mathbf{r}) \approx \mathbf{q}^* \cdot \mathbb{F}(\mathbf{r})$ in (10). We decompose the solution u of (3) into

$$u(\mathbf{r}) = u_1(\mathbf{r}) + \phi(\mathbf{r}), \quad \mathbf{r} \in \Omega_s.$$

Then, ϕ satisfies the following mixed boundary value problem:

$$\begin{cases} \nabla^2 \phi = 0 & \text{in } \Omega_s \\ \phi = 0 & \text{on } \partial\mathcal{E}_1 \cup \partial\mathcal{E}_2 \\ \frac{\partial \phi}{\partial \mathbf{n}} = -\frac{\partial u_1}{\partial \mathbf{n}} & \text{on } \partial\Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}. \end{cases} \quad (12)$$

The quantity $\nabla \phi$ can be viewed as a finite geometry effect. Due to the presence of the sharp darts \mathcal{E}_1 and \mathcal{E}_2 in Ω , the computation of ϕ is again a formidable job. In order to avoid this, we decompose ϕ again into two parts:

$$\phi(\mathbf{r}) = \phi_1(\mathbf{r}) + \phi_2(\mathbf{r})$$

where ϕ_1 is the solution of the following standard Neumann boundary problem:

$$\begin{cases} \nabla^2 \phi_1 = 0 & \text{in } \Omega \\ \mathbf{n} \cdot \nabla \phi_1 = -\mathbf{n} \cdot \nabla u_1 & \text{on } \partial\Omega, \quad \int_{\partial\Omega} \phi_1 = 0 \end{cases} \quad (13)$$

and ϕ_2 satisfies

$$\begin{cases} \nabla^2 \phi_2 = 0 & \text{in } \Omega_s \\ \phi_2 = -\phi_1 & \text{on } \partial\mathcal{E}_1 \cup \partial\mathcal{E}_2 \\ \mathbf{n} \cdot \nabla \phi_2 = 0 & \text{on } \partial\Omega \setminus \overline{\mathcal{E}_1 \cup \mathcal{E}_2}. \end{cases} \quad (14)$$

Note that the computation of ϕ_1 is easy via any standard numerical method without using a fine mesh.

Next, we consider the second term ϕ_2 . We conjecture that the ratio $\frac{|\nabla\phi_2|}{|\nabla u|}$ is very small in $\Omega_s \cap B$ where B is the ball with 30mm radius and centered at the origin. Then we can neglect the second term ϕ_2 in $\Omega_s \cap B$, the region of interest. Although this conjecture seems to be obvious, we are not able to rigorously prove it. Indeed, even the clear fact that $|\nabla u| > 0$ in $\Omega_s \cap B$ is very difficult to be proved in a three-dimensional case, while there are some related two-dimensional results in [1, 2, 3]. We think that these theoretical issues should be addressed in future studies, although those are obvious from physical insight.

Without a rigorous proof, we try to explain a reason why $\frac{|\nabla\phi_2|}{|\nabla u|}$ is very small in $\Omega_s \cap B$. According to the expression of u_1 in (10), the quantity $|\nabla u_1(\mathbf{r})|$ decays with a rate of the square of the distance from the dart. The solution ϕ_1 of (14) has the Neumann boundary condition $\frac{\partial\phi_1}{\partial\mathbf{n}} = -\frac{\partial u_1}{\partial\mathbf{n}}$ which is very small on $\partial\Omega$ due to the decaying property of u_1 . Hence, the contribution of $\nabla\phi_1$ to ∇u near the dart would be very small. We performed numerical computations of ϕ_1 with various $100\text{mm} \leq L \leq 300\text{mm}$ and the results showed that $|\nabla\phi_1| \approx 0$ near \mathcal{E}_1 and the difference of ϕ_1 between \mathcal{E}_1 and \mathcal{E}_2 are less than 0.01. These crude estimates yield that $\phi_1 \approx c_1$ on \mathcal{E}_1 and $\phi_1 \approx c_2$ on \mathcal{E}_2 and $0 < |c_1 - c_2| < 0.01$. Since ϕ_2 satisfies the same boundary value problem with u in (3) except some difference on $\partial\mathcal{E}_1 \cup \partial\mathcal{E}_2$, u can be approximated by

$$u(\mathbf{r}) \approx \frac{2}{c_1 - c_2} \phi_2(\mathbf{r}) + 1 - \frac{2c_1}{c_1 - c_2}.$$

Hence, $\frac{|\nabla\phi_2|}{|\nabla u|} \approx \frac{|c_1 - c_2|}{2} \leq 0.005$ in the region $\Omega_s \cap B$. We again stress that this estimate is not proven rigorously, so it is an open problem.

The above estimate means that ϕ_2 has less than 1% effect on \mathbf{J} in the region of interest. Hence, we can approximate

$$\nabla u \approx \nabla u_1 + \nabla\phi_1 \approx \nabla(\mathbf{q}^* \cdot \mathbb{F}(\mathbf{r})) + \nabla\phi_1. \quad (15)$$

The major advantage of this approximation is that the two terms of the right side of (15) are computable using a PC. Various numerical simulations show that $u_1 + \phi_1$ satisfies (3) with the Dirichlet conditions $u_1 + \phi_1 \approx 1$ on \mathcal{E}_1 and $u_1 + \phi_1 \approx -1$ on \mathcal{E}_2 . Hence, $u_1 + \phi_1$ can be viewed as an accurate numerical solution of (3).

3.2.2. Human body model with homogeneous conductivity. Now, let Ω be the human body model shown in figure 3(b) and let Ω_* be the hexahedral domain in (11). The three-dimensional model in figure 3(b) was generated by using the Visible Korean Human data set [23]. We still assume that the conductivity σ is homogeneous. When the origin is positioned at the pit of the stomach, most adult human bodies can contain the hexahedral domain Ω_* with an appropriate coordinate system. Then we are considering a worst case where the dart \mathcal{E}_1 strikes at the pit of the stomach. For simplicity, we assume $\Omega_* \cap \{z = 0\} \subset \Omega \cap \{z = 0\}$, which is reasonable. Since $\partial\Omega \cap \{z = 0\}$ is outside the hexahedron Ω_* , the boundary effect to \mathbf{J} near the dart is smaller than that is in the case of Ω_* only. For the human body model Ω under the assumption of a homogeneous conductivity, the approximation (15) also holds.

4. NUMERICAL EXPERIMENTS

4.1. Modelling of fishhook-shaped dart using MPSM. The MPSM provides an efficient way to model the fishhook-shaped dart by arranging point sources inside the dart. A sequence of source points \mathbf{a}_j and \mathbf{a}_j^- were chosen as in 3.1 by considering the voltage distribution due to the special geometry of the dart. Now, the shape modelling problem reduces to find appropriate point charges to enforce the boundary condition on the dart. In order to solve this, we employed the discrete minimization using

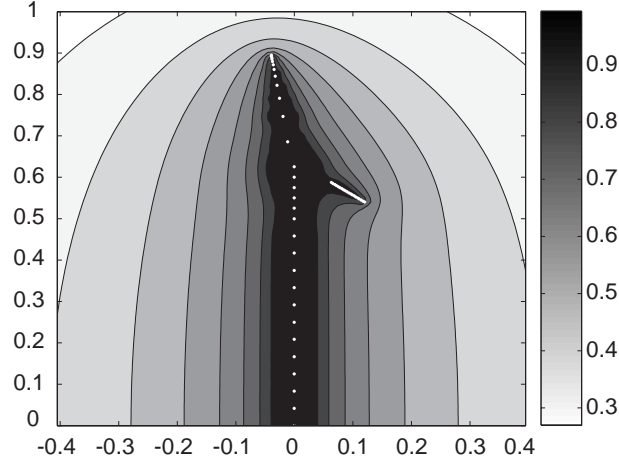


FIGURE 4. Modelling of the fishhook-shaped dart using the MPSM. White dots inside the dart are source points.

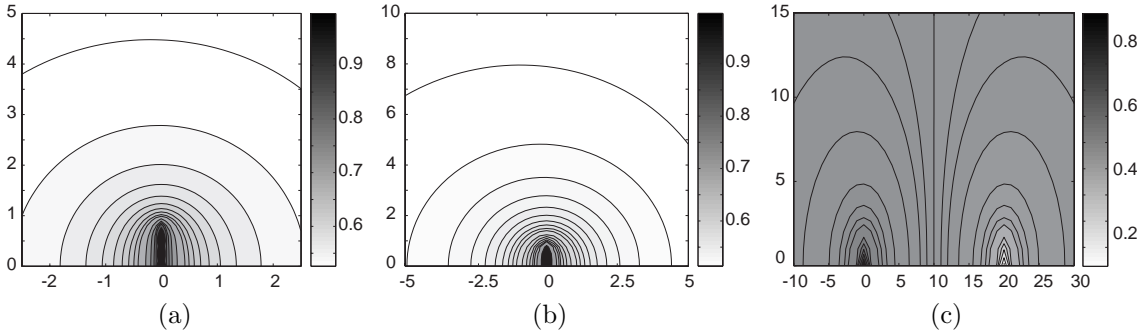


FIGURE 5. Distribution of voltage from the half-space model with different field-of-view: (a) $5 \times 5\text{cm}^2$, (b) $10 \times 10\text{cm}^2$ and (c) $40 \times 15\text{cm}^2$.

boundary points of the dart, \mathbf{r}_i given in 3.1. Then we approximated multiple-source function W and figure 4 shows the contour plot of the obtained voltage distribution. Equipotential surface of $V = 1$ denoted as black color matches well with the fishhook-shaped dart.

4.2. Distribution of voltage. Figure 5(a), (b) and (c) show distributions of voltage from the homogeneous half-space model with increasing field-of-view. Figure 6(a), (b) and (c) are the corresponding voltage distributions computed from the hexagonal model. Using the homogeneous human body model, voltage distributions were almost identical to those from the hexagonal model. As expected, the voltage distribution near each dart in the bounded domain changes little compared with that in the half-space. In the region far away from both darts, we can observe differences in voltage distributions. However, the gradients of voltage there are small.

4.3. Distribution of magnitude of current density. Figure 7(a), (b) and (c) are distributions of magnitude of current density from the homogeneous half-space model with increasing field-of-view. Figure 8(a), (b) and (c) are the corresponding distributions of magnitude of current density from

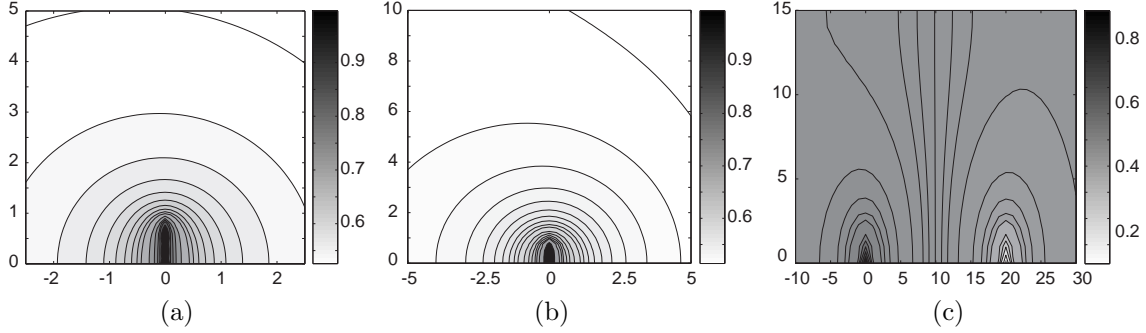


FIGURE 6. Distribution of voltage from the hexagonal model with different field-of-view: (a) $5 \times 5\text{cm}^2$, (b) $10 \times 10\text{cm}^2$ and (c) $40 \times 15\text{cm}^2$.

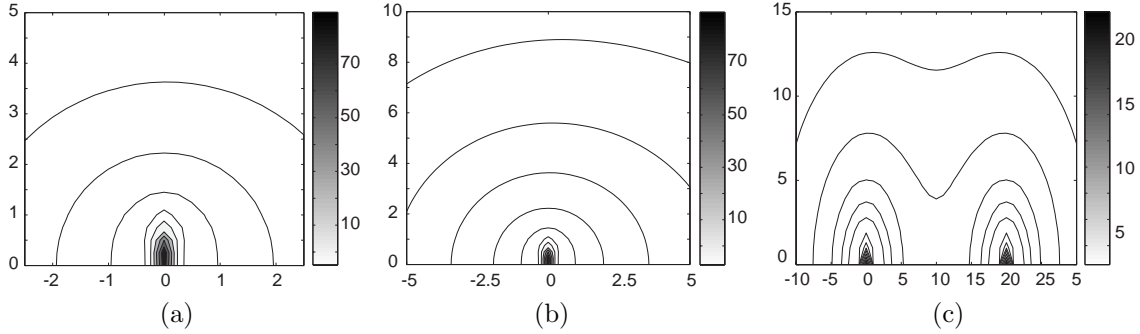


FIGURE 7. Distribution of magnitude of current density from the half-space model with different field-of-view: (a) $5 \times 5\text{cm}^2$, (b) $10 \times 10\text{cm}^2$ and (c) $40 \times 15\text{cm}^2$.

the hexagonal model. The homogeneous human body model produced almost identical results as in Figure 8(a), (b) and (c). Figure 9 shows how the magnitude of current density decays as we move along the straight line starting from the tip of the dart. In all three homogeneous models, the plots are almost identical within 13cm distance from the tip of the dart.

5. DISCUSSIONS ON REALISTIC HUMAN BODY MODEL AND ANIMAL EXPERIMENTS

The conductivity distribution of a realistic human body model should not be homogeneous since different tissues have different conductivity values. For a realistic human body model, therefore, we need a three-dimensional human anatomy data set with knowledge of conductivity distribution. As illustrated in figure 3(b), we are using the Visible Korean Human data set that includes segmented cross-sectional images of a Korean adult male cadaver [23]. We are currently conducting numerical simulations using this realistic human body model by assigning appropriate conductivity values to the segmented internal regions. Though this ongoing research requires a great deal of efforts and time, we speculate that the three-dimensional realistic human body model together with the MPSM suggested in this paper will be able to provide the most reliable numerical simulation results of EMD shocks. Especially we should focus on properly incorporating an inhomogeneous realistic conductivity distribution into the human body model since it is highly possible that such a conductivity distribution will alter voltage and current density distributions even in a region near the fishhook-shaped dart.

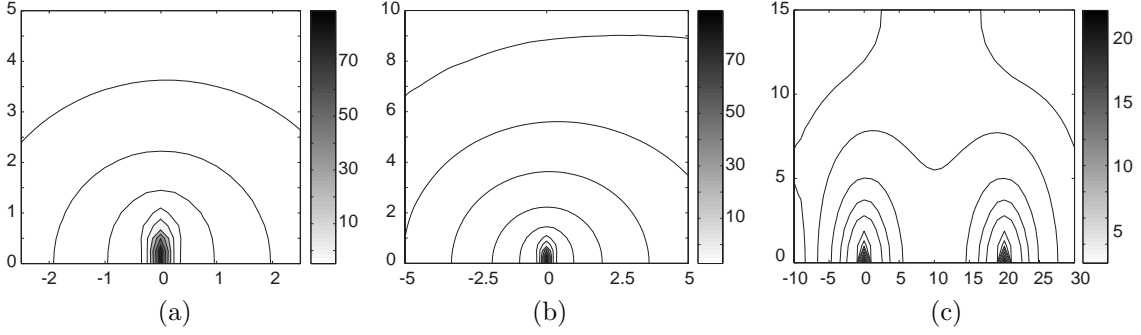


FIGURE 8. Distribution of magnitude of current density from the hexagonal model with different field-of-view: (a) $5 \times 5\text{cm}^2$, (b) $10 \times 10\text{cm}^2$ and (c) $40 \times 15\text{cm}^2$.

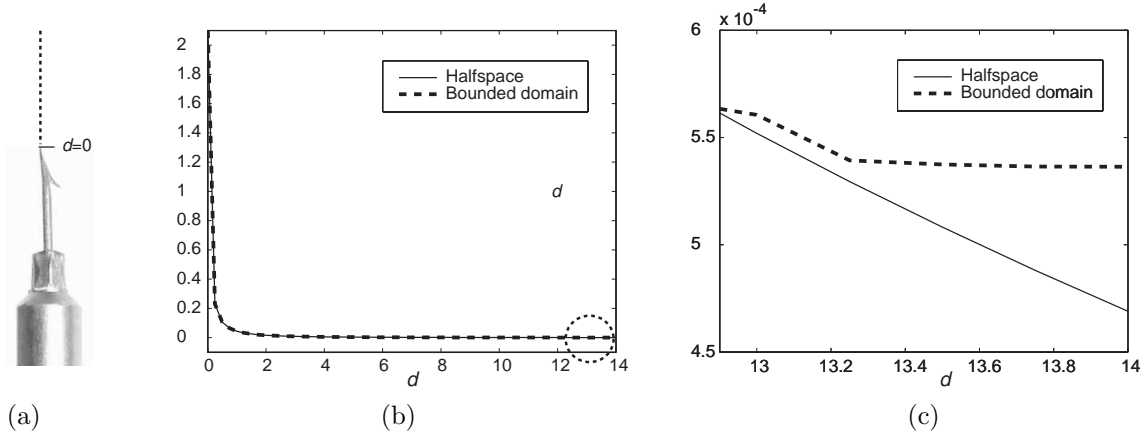


FIGURE 9. (a) We are interested in the magnitude of current density along the dotted line where d is the distance from the tip of the dart. (b) Decay of magnitude of current density where x -axis is the distance d from the tip of the dart. (c) is the magnified plot of the portion circled in (b).

Using the realistic human body model, we apply the approximation (10) inside a small region of pixels containing the dart to handle singularities at the sharp corners. For example,

$$\nabla u(r) \approx \nabla u_1(r) + \nabla \phi_1(r) \quad \text{in } \Omega_s \cap (D_1 \cup D_2) \quad (16)$$

where $D_1 = \{\mathbf{r} : |x| < \alpha, |y| < \alpha, 0 < z < \alpha + 10\}$, $D_2 = \{\mathbf{r} + (L, 0, 0) : \mathbf{r} \in D_1\}$, and α is a small number depending on a given pixel size of the segmented anatomy image data. For the computation of u in $\Omega \setminus \overline{D_1 \cup D_2}$, we may solve the following problem:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega \setminus \overline{D_1 \cup D_2} \\ \sigma \frac{\partial u}{\partial \mathbf{n}} = \sigma \frac{\partial u_1}{\partial \mathbf{n}} + \sigma \frac{\partial \phi_1}{\partial \mathbf{n}} & \text{on } \partial(D_1 \cup D_2) \cap \Omega \\ \sigma \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \setminus \overline{D_1 \cup D_2}. \end{cases} \quad (17)$$

Finally, let us discuss the assignment of conductivity values to different tissues inside the realistic human body model. Conductivity image reconstruction from a living human body has been a long-term goal in the area of electrical impedance tomography (EIT) whose corresponding mathematical problem is the inverse problem of identifying the elliptic coefficient from Dirichlet-to-Neumann map. In EIT, elegant mathematical theory and reconstruction algorithms have been developed [12, 13, 26, 29, 27, 28, 25]. However, in clinical environments, EIT has not reached yet to the stage providing reliable static conductivity values due to its ill-posed nature.

Recently, magnetic resonance electrical impedance tomography (MREIT) has been proposed to deal with the ill-posed nature of EIT [14, 17, 15, 19, 22, 21, 18, 20]. MREIT uses an MRI system to get a supplementary data, that is one component B_z of the magnetic flux density $\mathbf{B} = (B_x, B_y, B_z)$, where \mathbf{B} is governed by Amperé law $\nabla \times \mathbf{B} = -\sigma \nabla u$ and u is the solution of the elliptic PDE: $\nabla \cdot (\sigma \nabla u) = 0$ with an injected Neumann data (or current). Numerous numerical simulations and experiments have shown that MREIT can produce high-resolution conductivity images, provided that we inject enough amount of current (or L^1 -norm of Neumann data).

In our future experimental studies to validate numerical simulation results, we plan to first perform conductivity imaging of animal subjects using MREIT techniques. Once we obtain high-resolution conductivity images, then we can generate current density images by solving the direct problem with various position of darts using the method described in this paper. This kind of experimental validations of our numerical method using animal experiments will be invaluable for the interpretation of any numerical results using a realistic human body model since experimental studies using living human subjects are not possible.

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