Lessons From Random Matrix Theory for QCD at Finite Density

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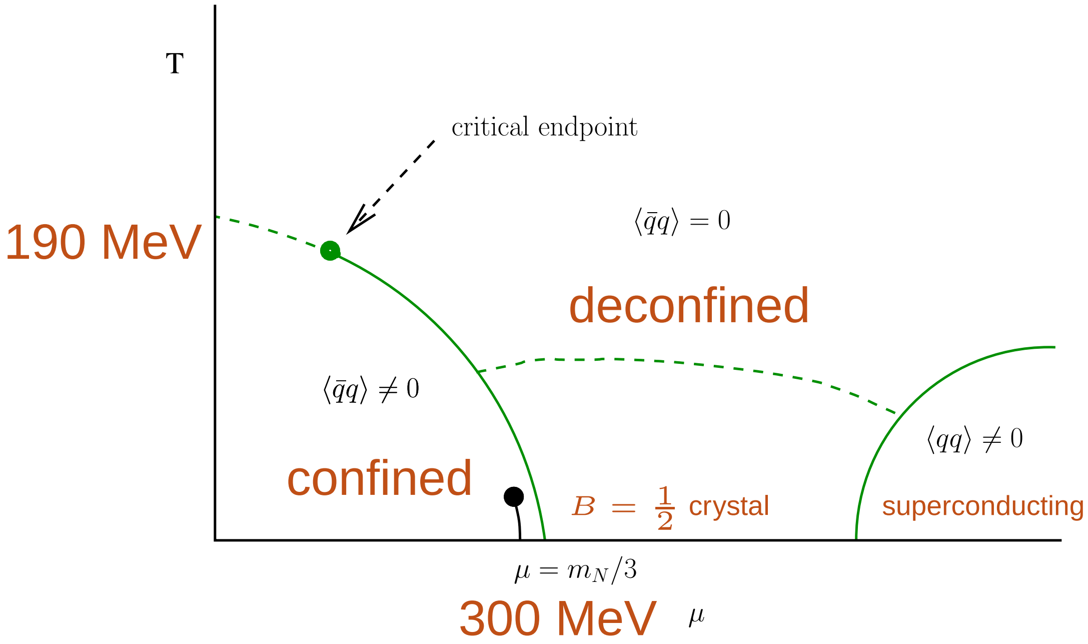


Schematic QCD phase diagram.

A transition to a phase with nonzero baryon density takes place when $3\mu > m_N - BE$ with BE the binding energy of nuclear matter per nucleon.

In the chirally restored phase at $T = 0$ we expect a crystal of $B = \frac{1}{2}$ objects in the large N_c limit.
Klebanov-1986, Jackson-JV-1988 , Goldhaber-Manton-1987

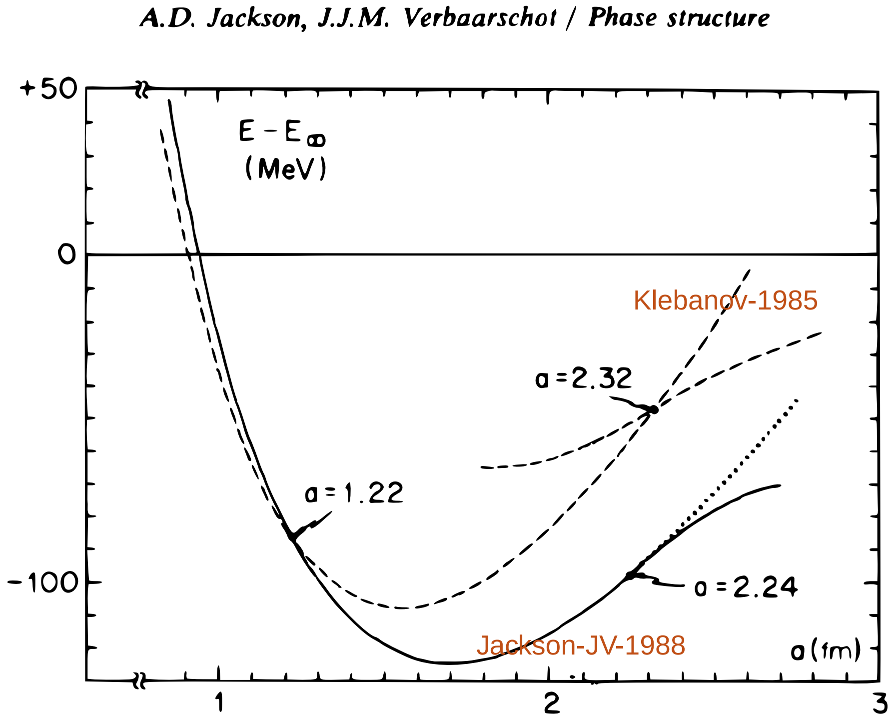
In the previous talk this phase was identified as the quarkionic phase.
McLerran-Pisarski-2007

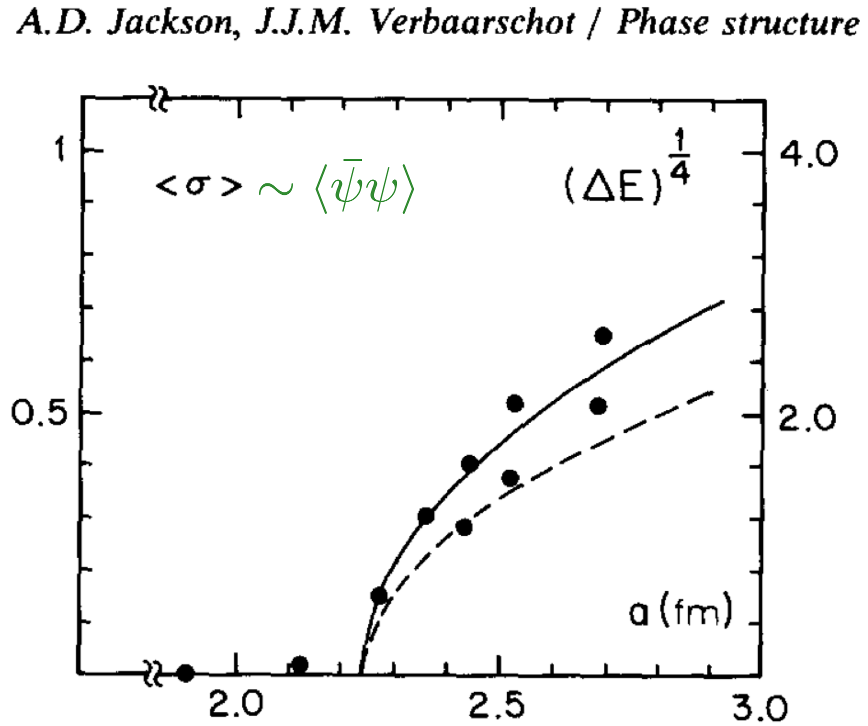
Our aim is to understand the QCD at $\mu \neq 0$ from first principles,

$$Z_{\rm QCD} = \langle \prod_f \det(D + m_f + \mu\gamma_0)\rangle.$$

The nonhermiticity of $D + \mu\gamma_0$ makes this very challenging task indeed.

Skyrme Crystal





Chiral symmetry restoration in a Skyrme crystal. Jackson-JV-1988

Notice binding energy is over 100 MeV, and we expect a melting temperature of similar order.

These results are expected to be accurate in the large N_c limit of QCD.

Phase Quenched QCD

$$
\begin{aligned}
Z_{|\mathrm{QCD}|} &= \langle |\det(D + m + \mu\gamma_0)|^2 \rangle \\
&= \langle \det(D + m + \mu\gamma_0) \det(-D + m + \mu\gamma_0) \rangle \\
&= \langle \det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0) \rangle
\end{aligned}
$$

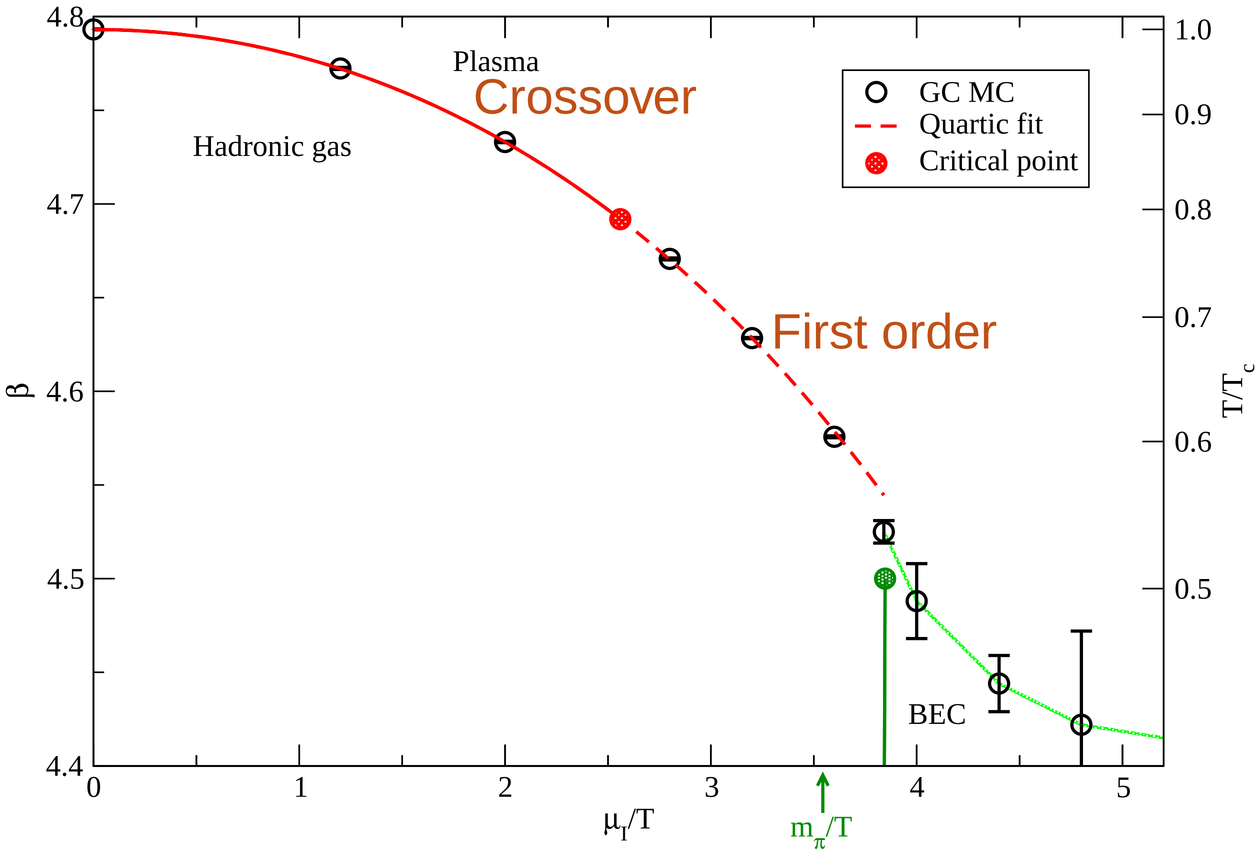
- √ The chemical potential of phase quenched QCD can be interpreted as and isospin chemical potential.

  Alford-Kapustin-Wilczek-1998

- √ It takes only $m_\pi - 2\mu$ to add a pion to the system and pions Bose condense at low temperature for $\mu > m_\pi/2$.

- √ Because the fermion determinant is complex, QCD at $\mu \neq 0$ cannot be simulated by probabilistic methods. That is why the phase quenched partition function has attracted a great deal of attention in the lattice community.

Phase Diagram of Phase Quenched QCD



Phase diagram of phase quenched QCD.

De Forcrand-Stephanov-Wenger-2007

At low temperature this phase diagram is described accurately by a mean field treatment of a chiral Lagrangian.

KSTVZ-2000, Toublan-JV-2000, Son-Stephanov-2000

Comments

- Phase diagram of QCD and phase quenched QCD are very similar for $\mu < m_\pi/2$. This can be derived Cohen-2004 in the large N_c limit.
- For $\mu > m_\pi/2$ the phase of the quark determinant completely changes the phase diagram. Pions condense in individual gauge configurations, but this effect is nullified by the phase of the fermion determinant after averaging.
- Strong nonperturbative effects persist in QCD for $\mu > m_N/3$.
- To better understand the effect of nonhermiticity, we will construct a model that can be solved analytically in the nonperturbative domain of QCD.

Random Matrix Model at $\mu \neq 0$

A chiral Random Matrix Theory (chRMT) is obtained by replacing the matrix elements of the Dirac operator be Gaussian random numbers (JV-1994, Shuryak-JV-1993):

$$D = \begin{pmatrix} 0 & iW^\dagger + \mu^\dagger \\ iW + \mu & 0 \end{pmatrix}$$

with W a complex $n \times (n+\nu)$ matrix. μ is a multiple of the identity (Stephanov-1996) or is an arbitrary random matrix (Osborn-2004) The partition function is the sector of topological charge ν is then given by

$$Z_\nu(m_f;\mu) = \langle \prod_f \det(D + m_f) \rangle.$$

This partition function has the global symmetries and transformation properties of QCD. At fixed θ -angle it is given by

$$Z(m_f,\theta;\mu) = \sum_\nu \mathcal{N}_\nu(\mu) e^{i\nu\theta} Z_\nu(m_f;\mu).$$

Validity of chRMT

In the microscopic domain of QCD,

$$m_\pi^2 \ll \frac{1}{\sqrt{V}}, \qquad \mu^2 \ll \frac{1}{\sqrt{V}}, \qquad V\Lambda_{\rm QCD}^4 \gg 1$$

$$Z_\nu^{\rm QCD}(m_f;\mu) \sim Z_\nu^{\rm chRMT}(m_f;\mu).$$

In this domain, both theories have the same static chiral Lagrangian.

Physically, this is the domain where the Compton wave length of the pion is much larger than the size of the box. Leutwyler-Smilga-1992

Since the kinetic terms of the chiral Lagrangian do not contribute in a mean field analysis, chRMT correctly describes mean field results for the low-energy limit of QCD for

$$m_f \ll \Lambda_{\rm QCD}, \qquad \mu \ll \Lambda_{\rm QCD}$$

Microscopic Limit

- The microscopic limit of QCD is the thermodynamic limit, $V \to \infty$, with fixed mV and fixed $\mu^2 V$.
- The large mV and $\mu^2 V$ joins smoothly with the thermodynamic limit at fixed m and μ .

Lessons for Phase Quenched QCD

Quenched Limit

Homogeneity of Dirac Eigenvalues

Width of Dirac Spectrum

Distribution of Small Dirac Eigenvalues

Lesson 1: Quenched Limit

The quenched limit is the limit where the fermion determinant is ignored in generating the statistical ensemble. For zero chemical potential this has been a reasonable approximation, but at nonzero chemical potential, the limit of no fermion determinant is not given by

$$\lim_{n \to 0} \langle (\det(D+m))^n \rangle,$$

but by

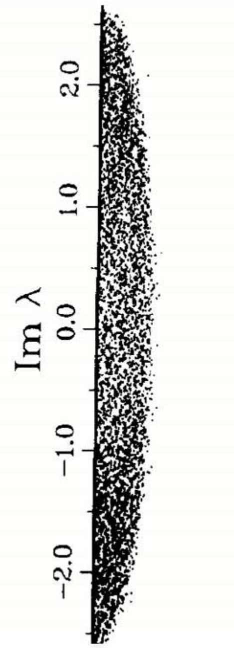
$$\lim_{n \to 0} \langle |\det(D+m)|^n \rangle,$$

i.e. the quenched limit of phase quenched QCD. Stephanov-1996

Quenching the quark determinant completely fails for $\mu \neq 0$. Instead of having a critical point at $\mu = m_N/3$, the critical point is at $\mu = m_\pi/2$. Barbour-et al.-1986, Gibbs-1986

Lesson 2: Homogeneity of Dirac Eigenvalues

At nonzero chemical potential, the Dirac operator is nonhermitean with eigenvalues that are scattered in the complex plane.



Quenched Dirac eigenvalues on a $4^3 \times 8$ lattice.
Barbour-Bhilil-Dagotto-Karsch-Moreo-Stone-Wyld-1986

- Eigenvalues behave as repulsive electric charges that are confined by a potential.
- Chiral condensate is given by

$$\Sigma(m) = \frac{1}{V} \sum_{\lambda_k} \frac{1}{m + i\lambda_k},$$

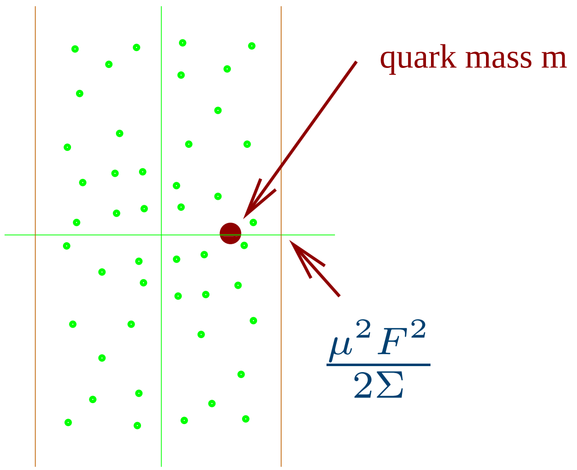
  and can be interpreted as the electric field at m of charges at λ_k .
- A homogeneous eigenvalue distribution is equivalent to a chiral condensate that increases linearly with m inside the domain of eigenvalues.

Lesson 2: Homogeneity of Dirac Eigenvalues

- This result is also obtained from a mean field analysis of the chiral Lagrangian of QCD. The linear mass dependence arises because the chiral condensate rotates into a pion condensate with decreasing quark mass. KSTVZ-2000
- The Dirac spectrum can be characterized by two parameters, its width and the density of eigenvalues. These two parameters determine Σ and F_π , the two low-energy constants of the the chiral Lagrangian.
- This property and variations thereof have been exploited to extract Σ and F_π from lattice QCD simulations. Et al-Wettig-1998, Osborn-Wettig-2006, Damgaard-Heller-Splittorff-Svetitsky-2005

Lesson 3: Width of the Dirac Spectrum

- The phase quenched partition function undergoes a phase transition to a Bose condensed phase at $\mu = m_\pi/2$.
- This has to be the point where the quark mass hits the boundary of the eigenvalues (Gibbs-1986).



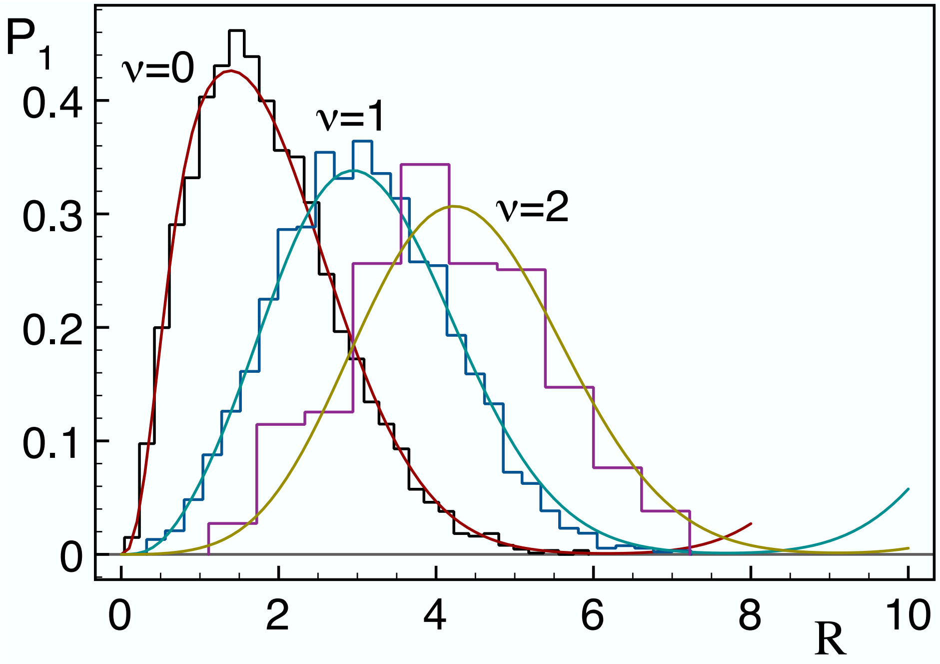
Scatter plot of Dirac eigenvalues

$$m_\pi^2 = \frac{2m\Sigma}{F^2} = 4\mu^2$$

For small enough chemical potential, the width, of course, increases linearly with μ .

In the thermodynamic limit the same eigenvalue distribution is found for each gauge field configuration.

Lesson 4: Distribution of Small Dirac Eigenvalues



Radial distribution of smallest Dirac eigenvalue for different topological charge sectors using a Dirac operator that satisfies the Ginsparg-Wilson relation.

Bloch-Wettig-2006, Akemann-Bloch-Shifrin-Wettig-2007

Lessons for Full QCD at $\mu \neq 0$

Hardness of Lattice QCD Simulations

Infrared Dominance

Failure of Banks-Casher Relation

Quark Mass and Average Phase Factor

Equality of Two Condensates

Tests of Algorithms

Lesson 5: Hardness of Lattice QCD Simulations

Average phase factor

$$\langle e^{2i\theta} \rangle_{\rm pq} = \frac{\langle {\det}^2(D+m+\mu\gamma_0)\rangle}{\langle |\det(D+m+\mu\gamma_0)|^2\rangle} \equiv \frac{Z_{\rm QCD}}{Z_{|\rm QCD|}}.$$

This implies that

$$\langle e^{2i\theta} \rangle_{\rm pq} = e^{-V(F_{\rm QCD}-F_{|\rm QCD|})}.$$

When the free energies of QCD and |QCD| are different, and this is the case at low temperatures, the cancellations in the full QCD partition function grow exponentially with the volume, and the required computer time as the square of that.

This raises the question whether there exists a region of parameter space where the average phase factor is finite.

Lesson 6: Infrared Dominance

For $\mu > m_\pi/2$ the phase of the fermion determinant determines the physics of the partition function. The phase is given by

$$\langle e^{2i\theta} \rangle = \left\langle \frac{\prod(i\lambda_k + m)}{\prod(-i\lambda_k^* + m)} \right\rangle$$

Since the eigenvalue density $\sim V\lambda^3$ it has been argued that the average phase factor is completely dominated by the contribution of the large eigenvalues and averages to zero.

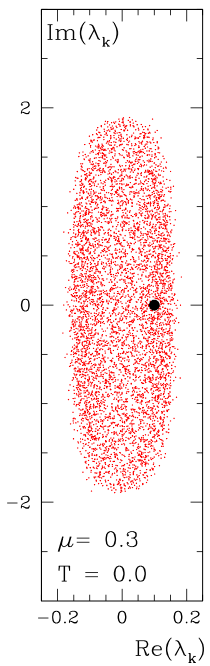
- √ In the microscopic domain, the average phase factor is determined by the infrared part of the Dirac spectrum.
- √ Because of the validity of chiral perturbation theory we expect that this extends to the case that $\mu \ll \lambda_{\rm QCD}$ and $m \ll \Lambda_{\rm QCD}$.
- √ Confirmed by lattice simulations at imaginary chemical potential.

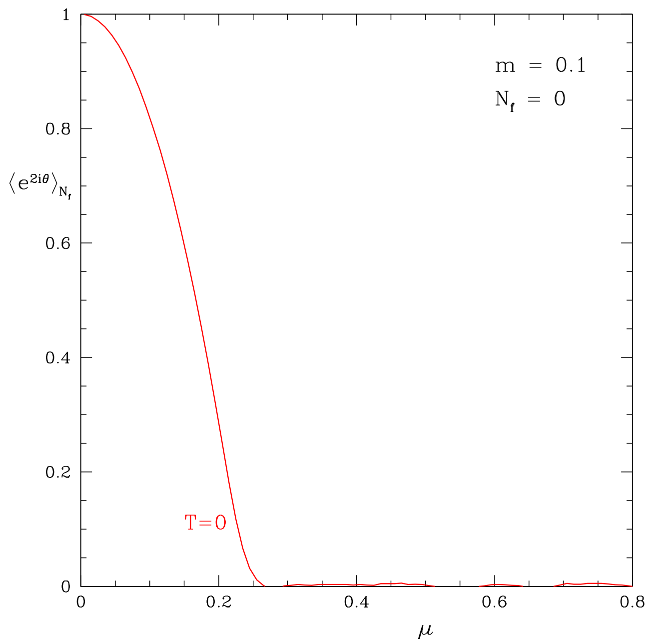
Splittorff-Svetitsky-2007

Lesson 7: Quark Mass and Average Phase Factor

The average phase factor vanishes if the quark mass is inside the domain of eigenvalues. This follows by defining the average phase factor as

$$\langle e^{2i\theta}\rangle_{\rm pq} = \frac{\langle \det^2(D+m+\mu\gamma_0)\rangle}{\langle |\det(D+m+\mu\gamma_0)|^2\rangle} \equiv \frac{Z_{\rm QCD}}{Z_{|\rm QCD|}}.$$





If the quark mass is inside the domain of eigenvalues, the average phase factor vanishes.

If the quark mass is outside the domain of eigenvalues, the free energy of phase quenched QCD is equal to the free energy of QCD (at low T) resulting in a finite phase factor.

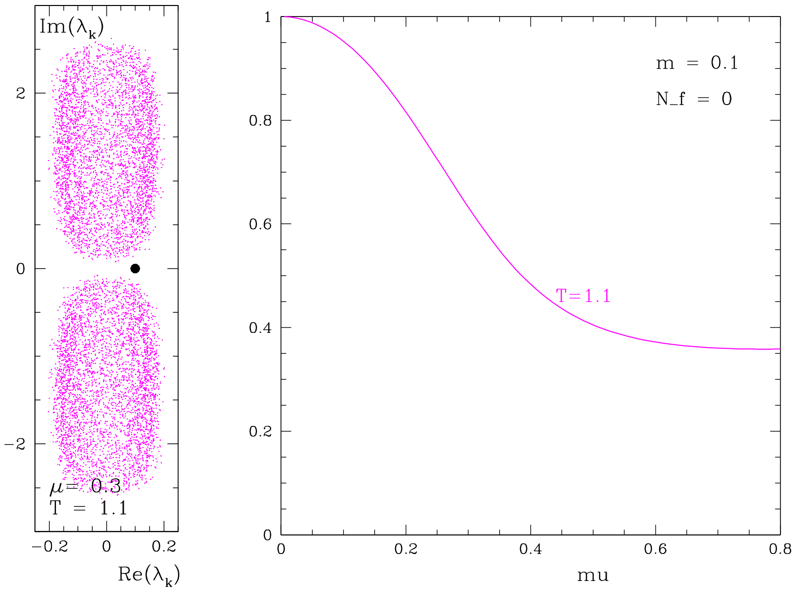
Lesson 7: Quark Mass and Average Phase Factor

- The average phase factor remains finite if the free energies of QCD and |QCD | are the same. A nontrivial dependence arises because of $1/V$ corrections, which give a finite contribution to the ratio of two partition functions.

- From a mean field treatment of a chiral Lagrangian we obtain

$$\langle e^{2i\theta} \rangle = \left(1 - \frac{\mu^2}{\mu_c^2}\right)^{N_f+1}$$ s

with $\mu_c = m_\pi/2$. Splittorff-JV-2006



For $T > T_c$ a gap develops in the Dirac spectrum. Then the average phase factor remains finite in the thermodynamic limit.

Ravagli-JV-2007

Lesson 8: Failure of Banks-Casher at $\mu \neq 0$

The Banks-Casher relation states that

$$\Sigma = \lim_{m \to 0} \lim_{V \to \infty} \frac{\pi \rho(m)}{V}.$$

This correctly gives a vanishing chiral condensate for the quenched theory.

However, for full QCD at $\mu \neq 0$ chiral symmetry is broken but there is no accumulation of Dirac eigenvalues on the imaginary axis. Nevertheless, the chiral condensate is given by the average trace of the inverse Dirac operator, and thus determined by the distribution of the Dirac eigenvalues.

An alternative mechanism is at work. Osborn-Splittorf-JV-2005/2008
(see talk by Kim Splittorff)

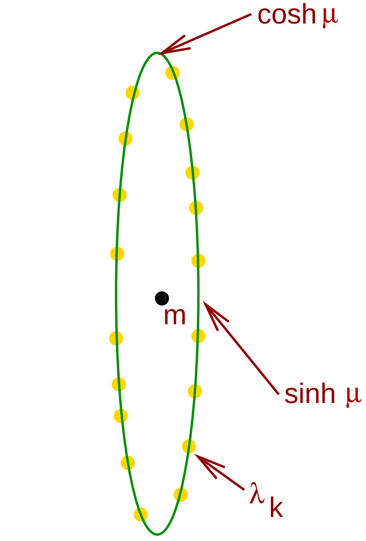
The same mechanism applies to QCD in 1d. Ravagli-JV-2007

QCD in one Dimension

Dirac operator:

$$D = \begin{pmatrix} mI & e^{\mu} & \dots & e^{-\mu}U^{\dagger} \\ -e^{-\mu} & mI & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & mI & e^{\mu} \\ -e^{\mu}U/2 & \cdots & -e^{-\mu} & mI \end{pmatrix}$$

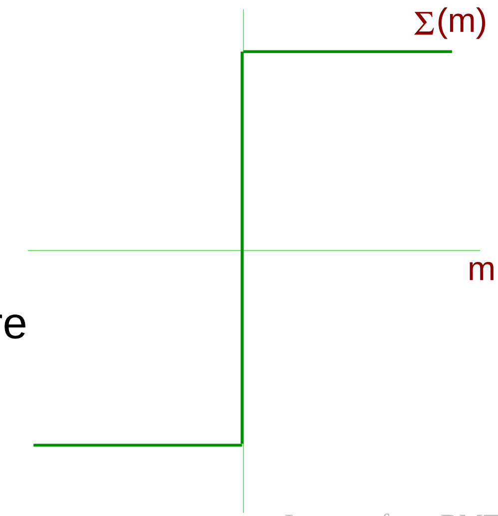
$U(N_c)$
matrix



Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k+m} \prod_k (\lambda_k + m) \right\rangle}{\langle \prod_k (\lambda_k + m) \rangle}$$

determinant with
a complex phase



The chiral condensate has a discontinuity in region where there are no eigenvalues

Ravagli-JV-2007

Lesson 9: Equality Two Condensates

The chiral condensate can be calculated in two ways

$$\Sigma^{(1)} = \lim_{m\to 0}\lim_{V\to\infty}\left\langle \frac{1}{V}\sum_k \frac{1}{\lambda_k + m}\right\rangle, \qquad \Sigma^{(2)} = \lim_{V\to\infty}\lim_{m\to 0}\left\langle \frac{1}{V}\sum_k \frac{1}{\lambda_k + m}\right\rangle$$

- √ $\Sigma^{(1)}$ is the chiral condensate obtained from spontaneous symmetry breaking. Its value does not depend on the total topological charge.
- √ $\Sigma^{(2)} = \frac{\langle \prod_{\lambda_k\neq 0}(\lambda_k+m)\rangle_{\nu=1}}{\langle \prod_{\lambda_k}(\lambda_k+m)\rangle_{\nu=0}}$ because of the exact zero mode.
- √ The equality $\Sigma^{(1)} = \Sigma^{(2)}$ requires a subtle reshuffling of the eigenvalues: For $\nu = 1$ the eigenvalues are shifted by half a level spacing w.r.t. $\nu = 0$ in order to satisfy the chiral Ward identity.
- √ For $\mu \neq 0$ we have that $\Sigma^{(1)} \neq \Sigma^{(2)}$ because the partition function is not correctly normalized. Lehner-Ohtani-JV-Wettig-2008

Lesson 10: Tests of Algorithms

Because random matrix models are exactly solvable and show all essential features related to nonhermiticity, they are an ideal tool to test algorithms.

- √ The density of states algorithm has developed and tested using chRMT at $\mu \neq 0$.

$$\frac{Z_{\rm QCD}}{Z_{|\rm QCD|}} = \int d\phi e^{i\phi} \frac{\langle \delta(\phi - \theta) |\det(D+m)|^2 \rangle}{\langle |\det(D+m)|^2 \rangle}$$

  Ambjorn-Anagnostopoulos-Nishimura-JV-2002, Ejiri-2007, Splittorff-JV-2007

- √ The radius of convergence has been determined for algorithms that rely on Taylor expansion or analytical continuation in μ.

  Stephanov-2006

Conclusions

- √ Exact analytical results for QCD at nonzero chemical potential can be derived from chiral random matrix theory – correlations of Dirac eigenvalues on the microscopic scale.

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- √ Perhaps the most surprising result of this enterprise is understanding the failure of the Banks-Casher relation. Kim Splittorff will explain this in the afternoon.
- √ Because exponential cancellations are required to calculate the QCD partition function at $\mu \neq 0$, one might worry that at finite volume these cancellations cannot be achieved because of $!/V$ -corrections. One way out might be that the cancellations are exact already at finite volume. This is what happens in chRMT, but we don't know if this is a general feature.

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- The obvious way to calculate oscillating integrals is to do a rotation in the complex plane. The natural choice is to evaluate the partition function for imaginary chemical potential.