Crystalline Condensate in the Chiral Gross-Neveu Model

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with : Gökçe Başar, arxiv:0803.1501, Phys Rev Lett **100**, 200404 (2008)
arxiv:0806.xxxx

Outline

- Gross-Neveu Model phase diagram
- chiral GN_2, or NJL_2
- gap equation -> nonlinear Schrödinger equation
- implications for NJL_2 phase diagram
- conclusions and outlook

Gross-Neveu Model

(Gross & Neveu, 1974)

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi}\, i\, \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2$$

GN$_2$

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}\, i\, \partial\!\!\!/ \psi + \frac{g^2}{2} \left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i \gamma^5 \psi\right)^2\right]$$

χGN$_2$ or NJL$_2$

- renormalizable
- asymptotically free
- large N$_F$
- dynamical mass generation

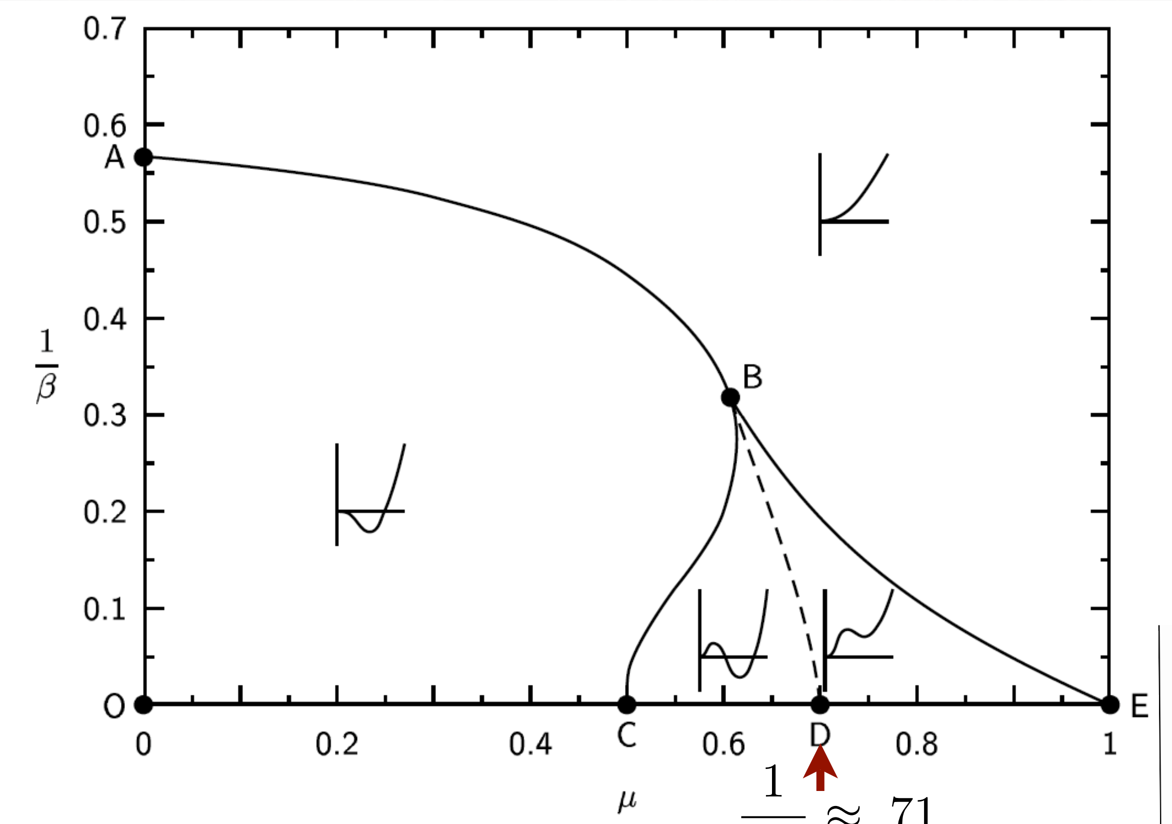
Dashen/Hasslacher/Neveu, 1975

- self-bound baryonic states

$$m_B = \frac{2}{\pi} m$$

- (T, μ) phase diagram for NJL$_2$?

Phase diagram of Gross-Neveu model



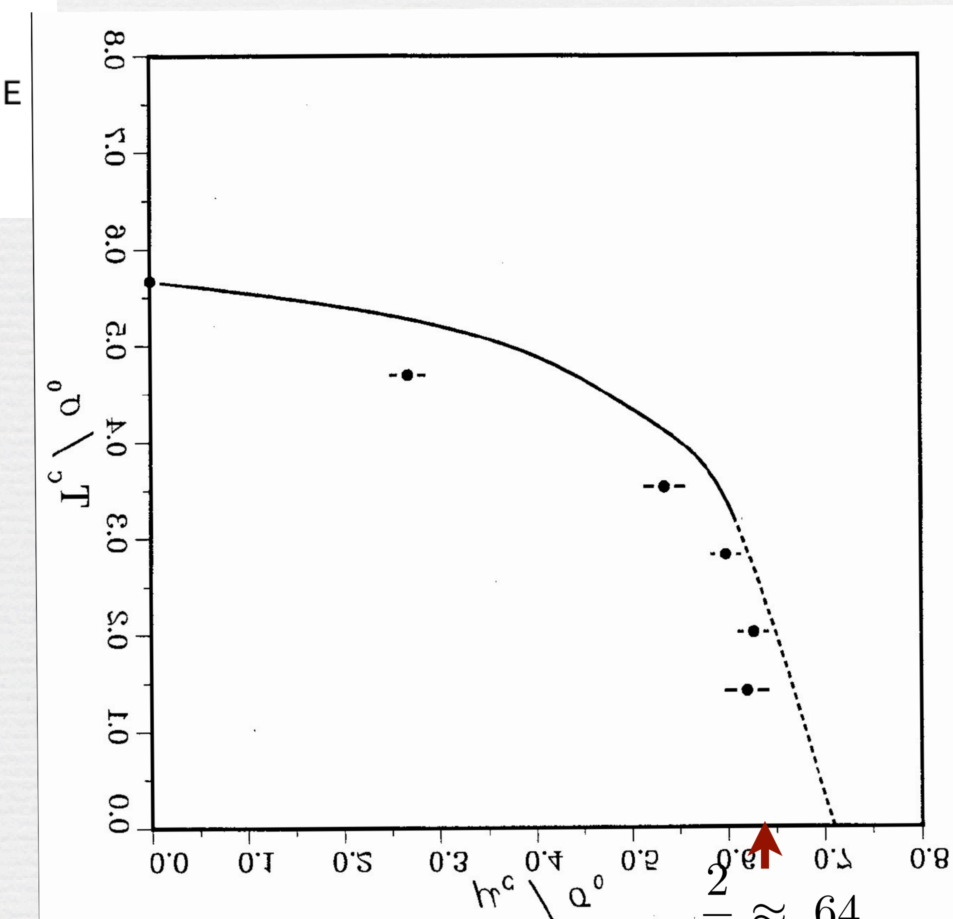
$\frac{1}{\sqrt{2}} \approx .71$

uniform condensate

Wolff, 1985

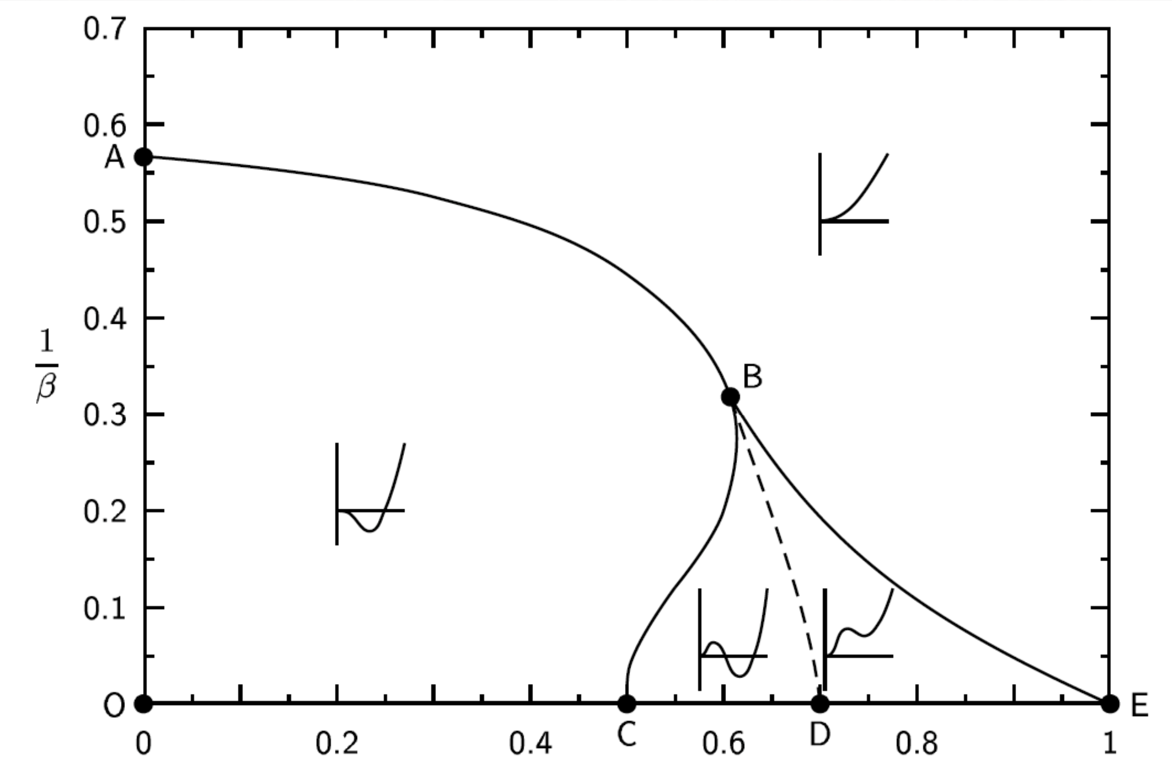
lattice analysis of GN_2

Karsch et al 1986



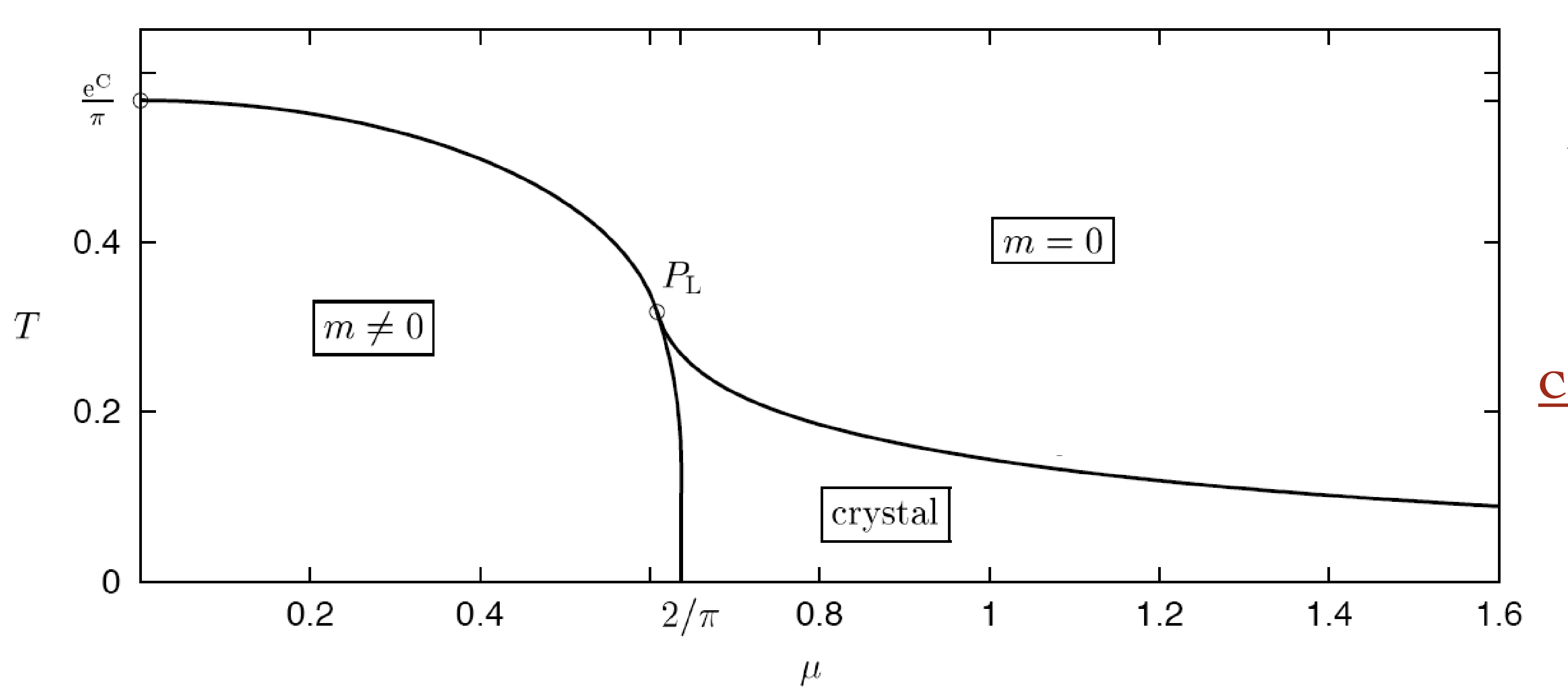
$\frac{2}{\pi} \approx .64$

Phase diagram of Gross-Neveu model



uniform condensate

Wolff, 1985

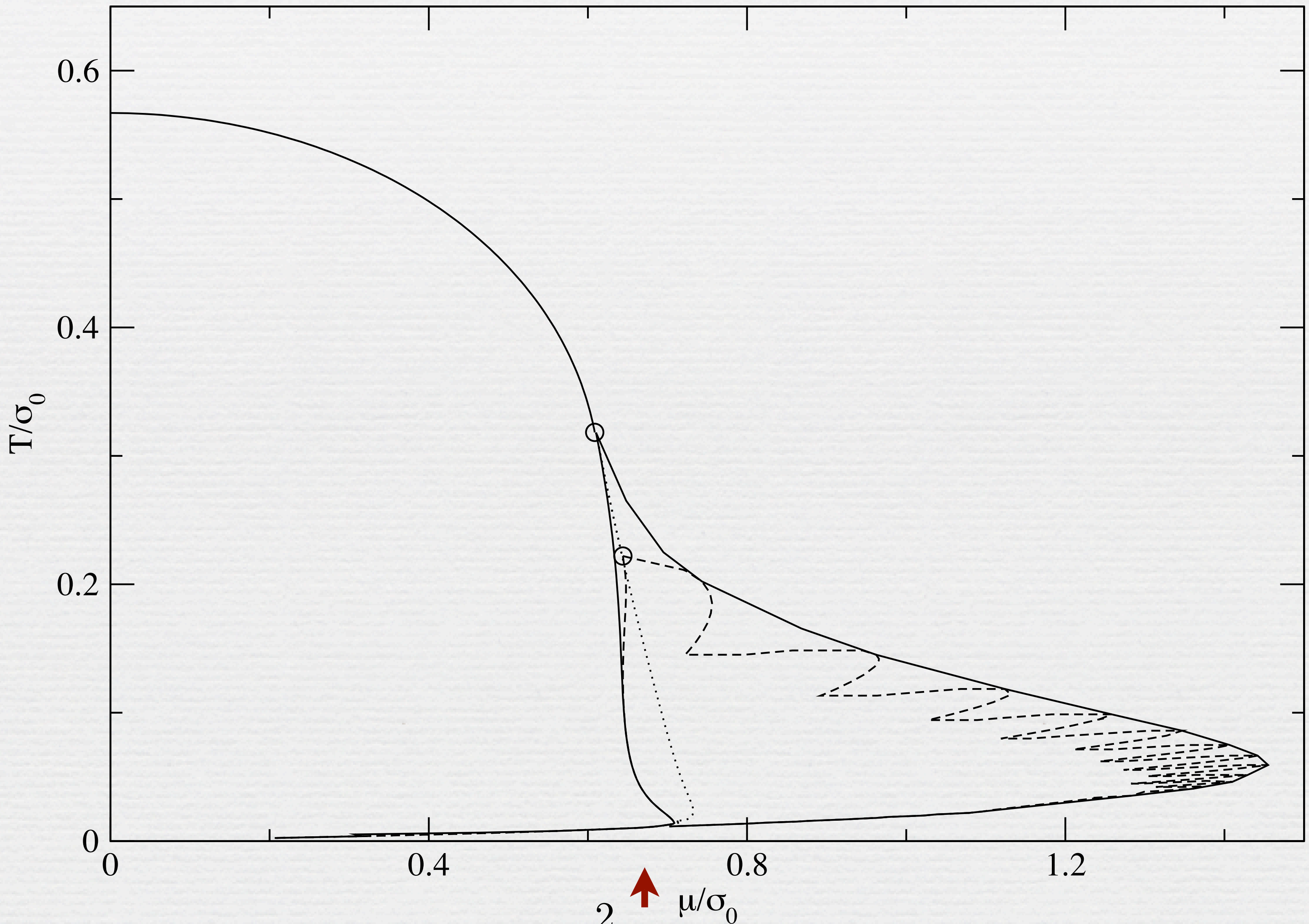


Thies & Urlichs, 2005

periodic, crystalline, phase

lattice GN_2 model

de Forcrand/Wenger 2006



$$\frac{2}{\pi} \approx .64$$

Gap Equation Approach

partition function $$Z = \int \mathcal{D}\psi \, \exp\left\{ -\int \left[\bar{\psi} \partial\!\!\!/ \psi - \frac{g^2}{2} \left(\bar{\psi}\psi \right)^2 \right] \right\}$$

effective potential $$V[\sigma] = \frac{1}{2g^2N} \sigma^2 - \ln \det \left[\partial\!\!\!/ + \sigma \right]$$

gap equation $$\frac{\sigma(x)}{g^2 N} = \frac{\delta}{\delta\sigma(x)} \ln \det \left[\partial\!\!\!/ + \sigma(x) \right]$$

Dashen/Hasslacher/Neveu, 1975 : inverse scattering

$\sigma^2 \pm \sigma$ reflectionless potentials

kink state $$\sigma(x) = m \tanh(mx)$$

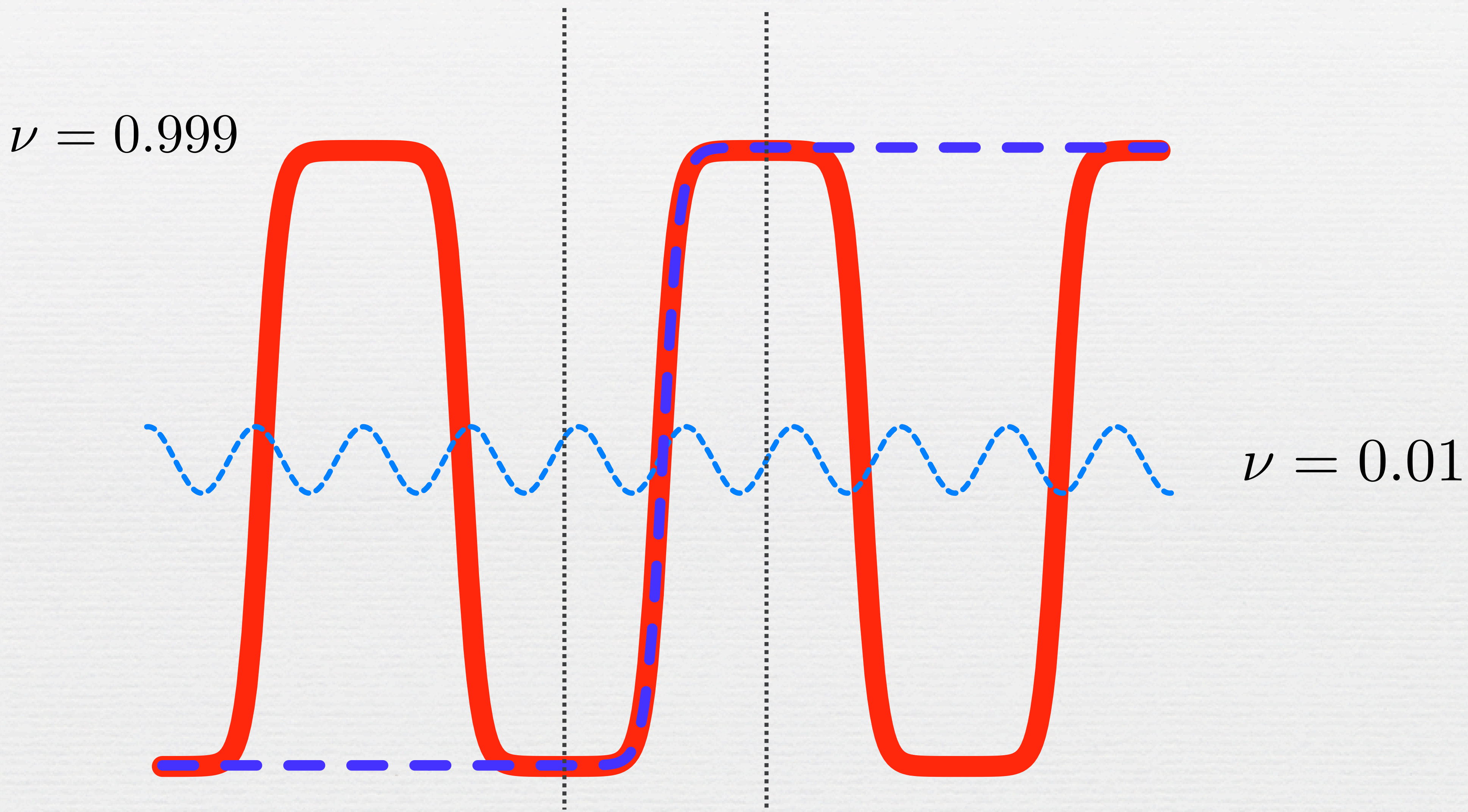
Thies/Urlichs, 2005 : crystalline condensate at finite density

$\sigma^2 \pm \sigma$ finite-gap potentials

kink crystal $$\sigma(x) = \nu\, m \, \frac{\mathrm{sn}(mx;\nu)\mathrm{cn}(mx;\nu)}{\mathrm{dn}(mx;\nu)}$$

kink crystal

$$\sigma(x) = \nu\, m\, \frac{\mathrm{sn}(mx;\nu)\mathrm{cn}(mx;\nu)}{\mathrm{dn}(mx;\nu)}$$



Condensed matter analogues

trans-polyacetylene = GN_2 Su, Schrieffer, Heeger, 1979

dimerization = discrete chiral symmetry of GN model

polaron crystal Brazovskii, 1980; Horovitz, 1981

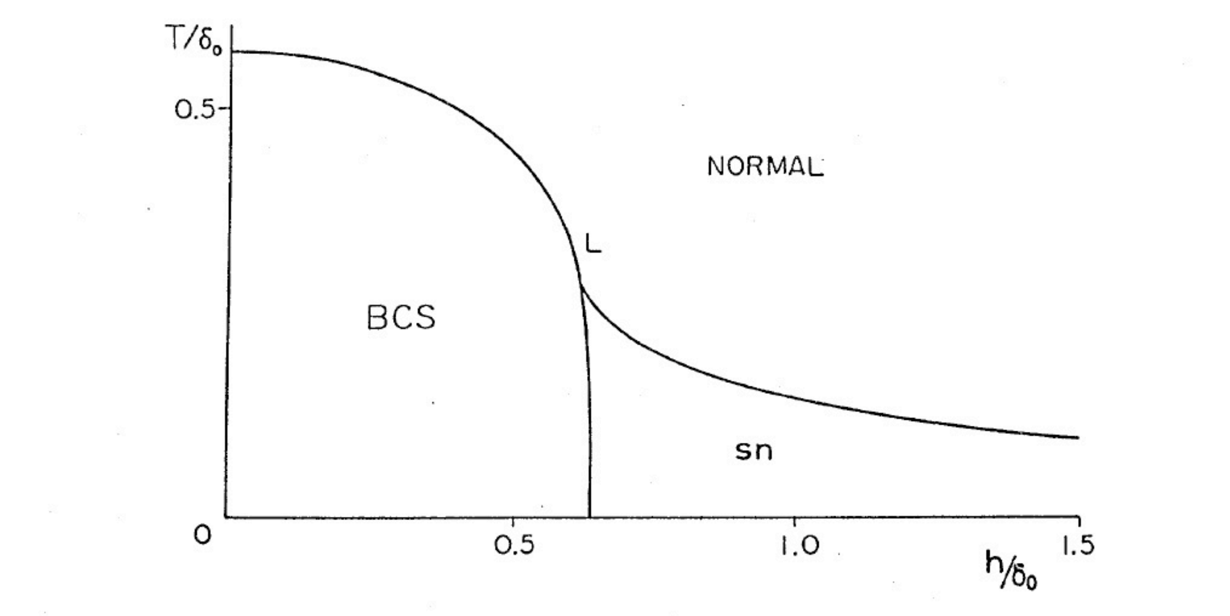
1 dim. Peierls-Fröhlich electron-phonon model

Mertsching/Fischbeck, 1981; Belokolos et al, 1981

inhomogeneous superconductors and ferromagnetism

Machida/Nakanishi, 1984

magnetic field = μ



Hartree-Fock approach to GN$_2$ gap equation

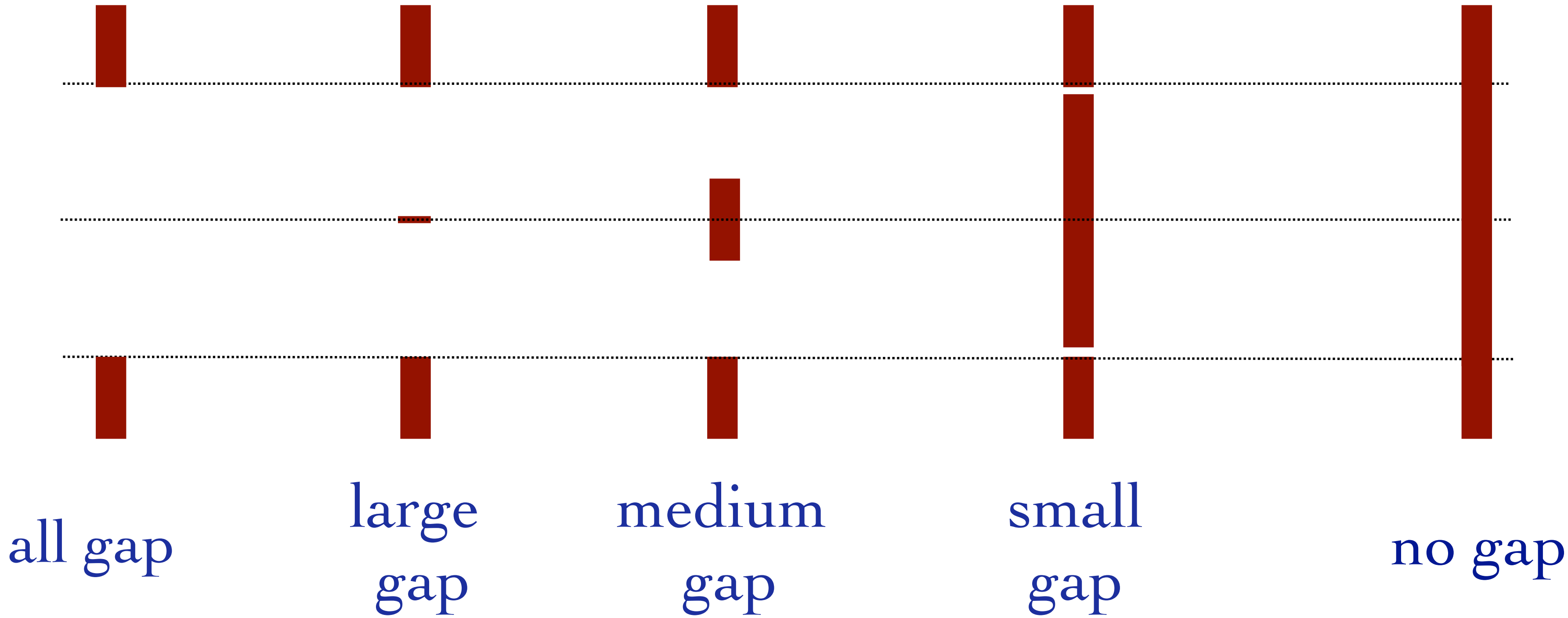
Bogoliubov/De Gennes equation : $H\psi = E\psi$

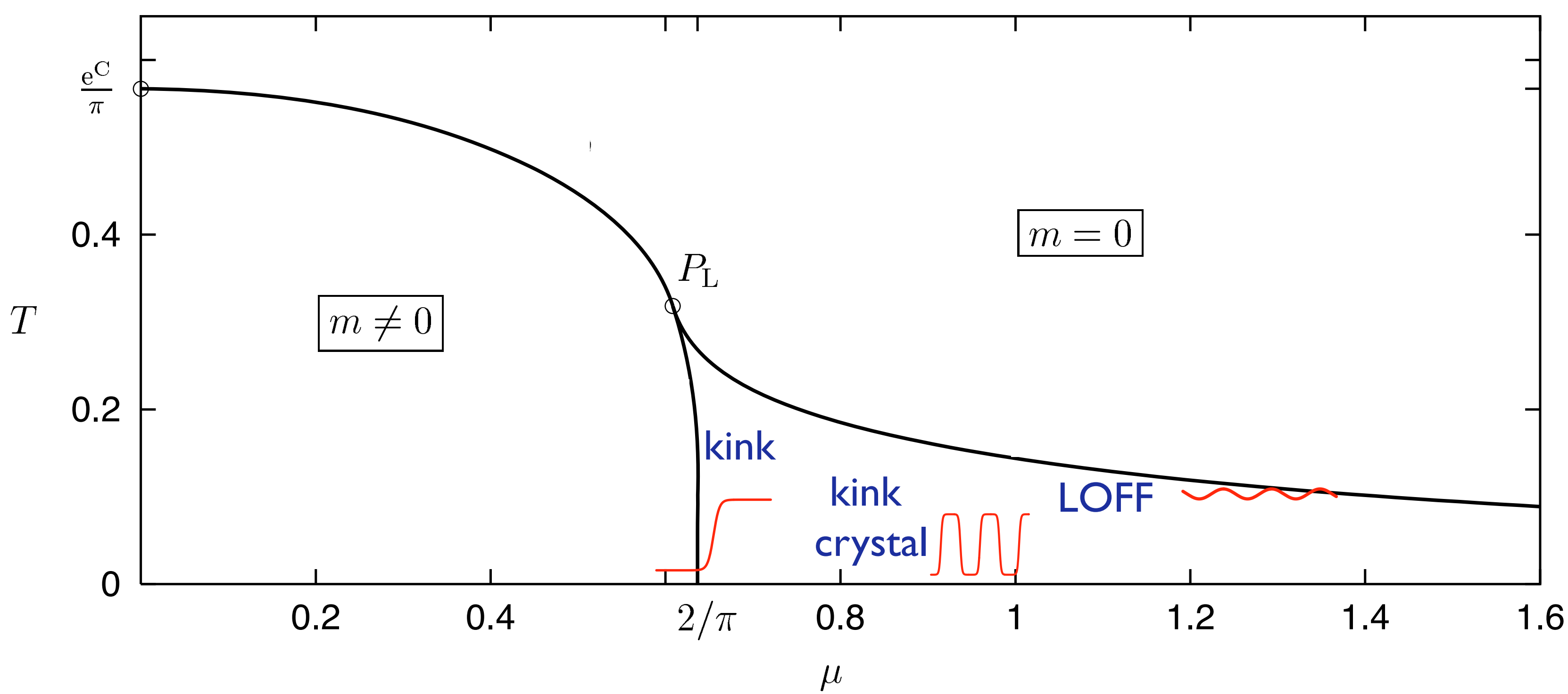
single particle states : $$\left(-i\gamma^5 \frac{d}{dx} + \gamma^0 \sigma(x)\right)\psi = E\psi$$

self-consistent condensate : $$\sigma(x) = -Ng^2 \langle \bar{\psi}(x)\psi(x)\rangle$$

$$\sigma(x) = \nu\, m\, \frac{\mathrm{sn}(mx;\nu)\mathrm{cn}(mx;\nu)}{\mathrm{dn}(mx;\nu)}$$

soluble : "finite-gap" spectrum


all gap
large gap
medium gap
small gap
no gap


$\frac{e^C}{\pi}$
0.4
0.2
0
T
P_L
$m \neq 0$
$m = 0$
kink
kink crystal
LOFF
0.2
0.4
$2/\pi$
0.8
1
1.2
1.4
1.6
μ

chiral Gross-Neveu or NJL$_2$

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - \frac{g^2}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\psi\right)^2\right]$$

scalar condensate σ pseudoscalar condensate π

$$\mathcal{L}_{\text{eff}} = \bar{\psi}\partial\!\!\!/\psi + \bar{\psi}\left(\sigma - i\pi\gamma^5\right)\psi + \frac{1}{2g^2}\left(\sigma^2 + \pi^2\right)$$

gap equation(s)

$$\frac{\sigma(x)}{g^2 N} = \frac{\delta}{\delta\sigma(x)} \ln \det\left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5\pi(x)\right)\right]$$

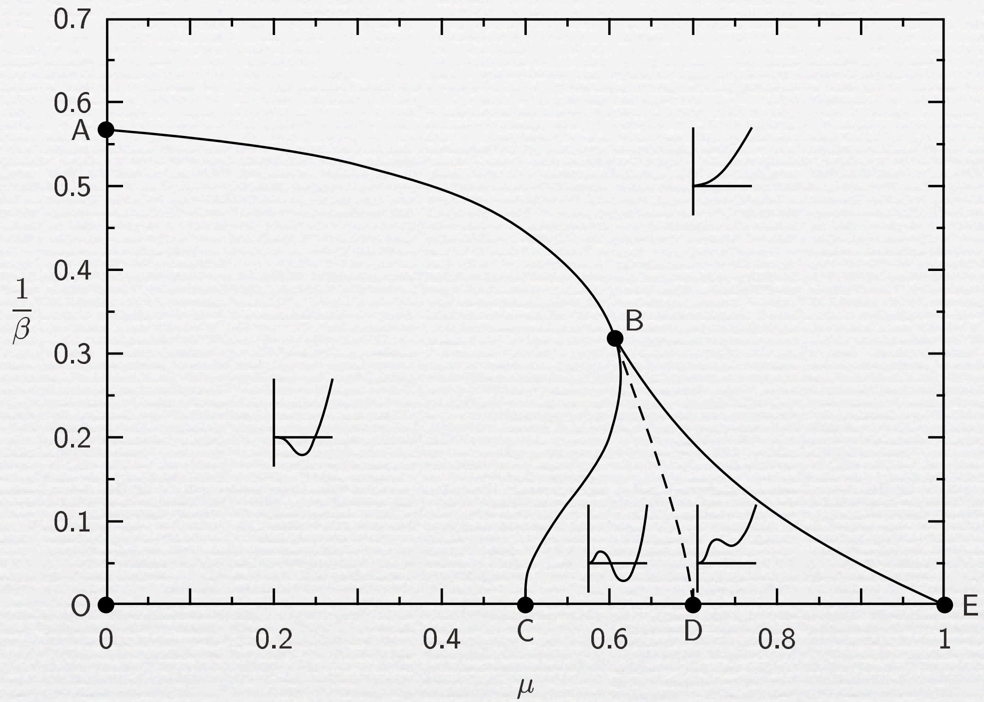
$$\frac{\pi(x)}{g^2 N} = \frac{\delta}{\delta\pi(x)} \ln \det\left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5\pi(x)\right)\right]$$

Hartree-Fock

$$\left(-i\gamma^5\frac{d}{dx} + \gamma^0\sigma(x) + i\gamma^1\pi(x)\right)\psi = E\psi$$

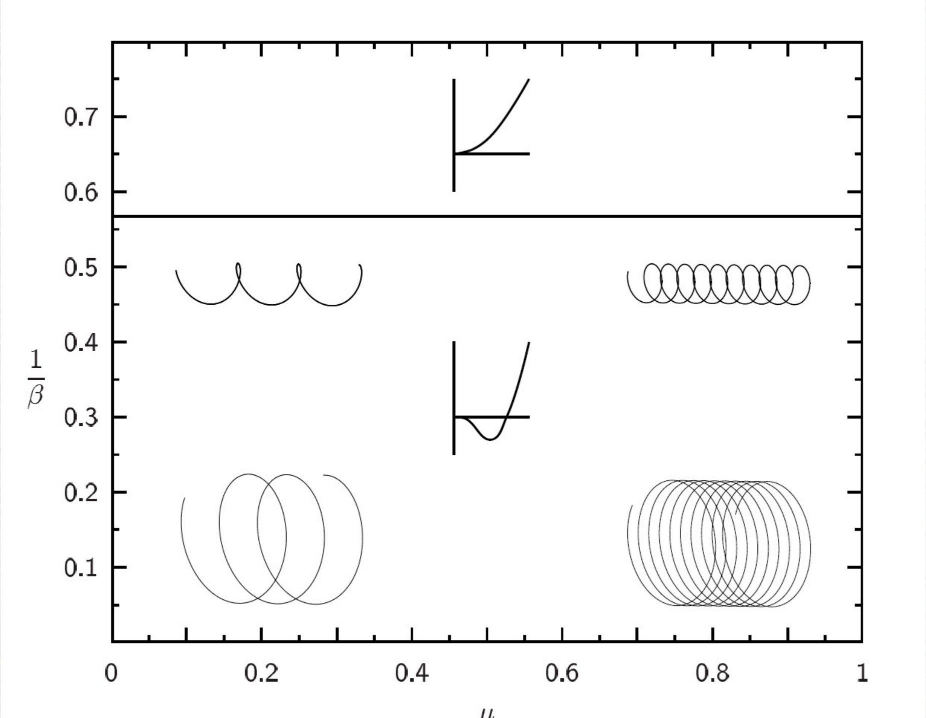
$$\sigma(x) = -Ng^2\langle\bar{\psi}(x)\psi(x)\rangle \qquad \pi(x) = -Ng^2\langle\bar{\psi}(x)i\gamma^5\psi(x)\rangle$$

phase diagram of chiral Gross-Neveu or NJL_2



Wolff, 1985

Barducci et al, 1995



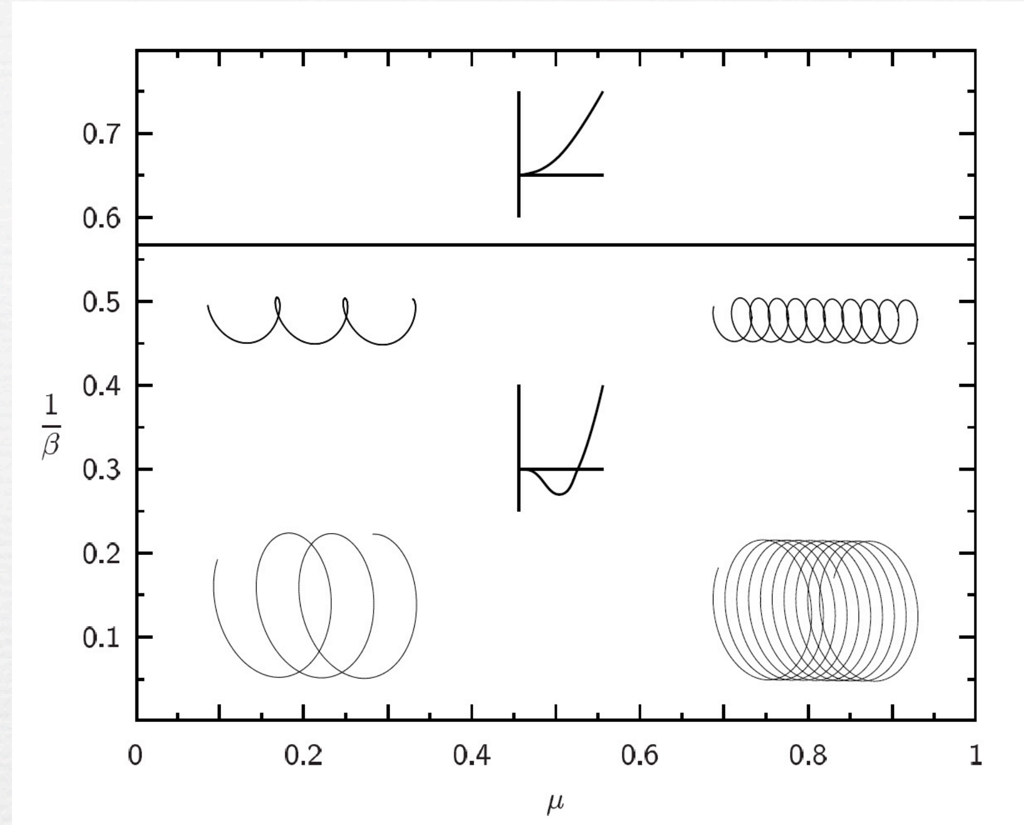
Schön/Thies, 2000

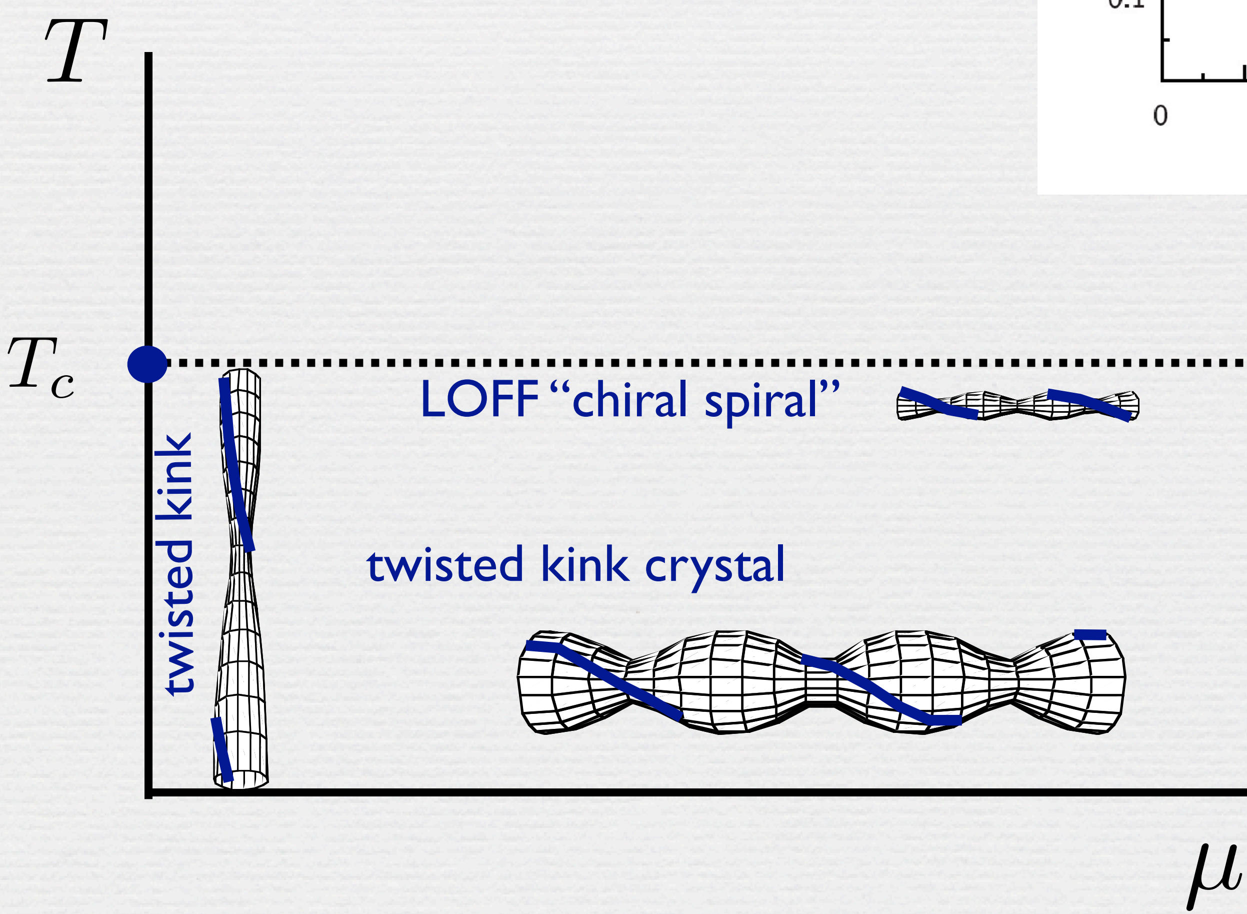
"chiral spiral"

$$\sigma(x) - i\pi(x) = A\, e^{2i\mu\, x}$$

Schön/Thies, 2000

“chiral spiral”





Basar, GD, 2008

twisted kink crystal

gap equation for NJL$_2$

Shei, 1976 : inverse scattering : reflectionless Dirac spectrum

$$\sigma(x) = m\left[\cos^2(\theta/2) + \sin^2(\theta/2)\tanh\left(m\,\sin\left(\theta/2\right)\,x\right)\right]$$

$$\pi(x) = \frac{m}{2}\,\sin(\theta)\left[1 - \tanh\left(m\,\sin\left(\theta/2\right)\,x\right)\right]$$

twisted kink state $\qquad 0 \leq \theta \leq 2\pi$

Basar & GD, 2008 : twisted kink crystal
finite-gap Dirac spectrum

twisted kink crystal

$$\sigma(x) - i\pi(x) = -A\frac{\sigma(A\,x + i\mathbf{K}' - i\theta/2)}{\sigma(A\,x + i\mathbf{K}')\sigma(i\theta/2)}\,e^{[iA\,x(-i\,\zeta(i\theta/2)+i\,\mathrm{ns}(i\theta/2))+i\,\theta\eta_3/2]}$$

$$0 \leq \theta \leq 4\mathbf{K}'$$

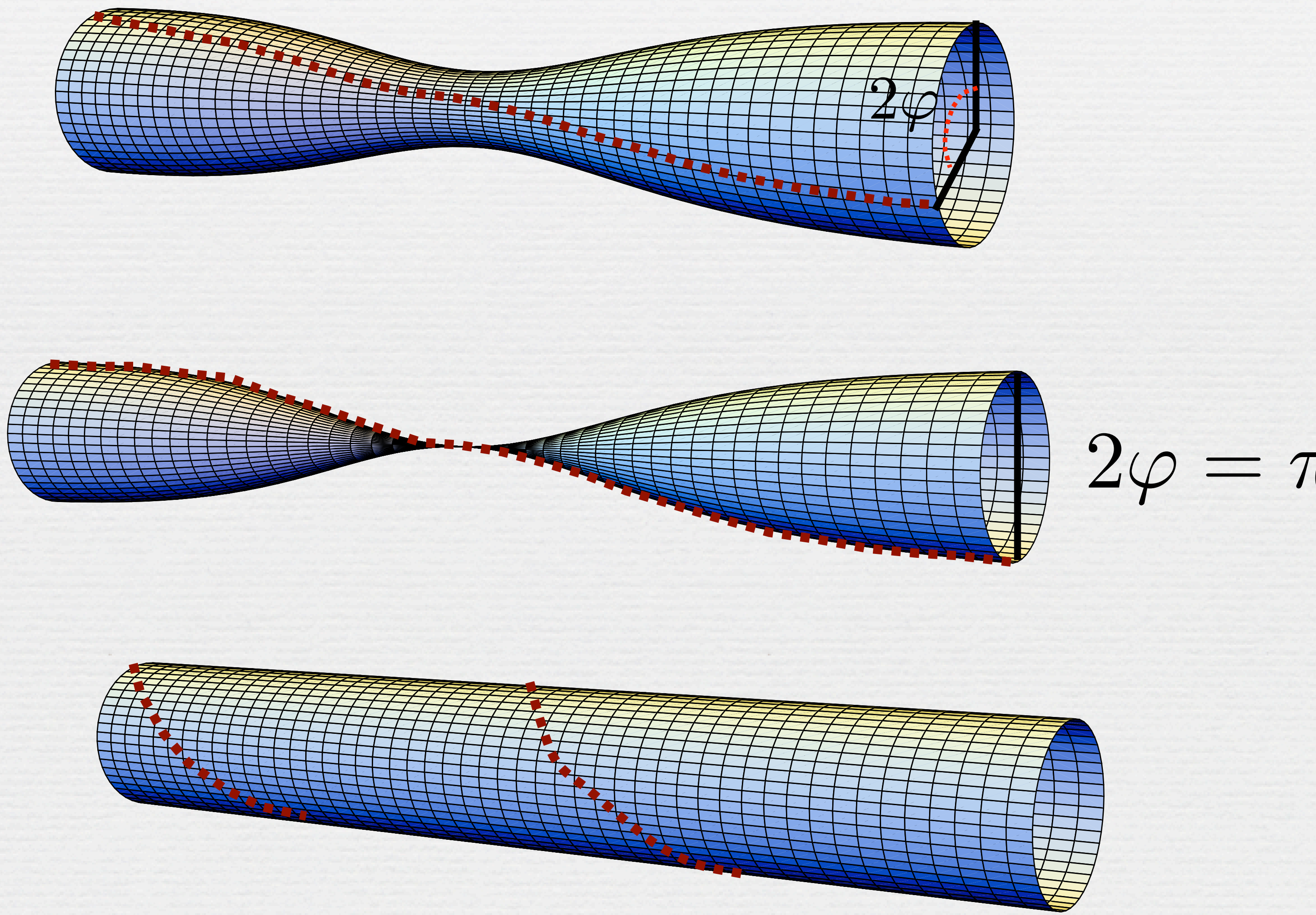
twisted kink crystal

$$\Delta(x) = \sigma(x) - i\pi(x) \qquad\qquad \Delta(x) = M(x)\, e^{i\chi(x)}$$

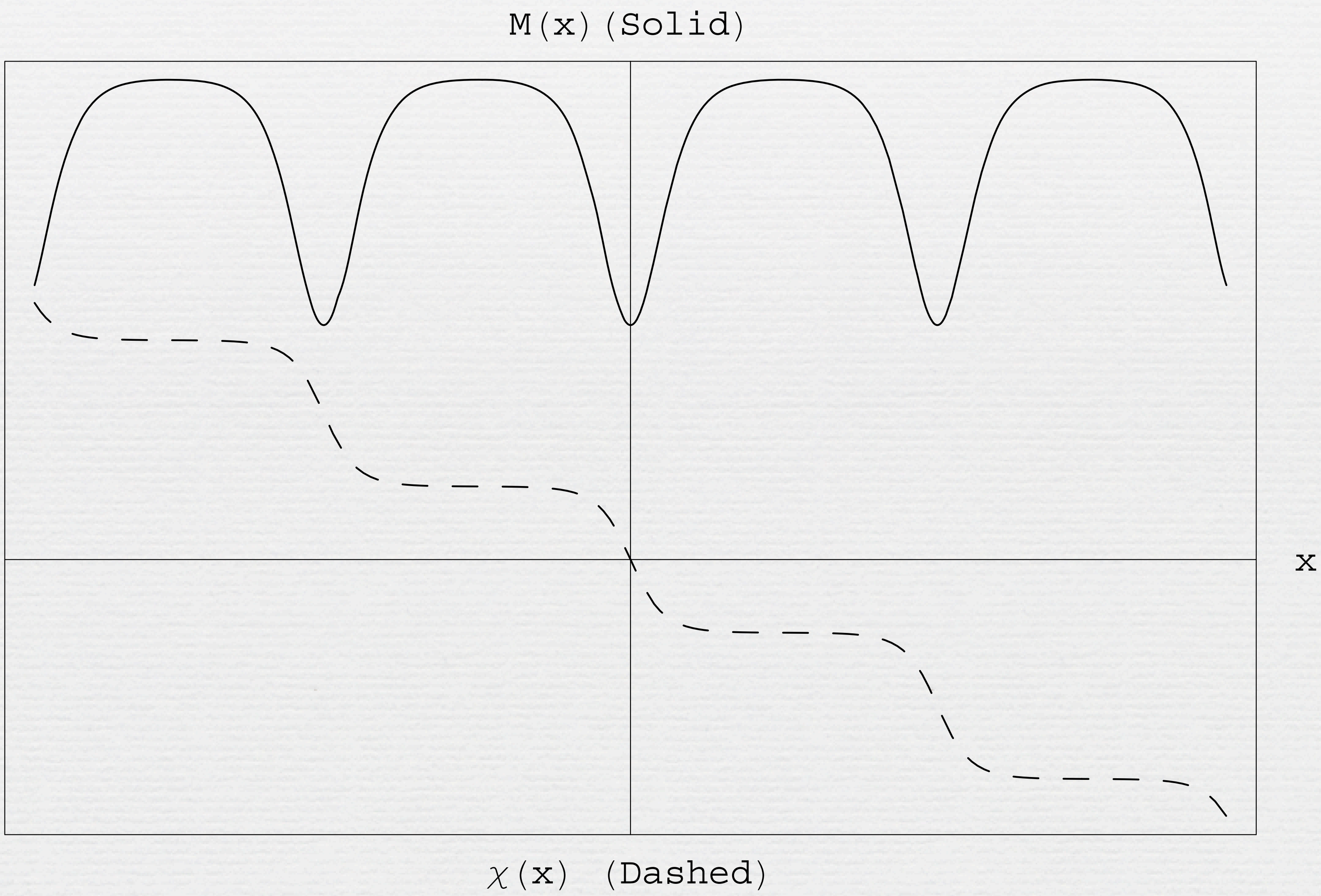
complex kink

real kink

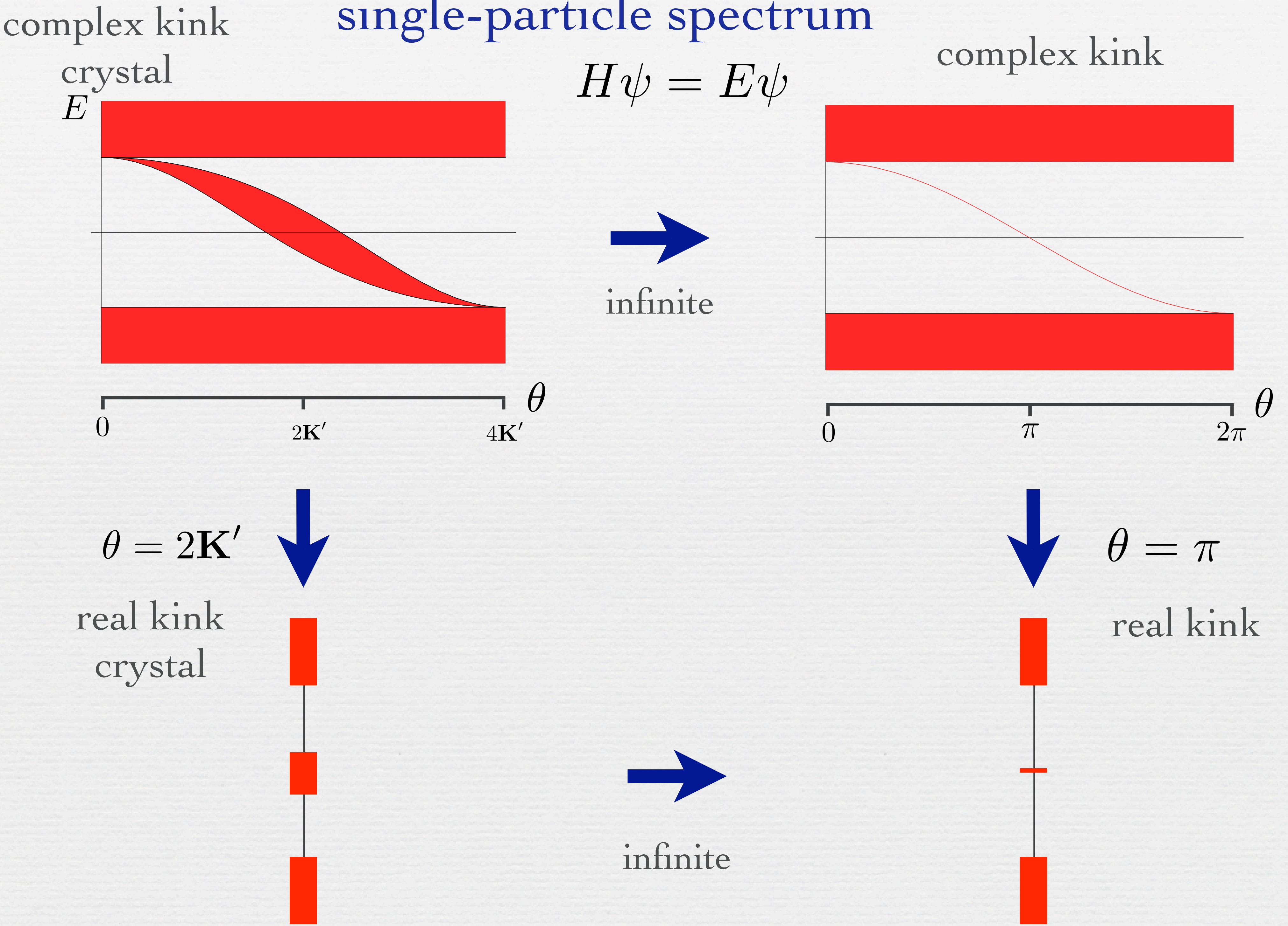
"chiral spiral"



$$\Delta(x) = \sigma(x) - i\pi(x) \qquad \Delta(x) = M(x)\, e^{i\chi(x)}$$



single-particle spectrum



solving the (complex) gap equation

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta \Delta^*(x)} \ln \det \left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5 \pi(x) \right) \right]$$

$$H = -i\gamma^5 \frac{d}{dx} + \gamma^0 \sigma(x) + i\gamma^1 \pi(x) = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}$$

Bogoliubov/de Gennes hamiltonian

resolvent : Gorkov Green's function

$$R(x; E) \equiv \langle x | \frac{1}{H - E} | x \rangle$$

spectral function

$$\rho(E) = \frac{1}{\pi} \mathrm{Im} \int dx \, \mathrm{tr}\, R(x; E + i\epsilon)$$

resolvent

$$R(x;E) \equiv \langle x | \frac{1}{H-E} | x \rangle$$

$$R = R^\dagger \qquad \mathrm{tr}\,(R\,\sigma_3) = 0 \qquad \det R = \frac{1}{4}$$

$$R'\sigma_3 = i\left[\begin{pmatrix} E & -\Delta \\ \Delta^* & -E \end{pmatrix}, R\,\sigma_3\right]$$

Dik'ii equation

Eilenberger equation

two views of gap equation

$$\ln\det\left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5\pi(x)\right)\right] = \int dE\,\rho(E)\,\frac{1}{\beta}\ln\left(1+e^{-\beta(E-\mu)}\right)$$

$$\Delta(x) = -Ng^2\int dE\,\mathrm{tr}\left[\left(\gamma^0\left(\mathbf{1}+\gamma^5\right)\right)R(x;E)\right]$$

$$R(x;E) = \mathcal{N}(E)\begin{pmatrix} a(E)+|\Delta(x)|^2 & b(E)\Delta(x) - i\Delta'(x) \\ b(E)\Delta^*(x) + i\Delta'^*(x) & a(E)+|\Delta(x)|^2 \end{pmatrix}$$

$$\Delta'' - 2|\Delta|^2\,\Delta + i\,(b-2E)\,\Delta' - 2\,(a-Eb)\,\Delta = 0$$

nonlinear Schrödinger equation (NLSE)

gap equation $\longrightarrow$ NLSE

$$\frac{\Delta(x)}{g^2 N} = \frac{\delta}{\delta \Delta^*(x)} \ln \det \left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5 \pi(x)\right)\right]$$

$\Big\downarrow$ ansatz, from gap equation

$$R(x;E) = \mathcal{N}(E) \begin{pmatrix} a(E) + |\Delta(x)|^2 & b(E)\Delta(x) - i\Delta'(x) \\ b(E)\Delta^*(x) + i\Delta'^*(x) & a(E) + |\Delta(x)|^2 \end{pmatrix}$$

$\Big\downarrow$ Eilenberger equation

$$\Delta'' - 2|\Delta|^2 \Delta + i\,(b - 2E)\,\Delta' - 2\,(a - Eb)\,\Delta = 0$$

NLSE : exactly soluble

NLSE : examples

$$\Delta'' - 2|\Delta|^2\,\Delta + i\,(b - 2E)\,\Delta' - 2\,(a - Eb)\,\Delta = 0$$

DHN kink $\Delta = \tanh(x)$ $\quad\Delta'' - 2\Delta^3 + 2\Delta = 0$

Thies's kink crystal $\Delta(x) = \nu\,\dfrac{\mathrm{sn}(x;\nu)\mathrm{cn}(x;\nu)}{\mathrm{dn}(x;\nu)}$

$$\Delta'' - 2\Delta^3 - 2(\nu - 2)\Delta = 0$$

Shei's twisted kink $\Delta(x) = \dfrac{\cosh\left(\sin(\theta/2)\,x - i\theta/2\right)}{\cosh\left(\sin(\theta/2)\,x\right)}\,e^{i\theta/2}$

$$\Delta'' - 2|\Delta|^2\Delta - 2i\,\cos(\theta/2)\Delta' + 2\Delta = 0$$

chiral spiral $\Delta = e^{2i\mu x}$ $\quad\Delta'' - 2|\Delta|^2\Delta - 4i\,\mu\Delta' - 2(2\mu^2 - 1)\Delta = 0$

twisted kink crystal $\Delta = A\,\dfrac{\sigma(A\,x + i\mathbf{K}' - i\theta/2)}{\sigma(A\,x + i\mathbf{K}')}\,e^{i\lambda\,x}$

$$\Delta'' - 2\,|\Delta|^2\,\Delta - i\,\left(2A\,i\,\mathrm{ns}(i\theta/2)\right)\Delta' - A^2\left(3\,\mathcal{P}(i\theta/2) - \mathrm{ns}^2(i\theta/2)\right)\Delta = 0$$

results from resolvent/NLSE approach

exact solution of gap equation

exact resolvent and spectral function

contains all previous cases, and new complex crystal

new exact crystalline solution to Eilenberger equation

new exact solution to Bogoliubov/de Gennes equation

Density of States

BdG equation : $H\psi = E\psi$ $\qquad H(x) = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}$

quasi-periodicity : $\Delta(x+L) = e^{2i\varphi}\Delta(x)$

$$H(x+L) = e^{i\varphi\gamma^5} H(x) e^{-i\varphi\gamma^5}$$

exact single particle "Bloch" states :

$$\psi_\pm(x+L) = e^{\pm ikL} e^{i\varphi\gamma^5}\psi_\pm(x)$$

density of states:

$$\frac{dk}{dE} = \frac{1}{L}\int_{\text{period}} \text{tr}\, R(x;E)$$

Implications for phase diagram of NJL_2

exact spectral function : exact free energy

(approximate) Ginzburg-Landau approach:

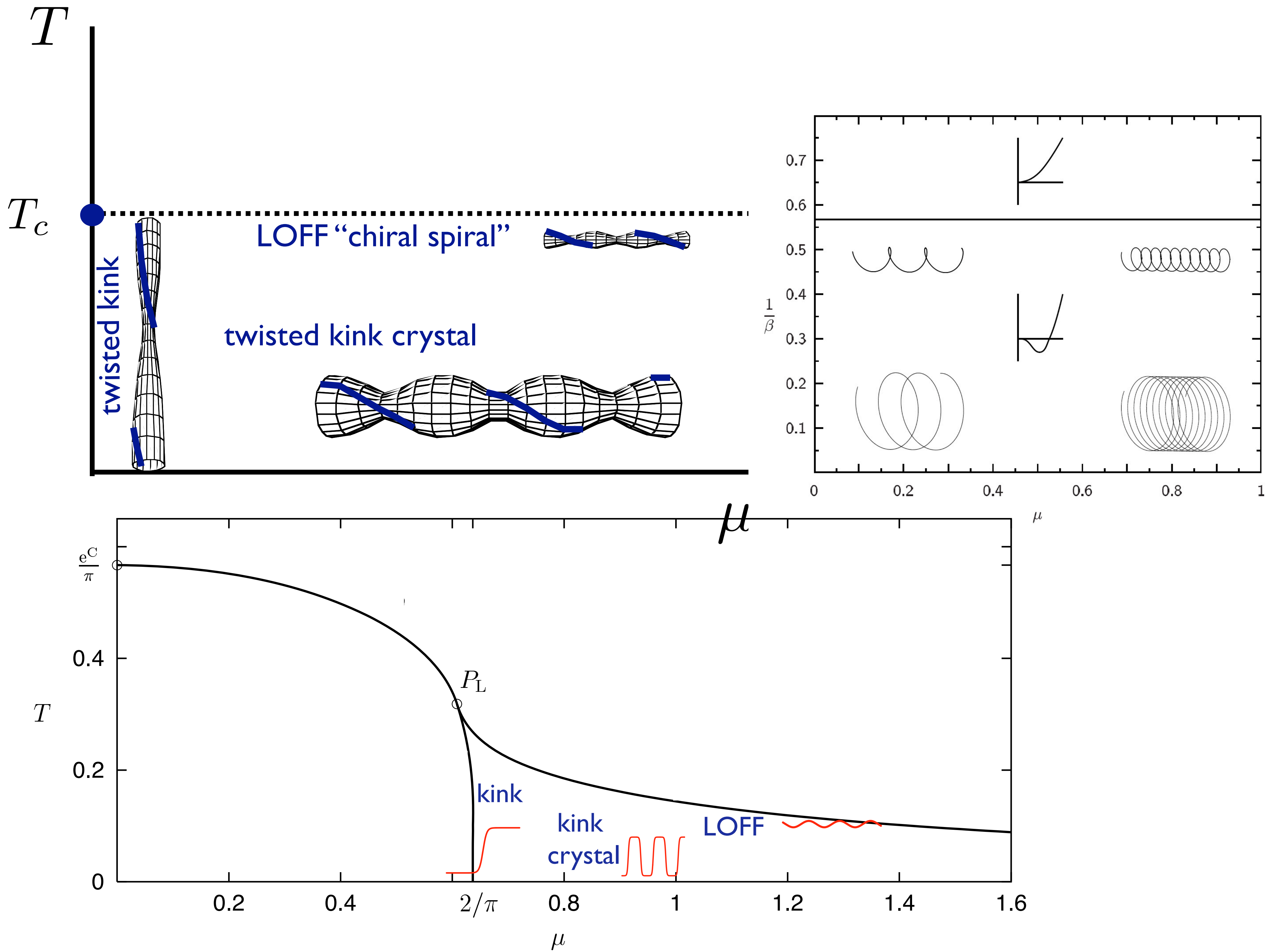
$$\begin{aligned}\mathcal{L}_{GL} &= c_0 + c_2|\Delta|^2 + c_3 \mathrm{Im}\left[\Delta(\Delta')^*\right] + c_4\left[|\Delta|^4 + |\Delta'|^2\right] \\ &\quad + c_5 \mathrm{Im}\left[\left(\Delta'' - 3|\Delta|^2\Delta\right)(\Delta')^*\right] \\ &\quad + c_6\left[2|\Delta|^6 + 8|\Delta|^2|\Delta'|^2 + 2\mathrm{Re}\left((\Delta')^2(\Delta^*)^2\right) + |\Delta''|^2\right]\end{aligned}$$

real (GN) case : tricritical point: $c_2=c_4=0$

complex (NJL) case : tricritical point: $c_2=c_3=0$

$$\Delta'' - 2|\Delta|^2\Delta - i\frac{c_3}{c_4}\Delta' - \frac{c_2}{c_4}\Delta = 0$$

NLSE !



T
T_c
LOFF “chiral spiral”
twisted kink
twisted kink crystal
μ
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1/β
0
0.2
0.4
0.6
0.8
1
μ
e^C/π
P_L
0.4
0.2
0
T
kink
kink crystal
LOFF
0.2
0.4
2/π
0.8
1
1.2
1.4
1.6
μ

gap equation to all-orders in Ginzburg-Landau

motivation : beyond 1 dim?

$$\mathcal{L}_{\rm GL} = \sum_n c_n(T,\mu)\, a_n(x)$$

$$\begin{aligned}\mathcal{L}_{\rm GL} \quad = \quad & c_0 + c_2|\Delta|^2 + c_3 \mathrm{Im}\left[\Delta(\Delta')^*\right] + c_4\left[|\Delta|^4 + |\Delta'|^2\right] \\ & + c_5 \mathrm{Im}\left[\left(\Delta'' - 3|\Delta|^2\Delta\right)(\Delta')^*\right] \\ & + c_6\left[2|\Delta|^6 + 8|\Delta|^2|\Delta'|^2 + 2\mathrm{Re}\left((\Delta')^2(\Delta^*)^2\right) + |\Delta''|^2\right] + \dots\end{aligned}$$

NLSE → entire hierarchy satisfied

$$[a_n(x)]_{NLSE} = \alpha_n |\Delta(x)|^2 + \beta_n$$

Conclusions

- complete solution of gap equation for chiral GN_2/NJL_2
- gap equation reduced to NLSE
- Ginzburg-Landau : chiral crystalline phase

- full, exact, thermodynamics & phase diagram ?
- lattice NJL_2 at finite density and temperature ?
- massive NJL_2 ?
- higher dimensions ?

verifying the gap equation

two views of gap equation

$$\ln\det\left[\partial\!\!\!/ + \left(\sigma(x) - i\gamma^5\pi(x)\right)\right] = \int dE\,\rho(E)\,\frac{1}{\beta}\ln\left(1 + e^{-\beta(E-\mu)}\right)$$

$$\Delta(x) = -Ng^2\int dE\,\mathrm{tr}\left[\left(\gamma^0\left(\mathbf{1}+\gamma^5\right)\right)R(x;E)\right]$$

$$R(x;E) = \mathcal{N}(E)\begin{pmatrix} a(E) + |\Delta(x)|^2 & b(E)\Delta(x) - i\Delta'(x) \\ b(E)\Delta^*(x) + i\Delta'^*(x) & a(E) + |\Delta(x)|^2 \end{pmatrix}$$

$$\int \mathcal{N}(E)dE = 0 \quad \longrightarrow \quad \frac{\theta}{4\mathbf{K}'} = \text{filling fraction}$$

cf. Shei, 1976 $\quad \dfrac{\theta}{2\pi} = \text{filling fraction}$

$$Ng^2\int \mathcal{N}(E)E\,dE = 1$$

vacuum gap equation

properties of resolvent

$$R(x;E) \equiv \langle x| \frac{1}{H-E} |x\rangle$$

resolvent in terms of independent spinor solutions

$$R = \frac{1}{2W}\left(\psi_1\, \psi_2^T + \psi_2\, \psi_1^T\right) \sigma_1$$

Wronskian $\quad W \equiv -i\, \psi_1^T \sigma_2 \psi_2$

$$R = R^\dagger \qquad \mathrm{tr}\,(R\,\sigma_3) = 0 \qquad \det R = \frac{1}{4}$$

$$R = \frac{1}{2}\begin{pmatrix} a & b \\ b^* & a \end{pmatrix} \qquad a^2 - |b|^2 = 1$$

$$H\psi = E\psi \qquad \psi' = i\sigma_3 \begin{pmatrix} E & -\Delta(x) \\ -\Delta^*(x) & E \end{pmatrix} \psi \equiv \mathcal{M}\,\psi$$

$$R'\sigma_3 = [\mathcal{M}, R\,\sigma_3]$$

Dik'ii equation

Eilenberger equation