A DISCREPANCY THEOREM FOR HARMONIC FUNCTIONS ON THE d DIMENSIONAL SPHERE WITH APPLICATIONS TO SCATTERINGS OF POINT CLOUDS

By

S.B. Damelin

IMA Preprint Series # 2093

(February 2006)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

A discrepancy theorem for harmonic functions on the d dimensional sphere with applications to scatterings of point clouds

S. B. Damelin*

2 February, 2006

Abstract

Let $d \geq 2$ be an integer, $S^d \subset \mathbb{R}^{d+1}$ the unit sphere and $d\sigma$ a finite signed measure on $\mathbb{R}^{d+1}$ whose positive and negative parts are supported on S^d. In this paper, we derive an error estimate for the quantity $\left|\int_{S^d} f d\sigma\right|$, for a class of harmonic functions $f : \mathbb{R}^{d+1} \to \mathbb{R}$. Our error estimate involves 2 sided bounds for a Newtonian potential with respect to $d\sigma$ away from its support. In particular, our main result allows us to study quadrature errors, for scatterings on the sphere with given mesh norm.

Keywords and Phrases: Discrepancy, Harmonic, Koksma-Hlawka, Numerical Integration, Potential, Quadrature, Measure, Smoothness, Sphere, Pseudo-differential operator.

1 Introduction

Integration and discrepancy are important problems in applied mathematics and approximation theory and in many applications, one needs to estimate the quantity $\sup_{f\in\mathcal{F}}\left|\int_{\mathcal{B}} f d\zeta\right|$ where $\mathcal{B} \subset \mathbb{R}^{d+1}$ is a bounded domain or manifold, $d \geq 2$ is an integer, $d\zeta : \mathbb{R}^{d+1} \to \mathbb{R}$ is a Borel measure with compact support in $\mathcal{B}$ and $\mathcal{F}$ is a suitable class of real valued functions on $\mathcal{B}$. Such problems above, arise naturally in many interdisciplinary areas of growing interest such as mathematical finance, imaging, geodesy, scattering and statistical learning theory. We refer the reader to the references listed and those cited therein, for a comprehensive account of this vast and interesting subject.

Our main objective in this paper is as follows. We derive an error estimate for the quantity $\left|\int_{S^d} f d\sigma\right|$, for a class of harmonic functions $f : \mathbb{R}^{d+1} \to \mathbb{R}$. Here

*Supported, in part by grants EP/C000285 and NSF-DMS-0439734. S. B. Damelin thanks the Institute for Mathematics and Applications for their hospitality.

and henceforth, S^d will denote the d dimensional sphere realized as a subset of $\mathbb{R}^{d+1}$ and $d\sigma$ will denote a finite signed measure on $\mathbb{R}^{d+1}$, whose positive and negative parts are supported on S^d. Our error estimate involves 2 sided bounds for a Newtonian potential with respect to $d\sigma$ away from its support. In particular, our main result allows us to study quadrature errors for scatterings on the sphere with given mesh norm.

2 Spherical harmonics, multipole expansions, potentials and pseudo-differential operators

In this section, we introduce needed notation and pertinent facts concerning spherical harmonics, multipole expansions, potentials and pseudo-differential operators which we use in the sequel. We also state our main results.

2.1 Spherical harmonics

In this subsection, we collect together some pertinent facts re spherical harmonics which we will need throughout.

Let $d \geq 2$ be given and define

$$S^d := \left\{x := (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + ... + x_{d+1}^2 = 1\right\}$$

to be the surface of the d dimensional unit sphere realized as a subset of $\mathbb{R}^{d+1}$ Euclidean space. By rS^d we shall mean the surface of a sphere of radius $0 < r_0 < r \leq 1$ for some fixed positive r_0. By $B^d(0, \delta)$, we will always mean the open ball of radius $0 < \delta < 1$ in $\mathbb{R}^{d+1}$ and by $\overline{B^d(0, \delta)}$ its closure with boundary δS^d. Once and for all, let $d\sigma_d := \frac{d\mu_d|_{S^d}}{w_d}$, denote normalized Lebesgue measure in $\mathbb{R}^{d+1}$ restricted to S^d where w_d denotes the volume of S^d given by

$$\frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$$

where Γ is the gamma function. Henceforth, $\mathcal{M}_d$, will denote the class of all finite signed measures $d\sigma$ on $\mathbb{R}^{d+1}$ whose positive and negative parts are supported on S^d. Given $f : \mathbb{R}^{d+1} \to \mathbb{R}$ and any $0 < r_0 < r \leq 1$, we have an associated norm

$$||f||_{L_p(rS^d)} := \begin{cases} \left(\int_{rS^d} |f(x)|^p d\mu_d(x)\right)^{1/p}, & 1 \leq p < \infty \\ \text{esssup}_{x \in rS^d} |f(x)|, & p = \infty. \end{cases}$$

The class of all measurable functions $f : rS^d \to \mathbb{R}$ for which $||f||_{L_p(rS^d)} < \infty$ will be denoted by $L_p(rS^d)$, with the usual understanding that functions that are equal almost everywhere are considered equal elements of $L_p(rS^d)$.

The usual inner product of 2 vectors $x, y \in \mathbb{R}^{d+1}$ will be denoted by $x.y$ so that

$$S^d := \{x \in \mathbb{R}^{d+1} : ||x|| = 1\}$$

where $||x|| = (x.x)^{1/2}$ is the Euclidean norm of $x \in \mathbb{R}^{d+1}$. We will often use the fact that for any $x, y \in S^d$, $x.y \in [-1, 1]$. The later fact follows easily from the Cauchy-Swartz inequality.

For a fixed integer $l \geq 0$, the restriction to S^d of a homogeneous harmonic polynomial of degree l, is called a spherical harmonic of degree l. The dimension of the space of spherical harmonics of degree l, which we denote by $Z(d, l)$, is given by:

$$Z(d,l) := \begin{cases} \frac{2l+d-1}{l+d-1}\binom{l+d-1}{l}, & l \geq 1 \\ 1, & l = 0. \end{cases} \tag{2.1}$$

Let

$$\{Y_{l,k} : l = 0, 1, ...; \; k = 1, ..., Z(d, l)\}$$

be a real orthonormal basis for $L^2(S^d)$.

We have for each $x, y \in S^d$, the well known

Addition formula:

$$\sum_{k=1}^{Z(d,l)} Y_{l,k}(x)Y_{l,k}(y) = \frac{Z(d,l)}{\omega_d} P_l(d+1, x.y), l = 0, 1, ..., \tag{2.2}$$

where $P_l(d + 1, .)$ is the Legendre polynomial of degree l in $d + 1$ dimensions over $[-1, 1]$.

The Legendre polynomials are normalized so that $P_l(d + 1, 1) = 1$ for each l and they satisfy the orthogonality relations:

$$\int_{-1}^{1} P_l(d+1,x)P_s(d+1,x)(1-x^2)^{d/2-1}dx = \frac{\omega_d \delta_{l,s}}{\omega_{d-1}Z(d,l)}.$$

We will need their relations to the Gegenbauer polynomials $P_l^{(d-1)/2}$ given by

$$P_l^{(d-1)/2}(x) = \binom{l+d-2}{l} P_l(d+1,x), \; l \geq 0, \; x \in [-1,1] \tag{2.3}$$

and the useful fact that

$$(1-2rt+r^2)^{-(d-1)/2} = \sum_{l=0}^{\infty} r^l P_l^{(d-1)/2}(t), \; 0 < r_0 < r < 1, \; |t| \leq 1. \tag{2.4}$$

In particular, see [3], if $x \in rS^d$ for some $r_0 < r < 1$ and $\eta \in S^d$, we may write $x = r\zeta$, $\zeta \in S^d$ and obtain the formula:

$$\frac{1}{||x-\eta||^{d-1}} = \frac{1}{(1+r^2-2r(\zeta.\eta))^{(d-1)/2}} = \sum_{l=0}^{\infty} r^l P_l^{(d-1)/2}(\zeta.\eta). \tag{2.5}$$

In the sequel, we will also make use of the following:

Funk-Hecke Formula: Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Then

$$\int_{S^d} f((\eta.\zeta))Y_{l,k}(\zeta)d\sigma_d(\zeta) = \lambda Y_{l,k}(\eta),\ \eta \in S^d \tag{2.6}$$

where $l = 0, 1, ...;\ k = 1, ..., Z(d, l)$ and

$$\lambda := \frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1} f(t)P_l(d+1, t)(1-t^2)^{(d-2)/2}dt.$$

Finally, once and for all, we adopt the following convention. Given $f : \mathbb{R}^{d+1} \to \mathbb{R}$ and $0 < r_0 < r \leq 1$, we let $f_r : S^d \to \mathbb{R}$ be the restriction of f to rS^d defined by

$$f_r(x) := f(rx), x \in S^d.$$

Then we write

$$\hat{f}_r(l, k) := \int_{S^d} f_r(x)Y_{l,k}(x)d\mu_d(x),\ l \geq 0, 1 \leq k \leq Z(d, l) \tag{2.7}$$

where

$$f_r(x) \sim \sum_{l=0}^{\infty}\sum_{k=1}^{Z(d,l)} \hat{f}_r(l, k)Y_{l,k}(x),\ x \in S^d \tag{2.8}$$

and the right hand side is the spherical expansion of f_r on S^d.

2.2 Measure of Discrepancy

As our measure of discrepancy, we will use the Newtonian potential with respect to $d\sigma \in \mathcal{M}_d$ which is given by

$$U^\sigma(x) := \int_{S^d} \frac{1}{||x-y||^{d-1}}d\sigma(y), x \in \mathbb{R}^{d+1}. \tag{2.9}$$

Henceforth, we will write

$$d\sigma := d\sigma^+ - d\sigma^-$$

where both measures $d\sigma^\pm$ are finite, non negative and supported on S^d. It is well known that $U^{\sigma^\pm}$ exist, are locally integrable and finite almost everywhere with respect to d dimensional Hausdorf measure. Moreover, $U^{\sigma^\pm}$ are superharmonic in $\mathbb{R}^{d+1}$ and harmonic outside the support of $\sigma^\pm$, see [1]. We will need to estimate U^σ away from the support of $d\sigma$ and to this end, we will need to take the balayage of $d\sigma^\pm$ onto a suitably defined region in $\mathbb{R}^{d+1}$. More precisely, given $0 < r_0 < r < 1$, we set

$$G := \mathbb{R}^{d+1}\backslash\overline{B^d(0, r)} \cup \{\infty\} \tag{2.10}$$

with compact boundary $\partial G = rS^d$. Observe that the supports of $d\sigma^{\pm}$ are contained in $\overline{G}$. Let $d\sigma_B^+$ denote the balayage of $d\sigma^+$ onto ∂G and let $d\sigma_B^-$ denote the balayage of $d\sigma^-$ onto ∂G. As G is regular with respect to the Dirichlet problem on $\mathbb{R}^{d+1}$, see [1, Theorems 4.2, 4.5], $d\sigma_B^{\pm}$ exist, are unique and have the following two additional properties:

$$U^{\sigma^{\pm}}(x) = U^{\sigma_B^{\pm}}(x),\, x \in rS^d \tag{2.11}$$

and

$$\int_{S^d} f d\sigma^{\pm} = \int_{\overline{G}} f d\sigma^{\pm} = \int_{\partial G} f d\sigma_B^{\pm} = \int_{rS^d} f d\sigma_B^{\pm} \tag{2.12}$$

for all functions $f : \mathbb{R}^{d+1} \to \mathbb{R}$, harmonic on G, continuous on $\overline{G}$ and satisfying $f(\infty) = 0$.

Here and throughout, we will adopt the convention that $C, C_1, \ldots$ will always denote positive, finite constants independent of f, x, r and n but possibly depending on other parameters such as d, ε, s, $||\sigma||$, α and r_0. These constants may also take on different values at different times. Throughout, lower case summation indices such as l, k, m and j will run through a subset of nonnegative integers unless stated otherwise.

2.3 Space of Functions

In this subsection, we define our approximation space.

Given $s > \frac{d}{2}$ and $0 < r_0 < r \leq 1$, we shall henceforth say that $f_r \in L_1(S^d)$ is an element of the space $H_s(S^d)$ if

$$\sum_{l=0}^{\infty} \sum_{k=1}^{Z(d,l)} (\hat{f}_r(l,k))^2 m_l^2 < \infty \tag{2.13}$$

where

$$m_l = m(l,d,s) := \begin{cases} \frac{l^s(2l+d-1)}{(d-1)\omega_d}, & l \geq 1 \\ 1, & l = 0 \end{cases} \tag{2.14}$$

We may define a norm on the space $H_s(S^d)$ by setting:

$$||f_r||_{H_s(S^d)} := \left(\sum_{l=0}^{\infty} \sum_{k=1}^{Z(d,l)} (\hat{f}_r(l,k))^2 m_l^2 \right)^{1/2}. \tag{2.15}$$

D-Operator A fundamental tool in our analysis will be the following operator which we now introduce. Given $f_r \in L_1(S^d)$, $0 < r_0 < r \leq 1$, we define (formally) the operator

$$D(f_r) := \sum_{l=0}^{\infty} \sum_{k=1}^{Z(d,l)} \frac{(2l+d-1)}{(d-1)\omega_d} \hat{f}_r(l,k) Y_{l,k}. \tag{2.16}$$

We begin with our first basic result concerning the class $H_s(S^d)$ and the operator D given by (2.16).

Proposition 1 Let $d \geq 2$, $0 < p \leq \infty$, $0 < r_0 < r \leq 1$ and $s > \frac{d}{2}$. Then the following hold true:

(i) Let $f_r \in H_s(S^d)$ and suppose f_r can be recovered pointwise by its spherical expansion on S^d. Then

$$||f_r||_{L_p(S^d)} \leq C^* ||f_r||_{H_s(S^d)} \tag{2.17}$$

where

$$C^* := \left[\sum_{l=1}^{\infty} \frac{e^d \omega_d (d-1)^2 l^{d-1-2s}}{(2l+d-1)^2} + \frac{1}{\omega_d} \right]^{1/2}.$$

(ii) Uniformly for $f_r \in H_s(S^d)$, we have

$$||D(f_r)||_{L_p(S^d)} \leq C^{**} ||f_r||_{H_s(S^d)} \tag{2.18}$$

where

$$C^{**} := \left[\sum_{l=1}^{\infty} \frac{e^d l^{d-1-2s}}{\omega_d} + \frac{1}{\omega_d^3} \right]^{1/2} \omega_d.$$

(iii) Suppose that $s > \frac{3d-2}{4}$ and let $f_r \in H_s(S^d)$. Suppose f_r can be recovered pointwise by its spherical expansion on S^d. Then f_r is Lipschitz of order 1 with Lipschitz constant

$$C||f_r||_{H_s(S^d)} \tag{2.19}$$

for some explicit positive constant C depending on d and s.

Henceforth, given fixed $d \geq 2$ and $0 < r_0 < r < 1$, $\mathcal{F}_r$ will denote the class of functions satisfying the following:

- $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is harmonic on G, continuous on $\overline{G}$ and vanishes at infinity.
- $$||D(f)||_{L_\infty(rS^d)} < \infty.$$

2.4 Discrepancy Results

Our main discrepancy result is given in:

Theorem 2: Let $d \geq 2$, $0 < r_0 < r < 1$ and choose $f \in \mathcal{F}_r$. Also let $1 \leq p, p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\sigma \in \mathcal{M}_d$. Suppose, moreover that

$$f_r(\eta) = \sum_{l,k} \hat{f}_r(l,k) Y_{l,k}(\eta),\ \eta \in S^d.$$

Then we have:

$$\left| \int_{S^d} f(x) d\sigma(x) \right| \leq \tag{2.20}$$
$$\leq \frac{1}{r_0} ||D(f)||_{L_p(rS^d)} ||U^\sigma||_{L_{p'}(rS^d)}.$$

Remark

(a) Our estimate consists of 2 contributions. The second contribution is a discrepancy estimate in terms of 2 sided bounds for a Newtonian potential with respect to $d\sigma$ on an inner sphere away from the support of $d\sigma$. As we show below, this later quantity may be estimated for certain measures defining point systems with given mesh norm. The first contribution, involves an L_p norm of $D(f)$ on the sphere rS^d.

If we now specialize the measures in Theorem 2, we obtain:

Corollary 3: Let $d \geq 2$, $0 < r_0 < r < 1$ and choose $f \in \mathcal{F}_r$ satisfying the condition of Theorem 2. Let E_o be a scattering of $n \geq 1$ distinct points $t_{k,n}$, $1 \leq k \leq n$ on S^d and $a_{k,n}$, $1 \leq k \leq n$, n real weights. Define

$$\nu_n(x) := \sum_{k=1}^{n} a_{k,n} \delta_{t_{k,n}}(x),\ x \in S^d \tag{2.21}$$

where $\delta_{t_k}(.)$ denotes Dirac mass and let μ be a Borel measure on $\mathbb{R}^{d+1}$ with support in S^d. Let $1 \leq p, p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\left| \int_{S^d} f d\mu - \sum_{k=1}^{n} a_{k,n} f(t_{k,n}) \right| \tag{2.22}$$
$$\leq \frac{1}{r_0} ||D(f)||_{L_p(rS^d)} ||U^{\mu-\nu_n}||_{L_{p'}(rS^d)}.$$

We now focus on estimating the discrepancy term $||U^\sigma||_{L_{p'}(rS^d)}$ in Theorem 2 for natural choices of σ. For simplicity we will consider the case $p' = \infty$. Let us recall quickly that given any finite scattering E_o of distinct points on S^d, the mesh norm of E_o is defined by

$$\delta_{E_o} := \max_{x \in S^d} \mathrm{dist}(x, E_o). \tag{2.23}$$

Moreover, if $\mathcal{R}$ denotes a finite partitioning of S^d, then the partition norm for $\mathcal{R}$ is defined by

$$||\mathcal{R}|| := \max_{R \in \mathcal{R}} (\mathrm{diam} R). \tag{2.24}$$

We have:

Theorem 4

(a) Let $d \geq 2$, $0 < r_0 < r < 1$ and E_o a scattering of $n \geq 1$ distinct points $t_{k,n}$, $1 \leq k \leq n$ on S^d. Suppose there exists a finite disjoint partitioning $\mathcal{R}$ of the sphere into n subsets $R_{k,n}$, $1 \leq k \leq n$ and with $S^q = \cup_{k=1}^n R_{k,n}$ with each $R_{k,n}$ containing exactly one $t_{k,n}$. Let μ be a finite Borel measure on $\mathbb{R}^{d+1}$ with support in S^d and suppose ν_n is given by (2.22) with $a_{k,n} = \mu(R_{k,n})$, $k = 1, ..., n$. Then setting $\sigma = \mu - \nu_n$, gives

$$||U^\sigma||_{L_\infty(rS^d)} \leq \frac{(d-1)||\mu||}{(1-r)^{d+1}}||\mathcal{R}||. \tag{2.25}$$

(b) Let $\epsilon > 0$, $d \geq 2$, E_o a finite scattering of $n \geq 1$ distinct points on S^d and let μ be a finite Borel measure on $\mathbb{R}^{d+1}$ with support in S^d. Suppose E_o is chosen so that δ_{E_o} is small enough to ensure that

$$\left(\frac{(d-1)8d\sqrt{2d(d+1)}||\mu||\delta_{E_o}}{\epsilon}\right) < 1. \tag{2.26}$$

Then there exists a reduction E of E_o and an explicit finite disjoint partitioning $\mathcal{R}$ of S^d with each $R \in \mathcal{R}$ containing one point of E and at least one point of E_o so that

$$\left(\frac{(d-1)||\mu||||R||}{\epsilon}\right) < 1. \tag{2.27}$$

Moreover, if $\mu_{\mathrm{card}E}$ is defined as in (2.21) with respect to E and with positive weights

$$a_{k,\mathrm{card}E},\ 1 \leq k \leq \mathrm{card}E$$

and in addition, we define the signed measure $\sigma = \mu - \mu_{\mathrm{card}E}$, then the following holds: For $0 < r_0 < r < 1 - \left(\frac{(d-1)||\mu||||R||}{\epsilon}\right)^{\frac{1}{d+1}}$, we have

$$||U^\sigma||_{L_\infty(rS^d)} \leq \epsilon. \tag{2.28}$$

Remark:

(a) Notice that if the mesh norm of E_o is small, (ie if for example the scattering E_o is not concentrated at one place on the sphere such as one of the poles), then we are able to control the estimate for the potential in Theorem 2 with the help of Theorem 4.

(b) We mention that it is possible to show that, there exists a disjoint (equal area) partitioning $\mathcal{R}$ of the sphere S^d into n parts such that each $R_{k,n} \in \mathcal{R}$, $1 \leq k \leq n$ satisfies

$$\mu_d(R_{k,n}) \sim \frac{1}{|R_{k,n}|} = 1/n,\ ||R_{k,n}|| \leq \frac{C}{n^{1/d}} \tag{2.29}$$

for some $C > 0$ independent of n but depending on d. Thus provided we have n points with exactly one point in each partition, we may apply Theorem 4 to equal weighted quadrature rules as well.

(c) Requiring information on mesh norm is natural since there are many interesting examples of point systems on the sphere where good estimates on mesh norm are available, for example extremal fundamental systems.

The remainder of this paper is devoted to the proofs of Theorems 2-4.

3 Proofs

In this section, we prove our results. We begin with the proof of Theorem 2. We find it instructive to briefly summarize the ideas of the proof. Let G be the region defined by (2.10). As $\sigma \in \mathcal{M}_d$, $\sigma^{\pm}$ are supported on $S^d \subset \overline{G}$. Moreover, the balayage of $\sigma^{\pm}$, $\sigma^{\pm}_B$ exist, are supported on $rS^d = \partial G$ and satisfy

$$\int_{S^d} f d\sigma^{\pm} = \int_{\overline{G}} f d\sigma^{\pm} = \int_{\partial G} f d\sigma^{\pm}_B = \int_{rS^d} f d\sigma^{\pm}_B.$$

Note that for the above string of equations to hold, it is enough that f be continuous on ∂G, harmonic on G and take the value 0 at infinity. Next, recalling the operator D defined by (2.16), we prove that

$$\int_{rS^d} f d\sigma^{\pm}_B = \frac{1}{r}\int_{rS^d} D(f) U^{\sigma^{\pm}_B} d\mu_d$$

so that using the property of balayage again, we have

$$\int_{S^d} f d\sigma^{\pm} = \frac{1}{r}\int_{rS^d} D(f) U^{\sigma^{\pm}} d\mu_d$$

Adding, we obtain

$$\int_{S^d} f d\sigma = \frac{1}{r}\int_{rS^d} D(f) U^{\sigma} d\mu_d.$$

Theorem 2 will follow from the following lemma which is of independent interest.

Lemma 1 Let $d \geq 2$, $\sigma \in \mathcal{M}_d$, $0 < r_0 < r < 1$ and assume f satisfies the assumptions of Theorem 2. Then the following holds.

$$\int_{S^d} r^{1-d} f d\sigma^{\pm} = \int_{rS^d} r^{-d} D(f) U^{\sigma^{\pm}_B} d\mu_d \tag{3.1}$$
$$= \int_{rS^d} r^{-d} D(f) U^{\sigma^{\pm}} d\mu_d.$$

Proof For notational convenience, we shall henceforth write

$$\sum_{l=0}^{\infty} \sum_{k=1}^{Z(d,l)} = \sum_{l,k}.$$

The first thing to do, is to calculate the coefficients of the expansion of $\sigma_B^{\pm}$. Thus to see this, lets recall, see (2.11), that we have

$$\int_{rS^d} \frac{d\sigma_B^{\pm}(y)}{||x-y||^{d-1}} = \int_{S^d} \frac{d\sigma^{\pm}(\eta)}{||x-\eta||^{d-1}},\ x \in rS^d.$$

For $x \in rS^d$, write $x = r\zeta$, $\zeta \in S^d$ and let $\eta \in S^d$. Then by (2.3) and (2.5)

$$\begin{aligned} \frac{1}{||x-\eta||^{d-1}} &= \frac{1}{(1+r^2-2r(\zeta.\eta))^{(d-1)/2}} \qquad (3.2) \\ &= \sum_l r^l P_l^{(d-1)/2}((\zeta.\eta)) \\ &= \sum_l r^l \binom{l+d-2}{l} P_l(d+1,(\zeta.\eta)) \\ &= \sum_{l,k} r^l \frac{\omega_d}{Z(d,l)} \binom{l+d-2}{l} Y_{l,k}(\zeta) Y_{l,k}(\eta) \\ &= \sum_{l,k} r^l \frac{\omega_d}{Z(d,l)} \binom{l+d-2}{l} Y_{l,k}(\eta) Y_{l,k}(\zeta). \end{aligned}$$

Now by (2.1)

$$\binom{l+d-2}{l} \frac{l+d-1}{2l+d-1} \frac{l!(d-1)!}{(l+d-1)!} = \frac{d-1}{2l+d-1}.$$

Thus we learn that

$$\frac{1}{||x-\eta||^{d-1}} = \sum_{l,k} \frac{(d-1)\omega_d}{2l+d-1} r^l Y_{l,k}(\eta) Y_{l,k}(\zeta). \qquad (3.3)$$

For a given l and k, let us set for convenience

$$\hat{\sigma}^{\pm}(l,k) := \int_{S^d} Y_{l,k}(\eta) d\sigma^{\pm}(\eta),$$

$$\sigma_B^{\pm}(r\eta) := \sigma_{B,r}^{\pm}(\eta),\ \eta \in S^d$$

and

$$\hat{\sigma}_{B,r}^{\pm}(l,k) := \int_{S^d} Y_{l,k}(\eta) d\sigma_B^{\pm}(r\eta).$$

Then we have

$$\int_{S^d} \frac{1}{||x-\eta||^{d-1}} d\sigma^{\pm}(\eta) = \sum_{l,k} \frac{(d-1)\omega_d}{2l+d-1} r^l \hat{\sigma}^{\pm}(l,k) Y_{l,k}(\zeta). \qquad (3.4)$$

Also, we claim that

$$\begin{aligned} \int_{rS^d} \frac{d\sigma_B^{\pm}(y)}{||x-y||^{d-1}} &= \int_{S^d} \frac{d\sigma_B^{\pm}(r\eta)}{||r\zeta - r\eta||^{d-1}} \qquad (3.5) \\ &= \sum_{l,k} \frac{(d-1)\omega_d}{(2l+d-1)r^{d-1}} \hat{\sigma}_{B,r}^{\pm}(l,k) Y_{l,k}(\zeta). \end{aligned}$$

To see (3.5), we proceed as follows. Consider the Newtonial kernel on S^d defined by way of the function

$$g(t) := 2^{(-d+1)/2}(1-t)^{(-d+1)/2},\ t \in [-1,1]$$

and define for each $\delta > 0$, a sequence of functions $g(t-\delta)$, $t \in [-1,1]$. This sequence coverges monotonically downward to $g(t)$ as $\delta \to 0^+$ for each fixed t. It is known, that the coefficients in the expansion of $g(t-\delta)$ (with respect to $P_l(t, d+1)$) converge to $\binom{l+d-2}{l}$ as $\delta \to 0^+$. Thus using the finiteness of the integral

$$\int_{S^d} \frac{d\sigma^{\pm}(\eta)}{||z-\eta||^{d-1}},\ z \notin S^d$$

and the monontone convergence theorem together with the calculations of (3.3) yield (3.5). Comparing coefficients in the two formulas (3.4) and (3.5) above, we learn that

$$\hat{\sigma}^{\pm}_{B,r}(l,k) = r^{l+d-1}\hat{\sigma}^{\pm}(l,k),\ k = 1, ..., Z(d,l),\ l \geq 0. \tag{3.6}$$

Armed with (3.6), we now recall (see (2.16)), that

$$D(f_r)(y) = \sum_{l,k} \frac{(2l+d-1)}{(d-1)\omega_d} \hat{f}_r(l,k) Y_{l,k}(y),\ y \in S^d.$$

Then we have on the one hand, by (3.6) and the property of balayage,

$$\begin{aligned} &\int_{S^d} f(y) d\sigma^{\pm}(y) = \int_{rS^d} f(y) d\sigma^{\pm}_B(y) \qquad (3.7) \\ &= \int_{S^d} f(r\eta) d\sigma^{\pm}_B(r\eta) = \int_{S^d} f_r(\eta) d\sigma^{\pm}_{B,r}(\eta) \\ &= \sum_{l,k} \hat{f}_r(l,k) \int_{S^d} Y_{l,k}(\eta) d\sigma^{\pm}_{B,r}(\eta) \\ &= \sum_{l,k} \hat{f}_r(l,k) \hat{\sigma}^{\pm}_{B,r}(l,k) \\ &= \sum_{l,k} r^{l+d-1} \hat{f}_r(l,k) \hat{\sigma}^{\pm}(l,k). \end{aligned}$$

On the other hand, using orthogonality and the definition of D, we see that

$$\begin{aligned} &\frac{1}{r}\int_{rS^d} D(f)(y) U^{\sigma^{\pm}}(y) d\mu_d(y) = \frac{1}{r}\int_{S^d} D(f)(r\eta) U^{\sigma^{\pm}}(r\eta) r^d d\mu_d(\eta) \qquad (3.8) \\ &= \frac{1}{r}\int_{S^d} D(f_r)(\eta) U^{\sigma^{\pm}}(r\eta) r^d d\mu_d(\eta) \\ &= \frac{1}{r}\int_{S^d} \left(\sum_{l,k} \hat{f}_r(l,k) \frac{(2l+d-1)}{(d-1)\omega_d} Y_{l,k}(\eta) \right) \left(\sum_{m,j} \frac{(d-1)\omega_d}{2m+d-1} r^m \hat{\sigma}^{\pm}(m,j) Y_{m,j}(\eta) r^d \right) d\mu_d(\eta) \\ &= \sum_{l,k} r^{l+d-1} \hat{f}_r(l,k) \hat{\sigma}^{\pm}(l,k). \end{aligned}$$

Comparing (3.7) and (3.8) gives Lemma 1. $\Box$

We remark that the proof of Lemma 1 shows that the expansion of f_r needed in the proof is only needed $d\sigma^{\pm}_{B,r} a.e.$

The Proof of Theorem 2 Theorem 2 follows from Lemma 1 and Minkowski's inequality. $\Box$

We now proceed with the

Proof of Proposition 1 We first establish (2.17). We may assume with loss of generality that $p = \infty$. Using the addition theorem, the Cauchy-Swartz inequality, and the estimate,

$$Z(d,l) \leq e^d l^{d-1},\ l \geq 0$$

we see that we have for $x \in S^d$,

$$\begin{aligned}
|f_r(x)|^2 &= \left| \sum_{l,k} \hat{f}_r(l,k) Y_{l,k}(x) \right|^2 \\
&\leq \left(\sum_{l,k} (\hat{f}_r(l,k))^2 m_l^2 \right) \left(\sum_{l,k} Y_{l,k}^2 m_l^{-2} \right) \\
&\leq ||f_r||^2_{H_s(S^d)} \left[\sum_{l=1}^{\infty} \frac{Z(l,d) m_l^{-2}}{\omega_d} + \frac{1}{\omega_d} \right] \\
&\leq ||f_r||^2_{H_s(S^d)} \left[\sum_{l=1}^{\infty} \frac{e^d \omega_d (d-1)^2 l^{d-1-2s}}{(2l+d-1)^2} + \frac{1}{\omega_d} \right].
\end{aligned}$$

Note that the last sum is finite since $s > \frac{d}{2}$. Thus (2.17) holds. To see (2.18), we proceed much as in (2.17). As $s > \frac{d}{2}$, we readily have for $x \in S^d$

$$\begin{aligned}
&\left| \sum_{l,k} \frac{(2l+d-1)}{(d-1)\omega_d} \hat{f}_r(l,k) Y_{l,k}(x) \right|^2 \\
&\leq \left(\sum_{l,k} (\hat{f}_r(l,k))^2 m_l^2 \right) \left(\sum_{l,k} \frac{(2l+d-1)^2}{(d-1)^2(\omega_d)^2} Y_{l,k}^2 m_l^{-2} \right) \\
&\leq ||f_r||^2_{H_s(S^d)} \left[\sum_{l=1}^{\infty} \frac{e^d l^{d-1-2s}}{\omega_d} + \frac{1}{\omega_d^3} \right].
\end{aligned}$$

It remains to show (2.19). In light of (2.17) and (2.18), it is easy to see that we have for any ζ and η on S^d the estimate:

$$\begin{aligned}
&|f_r(\eta) - f_r(\zeta)| \leq C ||f_r||_{H_s(S^d)} \times \qquad (3.9) \\
&\times \left[\sum_{l,k} \frac{(d-1)^2 \omega_d^2}{(2l+d-1)^2} (Y_{l,k}(\eta) - Y_{l,k}(\zeta))^2 l^{-2s} + \sum_k (Y_{0,k}(\eta) - Y_{0,k}(\zeta))^2 \right]^{1/2}.
\end{aligned}$$

Now it is a straightforward consequence of the addition formula, that for every fixed $l \geq 0$, we have

$$\begin{aligned}&\sum_{k=1}^{Z(d,l)} (Y_{l,k}(\zeta) - Y_{l,k}(\eta))^2 \\ &= \frac{2Z(d,l)}{\omega_d} \left(1 - P_l(\eta.\zeta, d+1)\right).\end{aligned}$$

Thus inserting the above estimate into (3.9) yields:

$$\begin{aligned}&|f_r(\eta) - f_r(\zeta)| \leq C_1 ||f_r||_{H_s(S^d)} \times \\ &\times \left[\sum_{l=1}^{\infty} l^{-2s-3+d} \left|1 - P_l(\eta.\zeta, d+1)\right| + C_2 \left|1 - P_0(\eta.\zeta, d+1)\right|\right]^{1/2}.\end{aligned} \tag{3.10}$$

We need to estimate the righthand side of (3.10). First, we recall, (see (2.3), [5, pp 63] and [5, pp 170]), that if $P_l^{(d/2-1,d/2-1)}$ is the Jacobi polynomial of degree $l \geq 0$, then we have for every $l \geq 0$ and $x \in [-1, 1]$,

$$P_l(x, d+1) = \binom{l + d/2 - 1}{l}^{-1} P_l^{(d/2-1,d/2-1)}(x)$$

and

$$(P_l^{(d/2-1,d/2-1)})'(1) \leq \begin{cases} Cl^{1+d/2}, & l > 0 \\ C_1, & l = 0 \end{cases}$$

Thus we deduce that for every $l \geq 1$,

$$\max_{|t|\leq 1} P_l'(t, d+1) = P_l'(1, d+1) \leq \begin{cases} Cl^2, & l > 0 \\ C_1, & l = 0 \end{cases} \tag{3.11}$$

We also have by the first mean value theorem and our normalization, that for every $l \geq 0$,

$$1 - P_l(\eta.\zeta, d+1) = \frac{1}{2} P_l'(t, d+1) 2(1 - \eta.\zeta) = \frac{1}{2} P_l'(t, d+1) ||\eta - \zeta||^2 \tag{3.12}$$

for some $t \in [-1, 1]$. Thus, we have

$$\sum_{l=1}^{\infty} l^{-2s-3+d} |1 - P_l(\eta.\zeta, d+1)| \leq C \left\{\sum_{l=1}^{\infty} l^{-2s-3+d}\right\} ||\eta - \zeta||^2.$$

(2.19) now follows. $\Box$

The Proof of Theorem 3 This follows in a straightforward way from Theorem 2. $\Box$

4 Proof of Theorem 4

In this last section, we establish Theorem 4.

Proof of Theorem 4 Let us write $d\sigma_n = d\sigma := d\mu - d\nu_n$ and take $x \in S^d$. First we observe that we have

$$-U^{-\nu_n}(rx) = \int_{S^d} \frac{1}{||rx-y||^{d-1}} d\nu_n(y) \tag{4.1}$$

$$= \int_{S^d} \frac{1}{||rx-y||^{d-1}} \sum_{k=1}^{n} a_{k,n} d\delta_{t_{k,n}}(y) = \sum_{k=1}^{n} \frac{a_{k,n}}{||rx-t_k||^{d-1}}$$

$$= \sum_{k=1}^{n} \frac{\mu(R_{k,n})}{||rx-t_{k,n}||^{d-1}} = \sum_{k=1}^{n} \int_{R_{k,n}} \frac{1}{||rx-t_{k,n}||^{d-1}} d\mu(y).$$

On the other hand,

$$U^{\mu}(rx) = \int_{S^d} \frac{d\mu(y)}{||rx-y||^{d-1}} = \sum_{k=1}^{n} \int_{R_{k,n}} \frac{d\mu(y)}{||rx-y||^{d-1}}. \tag{4.2}$$

Thus, using (4.1) and (4.2), we have

$$|U^{\sigma}(rx)| \le \sum_{k=1}^{n} \int_{R_{k,n}} \left| \frac{1}{||rx-y||^{d-1}} - \frac{1}{||rx-t_k||^{d-1}} \right| d\mu(y). \tag{4.3}$$

Now let us estimate the integrand in (4.3). We first write for $x, y \in S^d$:

$$||rx-y||^{d-1} = (r^2 - 2r(x.y) + 1)^{(d-1)/2}.$$

A simple calculation also shows that

$$\begin{aligned} &\frac{d}{d\theta}(1+r^2-2r\theta)^{-(d-1)/2} \\ &= (d-1)r((1+r^2-2r\theta)^{-(d+1)/2} \\ &\le (d-1)(1-r)^{-(d+1)} \end{aligned}$$

for any $-1 \le \theta \le 1$. Thus an application of the mean value theorem to (4.3) easily yields

$$\begin{aligned} &|U^{\sigma}(rx)| \le (d-1)||\mu||(1-r)^{-(d+1)}||y-t_k|| \\ &\le (d-1)||\mu||(1-r)^{-(d+1)}||\mathcal{R}||. \end{aligned}$$

Theorem 4(a) is then proved. Next, it is known that, we can find a reduction E of E_o and an explicit finite disjoint partitioning $\mathcal{R}$ of S^d with each $R \in \mathcal{R}$ containing one point of E and at least one point of E_o. Moreover,

$$\delta_{E_o} \le \delta_E < ||\mathcal{R}|| \le 8d\sqrt{2d(d+1)}\delta_{E_o}. \tag{4.4}$$

Then (2.27) follows from (2.26). Moreover, (2.28) then follows from (2.25) and (2.27). This completes the proof of Theorem 4. □

References

[1] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, New York 1972.

[2] J. Levesley and X. Sun, *Approximation in rough native spaces by shifts of smooth kernels on spheres*, J. Approximation Theory 133 (2005), 269-283.

[3] C. Müller, *Spherical harmonics*, Lecture Notes in Mathematics, **17**, Springer, Berlin, New York, 1966.

[4] X. Sun, *Strictly positive definite functions on the unit circle*, Mathematics of Computation 74 (2005), 709-721.

[5] G. Szeg"o, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, American Mathematical Society, Providence RI, 1939.

Institute for Mathematics and its Applications, University of Minnesota, 400 Lind Hall, 207 Church Hill, S.E, Minneapolis, MN 55455
Email address: damelin@ima.umn.edu
Homepage: http://www.ima.umn.edu/ damelin