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By

S.B. Damelin

J. Levesley

and

X. Sun

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

Energy estimates and the Weyl criterion on compact homogeneous manifolds

S.B Damelin[1], J. Levesley[2] and X. Sun[3]

[1] Institute for Mathematics and its Applications, University of Minnesota, 400 Lind Hall, 207 Church Hill, Minneapolis, MN 55455, U.S.A.
damelin@ima.umn.edu
This author is supported, in part, by EP/C000285 and NSF-DMS-0439734.
[2] Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK.
jl1@mcs.le.ac.uk
[3] Department of Mathematics, 10M Cheek Hall, Missouri State University Springfield, MO 65897, U.S.A XSun@MissouriState.edu

Summary. The purpose of this paper is to demonstrate how many results concerning approximation, integration, and density on the sphere can be generalised to a much wider range of manifolds M, namely the compact homogeneous manifolds. The essential ingredient is that invariant kernels (the generalisation of zonal or radial kernels) have a spectral decomposition in terms of projection kernels onto invariant polynomial subspaces. In particular, we establish a Weyl criterion for such manifolds M and announce an energy result which generalizes work of Damelin and Grabner.

Keywords and Phrases: Compact Homogeneous Manifold, Energy, Harmonic, Invariant Kernels, Invariant Polynomial Subspaces, Numerical Integration, Projection Kernels, Uniform Distribution, Weyl

1 Introduction

Let M be a $d \geq 1$ dimensional homogeneous space of a compact Lie group G embedded in $\mathbb{R}^{d+r}$ for some $r \geq 0$. Then (see [4]), we may assume that $G \subset O(d+r)$, the orthogonal group on $\mathbb{R}^{d+r}$. Thus $M = \{gp : g \in G\}$ where $p \in M$ is a non-zero vector in $\mathbb{R}^{d+r}$. For technical reasons, we will assume that M is *reflexive*. That is, for any given $x, y \in M$, there exists $g \in G$ such that $gx = y$ and $gy = x$.

Let $d(x, y)$ be the geodesic distance between $x, y, \in M$ induced by the embedding of M in $\mathbb{R}^{d+r}$ (see [5] for details). On the spheres, this corresponds to the usual geodesic distance. A real valued function $\mathcal{K}(x, y)$ defined on $M \times M$ is called a positive definite kernel on M, if for every finite subset $Y \subset M$, and arbitrary real numbers $c_y, y \in Y$, we have

$$\sum_{x\in Y}\sum_{y\in Y} c_x c_y \kappa(x,y) \geq 0.$$

If the above inequality becomes strict whenever the points y are distinct, and not all the c_y are zero, then the kernel κ is called strictly positive definite. A kernel κ is called G-invariant (or zonal) if $\kappa(gx,gy)=\kappa(x,y)$ for all $x,y\in M$ and $g\in G$. For example, if $M:=S^d$, the d dimensional sphere realized as a subset of $\mathbb{R}^{d+1}$ and $G:=O(d+1)$, then all the G invariant kernels have the form $\phi(xy)$, where $\phi:[-1,1]\to\mathbb{R}$, and where xy denotes the usual inner product of x and y on $L_2(M)$. A kernel of the form $\phi(xy)$ is often called a zonal kernel on the sphere in the literature.

Let μ be a G–invariant measure on M (which may be taken as an appropriately normalized 'surface' measure). Then, for real valued $f,g: L_2(M)\to\mathbb{R}$, we define an inner product with respect to μ:

$$[f,g]=[f,g]_\mu := \int_M fg d\mu$$

and let $L_2(M)_\mu$ denote the space of all those functions from $L_2(M)$ into $\mathbb{R}$ for which the above inner product is finite. In the usual way, we identify 2 functions as being equal in $L_2(M)_\mu$, if they are equal almost everywhere μ wise.

Let $n\geq 0$ and P_n be the space of polynomials in $d+r$ variables of degree n restricted on M. Here, multiplication is taken pointwise on $\mathbb{R}^{d+r}$. The *harmonic polynomials* of degree n on M are $H_n := P_n \bigcap P_{n-1}^{\perp}$. We may always (uniquely) decompose H_n into irreducible G-invariant subspaces $H_{n,k}$, $k=1,\ldots,\nu_n$. Indeed, the uniqueness of the decomposition follows by the minimality of the G invariant space, since a different decomposition would give subspaces contained in minimal ones leading to a contradiction.

Now, for $n\geq 0, k\geq 1$, let $Y_{n,k}^1,\ldots,Y_{n,k}^{d_{n,k}}$ be any orthonormal basis for $H_{n,k}$, and set

$$Q_{n,k}(x,y) := \sum_{j=1}^{d_{n,k}} Y_{n,k}^j(x) Y_{n,k}^j(y).$$

Then $Q_{n,k}$ is the unique G-invariant kernel for the orthogonal projection $T_{n,k}$ of $L_2(M)_\mu$ onto $H_{n,k}$ acting as

$$T_{n,k}f(x) = \int_M Q_{n,k}(x,y) f(y)\, d\mu(y).$$

For what follows, we require the following easily proved facts:

Lemma 1. *Let y,z be fixed points in M. Then*

a.
$$\int_M Q_{n,k}(y,x) Q_{n,k}(x,z) d\mu(x) = Q_{n,k}(y,z).$$

b. $Q_{n,k}(x,x) = d_{n,k}$.
c. If κ *is a G-invariant kernel, then* $\kappa(x,y) = \kappa(y,x)$.

Proof: Part (a) follows directly from the fact that $Q_{n,k}$ is the kernel for projection onto $H_{n,k}$.

Part (b) is a consequence of the equation

$$Q_{n,k}(x,x) := \sum_{j=1}^{d_{n,k}} Y_{n,k}^j(x) Y_{n,k}^j(x).$$

Indeed, since $Q_{n,k}$ is G-invariant, it is $Q_{n,k}(x,x)$ constant on M. Integrating the last equation over M and using the orthonormality of the $Y_{n,k}^j$, we then arrive at the required result.

The proof of Part (c) needs the reflexivity of M. Indeed, pick a $g \in G$ so that $gx = y$ and $gy = x$. Then

$$\kappa(x,y) = \kappa(gy, gx) = \kappa(y,x)$$

using the G-invariance of κ. □

An important consequence of the above construction is that each irreducible subspace is generated by the translates of a fixed element. For this result on the sphere S^d, see, for instance, [1].

Proposition 1. *Let* $0 \neq Y \in H_{n,k}$. *Then* $H_{n,k} = \{Y(g\cdot); g \in G\}$.

Proof: It is clear that $V = \{Y(g\cdot); g \in G\}$ is a G-invariant subspace of $H_{n,k}$, and since Y is not zero this is a non trivial subspace. But $H_{n,k}$ is irreducible, so that V cannot be a proper subspace of $H_{n,k}$. Thus $V = H_{n,k}$. □

Any G–invariant kernel κ, has an associated integral operator which we define by

$$T_\kappa f(x) = \int_M \kappa(x,y) f(y)\, d\mu(y).$$

Then we have

Lemma 2. *Let* κ_1 *and* κ_2 *be G-invariant. If M is a reflexive space,* $T_{\kappa_1} T_{\kappa_2} = T_{\kappa_1} T_{\kappa_2}$.

Proof: Let $f \in L_2(M)_\mu$. Then

$$\begin{aligned}
[T_{\kappa_1} T_{\kappa_2} f](x) &= \int_M \kappa_1(x,y) \left\{ \int_M \kappa_2(y,z) f(z) d\mu(z) \right\} d\mu(y) \\
&= \int_M f(z) \left\{ \int_M \kappa_1(x,y) \kappa_2(y,z) d\mu(y) \right\} d\mu(z).
\end{aligned}$$

Since the manifold is reflexive there is a $g \in G$ which interchanges x and z. Thus,

$$\int_M \kappa_1(x,y)\kappa_2(y,z)d\mu(y) = \int_M \kappa_1(z,y)\kappa_2(y,x)d\mu(y),$$

so that

$$\begin{aligned}
[T_{\kappa_1}T_{\kappa_2}f](x) &= \int_M f(z)\left\{\int_M \kappa_1(z,y)\kappa_2(y,x)d\mu(y)\right\}d\mu(z) \\
&= \int_M \kappa_2(x,y)\left\{\int_M \kappa_1(y,z)f(z)d\mu(z)\right\}d\mu(y) \\
&= [T_{\kappa_2}T_{\kappa_1}f](x),
\end{aligned}$$

where the penultimate step uses Lemma 1 (c). The changes of order of integration are easy to justify since the kernels are continuous and $f \in L_2(M)_\mu$. $\square$

We are now able to show that zonal kernels have a spectral decomposition in terms of projection kernels onto invariant polynomial subspaces. This is contained in

Theorem 1. *If M is a reflexive manifold, then any G–invariant kernel κ has the spectral decomposition*

$$\kappa(x,y) = \sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)Q_{n,k}(x,y),$$

where

$$a_{n,k}(\kappa) = \frac{1}{d_{n,k}}\int_M \kappa(x,y)Q_{n,k}(x,y)\,d\mu(y).$$

Here convergence is understood operatorwise in $L_2(M)_\mu$.

Proof: If $Y \in H_{n,k}$ then $T_{p_{n,k}}Y = Y$. Thus

$$\begin{aligned}
T_\kappa Y &= T_\kappa(T_{Q_{n,k}}Y) \\
&= T_{Q_{n,k}}(T_\kappa Y) \in H_{n,k},
\end{aligned}$$

since $T_{Q_{n,k}}$ is the orthogonal projection onto $H_{n,k}$. Here we have used Lemma 2.

Since T_κ is a symmetric operator, it can be represented on the finite dimensional subspace by a symmetric matrix. Either this matrix is the zero matrix, in which case $a_{n,k}(\kappa) = 0$, or T_κ has non trivial range. Since the matrix is symmetric, it must have a non-zero real eigenvalue, γ say, with associated eigenvector Y.

However, if Y is an eigenvector then, so is $Y(g\cdot)$ for any $g \in G$ since

$$\begin{aligned}
[T_\kappa Y(g\cdot)](x) &= \int_M \kappa(x,y)Y(gy)d\mu(y) \\
&= \int_M \kappa(x,g^{-1}y)Y(y)d\mu(g^{-1}y) \\
&= \int_M \kappa(gx,y)Y(y)d\mu(y),
\end{aligned}$$

using the G-invariance of both κ and μ. But Y is an eigenvector of T_κ, so that

$$[T_\kappa Y(g\cdot)](x) = \gamma Y(gx).$$

But, using Proposition 1 we see that $H_{n,k}$ is an eigenspace for T_κ with single eigenvalue γ. We can compute γ by evaluating T_κ on $Q_{n,k}(\cdot, y)$ for fixed y:

$$\int_M \kappa(z,x) Q_{n,k}(x,y) d\mu(x) = \gamma Q_{n,k}(z,y).$$

Setting $z = y$ and using Lemma 1 (b) we have

$$\gamma = \frac{1}{d_{n,k}} \int_M \kappa(y,x) Q_{n,k}(x,y) d\mu(x),$$

and the appropriate form for γ follows using the symmetry of G-invariant kernels (Lemma 1 (c)). $\square$

Henceforth, we assume that $a_{n,k}(\kappa) \geq 0$ for all n, k, and

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} d_{n,k} a_{n,k}(\kappa) < \infty. \tag{1}$$

Thus κ is bounded on $M \times M$, is positive definite there and hence absolutely integrable on M, ie, there exists finte C such that uniformly for all $x \in M$,

$$\int_M |\kappa(x,y)| d\mu(y) \leq C$$

In what follows, such kernels will be called *admissible*. The archetype for admissible kernels is the *Riesz kernel*

$$\kappa(x,y) = \|x-y\|^{-s}, \quad s > 0, \quad x, y \in M,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{d+r}$.

We will also need to assume the following simple consequence of Theorem 4. See [3] for the proof.

Lemma 3. *(Funk-Hecke formula) For any $x, y \in M$,*

$$\int_M Q_{n,k}(x,y) Q_{n,k}(y,z) d\mu(y) = Q_{n,k}(x,z).$$

2 The Weyl criterion

In this section, we will prove the equivalence of two definitions of uniform distribution of points. Following is our main result of this section.

Theorem 2. *The following two definitions for a uniformly distributed sequence are equivalent.*

a. A sequence $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed on M if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} Y_{n,k}^{j}(x_l) = 0$$

for all $n \geq 0$ and $1 \leq k \leq \nu_n$, $1 \leq j \leq d_{n,k}$.

b. Let κ be a positive definite zonal kernel on M. A sequence $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed on M if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \kappa(x_l, y) = a_{0,0}(\kappa),$$

holds true uniformly for $y \in M$.

Proof: Using the series expansion for κ we have

$$\frac{1}{N} \sum_{l=1}^{N} \kappa(x_l, y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \sum_{j=1}^{d_{n,k}} Y_{n,k}^{j}(y) \left(\frac{1}{N} \sum_{l=1}^{N} Y_{n,k}^{j}(x_l) \right). \tag{2}$$

Suppose $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed by (a). Now, the right hand side of the last equation is majorised by

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \frac{1}{N} \sum_{l=1}^{N} |Q_{n,k}(x_l, y)| \leq \sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} d_{n,k} a_{n,k}(\kappa),$$

using Lemma 1 (a). This is bounded from (1).

Therefore, using the dominated convergence theorem we can pass the limit over N through the sum to give

$$\begin{aligned} &\lim_{N\to\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \sum_{j=1}^{d_{n,k}} Y_{n,k}^{j}(y) \left(\frac{1}{N} \sum_{l=1}^{N} Y_{n,k}^{j}(x_l) \right) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \sum_{j=1}^{d_{n,k}} Y_{n,k}^{j}(y) \lim_{N\to\infty} \left(\frac{1}{N} \sum_{l=1}^{N} Y_{n,k}^{j}(x_l) \right) \\ &= 0, \end{aligned}$$

by assumption. Thus

$$\frac{1}{N} \sum_{l=1}^{m} \kappa(x_l, y) = a_{0,0}(\kappa),$$

uniformly in y, so that $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed by (b).

Conversely suppose that $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed by (b). Then, as in (2), we have

$$\frac{1}{N^2}\sum_{m=1}^{N}\sum_{l=1}^{N}\kappa(x_m,x_l)=\sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n}a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}}\left(\frac{1}{N}\sum_{l=1}^{N}Y_{n,k}^{j}(x_l)\right)^2.$$

Now, for each x_m, by hypothesis

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\phi(x_m,x_l)\to\int_M\phi(x_m,x)d\mu(x)=a_{0,0}(\kappa).$$

Thus,

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{l=1}^{N}\phi(x_l,x_j)\to\int_M\phi(x,x_j)d\mu(x)=a_{0,0}(\kappa).$$

Therefore

$$\lim_{N\to\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\nu_n}a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}}\left(\frac{1}{N}\sum_{l=1}^{N}Y_{n,k}^{j}(x_l)\right)^2=0,$$

and since $a_{n,k}(\kappa)>0$, $n\in\mathbf{N}$ and $1\le k\le\nu_n$, it must be that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}Y_{n,k}^{j}(x_l)=0,$$

so that $\{x_l : l\in\mathbf{N}\}$ is uniformly distributed by (a). $\square$

3 Energy on manifolds

In this last section, we are interested in studying errors of numerical integration of continuous functions $f : M \to \mathbb{R}$ over a countable point set $Z \subset M$ of cardinality $N \geq 1$. In particular, we seek a generalization of results of Damelin and Grabner in [2]. More precisely, given an admissible kernel κ and such a point set Z, we define the discrete energy

$$E_\kappa(Z)=\frac{1}{N^2}\sum_{\substack{y,z\in Z\\ y\neq z}}\kappa(y,z)$$

and for the normalized G invariant measure μ on M, denote by

$$R(f,Z,\mu):=\left|\int_M f\,d\mu-\frac{1}{N}\sum_{y\in Z}f(y)\right|$$

the error of numerical integration of f with respect to μ over M.

For an admissible kernel κ and probability measure ν on M, we define the energy integral

$$\mathcal{E}_\kappa(\nu) = \int_M \int_M \kappa(x,y)d\nu(x)d\nu(y).$$

We have

Lemma 4. *The energy integral $\mathcal{E}_\kappa(\nu)$ is uniquely minimised by the normalized G invariant measure μ.*

Proof: Since κ is positive definite, $\mathcal{E}_\kappa(\nu) \geq 0$ for every Borel probability measure ν. Also, a simple computation shows that $\mathcal{E}_\kappa(\mu) = a_{0,0}(\kappa)$.

Next, we observe that we have using Lemma 3, for all probability measures σ on M,

$$\begin{aligned}
\mathcal{E}_\kappa(\sigma) &= \int_M \int_M \left\{ \sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) P_{n,k}(x,z) \right\} d\sigma(x)d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \int_M P_{n,k}(x,z)d\sigma(x)d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \int_M \int_M P_{n,k}(x,y)P_{n,k}(y,z)d\mu(y)d\sigma(x)d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{j=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \left\{ \int_M P_{n,k}(x,y)d\sigma(x) \right\}^2 d\mu(y),
\end{aligned}$$

using the symmetry of the kernel $P_{n,k}(x,y)$.

Thus, for ν to minimise $\mathcal{E}_\kappa$,

$$\int_M P_{n,k}(x,y)d\nu(x) = 0, \quad k = 1,\ldots,\nu_n, \quad n = 1,\ldots.$$

Hence, since μ also annihilates all polynomials of degree ≥ 0, $\nu - \mu$ annihilates all polynomials. Because the polynomials are dense in the continuous functions, we see that $\nu - \mu$ is zero and the result is proved. $\square$

Heuristicly, one expects that a point distribution Z of minimal energy gives a discrete approximation to the measure μ, in the sense that the integral with respect to the measure is approximated by a discrete sum over the points of Z. For the sphere, this was shown by Damelin and Grabner in [2] for Riesz kernels. The essence of our main result below is that we are able to formulate a general analogous result which works on M and for a subclass of admissible kernels κ. To describe this result, we need some more notation.

Let σ_α be a sequence of kernels converging to the δ distribution (the distribution for which all Fourier coefficients are unity) as $\alpha \to 0$. Let κ be admissible and for $\alpha < \alpha_0$ for some fixed α_0, we wish the convolution $\kappa_\alpha = \kappa * \sigma_\alpha$ to have the following properties:

a. κ_α is positive definite
b. $\kappa_\alpha(x,y) \leq \kappa(x,y)$ for all $x, y \in M$.

If the above construction is possible, we say that κ is *strongly admissible*. Besides Riesz kernels on d dimensional spheres see [2, 3], we have as a futher natural example on the 2-torus embedded in $\mathbb{R}^4$, strongly admissible kernels defined as products of univariate kernels:

$$\kappa(x,y) = \rho(x_1,y_1)\rho(x_2,y_2), \quad x_1,y_1,x_2,y_2 \in S^1,$$

where

$$\rho(s,t) = |1-st|^{-1/2},\ s,t \in (-1,1).$$

See [3] for further details. We are now able to announce our main result of this section. See [3] for the proof and further results.

Theorem 3. *Let κ be strongly admissible on M and $Z \subset M$ be a point subset of cardinality $N \geq 1$. Fix $x \in M$. If q is a polynomial of degree at most $n \geq 0$ on M then, for $\alpha < \alpha_0$,*

$$|R(f,Z,\mu)| \leq \max_{j\leq n,\, l\leq h_j} \frac{1}{(a_{j,l}(\kappa_\alpha))^{1/2}} \|q\|_2 \left(E_\kappa(Z) + \frac{1}{N}\kappa_\alpha(x,x) - a_{0,0}(\kappa_\alpha)\right)^{1/2}.$$

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