ASYMPTOTIC STABILITY OF A FLUID-STRUCTURE SEMIGROUP

By

George Avalos

IMA Preprint Series # 2077

(November 2005)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

Asymptotic Stability of a Fluid-Structure Semigroup

George Avalos
Department of Mathematics
University of Nebraska-Lincoln, 68588

November 6, 2005

Abstract

The strong stability problem for a fluid-structure interactive partial differential equation (PDE) is considered. The PDE comprises a coupling of the linearized Stokes equations to the classical system of elasticity, with the coupling occurring on the boundary interface between the fluid and solid media. It is now known that this PDE may be modeled by a C_0-semigroup of contractions on an appropriate Hilbert space. However, because of the nature of the unbounded coupling between fluid and structure, the resolvent of the semigroup generator will *not* be a compact operator. In consequence, the classical solution to the stability problem, by means of the Nagy-Foias decomposition, will not avail here. Moreover, it is not practicable to explicitly write down the resolvent of the fluid-structure generator; this situation thus makes it problematic to use the wellknown semigroup stability result of Arendt-Batty and Lyubich-Phong. Instead, our proof of strong stability for the fluid-structure PDE will depend on an appropriate usage of a recently derived abstract stability result of Y. Tomilov.

1 Statement of the Problem

In this paper, we show how a recently derived abstract operator theoretic result can be used to ascertain the asymptotic decay of solutions for a so-called "transmission hyperbolic-parabolic problem". A simplified version of this model and its relevance to biological modeling is discussed [11]; see also [7] and [8] for related PDE's. Because of the non-compactness of the resolvent for the associated semigroup generator–see 4 below– the classical stability treatments involving the Nagy-Foias decomposition and Lasalle Invariance Principle are *not* applicable (see [10] and references therein). Nor does the resolvent of this fluid-structure semigroup admit an explicit, working expression which might allow an appeal to now the wellknown abstract stability results in [1] and [13]. Instead, we will use the recently derived stability result posted in [15] and [6] (in Theorem 2 below; see also a precursor of this result in [5]). This stability result is formulated as a necessary resolvent criterion; however, to use this result one does not actually need to know what the resolvent looks like. The methodology for the use of Tomilov's abstract stability result was first developed in [3].

We describe the partial differential equation (PDE) under consideration here: Let Ω_f and Ω_s be bounded open sets, with smooth boundaries Γ_f and Γ_s, respectively; these geometries are configured as in Figure 1: On the "fluid portion" Ω_f of the geometry, we define the following spaces:

$$\begin{aligned} Null(div) &= \left\{u \in [L^2(\Omega_f)]^3 : div\ u = 0\right\}; \\ V &= \left\{\phi \in [H^1(\Omega_f)]^3 \cap Null(div) : div\ \phi = 0 \ \text{ and } \ \phi|_{\Gamma_f} = 0\right\}. \end{aligned}$$

With respect to the "solid portion" Ω_s of the geometry, we define the following classic operators which mathematically realize the 3-D system of elasticity (see e.g., [9]):

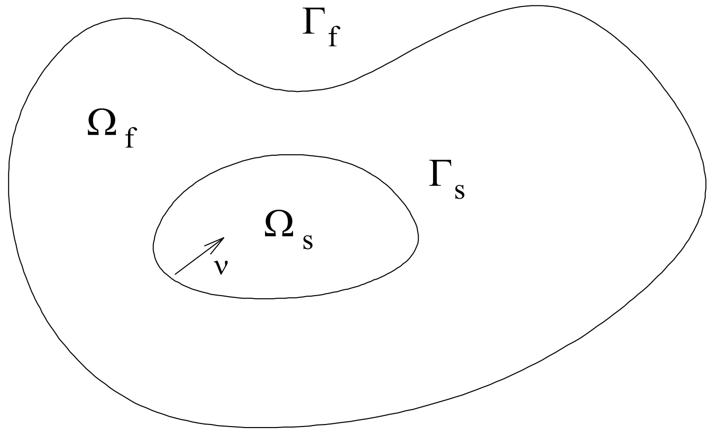

Figure 1: The Geometry of the Problem

1. For $w = [w_1, w_2, w_3]$, the *strain tensor* $\{\epsilon_{ij}\}$ is given by

$$\epsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \quad 1 \leq i, j \leq 3.$$

2. Subsequently, the *stress tensor* is described by means of Hooke's Law:

$$\sigma_{ij}(w) = \lambda\left(\sum_{k=1}^{3} \epsilon_{kk}(w)\right)\delta_{ij} + 2\mu\epsilon_{ij}(w), \;\; 1 \leq i, j \leq 3,$$

   where $\lambda \geq 0$ and $\mu > 0$ are the so-called *Lame's coefficients* of the system. Moreover, δ_{ij} denotes as usual the Kronecker delta; i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

   Letting

$$\sigma(w) = (\sigma_{ij}(w))_{i,j=1}^{3}, \;\; \epsilon(w) = (\epsilon_{ij}(w))_{i,j=1}^{3},$$

   then by virtue of Korn's inequality, $[H^1(\Omega_s)]^3$ may be endowed with the following inner-product, equivalent to the usual $[H^1(\Omega_s)]^3$-norm:

$$(w, \tilde{w})_{[H^1(\Omega_s)]^3} = (\epsilon(w), \sigma(\tilde{w}))_{\Omega_s} + (w, \tilde{w})_{\Omega_s}. \tag{1}$$

3. With this nomenclature, we denote the Hilbert space $\mathbf{H}$ (of wellposedness) as

$$\begin{aligned} \mathbf{H} &\equiv Null(div) \times [H^1(\Omega_s)]^3 \times [L^2(\Omega_s)]^3; \\ \left(\begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \tilde{u}_0 \\ \tilde{w}_0 \\ \tilde{w}_1 \end{bmatrix}\right)_{\mathbf{H}} &\equiv (u_0, \tilde{u}_0)_{\Omega_f} + (\epsilon(w_0), \sigma(\tilde{w}_0))_{\Omega_s} + (w_0, \tilde{w}_0)_{\Omega_s} + (w_1, \tilde{w}_1)_{\Omega_s}. \end{aligned}$$

   (Here, $(\cdot,\cdot)_{\Omega_f}$ and $(\cdot,\cdot)_{\Omega s}$ denote the respective L^2-norms on the two geometries.)

We will discern strong stability properties of functions $[u(t), w(t), w_t(t)] \in C([0, T]; \mathbf{H})$ which solve the following problem:

$$\begin{cases} (u_t, \phi)_{\Omega_f} + (\nabla u, \nabla \phi)_{\Omega_f} - \langle \sigma(w)\cdot\nu, \phi\rangle_{\Gamma_s} = 0 \ \text{ on } (0,\infty), \text{ for all } \phi \in V; \\ div u = 0 \ \text{ in } (0,\infty)\times\Omega_f \\ u|_{\Gamma_f} = 0 \ \text{ on } (0,\infty)\times\Gamma_f \\ w_{tt} - div\sigma(w) + w = 0 \ \text{ in } (0,\infty)\times\Omega_s \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \ \text{ on } (0,\infty)\times\Gamma_s \\ [u(0), w(0), w_t(0)] = [u_0, w_0, w_1] \in \mathbf{H}. \end{cases} \tag{2}$$

Remark 1 *The fluid variational relation in (2) is the weak formulation of the following coupled Stokes flow-elasticity system, with inhomogeneous Neumann boundary data:*

$$\begin{cases} u_t - \Delta u + \nabla p = 0 \ \textit{ on } (0,\infty)\times\Omega_f; \\ div u = 0 \ \textit{ in } (0,\infty)\times\Omega_f \\ u|_{\Gamma_f} = 0 \ \textit{ on } (0,\infty)\times\Gamma_f \\ \dfrac{\partial u}{\partial\nu} = \sigma(w)\cdot\nu - p\nu \ \textit{ in } (0,\infty)\times\Gamma_s \\ w_{tt} - div\sigma(w) + w = 0 \ \textit{ in } (0,\infty)\times\Omega_s \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \ \textit{ on } (0,\infty)\times\Gamma_s \\ [u(0), w(0), w_t(0)] = [u_0, w_0, w_1] \in \mathbf{H}. \end{cases} \tag{3}$$

In fact, because of the "hidden regularity" enjoyed by the displacement w–i.e., $\sigma(w)\cdot\nu|_{\Gamma_s} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_s))$ (see [4])–one can justify that weak solutions of (2) are classical solutions of (3), in the sense of distributions.

Because of the recent wellposedness result in [4], we have continuity of the solution map in the space $\mathbf{H}$ of wellposedness; i.e.,

$$[u_0, w_0, w_1] \in \mathbf{H} \Rightarrow [u(t), w(t), w_t(t)] \in C([0,T];\mathbf{H}).$$

In fact, this problem admits (a nonpedestrian) semigroup formulation: To wit, as in [4] we define the operator $A : V \times [H^{-\frac{1}{2}}(\Gamma)]^3 \to V'$ by

$$\langle A(u,z), \phi\rangle_{V'\times V} = (\nabla u, \nabla\phi)_{L^2(\Omega_f)} - \langle \sigma(z)\cdot\nu, \phi\rangle_{\Gamma_s} \quad \text{for all } \phi \in V$$

We subsequently define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ by

$$\begin{aligned} \mathcal{A}\begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} &= \begin{bmatrix} -A(u_0, \sigma(w_0)) \\ w_1 \\ div(w_0) - w_0 \end{bmatrix}, \\ D(\mathcal{A}) &= \left\{ [u_0, w_0, w_1] \in \mathbf{H} : u_0 \in V, \ [A(u_0, \sigma(w_0)), w_1, div\sigma(w_0)] \in \mathbf{H}, \ u_0|_{\Gamma_s} = w_1|_{\Gamma_s} \right\}. \end{aligned} \tag{4}$$

It is shown in [4] that $\mathcal{A}$ generates a C_0-semigroup of contractions $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ on $\mathbf{H}$. Thus the weak solution to (2) is given by

$$\begin{bmatrix} u(t) \\ w(t) \\ w_t(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} \in C([0,T];\mathbf{H}).$$

From (2), we readily see that this semigroup is dissipative; in particular, for all $0 \leq s < t$, we have

$$\left\| \begin{bmatrix} u(s) \\ w(s) \\ w_t(s) \end{bmatrix} \right\|_{\mathbf{H}}^2 = \left\| \begin{bmatrix} u(t) \\ w(t) \\ w_t(t) \end{bmatrix} \right\|_{\mathbf{H}}^2 + 2\int_s^t |\nabla u|^2 \, d\tau. \tag{5}$$

This dissipation naturally gives rise to the question of strong stability; recall that a C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ is said to be *strongly stable* if for every $\mathbf{x} \in \mathbf{H}$, $\lim_{t\to\infty} e^{\mathcal{A}t}\mathbf{x} = 0$. Moreover, one can easily infer that zero is not an eigenvalue of $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, as defined in (4). Thus our problem of asymptotic behaviour for the fluid-structure dynamics avoids the issue of "steady states".

As we said at the outset, the definition of the domain $D(\mathcal{A})$ means that the resolvent $\mathcal{R}(\lambda; \mathcal{A})$ is not compact as a mapping into $\mathbf{H}$. Nor does $\mathcal{R}(\lambda; \mathcal{A})$ admit of an explicit representation. Thus, the usual stability solutions outlined in [10], [1] and [13] are not applicable. Instead, we will appeal to the following operator theoretic result:

Theorem 2 *(see See Theorem 8.7 of [6]; see also p. 75-76 of [15]) Let $\mathcal{A}$ generate a C_0-semigroup of completely non-unitary contractions on a Hilbert space $\mathbf{H}$. If there exists a dense set $M \subset \mathbf{H}$ such that*

$$\lim_{\alpha\to 0^+} \sqrt{\alpha}\mathcal{R}(\alpha + i\beta; \mathcal{A})x = 0 \quad \text{for every } x \in M \text{ and almost every } \beta \in \mathbb{R},$$

then the semigroup is strongly stable.

(We recall that a contraction C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is *completely non-unitary* (c.n.u.) if $\mathbf{H}$ has no nontrivial reducing subspace for $e^{\mathcal{A}(\cdot)}$ on which $e^{\mathcal{A}(\cdot)}$ is unitary.)

Through the agency of this abstract result, we will show the following:

Theorem 3 *The fluid-structure semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ generated by $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ (as defined in (4)) is strongly stable.*

2 Proof of Theorem 3

2.1 A Preliminary Result

The proof will follow the algorthim devised in [3]. In what follows, we will have need of the following elliptic operator $\mathring{\mathbf{A}}$: $D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3$, defined on the solid portion of the geometry Ω_s:

$$\mathring{\mathbf{A}}\omega = -div\sigma(\omega) + \omega; \quad D(\mathring{\mathbf{A}}) = \left[H^2(\Omega_s) \cap H_0^1(\Omega_s)\right]^3. \tag{6}$$

By Korn's inequality, $\mathring{\mathbf{A}}$ is positive definite and self-adjoint, with compact resolvent.

To justify the invocation of Theorem 2, we must first show the following:

Proposition 4 *The contraction semigroup $\{e^{\mathcal{A}t}\} \subset \mathcal{L}(\mathbf{H})$ of the generator defined in (4) is completely non-unitary.*

Proof of Proposition 4: Let $\mathbf{H}_u$ denote a subspace of $\mathbf{H}$ on which $\{e^{\mathcal{A}t}\}$ is unitary. Then by Stone's Theorem $i\mathcal{A}|_{\mathbf{H}_u}$ is *self-adjoint*. Thus, if λ is a (real) eigenvalue of $i\mathcal{A}|_{\mathbf{H}_u}$, corresponding to eigenfunction $[u_0, w_0, w_1]$ in $\mathbf{H}_u$, we have from (4) the following relations:

$$\begin{aligned}
-\left(\nabla \operatorname{Re} u_0, \nabla\phi\right)_{L^2(\Omega_f)} + \langle\sigma(\operatorname{Re} w_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= \lambda\left(\operatorname{Im} u_0, \nabla\phi\right)_{L^2(\Omega_f)} \quad \text{for all (real-valued) } \phi \in V; &(7)\\
\left(\operatorname{Im} u_0, \nabla\phi\right)_{L^2(\Omega_f)} - \langle\sigma(\operatorname{Im} u_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= \lambda\left(\operatorname{Re} u_0, \nabla\phi\right)_{L^2(\Omega_f)} \quad \text{for all (real-valued) } \phi \in V; &(8)\\
\operatorname{Re} w_1 &= \lambda \operatorname{Im} w_0 \quad\text{and}\quad \operatorname{Im} w_1 = -\lambda\operatorname{Re} w_0; &(9)\\
div(\sigma(\operatorname{Re} w_0)) - \operatorname{Re} w_0 &= \lambda \operatorname{Im} w_1; &(10)\\
-div(\sigma(\operatorname{Im} w_0)) + \operatorname{Im} w_0 &= \lambda \operatorname{Re} w_1. &(11)
\end{aligned}$$

We now: (i) take $\phi \equiv -\operatorname{Re} u_0$ in (7); (ii) take $\phi \equiv -\operatorname{Re} u_0$ in (8); (iii) multiply both sides of (10) by $-\operatorname{Re} w_1$ and integrate; (iii) multiply both sides of (11) by $\operatorname{Im} w_1$ and integrate. Upon an addition of these relations, we then have,

$$\|\nabla \operatorname{Re} u_0\|_{\Omega_f}^2 + \|\nabla \operatorname{Im} u_0\|_{\Omega_f}^2 = 0 \tag{12}$$

(in obtaining this relation, we have also implicitly used (9) and the fact that $u_0|_{\Gamma_s} = w_1|_{\Gamma_s}$). By Poincaré's inequality, we have then that

$$\operatorname{Re} u_0 = \operatorname{Im} u_0 = 0. \tag{13}$$

In turn, from (7) and (8) and the definition of $D(\mathcal{A})$ we have that

$$\begin{aligned} \sigma(\operatorname{Re} w_0)\cdot\nu &= 0 \text{ on } \Gamma_s; \\ \sigma(\operatorname{Im} w_0)\cdot\nu &= 0 \text{ on } \Gamma_s. \end{aligned} \tag{14}$$

In turn, since $w_1|_{\Gamma_s} = 0$ from the definition of $D(\mathcal{A})$, then using (9), (10) and (14), we have that $\operatorname{Re} w_0$ satisfies

$$\begin{aligned} \left(\lambda^2 - \mathring{\mathbf{A}}\right) \operatorname{Re} w_0 &= 0 \text{ in } \Omega_s; \\ \operatorname{Re} w_0 &= 0 \text{ on } \Gamma_s, \end{aligned}$$

where $\mathring{\mathbf{A}}$: $D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3$ is as defined in (6). From elliptic theory we have consequently that $\operatorname{Re} w_0 = 0$. In turn, from (9) and the fact that $\lambda \neq 0$, we have that $\operatorname{Im} w_1 = 0$. In the same way, $\operatorname{Im} w_0 = 0$ and $\operatorname{Re} w_1 = 0$. These consequences and (13) now complete the proof of Proposition 4. $\square$

As $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is c.n.u., we can now apply Tomilov's resolvent criterion. In fact, we shall invoke Theorem 2 with therein, $M = \mathbf{H}$ and $\beta \in \mathbb{R}^2 \setminus \mathcal{S}$, where

$$\mathcal{S} = \left\{\beta \in \mathbb{R} : \beta^2 \text{ is an eigenvalue of } \mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3\right\} \tag{15}$$

(so $\mathcal{S}$ is a countable set).

2.2 Proof proper of Theorem 3

Step 1 (A priori bounds for the damping mechanism)

With $\lambda = \alpha + i\beta$, where $\beta \in \mathbb{R}^2 \setminus \mathcal{S}$, we look at the resolvent equation

$$(\lambda I - \mathcal{A}) \begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} f_0 \\ g_0 \\ g_1 \end{bmatrix} \in \mathbf{H}. \tag{16}$$

Since $\beta = 0$ is an easy case, as then there is no coupling between real and imaginary parts, we also assume throughout that $\beta \neq 0$. By Theorem 2, it is enough to show that

$$\lim_{\alpha\to 0^+} \sqrt{\alpha}\, \|[u_0(\alpha+i\beta)), w_0(\alpha+i\beta), w_1(\alpha+i\beta)]\|_{\mathbf{H}} = 0. \tag{17}$$

Componentwise, (16) gives the following relations:

$$\begin{aligned} \lambda\,(u_0,\phi)_{\Omega_f} + (\nabla u_0, \nabla\phi)_{\Omega_f} - \langle \sigma(w_0)\cdot\nu, \phi\rangle_{-\frac{1}{2}\times\frac{1}{2}} &= (f_0,\phi)_{\Omega_f} \quad \text{for every } \phi \in V; \\ \alpha \operatorname{Re} w_0 - \beta \operatorname{Im} w_0 - \operatorname{Re} w_1 &= \operatorname{Re} g_0 \in [H^1(\Omega_s)]^3; \\ \alpha \operatorname{Im} w_0 + \beta \operatorname{Re} w_0 - \operatorname{Im} w_1 &= \operatorname{Im} g_0 \in [H^1(\Omega_s)]^3; \\ \lambda w_1 + w_0 - div\sigma(w_0) &= g_1 \in [L^2(\Omega_s)]^3. \end{aligned} \tag{18}$$

Subsequently distinguishing real and imaginary parts gives then

$$\begin{aligned}
(\alpha \operatorname{Re} u_0 - \beta \operatorname{Im} u_0, \phi)_{\Omega_f} + (\nabla \operatorname{Re} u_0, \nabla\phi)_{\Omega_f} - \langle \sigma(\operatorname{Re} w_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= (\operatorname{Re} f_0, \phi)_{\Omega_f} \quad \text{for every } \phi \in V; &(19)\\
(\alpha \operatorname{Im} u_0 + \beta \operatorname{Re} u_0, \phi)_{\Omega_f} + (\nabla \operatorname{Im} u_0, \nabla\phi)_{\Omega_f} - \langle \sigma(\operatorname{Im} w_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= (\operatorname{Im} f_0, \phi)_{\Omega_f} \quad \text{for every } \phi \in V; &(20)\\
(\alpha^2+1)\operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \beta^2 \operatorname{Re} w_0 - div\sigma(\operatorname{Re} w_0) &= \operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0; &(21)\\
(\alpha^2+1)\operatorname{Im} w_0 + 2\alpha\beta \operatorname{Re} w_0 - \beta^2 \operatorname{Im} w_0 - div\sigma(\operatorname{Im} w_0) &= \operatorname{Im} g_1 + \alpha \operatorname{Im} g_0 + \beta \operatorname{Re} g_0. &(22)
\end{aligned}$$

(Part 1) We now multiply (21) by $-\beta \operatorname{Im} w_0$, multiply (22) by $\beta \operatorname{Re} w_0$, and integrate the two subsequent relations. Integrating by parts and adding the two gives

$$2\alpha\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s} - \beta\langle \sigma(\operatorname{Re} w_0)\cdot\nu, \operatorname{Im} w_0\rangle_{\Gamma_s} + \beta\langle \sigma(\operatorname{Im} w_0)\cdot\nu, \operatorname{Re} w_0\rangle_{\Gamma_s} = F^{(1)}_\alpha, \qquad (23)$$

where

$$F^{(1)}_\alpha = -\beta\left(\operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0, \operatorname{Im} w_0\right)_{\Omega_s} + \beta\left(\operatorname{Im} g_1 + \alpha \operatorname{Im} g_0 + \beta \operatorname{Re} g_0, \operatorname{Re} w_0\right)_{\Omega_s}. \qquad (24)$$

Using the second and third relations in (18) to rewrite the boundary terms in (23), we have then

$$\begin{aligned}
&2\alpha^2\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha^2\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s}\\
&+\alpha\langle \sigma(\operatorname{Re} w_0)\cdot\nu, \operatorname{Re} w_1 - \alpha \operatorname{Re} w_0 + \operatorname{Re} g_0\rangle_{\Gamma_s} + \alpha\langle \sigma(\operatorname{Im} w_0)\cdot\nu, \operatorname{Im} w_1 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0\rangle_{\Gamma_s}\\
= \;& \alpha F^{(1)}_\alpha. \qquad (25)
\end{aligned}$$

Moreover, we take $\phi \equiv \alpha \operatorname{Re} u$ in (19); we take $\phi \equiv \alpha \operatorname{Im} u$ in (20). Integrating in space and adding the subsequent relations, we then obtain,

$$\begin{aligned}
&\alpha^2 \|\operatorname{Re} u_0\|^2_{\Omega_f} + \alpha^2 \|\operatorname{Im} u_0\|^2_{\Omega_f} + \alpha\|\nabla \operatorname{Re} u_0\|^2_{\Omega_f} + \alpha\|\nabla \operatorname{Im} u_0\|^2_{\Omega_f}\\
&\quad - \alpha\langle \sigma(\operatorname{Re} w_0)\cdot\nu, \operatorname{Re} u_0\rangle_{\Gamma_s} - \alpha\langle \sigma(\operatorname{Im} w_0)\cdot\nu, \operatorname{Im} u_0\rangle_{\Gamma_s} = F^{(2)}_\alpha, \qquad (26)
\end{aligned}$$

where

$$F^{(2)}_\alpha = \alpha\left(\operatorname{Re} f_0, \operatorname{Re} u_0\right)_{\Omega_f} + \alpha\beta\left(\operatorname{Im} f_0, \operatorname{Im} u_0\right)_{\Omega_f}. \qquad (27)$$

Adding the relations (25) and (26) and using the boundary condition $w_1|_{\Gamma_s} = u_0|_{\Gamma_s}$, we have

$$\begin{aligned}
&\alpha\|\nabla \operatorname{Im} u_0\|^2_{\Omega_f} + \alpha\|\nabla \operatorname{Re} u_0\|^2_{\Omega_f} + 2\alpha^2\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha^2\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s} + \alpha^2\|\operatorname{Re} u_0\|^2_{\Omega_f} + \alpha^2\|\operatorname{Im} u_0\|^2_{\Omega_f}\\
= \;& \alpha\left(\langle \sigma(\operatorname{Re} w_0)\cdot\nu, \alpha \operatorname{Re} w_0 - \operatorname{Re} g_0\rangle_{\Gamma_s} + \langle \sigma(\operatorname{Im} w_0)\cdot\nu, \alpha \operatorname{Im} w_0 - \operatorname{Im} g_0\rangle_{\Gamma_s}\right) + \alpha F^{(1)}_\alpha + F^{(2)}_\alpha. \qquad (28)
\end{aligned}$$

Now to estimate the first term on the right hand side, we can refer to the abstract trace result in Théorem 1, p. 307 of [2], in order to justify the following estimate:

$$\begin{aligned}
&\|\sigma(\operatorname{Re} w_0)\cdot\nu\|_{-\frac{1}{2},\Gamma_s}\\
\le\;& C\left(\|\operatorname{Re} w_0\|_{1,\Omega_s} + \|div\sigma(\operatorname{Re} w_0)\|_{\Omega_s}\right)\\
=\;& C\left(\|\operatorname{Re} w_0\|_{1,\Omega_s} + \left\|(\alpha^2+1)\operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \beta^2 \operatorname{Re} w_0 - \operatorname{Re} g_1 - \alpha \operatorname{Re} g_0 + \beta \operatorname{Im} g_0\right\|_{\Omega_s}\right), \qquad (29)
\end{aligned}$$

where we have also used the relation (21).

Moreover, we also use the following basic result from semigroup theory: *Given Banach space* X, if $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ *is the infinitesimal generator of a contraction semigroup, then for all* $\lambda = \alpha + i\beta$, with $\alpha > 0$, we have the estimate

$$\|\mathcal{R}(\lambda;\mathcal{A})\|_X \le \frac{1}{\alpha} \qquad (30)$$

(see; e.g., p. 11 of [14]).Using the estimates (29), (30) along with the Sobolev Trace Theorem, we have now

$$\left|\alpha\left\langle\sigma(\operatorname{Re}w_0)\cdot\nu,\alpha\operatorname{Re}w_0-\operatorname{Re}g_0\right\rangle_{\Gamma_s}\right|\le C_\beta\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}. \tag{31}$$

In the exact same way, we have

$$\begin{aligned}
&\|\sigma(\operatorname{Im}w_0)\cdot\nu\|_{-\frac{1}{2},\Gamma_s}\\
\le\ & C\left(\|\operatorname{Im}w_0\|_{1,\Omega_s}+\|div\sigma(\operatorname{Im}w_0)\|_{\Omega_s}\right)\\
=\ & C\left(\|\operatorname{Re}w_0\|_{1,\Omega_s}+\left\|(\alpha^2+1)\operatorname{Re}w_0+2\alpha\beta\operatorname{Re}w_0-\beta^2\operatorname{Im}w_0-\operatorname{Im}g_1-\alpha\operatorname{Im}g_0-\beta\operatorname{Re}g_0\right\|_{\Omega_s}\right);
\end{aligned} \tag{32}$$

which along with (30) gives rise to

$$\left|\alpha\left\langle\sigma(\operatorname{Im}w_0)\cdot\nu,\alpha\operatorname{Im}w_0-\operatorname{Im}g_0\right\rangle_{\Gamma_s}\right|\le C_\beta\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}. \tag{33}$$

Combining (28) with (31), (33) and the resolvent estimate (30), we have finally the following estimate for the gradient of the fluid component:

$$\begin{aligned}
&\alpha\|\nabla\operatorname{Im}u_0\|^2_{\Omega_f}+\alpha\|\nabla\operatorname{Re}u_0\|^2_{\Omega_f}+2\alpha^2\beta^2\|\operatorname{Im}w_0\|^2_{\Omega_s}+2\alpha^2\beta^2\|\operatorname{Re}w_0\|^2_{\Omega_s}+\alpha^2\|\operatorname{Re}u_0\|^2_{\Omega_f}+\alpha^2\|\operatorname{Im}u_0\|^2_{\Omega_f}\\
\le\ & C_\beta\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}.
\end{aligned} \tag{34}$$

This estimate and Poincaré's inequality gives, in turn,

Lemma 5 *Given initial data $[f_0,g_0,g_1]\in\mathbf{H}$, the fluid variable of the quantity in (16), with $\beta\in\mathbb{R}^2\diagdown\mathcal{S}$, satisfies the estimate*

$$\sqrt{\alpha}\|u_0\|_{\Omega_f}+\sqrt{\alpha}\|\nabla u_0\|_{\Omega_f}\le C_\beta\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}, \tag{35}$$

where C_β is independent of α (small).

Step 2 (a priori bounds in a lower topology).

The estimate (35) can in turn be used to derive the following:

Lemma 6 *For $\alpha>0$, the elastic component $[\operatorname{Re}w_0,\operatorname{Im}w_0]$ of (16) obeys the following estimate:*

$$\left\|\left[\sqrt{\alpha}\operatorname{Re}w_0,\sqrt{\alpha}\operatorname{Im}w_0\right]\right\|_{\Omega_s}\le C_\beta\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}, \tag{36}$$

where the constant C is independent of α (small).

Proof of Lemma 6: We define the elliptic operator $D:[L^2(\Gamma_s)]^3\to[L^2(\Omega_s)]^3$ by $Df=g$ if and only if g satisfies

$$\begin{aligned}
-div\sigma(g)+g &= 0 \text{ on } \Omega_s\\
g|_{\Gamma_s} &= f \text{ on } \Gamma_s.
\end{aligned}$$

By elliptic theory, see e.g., [12], we have $D\in\mathcal{L}([L^2(\Gamma_s)]^3,[H^{\frac{1}{2}}(\Omega_s)]^3)$. Accordingly, we have for any H^1-function ω on Ω_s,

$$-div\sigma(\omega)+w=\mathbf{\mathring{A}}\omega-\mathbf{\mathring{A}}D(\omega|_{\Gamma_s}). \tag{37}$$

Applying the expression (37) into the relation (21) we have

$$\left(\beta^2-\mathbf{\mathring{A}}\right)\operatorname{Re}w_0=-\mathbf{\mathring{A}}D(\operatorname{Re}w_0|_{\Gamma_s})+\alpha^2\operatorname{Re}w_0-2\alpha\beta\operatorname{Im}w_0-(\operatorname{Re}g_1+\alpha\operatorname{Re}g_0-\beta\operatorname{Im}g_0).$$

Multiplying both sides of this relation by $\alpha\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\operatorname{Re}w_0$, and subsequently integrating, gives now

$$\begin{aligned}\alpha\left\|\operatorname{Re}w_0\right\|^2_{\Omega_s} &= -\alpha\left(\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\mathbf{\mathring{A}}D(\operatorname{Re}w_0|_{\Gamma_s}),\operatorname{Re}w_0\right)_{\Omega_s}+\alpha\left(\alpha^2\operatorname{Re}w_0-2\alpha\beta\operatorname{Im}w_0,\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\operatorname{Re}w_0\right)_{\Omega_s}\\ &\quad -\alpha\left(\operatorname{Re}g_1+\alpha\operatorname{Re}g_0-\beta\operatorname{Im}g_0,\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\operatorname{Re}w_0\right)_{\Omega_s}. \end{aligned} \tag{38}$$

To handle the first term on the right hand side of (38), we use again the third relation in (18) to have

$$-\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\mathbf{\mathring{A}}D(\operatorname{Re}w_0|_{\Gamma_s})=\frac{1}{\beta}\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\mathbf{\mathring{A}}D([\alpha\operatorname{Im}w_0-\operatorname{Im}w_1-\operatorname{Im}g_0]_{\Gamma_s}).$$

Using $w_1|_{\Gamma_s}=u_0|_{\Gamma_s}$, Lemma 5 and the resolvent estimate (30), we have then

$$\left\|\mathcal{R}(\beta^2;\mathbf{\mathring{A}})\mathbf{\mathring{A}}D(\operatorname{Re}w_0|_{\Gamma_s})\right\|_{\Omega_s}\le C_\beta\left(\|\nabla u_0\|_{1,\Omega_s}+\|[f_0,g_0,g_1]\|_{\mathbf{H}}\right). \tag{39}$$

Applying this estimate to the right hand side of (38), followed by use of Lemma 5 and the estimates (30), $ab\le\delta a^2+C_\delta b^2$, give now

$$\alpha\left\|\operatorname{Re}w_0\right\|^2_{\Omega_s}\le C_{\beta,\delta}\left\|[f_0,g_0,g_1]\right\|^2_{\mathbf{H}}. \tag{40}$$

By these same means, we can use (22) to obtain also

$$\alpha\left\|\operatorname{Im}w_0\right\|^2_{\Omega_s}\le C_{\beta}\left\|[f_0,g_0,g_1]\right\|^2_{\mathbf{H}}. \tag{41}$$

□

Step 3 (a priori bounds in finite energy topology) Multiplying (21) by $\alpha\operatorname{Re}w_0$, and subsequently integrating gives

$$\begin{aligned}&\alpha\left(\sigma(\operatorname{Re}w_0),\epsilon(\operatorname{Re}w_0)\right)_{\Omega_s}+\alpha\left(\operatorname{Re}w_0,\operatorname{Re}w_0\right)_{\Omega_s}=\\ &-\alpha\left\langle\sigma(\operatorname{Re}w_0)\cdot\nu,\operatorname{Re}w_0\right\rangle_{\Gamma_s}+\alpha\beta^2\left\|\operatorname{Re}w_0\right\|^2_{\Omega_s}\\ &-\alpha\left(\alpha^2\operatorname{Re}w_0-2\alpha\beta\operatorname{Im}w_0-\operatorname{Re}g_1-\alpha\operatorname{Re}g_0+\beta\operatorname{Im}g_0,\operatorname{Re}w_0\right)_{\Omega_s}.\end{aligned} \tag{42}$$

Using the third resolvent relation in (18), we have

$$\alpha\left\langle\sigma(\operatorname{Re}w_0)\cdot\nu,\operatorname{Re}w_0\right\rangle_{\Gamma_s}=\frac{\alpha}{\beta}\left\langle\sigma(\operatorname{Re}w_0)\cdot\nu,\operatorname{Im}u_0-\alpha\operatorname{Im}w_0+\operatorname{Im}g_0\right\rangle_{\Gamma_s}; \tag{43}$$

applying the estimates in (29), (35) (36) and (30), we have then

$$\left|\alpha\left\langle\sigma(\operatorname{Re}w_0)\cdot\nu,\operatorname{Re}w_0\right\rangle_{\Gamma_s}\right|\le\delta\alpha\left\|\operatorname{Re}w_0\right\|^2_{1,\Omega_s}+C_{\beta,\delta}\left\|[f_0,g_0,g_1]\right\|^2_{\mathbf{H}}.$$

Applying this estimate to the right hand side of (42) and subsequently invoking estimate (36) gives now,

$$\alpha\left(\sigma(\operatorname{Re}w_0),\epsilon(\operatorname{Re}w_0)\right)_{\Omega_s}+\alpha\left(\operatorname{Re}w_0,\operatorname{Re}w_0\right)_{\Omega_s}\le C_{\beta,\delta}\left\|[f_0,g_0,g_1]\right\|^2_{\mathbf{H}},$$

where in the last step we have also used the equivalence of the H^1-norm with the inner product given in (1). The analogous steps will give us a priori energy bounds for $\operatorname{Im}w_0$. That is, we can multiply both sides of (22) by $\alpha\operatorname{Im}w_0$ to obtain the relation

$$\begin{aligned}&\alpha\left(\sigma(\operatorname{Im}w_0),\epsilon(\operatorname{Im}w_0)\right)_{\Omega_s}+\alpha\left(\operatorname{Im}w_0,\operatorname{Im}w_0\right)_{\Omega_s}=\\ &-\alpha\left\langle\sigma(\operatorname{Im}w_0)\cdot\nu,\operatorname{Im}w_0\right\rangle_{\Gamma_s}+\alpha\beta^2\left\|\operatorname{Im}w_0\right\|^2_{\Omega_s}\\ &-\alpha\left(\alpha^2\operatorname{Im}w_0+2\alpha\beta\operatorname{Im}w_0-\operatorname{Im}g_1-\alpha\operatorname{Im}g_0-\beta\operatorname{Re}g_0,\operatorname{Im}w_0\right)_{\Omega_s}.\end{aligned} \tag{44}$$

Subsequently, we estimate the right hand side of this expression by using the second resolvent relation in (18), and then (29), (35) (36) and (30). In sum, we have shown the following:

Proposition 7 *For $\alpha > 0$, the solution component $[\operatorname{Re} w_0, \operatorname{Im} w_0]$ obeys the following estimate:*

$$\left\| \left[\sqrt{\alpha} \operatorname{Re} w_0, \sqrt{\alpha} \operatorname{Im} w_0 \right] \right\|_{1,\Omega_s} \leq C_\beta \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}, \tag{45}$$

where the constant C is independent of α (small).

Step 4. Conclusion of the proof of Theorem 3.

We first note that the a priori bounds we have obtained imply that $\sqrt{\alpha} \operatorname{Re} u_0$ and $\sqrt{\alpha} \operatorname{Im} u_0$ each converge to zero strongly in $\left[H^1(\Omega_f)\right]^3$. In fact, letting α tend to zero in (28), we have after using the estimates (29), (32) (35) and (45),

$$\begin{aligned} \lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Re} u_0 &= 0 \text{ in } \left[H^1(\Omega_f)\right]^3 ; \\ \lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Im} u_0 &= 0 \text{ in } \left[H^1(\Omega_f)\right]^3 \end{aligned} \tag{46}$$

(we also implicitly used Poincaré's inequality).

Next, we use the elliptic operator defined in (6) so as to rewrite the relation in (21) as

$$\left(\beta^2 - \mathring{\mathbf{A}}\right) \operatorname{Re} w_0 = -\mathring{\mathbf{A}} D(\operatorname{Re} w_0|_{\Gamma_s}) + \alpha^2 \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \operatorname{Re} g_1 - \alpha \operatorname{Re} g_0 + \beta \operatorname{Im} g_0.$$

Using the fact that $\beta \in \mathbb{R} \diagdown \mathcal{S}$ and the third relation in (18) we have now

$$\begin{aligned} \sqrt{\alpha} \operatorname{Re} w_0 &= -\frac{\sqrt{\alpha}}{\beta} \mathcal{R}(\beta^2; \mathring{\mathbf{A}}) \mathring{\mathbf{A}} D([\operatorname{Im} u_0 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0]_{\Gamma_s}) \\ &\quad + \sqrt{\alpha} \mathcal{R}(\beta^2; \mathring{\mathbf{A}}) \left[\alpha^2 \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \operatorname{Re} g_1 - \alpha \operatorname{Re} g_0 + \beta \operatorname{Im} g_0 \right]. \end{aligned}$$

Using now the second limit in (46), the apriori bounds (45), the fact that $\operatorname{Im} w_0|_{\Gamma_s} \in \left[H^{\frac{1}{2}}(\Gamma_s)\right] \hookrightarrow \left[L^2(\Gamma_s)\right]^3$ with compact inclusion, and the compactness of $\mathcal{R}(\beta^2;\mathring{\mathbf{A}})$, we have now

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Re} w_0 = 0 \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3. \tag{47}$$

Using in the same way the relation

$$\left(\beta^2 - \mathring{\mathbf{A}}\right) \operatorname{Im} w_0 = -\mathring{\mathbf{A}} D(\operatorname{Im} w_0|_{\Gamma_s}) + \alpha^2 \operatorname{Im} w_0 + 2\alpha\beta \operatorname{Re} w_0 - \operatorname{Im} g_1 + \alpha \operatorname{Im} g_0 - \beta \operatorname{Re} g_0,$$

from (22), we will have

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Im} w_0 = 0 \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3. \tag{48}$$

Combining (47) and (48) with the second and third relations of (18) give, in turn,

$$\begin{aligned} \lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Re} w_1 &= 0 \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3 ; \\ \lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Im} w_1 &= 0 \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3. \end{aligned} \tag{49}$$

Finally, we appeal to the elastic energy relation (42). Letting α tend to zero on both sides of (42), while using (43), (29), (45), (46) and (47), we have finally

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Re} w_0 = 0 \textit{strongly} \text{ in } \left[H^1(\Omega_s)\right]^3. \tag{50}$$

In the same way, letting α go to zero on both sides of (44) gives

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \operatorname{Im} w_0 = 0 \textit{strongly} \text{ in } \left[H^1(\Omega_s)\right]^3. \tag{51}$$

The relations (46), (49), (50) and (51) now establish the limit (17). The proof of Theorem 3 is now complete.

References

[1] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one parameter semigroups*, Trans. Amer. Math. Soc. **306** No. 8 (1988), pp. 837-852.

[2] J. P. Aubin, *Analyse Fonctionnelle Appliquée, Tome 2*, Presses Universitares de France (1987).

[3] G. Avalos, “Strong stability of PDE semigroups via a resolvent criterion of Y. Tomilov”, IMA Preprint Series #2042 (University of Minnesota) (May 2005).

[4] V. Barbu, I. Lasiecka and R. Triggiani, “Concerning a fluid-structure transmission problem”, preprint (1995).

[5] K. N. Boyadzhiev and N. Levan, “Strong stability of Hilbert space contraction semigroups”, Stud. Sci. Math. Hung. **30** (1995), pp. 162-182.

[6] R. Chill and Y. Tomilov, *Stability of operator semigroups: ideas and results*, preprint (2005).

[7] Q. Du, M. Gunzburger, L. Hou, and J. Lee, “Analysis of a linear fluid-structure problem”, Disc. Cont. Dyn. Syst., 9 (2003), pp. 633-650.

[8] G. Hsiao, R. Kleinman and G. Roach, “Weak solutions of fluid-structure interaction problems, Math. Nachr., 218 (2000), pp. 139-163.

[9] S. Kesavan, *Topics in Functional Analysis*, John Wiley & Sons, New York (1989).

[10] N. Levan, “The stabilization problem: a Hilbert space operator decomposition approach”, IEEE Trans. Circuits and Systems (AS-2519) (1978), pp. 721-727.

[11] J. L. Lions, “Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéares”, Dunod, Paris (1969).

[12] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications I*, Springer-Verlag (1972).

[13] Y. I. Lyubich and V. Q. Phong, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Matematica, LXXXVII (1988), pp. 37-42.

[14] A. Pazy, “Semigroups of Linear Operators and Applications to Partial Differential Equations”, Springer-Verlag, New York (1983).

[15] Y. Tomilov, *A resolvent approach to stability of operator semigroups*, J. Operator Theory **46** (2001), pp. 63-98.