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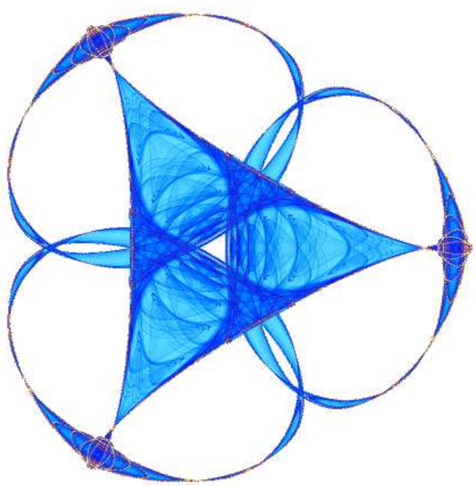

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BOUNDARY CONDITIONS FOR THE EINSTEIN-CHRISTOFFEL FORMULATION OF EINSTEIN'S EQUATIONS

DOUGLAS N. ARNOLD AND NICOLAE TARFULEA

Abstract. Specifying boundary conditions continues to be a challenge in numerical relativity in order to obtain a long time convergent numerical simulation of Einstein's equations in domains with artificial boundaries. In this paper, we address this problem for the Einstein–Christoffel (EC) symmetric hyperbolic formulation of Einstein's equations linearized around flat spacetime. First, we prescribe simple boundary conditions that make the problem well posed and preserve the constraints. Next, we indicate boundary conditions for a system that extends the linearized EC system by including the momentum constraints and whose solution solves Einstein's equations in a bounded domain. Finally, we extend our results to the case of inhomogeneous boundary conditions.

1. Introduction

In the Arnowitt–Deser–Misner or ADM decomposition, Einstein's equations split into a set of evolution equations and a set of constraint equations (see Section 2), and what one does to construct a solution consists of first specifying the initial data that satisfies the constraints and then applying the evolution equations to compute the solution for later times. The problem of well-posedness in the analytic sense has been intensely studied, with the result that there is a great deal of choice of formulations available for analytic studies (see [15], [47], [28], [6], [31], [4], [7], [8], [1], [11], [13], [14], [20], [21], [23], [26], [39], [40], among others). However, in numerical relativity, one usually solves the Einstein equations in a bounded domain (cubic boxes are commonly used) and the question that arises is what boundary conditions to provide at the artificial boundary. In general, most numerical approaches have been made using carefully chosen initial data that satisfies the constraints. On the other hand, finding appropriate boundary conditions that lead to well-posedness and consistent with constraints is a difficult problem and subject to intense investigations in the recent years. In 1998, Stewart [48] has addressed this subject within Frittelli–Reula formulation [28] linearized around flat space with unit lapse and zero shift in the quarter plane. Both main system and constraints propagate as first order strongly hyperbolic systems. This implies that vanishing values of the constraints at $t = 0$ will propagate along characteristics. One wants the values of the incoming constraints at the boundary to vanish. However, one can not just impose them to vanish along the boundaries since the constraints involve derivatives of the fields across the boundary, not just the values of the fields themselves. If the Laplace–Fourier transforms are used, the linearity of the differential equations gives algebraic equations for the transforms of the fields. Stewart deduces

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boundary conditions for the main system in terms of Laplace–Fourier transforms that preserve the constraints by imposing the incoming modes for the system of constraints to vanish and translating these conditions in terms of Laplace–Fourier transforms of the main system variables. In 1999, a well posed initial-boundary value formulation was given by Friedrich and Nagy [22] in terms of a tetrad-based Einstein–Bianchi formulation. In view of our work which is to be presented here, of particular interest are the more recent investigations regarding special boundary conditions that prevent the influx of constraint violating modes into the computational domain for various hyperbolic formulations of Einstein's equations (see [2], [3], [5], [12], [16], [17], [25], [27], [33], [30], [41], [43], [42], [51], [52], among others). Of course, specifying constraint-preserving boundary conditions for a certain formulation of Einstein's equations does not solve entirely the complicated problem of numerical relativity. There are other aspects that have to be addressed in order to obtain good numerical simulations, as for example, the existence of bulk constraint violations, in which existing violations are amplified by the evolution equations (see [18], [19], [35], [44], and references therein). A review of some work done in this direction can be found in the introductory section of [33]. Before we end this very brief review, it should also be mentioned the work done on boundary conditions for Einstein's equations in harmonic coordinates, when Einstein's equations become a system of second order hyperbolic equations for the metric components. The question of the constraints preservation does not appear here, as it is hidden in the gauge choice, i.e., the constraints have to be satisfied only at the initial surface, the harmonic gauge guarantees their preservation in time (see [37], [49], [50], and references therein).

In this paper we address the boundary conditions problem for the classical Einstein–Christoffel or EC equations derived in [6], linearized with respect to the flat Minkowski spacetime, and with arbitrary lapse density and shift perturbations. By exploiting the propagation of both main variables and constraints, we are able to prescribe two distinct sets of homogeneous boundary conditions that make the linearized EC problem well-posed and preserve the constraints. The main point is to produce solutions for the linearized ADM problem in bounded domains through the equivalence between it and the linearized EC problem. We also show an alternative way of producing solutions for the linearized ADM problem by solving an extended EC system which incorporates the momentum constraints as dynamical variables. Finally, we analyze the case of inhomogeneous boundary conditions. Much of this material appeared in the thesis of the second author [51].

The organization of this paper is as follows: in Section 2 we introduce Einstein's equations and their ADM equations for vacuum spacetime. In Section 3, by densitizing the lapse, linearizing, and defining a set of new variables, we derive the linearized EC first order symmetric hyperbolic formulation around flat spacetime. The equivalence of this formulation with the linearized ADM is proven in the Cauchy problem case. In Section 4 we indicate two distinct sets of well-posed constraint-preserving boundary conditions for the linearized EC. We prove that the linearized EC together with these boundary conditions is equivalent with linearized ADM on polyhedral domains. In Section 5 we indicate boundary conditions for an extended unconstrained system equivalent to the linearized ADM decomposition. Finally, in Section 6 we extend our results to the case of inhomogeneous boundary conditions.

2. Einstein's Equations and the ADM Decomposition

In general relativity, spacetime is a 4-dimensional manifold M of events endowed with a pseudo-Riemannian metric $g_{\alpha\beta}$ that determines the length of the line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. This metric determines curvature on the manifold, and Einstein's equations relate the curvature at a point of spacetime to the mass-energy there: $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, where $G_{\alpha\beta}$ is the *Einstein tensor*, i.e., the *trace-reversed* Ricci tensor $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$, and $T_{\alpha\beta}$ is the *energy-momentum tensor*. In what follows we will restrict ourselfs to the case of vacuum spacetime, that is $T_{\alpha\beta} = 0$. Einstein's equations can be viewed as equations for geometries, that is, their solutions are equivalent classes under spacetime diffeomorphisms of metric tensors. To break this diffeomorphisms invariance, Einstein's equations must be first transformed into a system having a well-posed Cauchy problem. In other words, the spacetime is foliated and each slice Σ_t is characterized by its intrinsic geometry γ_{ij} and extrinsic curvature K_{ij}, which is essentially the "velocity" of γ_{ij} in the unit normal direction to the slice. Subsequent slices are connected via the lapse function N and shift vector β^i corresponding to the ADM decomposition [10] (also [53]) of the line element

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{1}$$

This decomposition allows one to express six of the ten components of Einstein's equations in vacuum as a constrained system of evolution equations for the metric γ_{ij} and the extrinsic curvature K_{ij}:

$$\begin{gathered}
\dot{\gamma}_{ij} = -2NK_{ij} + 2\nabla_{(i}\beta_{j)}, \\
\dot{K}_{ij} = N[R_{ij} + (K^l_l)K_{ij} - 2K_{il}K^l_j] + \beta^l \nabla_l K_{ij} + K_{il}\nabla_j \beta^l + K_{lj}\nabla_i\beta^l - \nabla_i\nabla_j N, \\
R^i_i + (K^i_i)^2 - K_{ij}K^{ij} = 0, \\
\nabla^j K_{ij} - \nabla_i K^j_j = 0.
\end{gathered} \tag{2}$$

where we use a dot to denote time differentiation and ∇_j for the covariant derivative associated to γ_{ij}. The spatial Ricci tensor R_{ij} has components given by second order spatial differential operators applied to the spatial metric components γ_{ij}. Indices are raised and traces taken with respect to the spatial metric γ_{ij}, and paranthesized indices are used to denote the symmetric part of a tensor.

3. Linearized Einstein–Christoffel

The Einstein–Christoffel or EC formulation [6] is derived from the ADM system with a *densitized lapse*. That is, we replace the lapse N in (2) with $\alpha\sqrt{\gamma}$ where α denotes the lapse density and $\gamma := \det \gamma_{ij}$. A trivial solution to this system is Minkowski spacetime in Cartesian coordinates, given by $\gamma_{ij} = \delta_{ij}$, $K_{ij} = 0$, $\beta^i = 0$, $\alpha = 1$. In the remainder of the paper we will consider the problem linearized about this solution. To derive the linearization, we write $\gamma_{ij} = \delta_{ij} + \bar{g}_{ij}$, $K_{ij} = \bar{K}_{ij}$, $\beta^i = \bar{\beta}^i$, $\alpha = 1 + \bar{\alpha}$, where the bars indicate perturbations, assumed to be small. If we substitute these expressions into (2) (with $N = \alpha\sqrt{\gamma}$), and ignore terms which are at least quadratic in the perturbations and their derivatives, then we obtain a linear system for the perturbations. Dropping the bars, the system is

$$\dot{g}_{ij} = -2K_{ij} + 2\partial_{(i}\beta_{j)}, \tag{3}$$

$$\dot{K}_{ij} = \partial^l\partial_{(j}g_{i)l} - \frac{1}{2}\partial^l\partial_l g_{ij} - \partial_i\partial_j g^l_l - \partial_i\partial_j\alpha, \tag{4}$$

$$C := \partial^j(\partial^l g_{lj} - \partial_j g^l_l) = 0, \tag{5}$$

$$C_j := \partial^l K_{lj} - \partial_j K^l_l = 0, \tag{6}$$

where we use a dot to denote time differentiation.

Remark. For the linear system the effect of densitizing the lapse is to change the coefficient of the term $\partial_i\partial_j g^l_l$ in (4). Had we not densitized, the coefficient would have been $-1/2$ instead of -1, and the derivation of the linearized EC formulation below would not be possible.

The usual approach to solving the system (3)–(6) is to begin with initial data $g_{ij}(0)$ and $K_{ij}(0)$ defined on $\mathbb{R}^3$ and satisfying the constraint equations (5), (6), and to define g_{ij} and K_{ij} for $t > 0$ via the Cauchy problem for the evolution equations (3), (4). It can be easily shown that the constraints are then satisfied for all times. Indeed, if we apply the Hamiltonian constraint operator defined in (5) to the evolution equation (3) and apply the momentum constraint operator defined in (6) to the evolution equation (4), we obtain the first order symmetric hyperbolic system

$$\dot{C} = -2\partial^j C_j, \quad \dot{C}_j = -\frac{1}{2}\partial_j C.$$

Thus if C and C_j vanish at $t = 0$, they vanish for all time.

The linearized EC formulation provides an alternate approach to obtaining a solution of (3)–(6) with the given initial data, based on solving a system with better hyperbolicity properties. If g_{ij}, K_{ij} solve (3)–(6), define

$$f_{kij} = \frac{1}{2}[\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g^l_l)\delta_{jk} - (\partial^l g_{lj} - \partial_j g^l_l)\delta_{ik}]. \tag{7}$$

Then $-\partial^k f_{kij}$ coincides with the first three terms of the right-hand side of (4), so

$$\dot{K}_{ij} = -\partial^k f_{kij} - \partial_i\partial_j\alpha. \tag{8}$$

Differentiating (7) in time, substituting (3), and using the constraint equation (6), we obtain

$$\dot{f}_{kij} = -\partial_k K_{ij} + L_{kij}, \tag{9}$$

where

$$L_{kij} = \partial_k\partial_{(i}\beta_{j)} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik} \tag{10}$$

The evolution equations (8) and (9) for K_{ij} and f_{kij}, together with the evolution equation (3) for g_{ij}, form the linearized EC system. As initial data for this system we use the given initial values of g_{ij} and K_{ij} and derive the initial values for f_{kij} from those of g_{ij} based on (7):

$$f_{kij}(0) = \frac{1}{2}\{\partial_k g_{ij}(0) - [\partial^l g_{li}(0) - \partial_i g^l_l(0)]\delta_{jk} - [\partial^l g_{lj}(0) - \partial_j g^l_l(0)]\delta_{ik}\}. \tag{11}$$

In this paper we study the preservation of constraints by the linearized EC system and the closely related question of the equivalence of that system and the linearized ADM system. Our main interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed, but first we consider the result for the pure Cauchy problem in the remainder of this section.

Suppose that K_{ij} and f_{kij} satisfy the evolution equations (8) and (9) (which decouple from (3)). If K_{ij} satisfies the momentum constraint (6) for all time, then from (8) we obtain a constraint which must be satisfied by f_{kij}:

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l) = 0. \tag{12}$$

The following theorem shows that the pair of constraints (6), (12) is preserved by the linearized EC evolution.

Theorem 1. *Let initial data $K_{ij}(0)$, $f_{kij}(0)$ be given satisfying the constraints* (6) *and* (12)*. Then the unique solution of the evolution equations* (8)*,* (9) *satisfy* (6) *and* (12) *for all time.*

Proof. It is immediate from the evolution equations that each component K_{ij} satisfies the inhomogeneous wave equation

$$\ddot{K}_{ij} = \partial^k\partial_k K_{ij} - \partial^k L_{kij} - \partial_i\partial_j\dot{\alpha}.$$

Applying the momentum constraint operator defined in (6), we see that each component C_j satisfies the homogeneous wave equation

$$\ddot{C}_j = \partial^k\partial_k C_j. \tag{13}$$

Now $C_j = 0$ at the initial time by assumption, so if we can show that $\dot{C}_j = 0$ at the initial time, we can conclude that C_j vanishes for all time. But, from (8) and the definition of C_j,

$$\dot{C}_j = -\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l), \tag{14}$$

which vanishes at the initial time by assumption. Thus we have shown C_j vanishes for all time, i.e., (6) holds. In view of (14), (12) holds as well. □

In view of this theorem it is straightforward to establish the key result that for given initial data satisfying the constraints, the unique solution of the linearized EC evolution equations satisfies the linearized ADM system, and so the linearized ADM system and the linearized EC system are equivalent.

Theorem 2. *Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (5) *and momentum constraint* (6)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (11)*. Then the unique solution of the linearized EC evolution equations* (3)*,* (8)*,* (9) *satisfies the linearized ADM system* (3)*–*(6)*.*

Proof. First we show that the initial data $f_{kij}(0)$ defined in (7) satisfies the constraint (12). Applying the constraint operator in (12) to (7) we find

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l) = \frac{1}{2}\partial_j(\partial^l\partial^k g_{kl} - \partial^k\partial_k g_l^l) = \frac{1}{2}\partial_j C,$$

which vanishes at time 0 by (5). From Theorem 1, we conclude that $C_j = 0$ for all time, i.e., (6) holds. Then from (3) and (6) we see that $\dot{C} = -2\partial^j C_j = 0$, and, since C vanishes at initial time by assumption, C vanishes for all time, i.e., (5) holds as well.

It remains to verify (4). From (9) and (3) we have

$$\dot{f}_{kij} = \frac{1}{2}\partial_k\dot{g}_{ij} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik}.$$

Applying the momentum constraint operator to (3) and using (6), it follows that

$$\frac{1}{2}(\partial^l \dot{g}_{li} - \partial_i \dot{g}_l^l) = \partial^l \partial_{[l}\beta_{i]},$$

so $f_{kij} - [\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g_l^l)\delta_{kj} - (\partial^l g_{lj} - \partial_j g_l^l)\delta_{ki}]/2$ does not depend on time. In view of (11), we have (7).

Substituting (7) in (8) gives (4), as desired. □

4. Maximal Nonnegative Constraint Preserving Boundary Conditions

In this, the main section of the paper, we provide maximal nonnegative boundary conditions for the linearized EC system which are constraint-preserving in the sense that the analogue of Theorem 1 is true for the initial–boundary value problem. This will then imply the analogue of Theorem 2. We assume that Ω is a polyhedral domain.

Consider an arbitrary face of $\partial\Omega$ and let n^i denote its exterior unit normal. Denote by m^i and l^i two additional vectors which together n^i form an orthonormal basis. The projection operator orthogonal to n^i is then given by $\tau_i^j := m_i m^j + l_i l^j$ (and does not depend on the particular choice of these tangential vectors). Note that

$$\delta_i^j = n_i n^j + \tau_i^j, \quad \tau_i^j \tau_j^k = \tau_i^k. \tag{15}$$

Consequently,

$$v_l w^l = n^j v_j n_i w^i + \tau_l^j v_j \tau_i^l w^i \quad \text{for all } v_l,\ w^l. \tag{16}$$

First we consider the following boundary conditions on the face:

$$n^i m^j K_{ij} = n^i l^j K_{ij} = n^k n^i n^j f_{kij} = n^k m^i m^j f_{kij} = n^k l^i l^j f_{kij} = n^k m^i l^j f_{kij} = 0. \tag{17}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k \tau^{il} \tau^{jm} f_{kij} = 0, \tag{18}$$

and so do not depend on the choice of basis for the tangent space. We begin by showing that these boundary conditions are maximal nonnegative for the hyperbolic system (8), (9), and so, according to the classical theory of [24] and [34] (also [29], [32], [36], [38], [45], [46], among others), the initial–boundary value problem is well-posed.

We recall the definition of maximal nonnegative boundary conditions. Let V denote the vector space of pairs of constant tensors (K_{ij}, f_{kij}) both symmetric with respect to the indices i and j. Thus $\dim V = 24$. The boundary operator A_n associated to the evolution equations (8), (9) is the symmetric linear operator $V \to V$ given by

$$\tilde{K}_{ij} = n^k f_{kij}, \quad \tilde{f}_{kij} = n_k K_{ij}. \tag{19}$$

A subspace N of V is called nonnegative for A_n if

$$K_{ij}\tilde{K}^{ij} + f_{kij}\tilde{f}^{kij} \geq 0 \tag{20}$$

whenever $(K_{ij}, f_{kij}) \in N$ and $(\tilde{K}_{ij}, \tilde{f}_{kij})$ is defined by (19). The subspace is maximal nonnegative if also no larger subspace has this property. Since A_n has six positive, 12 zero, and six negative eigenvalues, a nonnegative subspace is maximal nonnegative if and only if it has dimension 18. Our claim is that the subspace N defined by (17) is maximal nonnegative. The dimension is clearly 18. In view of (19), the verification of (20) reduces to showing that $n^k f_{kij} K^{ij} \geq 0$ whenever (17) holds. In fact, $n^k f_{kij} K^{ij} = 0$, that is, $n^k f_{kij}$ and K_{ij} are orthogonal (when (17) holds). To see this, we use orthogonal expansions of each based on the normal and tangential components:

$$K_{ij} = n^l n_i n^m n_j K_{lm} + n^l n_i \tau_j^m K_{lm} + \tau_i^l n^m n_j K_{lm} + \tau_i^l \tau_j^m K_{lm}, \tag{21}$$

$$n^k f_{kij} = n^l n_i n^m n_j n^k f_{klm} + n^l n_i \tau_j^m n^k f_{klm} + \tau_i^l n^m n_j n^k f_{klm} + \tau_i^l \tau_j^m n^k f_{klm}. \tag{22}$$

In view of the boundary conditions (in the form (18)), the two inner terms on the right-hand side of (21) and the two outer terms on the right-hand side of (22) vanish, and so the orthogonality is evident.

Next we show that the boundary conditions are constraint-preserving. This is based on the following lemma.

Lemma 3. *Suppose that α and β^i vanish. Let K_{ij}, f_{kij} be a solution to the homogeneous hyperbolic system* (8), (9) *and suppose that the boundary conditions* (17) *are satisfied on some face of $\partial\Omega$. Let C_j be defined by* (6). *Then*

$$\dot{C}_j n^l \partial_l C^j = 0 \tag{23}$$

on the face.

Proof. In fact we shall show that $n^j C_j = 0$ (so also $n^j \dot{C}_j = 0$) and $\tau_j^p n^l \partial_l C^j = 0$, which, by (16) implies (23). First note that

$$C_j = (\delta_j^m \delta^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im} = (\delta_j^m n^i n^k + \delta_j^m \tau^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im},$$

where we have used the first identity in (15). Contracting with n^j gives

$$\begin{aligned} n^j C_j &= (n^m n^i n^k + n^m \tau^{ik} - n^k \delta^{im}) \partial_k K_{im} \\ &= -n^m n^i n^k \dot{f}_{kim} + \tau^{il} \tau_l^k n^m \partial_k K_{im} + n^k \delta^{im} \dot{f}_{kim} \end{aligned}$$

where now we have used the equation (9) (with $\beta_i = 0$) for the first and last term and the second identity in (15) for the middle term. From the boundary conditions we know that $n^m n^i n^k f_{kim} = 0$, and so the first term on the right-hand side vanishes. Similarly, we know that $\tau^{il} n^m K_{im} = 0$ on the boundary face, and so the second term vanishes as well (since the differential operator $\tau_l^k \partial_k$ is purely tangential). Finally, $n^k \delta^{im} f_{kim} = n^k (n^i n^m + l^i l^m + m^i m^m) f_{kim} = 0$, and so the third term vanishes. We have established that $n^j C_j = 0$ holds on the face.

To show that $\tau_j^p n^l \partial_l C^j = 0$ on the face, we start with the identity

$$\tau_j^p n^l \delta^{mj} \delta^{ik} = \tau^{pm} (n^i n^k + \tau^{ik}) n^l = \tau^{pm} n^i (\delta^{kl} - \tau^{kl}) + \tau^{pm} \tau^{ik} n^l.$$

Similarly

$$\tau_j^p n^l \delta^{kj} \delta^{im} = \tau^{pk} n^l n^i n^m + \tau^{pk} \tau^{im} n^l.$$

Therefore,

$$\begin{aligned} \tau_j^p n^l \partial_l C^j &= \tau_j^p n^l \partial_l (\delta^{mj} \delta^{ik} - \delta^{kj} \delta^{im}) \partial_k K_{im} \\ &= (\tau^{pm} n^i \delta^{kl} - \tau^{pm} n^i \tau^{kl} + \tau^{pm} \tau^{ik} n^l - \tau^{pk} n^l n^i n^m - \tau^{pk} \tau^{im} n^l) \partial_k \partial_l K_{im}. \end{aligned}$$

For the last three terms, we again use (9) to replace $\partial_l K_{im}$ with $-\dot{f}_{lim}$ and argue as before to see that these terms vanish. For the first term we notice that $\delta^{kl} \partial_k \partial_l K_{im} = \partial^k \partial_k K_{im} = \ddot{K}_{im}$ (from (8) and (9) with vanishing α and β^i). Since $\tau^{pm} n^i K_{im}$ vanishes on the boundary, this term vanishes. Finally we recognize that the second term is the tangential Laplacian, $\tau^{kl} \partial_k \partial_l$ applied to the quantity $n^i \tau^{pm} K_{im}$, which vanishes. This concludes the proof of (23). □

The next theorem asserts that the boundary conditions are constraint-preserving.

Theorem 4. *Let Ω be a polyhedral domain. Given initial data $K_{ij}(0)$, $f_{kij}(0)$ on Ω satisfying the constraints* (6) *and* (12)*, define K_{ij} and f_{kij} for positive time by the evolution equations* (8)*,* (9) *and the boundary conditions* (17)*. Then the constraints* (6) *and* (12) *are satisfied for all time.*

Proof. Exactly as for Theorem 1 we find that C_j satisfies the wave equation (13) and both C_j and $\dot{C}_j$ vanish at the initial time. Define the usual energy

$$E(t) = \frac{1}{2}\int_\Omega (\dot{C}_j\dot{C}^j + \partial^l C_j \partial_l C^j)\, dx.$$

Clearly $E(0) = 0$. From (13) and integration by parts

$$\dot{E} = \int_{\partial\Omega} \dot{C}_j n^l \partial_l C^j \, d\sigma. \tag{24}$$

Therefore, if $\alpha = 0$ and $\beta^i = 0$, we can invoke Lemma 3, and conclude that E is constant in time. Hence E vanishes identically. Thus C_j is constant, and, since it vanishes at time 0, it vanishes for all time. This establishes the theorem under the additional assumption that α and β^i vanish.

To extend to the case of general α and β^i we use Duhamel's principle. Let $S(t)$ denote the solution operator associated to the homogeneous boundary value problem. That is, given functions $\kappa_{ij}(0)$, $\phi_{kij}(0)$ on Ω, define $S(t)(\kappa_{ij}(0), \phi_{kij}(0)) = (\kappa_{ij}(t), \phi_{kij}(t))$ where κ_{ij}, ϕ_{kij} is the solution to the homogeneous evolution equations

$$\dot{\kappa}_{ij} = -\partial^k \phi_{kij}, \quad \dot{\phi}_{kij} = -\partial_k \kappa_{ij},$$

satisfying the boundary conditions and assuming the given initial values. Then Duhamel's principle represents the solution K_{ij}, f_{kij} of the inhomogeneous initial-boundary value problem (8), (9), (17) as

$$(K_{ij}(t), f_{kij}(t)) = S(t)(K_{ij}(0), f_{kij}(0)) + \int_0^t S(t-s)(-\partial_i\partial_j\alpha(s), L_{kij}(s))\, ds. \tag{25}$$

Now it is easy to check that the momentum constraint (6) is satisfied when K_{ij} is replaced by $-\partial_i\partial_j\alpha(s)$ (for any smooth function α), and the constraint (12) is satisfied when f_{kij} is replaced by $L_{kij}(s)$ defined by (10) (for any smooth function β^i). Hence the integrand in (25) satisfies the constraints by the result for the homogeneous case, as does the first term on the right-hand side, and thus the constraints are indeed satisfied by K_{ij}, f_{kij}. □

The analogue of Theorem 2 for the initial–boundary value problem follows from the preceding theorem exactly as before.

Theorem 5. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (5) *and momentum constraint* (6)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (11)*. Then the unique solution of the linearized EC initial–boundary value problem* (3)*,* (8)*,* (9)*, together with the boundary conditions* (17) *satisfies the linearized ADM system* (3)*–*(6) *in Ω.*

We close this section by noting a second set of boundary conditions which are maximal nonnegative and constraint-preserving. These are

$$n^i n^j K_{ij} = m^i m^j K_{ij} = l^i l^j K_{ij} = m^i l^j K_{ij} = n^k n^i m^j f_{kij} = n^k n^i l^j f_{kij} = 0, \tag{26}$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il}\tau^{jm} K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} = 0.$$

Now when we make an orthogonal expansion as in (21), (22), the outer terms on the right-hand side of the first equation and the inner terms on the right-hand side of the second equation vanish (it was the reverse before), so we again have the necessary orthogonality to demonstrate that the boundary conditions are maximal nonnegative. Similarly, to prove the analogue of Lemma 3, for these boundary conditions we show that the tangential component of $\dot{C}_j$ vanishes and the normal component of $n^l \partial_l C^j$ vanishes (it was the reverse before). Otherwise the analysis is essentially the same as for the boundary conditions (17).

5. Extended EC System

In this section we indicate an extended initial boundary value problem whose solution solves the linearized ADM system (3)–(6) in Ω. This approach could present advantages from the numerical point of view since the momentum constraint is "built-in," and so controlled for all time. The new system consists of (3), (9), and two new sets of equations corresponding to (8)

$$\dot{K}_{ij} = -\partial^k f_{kij} + \frac{1}{2}(\partial_i p_j + \partial_j p_i) - \partial^k p_k \delta_{ij} - \partial_i \partial_j \alpha, \tag{27}$$

and to a new three dimensional vector field p_i defined by

$$\dot{p}_i = \partial^l K_{li} - \partial_i K^l_l. \tag{28}$$

Observe that the additional terms that appear on the right-hand side of (27) compared with (8) are nothing but the negative components of the formal adjoint of the momentum constraint operator applied to p_i.

Let $\tilde{V}$ be the vector space of quadruples of constant tensors $(g_{ij}, K_{ij}, f_{kij}, p_k)$ symmetric with respect to the indices i and j. Thus $\dim \tilde{V} = 33$. The boundary operator $\tilde{A}_n : \tilde{V} \to \tilde{V}$ in this case is given by

$$\tilde{g}_{ij} = 0,\ \tilde{K}_{ij} = n^k f_{kij} - \frac{1}{2}(n_i p_j + n_j p_i) + n^k p_k \delta_{ij},\ \tilde{f}_{kij} = n_k K_{ij},\ \tilde{p}_i = -n^l K_{il} + n_i K^l_l. \tag{29}$$

The boundary operator $\tilde{A}_n$ associated to the evolution equations (3), (27), (9), and (28) has six positive, 21 zero, and six negative eigenvalues. Therefore, a nonnegative subspace is maximal nonnegative if and only if it has dimension 27. We claim that the following boundary conditions are maximal nonnegative for (3), (27), (9), and (28)

$$\begin{gathered} n^i m^j K_{ij} = n^i l^j K_{ij} = n^k n^i n^j f_{kij} = n^k (m^i m^j f_{kij} + p_k) = \\ n^k (l^i l^j f_{kij} + p_k) = n^k m^i l^j f_{kij} = 0. \end{gathered} \tag{30}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k (\tau^{il}\tau^{jm} f_{kij} + \tau^{lm} p_k) = 0, \tag{31}$$

and so do not depend on the choice of basis for the tangent space.

Let us prove the claim that the subspace $\tilde{N}$ defined by (30) is maximal nonnegative. Obviously, $\dim \tilde{N} = 27$. Hence, it remains to be proven that $\tilde{N}$ is also nonnegative. In view of (29), the verification of non-negativity of $\tilde{N}$ reduces to showing that

$$n^k f_{kij} K^{ij} - n^i p^j K_{ij} + n^k p_k K^l_l \geq 0 \tag{32}$$

whenever (30) holds. In fact, we can prove that the left-hand side of (32) vanishes pending (30) holds. From the boundary conditions (in the form (31)) and the orthogonal expansions (21) and (22) of K_{ij} and f_{kij}, respectively, the first term on the right-hand side of (32) reduces to $n^k\tau^{il}\tau^{jm}f_{kij}K_{lm} = -n^k p_k \tau^{lm}K_{lm}$. Then, combining the first and third terms of the left-hand side of (32) gives $-n^k p_k\tau^{ij}K_{ij}+ n^k p_k\delta^{ij}K_{ij} = n^k p_k n^i n^j K_{ij}$. Finally, by using the orthogonal decomposition $p^j = n^k p_k n^j + \tau^{kj}p_k$ and the first part of the boundary conditions (31) the second term of the left-hand side of (32) is $-n^k p_k n^i n^j K_{ij} - p_k n^i \tau^{kj} K_{ij} = -n^k p_k n^i n^j K_{ij}$, which is precisely the negative sum of the first and third terms of the left-hand side of (32). This concludes the proof of (32).

Theorem 6. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (5) *and momentum constraints* (6)*, respectively, $f_{kij}(0)$ is defined by* (7)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3), (27), (9), *and* (28), *together with the boundary conditions* (30), *satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3)–(6) *in Ω.*

Proof. Observe that the solution of the initial boundary value problem (3), (8), (9), and (17) (boundary conditions), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3), (27), (9), and (28), together with the boundary conditions (30). The conclusion follows from Theorem 5. □

We close by indicating a second set of maximal nonnegative boundary conditions (corresponding to (26)) for (3), (27), (9), and (28) for which Theorem 6 holds as well. These are

$$\begin{aligned} n^i n^j K_{ij} = m^i m^j K_{ij} = m^i l^j K_{ij} = l^i l^j K_{ij} = \\ n^k n^i m^j f_{kij} - m^k p_k = n^k n^i l^j f_{kij} - l^k p_k = 0, \end{aligned} \tag{33}$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il}\tau^{jm}K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} - \tau^{kl}p_k = 0. \tag{34}$$

6. Inhomogeneous Boundary Conditions

In this section we provide well-posed constraint-preserving *inhomogeneous* boundary conditions for (3), (8), and (9) corresponding to the two sets of boundary conditions (17), and (26), respectively. The first set of inhomogeneous boundary conditions corresponds to (17) and can be written in the following form

$$n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} = n^k n^i n^j \tilde{f}_{kij} = n^k m^i m^j \tilde{f}_{kij} = n^k l^i l^j \tilde{f}_{kij} = n^k m^i l^j \tilde{f}_{kij} = 0, \tag{35}$$

where $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given in $\overline{\Omega}$ for all time and satisfying the constraints (6) and (12), respectively.

The analogue of Theorem 4 for the inhomogeneous boundary conditions (35) is true.

Theorem 7. *Let Ω be a polyhedral domain. Given initial data $K_{ij}(0)$, $f_{kij}(0)$ on Ω satisfying the constraints* (6) *and* (12)*, define K_{ij} and f_{kij} for positive time by the evolution equations* (8), (9) *and the boundary conditions* (35)*. Then the constraints* (6) *and* (12) *are satisfied for all time.*

Proof. Observe that $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ satisfy (8) and (9) with the forcing terms replaced by $-\partial_i\partial_j\alpha - \partial^k F_{kij} - \dot{\kappa}_{ij}$ and $L_{kij} - \partial_k \kappa_{ij} - \dot{F}_{kij}$, respectively. Exactly as in Theorem 4, it follows that $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ satisfy (6) and (12), respectively, for all time. Thus, K_{ij} and f_{kij} satisfy (6) and (12), respectively, for all time. □

The analogue of Theorem 5 for the case of the inhomogeneous boundary conditions (35) follows from the preceding theorem by using the same arguments as in the proof of Theorem 2.

Theorem 8. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (5) *and momentum constraint* (6)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (11)*. Then the unique solution of the linearized EC initial–boundary value problem* (3)*,* (8)*,* (9)*, together with the inhomogeneous boundary conditions* (35) *satisfies the linearized ADM system* (3)*–*(6) *in Ω.*

Note that there is a second set of inhomogeneous boundary conditions corresponding to (26) for which Theorem 7 and Theorem 8 remain valid. These are

$$n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} = l^i l^j \tilde{K}_{ij} = m^i l^j \tilde{K}_{ij} = n^k n^i m^j \tilde{f}_{kij} = n^k n^i l^j \tilde{f}_{kij} = 0, \quad (36)$$

where again $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given and satisfying the constraints (6) and (12), respectively.

Similar considerations can be made for the extended system introduced in the previous section. There are two sets of *inhomogeneous* boundary conditions for which the extended system produces solutions of the linearized ADM system (3)–(6) on a polyhedral domain Ω. These are

$$\begin{gathered} n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} = n^k n^i n^j \tilde{f}_{kij} = n^k (m^i m^j \tilde{f}_{kij} + p_k) = n^k (l^i l^j \tilde{f}_{kij} + p_k) = \\ n^k m^i l^j \tilde{f}_{kij} = 0 \end{gathered} \quad (37)$$

and

$$\begin{gathered} n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} = m^i l^j \tilde{K}_{ij} = l^i l^j \tilde{K}_{ij} = n^k n^i m^j \tilde{f}_{kij} - m^k p_k = \\ n^k n^i l^j \tilde{f}_{kij} - l^k p_k = 0, \end{gathered} \quad (38)$$

where $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ are defined as before.

The next theorem is an extension of Theorem 6 to the case of inhomogeneous boundary conditions.

Theorem 9. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (5) *and momentum constraints* (6)*, respectively, $f_{kij}(0)$ is defined by* (7)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3)*,* (27)*,* (9)*, and* (28)*, together with the inhomogeneous boundary conditions* (37) *(or* (38)*), satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3)*–*(6) *in Ω.*

Proof. Note that the solution of the initial boundary value problem (3), (8), (9), and (35) (or (36), respectively), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3), (27), (9), and (28), together with the boundary conditions (37) (or (38), respectively). The conclusion follows from Theorem 8. □

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