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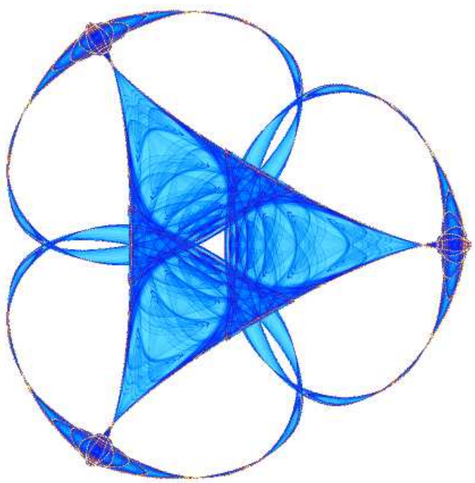

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ALGORITHMS FOR DIFFERENTIAL INVARIANTS OF SYMMETRY GROUPS OF DIFFERENTIAL EQUATIONS

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Abstract. We present new computational algorithms, based on equivariant moving frames, for classifying the differential invariants of Lie symmetry pseudo-groups of differential equations and establishing the structure of the induced differential invariant algebra. The Korteweg–deVries and Kadomtsev–Petviashvili equations are studied to illustrate these methods.

1. Introduction

Differential invariants play a fundamental role in wide range of applications including equivalence problems for geometric structures, classification of invariant differential equations and variational problems arising in the construction of physical theories, solution methods for ordinary and partial differential equations, computer vision, the design of numerical algorithms, and so on. The focus of this paper is the differential and algebraic structure of the space of differential invariants of Lie group and pseudo-group actions, with particular emphasis on those arising as symmetry groups of systems of differential equations. The most basic feature is an analog of the Hilbert Basis Theorem for such differential invariant algebras. The moving frame methods developed in [4, 27, 28, 29] provide a constructive algorithm for determining the fundamental generating differential invariants and exposing their differential algebraic structure. In this paper, we develop and implement the moving frame calculus in the context of two representative examples: the symmetry (pseudo-)groups of the Korteweg–deVries (KdV) and Kadomtsev–Petviashvili (KP) equations.

Let M be a smooth m-dimensional manifold[1], known as the *total space*. We will study the action of finite-dimensional Lie groups and infinite-dimensional Lie pseudo-groups $\mathcal{G}$ on p-dimensional submanifolds $N \subset M$. In many applications, the total space is coordinatized by the independent and dependent variables for a system of differential equations, the submanifolds are the graphs of solutions $u = f(x)$ and the (pseudo-) group $\mathcal{G}$ is the symmetry group of the system.

For $0 \le n \le \infty$, let $\mathrm{J}^n(M,p)$ denote the n^{th} order (extended) jet bundle for p-dimensional submanifolds of M, whose local coordinates $(x, u^{(n)})$ consist of the independent variables x^i, the dependent variables u^α, and their derivatives $u^\alpha_J = \partial^{\# J} u^\alpha / \partial x^J$ for $0 \le \# J \le n$. The action of $\mathcal{G}$ on submanifolds of M induces an action on $\mathrm{J}^n(M,p)$, known as its n^{th} *prolongation*. As usual, a *differential invariant* is a function[2] $I\colon \mathrm{J}^n(M,p) \to \mathbb{R}$ that is not affected by the prolonged action. We let $\mathcal{I}(\mathcal{G})$ denote the algebra[3] of differential invariants. The over-riding goal of this

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[1] In many applications, M is endowed with a bundle structure, but this is unnecessary in the general geometrical formulation of symmetry groups of differential equations, [24].

[2] Throughout, we allow the domain of any map to be a (proper) open subset of its source space. Thus, a differential invariant need only be defined on an open subset of the jet space.

[3] In our geometric approach to the subject, the term "algebra" is to be taken in a loose sense. We classify differential invariants up to functional dependency, [23]. Keep in mind that differential invariants may only be locally defined, and so functional combinations must respect the various domains of definition. A more technically precise development would recast everything in the language of sheaves, [33, 13]. However, as our primary target

research is to determine the detailed structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ when $\mathcal{G}$ is a symmetry (pseudo-)group of a system of differential equations.

The first step in this program is to establish a differential invariant version of the Hilbert Basis Theorem — that the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is finitely generated. However, "generated" does not mean in the algebraic sense — keep in mind that we are working up to functional independence, and so algebraic dependencies among differential invariants are automatically accounted for — but rather that, under suitable hypotheses, the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is (locally) generated by a finite number of differential invariants and their invariant derivatives. The undefined terms in our version of the Basis Theorem can be found in the body of the paper and in [10, 28].

Theorem 1.1. *Let $\mathcal{G}$ be a Lie pseudo-group acting on M that acts regularly and locally freely on an open subset $V \subset \mathrm{J}^n(M,p)$ for all sufficiently large n. Then, locally on V, the differential invariant algebra $\mathcal{I}(\mathcal{G})$ admits a finite generating set $I_1, \ldots, I_k$, and invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, so that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives: $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1}\mathcal{D}_{j_2}\cdots\mathcal{D}_{j_i} I_\kappa$, $\kappa = 1, \ldots, k$, $i = \#J \geq 0$.*

The original version of the Differential Invariant Basis Theorem is due to Tresse, [31]. Proofs in the case of finite-dimensional Lie groups can be found in [24, 26, 30]. For infinite-dimensional pseudo-groups, a rigorous modern formulation, based on the machinery of Spencer cohomology, can be found in Kumpera, [13]. Our version emphasizes the role of freeness, as defined below; indeed, Kumpera's cohomological growth bounds are closely tied to the local freeness of the prolonged pseudo-group action. In [29], we will present a fully constructive proof of the general result based on moving frames and Gröbner basis techniques. Generalizing the Basis Theorem to non-freely acting pseudo-groups will be the subject of future research.

In general, the differential-algebraic structure of $\mathcal{I}(\mathcal{G})$ is subject to the following complications:

- While the number p of independent invariant differential operators is fixed *a priori* by the the dimension of the submanifolds (or, equivalently, by the number of independent variables), the number k of generating differential invariants and their minimum order in the jet variables depend on the pseudo-group and are difficult to predict in advance.
- The invariant differential operators do not necessarily commute. Thus, effective computations in $\mathcal{I}(\mathcal{G})$ will, of necessity, rely on the methods from noncommutative differential algebra, [11].
- In general, the differentiated invariants are not necessarily functionally independent, and are subject to certain functional relations or *syzygies*

$$\mathcal{S}(\,\ldots\,, \mathcal{D}_J I_\kappa,\,\ldots\,) \equiv 0.$$

  Finding and classifying these syzygies is essential to understanding the structure of, as well as computing in, the differential invariant algebra $\mathcal{I}(\mathcal{G})$.

A well-known example of a differential invariant syzygy is the Codazzi equation relating derivatives of the principal curvatures (or, equivalently, the Gauss and mean curvatures), which are the generating differential invariants in the geometry of surfaces $S \subset M = \mathbb{R}^3$ under the action of the Euclidean group, [2, 12].

The structure of the differential invariant algebra is completely revealed by appealing to a new, equivariant approach to Cartan's method of *moving frames* that was initiated in [9, 10], and then developed through a series of papers, including [12, 27, 28]. The construction of moving frames

audience is oriented towards applications, we will refrain from this additional technicality, and proceed to work locally on suitable open subsets of the indicated manifolds and bundles.

for finite dimensional group actions can be effectively extended to the infinite-dimensional case by allowing the pseudo-group jet bundle coordinates to assume the role of the group parameters. Once a moving frame is fixed, the task of explicit construction, via *invariantization*, of differential invariants of all orders, as well as invariant differential forms, invariant differential operators, etc., becomes a routine algorithmic procedure. The resulting *recurrence formulas*, relating normalized and differentiated invariants, can then be used to prescribe a minimal generating set of differential invariants, and, once the commutation formulas for the invariant differential operators have been established, complete classification of the syzygies among the differentiated invariants. This procedure relies essentially on the associated Maurer–Cartan forms, which, for Lie pseudo-groups (and groups) are realized as invariant contact forms on the pseudo-group jet bundle, [27, 4].

In [4], we formulated an algorithm for obtaining the structure equations of symmetry (pseudo-) groups directly from the infinitesimal determining equations. In a similar fashion, we show here how to uncover the structure of their differential invariant algebras. Remarkably, our algorithms require only linear algebra and differentiation, and do not require any explicit formulas for either the moving frame, or the differential invariants and invariant differential operators, or even the Maurer–Cartan forms! Our methods will be illustrated by the well-studied examples of the Korteweg–deVries (KdV) and Kadomtsev–Petviashvili (KP) equations, [1] — although it should be emphasized that these examples were chosen due to their familiarity, and not their remarkable soliton properties. Our algorithms are applicable to arbitrary systems of differential equations. The theoretical foundations underlying these computational methods were established in the earlier papers [27, 28], and in [29]. In this paper, we quote the basic theorems from these references, where detailed proofs can be found. In the interests of brevity, the general algorithms will be illustrated mainly on the running example of the KdV equation; the technically more complicated case of the KP equation will be deferred until the end of the paper.

Applications in case of the finite-dimensional symmetry groups can be found in many of the references, including [12, 18, 19, 20, 25]. These include solution to equivalence and symmetry problems, invariant variational problems arising in differential geometry and mathematical physics, solution of symmetric overdetermined systems of partial differential equations, Poisson geometry and integrability of flows in homogeneous spaces, as well as applications in computer vision and numerical analysis, to name a few. Elsewhere, we will develop the corresponding applications of our methods in the case of infinite-dimensional pseudo-groups, as well as the construction of explicit solutions to nonlinear partial differential equations by Vessiot's group splitting method, [21, 22, 30, 32].

2. Preliminaries

Throughout, we will use the basic framework and notation of [23, 24] without further comment. We are concerned with the point[4] symmetry group of a system of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k, \tag{1}$$

involving p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$ and their derivatives u^α_J up to some finite order n. We regard $z = (x, u)$ as local coordinates on the *total space* M, a manifold of dimension $m = p + q$, and so the system defines a subvariety $\mathcal{S}_\Delta \subset \mathrm{J}^n(M, p)$ of the n^{th} order (extended) jet bundle of p-dimensional submanifolds of M, that is, graphs of functions $u = f(x)$. To avoid unnecessary technicalities, the system (1) is assumed to be locally solvable, [23], and define a regular submanifold of $\mathrm{J}^n(M, p)$.

[4] In this paper, we restrict our attention to point symmetries. Extensions of our methods to, say, projectable (fiber-preserving) or contact symmetry groups, [24], are straightforward.

Let $\mathcal{X}(M)$ denote the space of smooth vector fields

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha}, \tag{2}$$

on M. Let

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha} \tag{3}$$

denote the n^{th} order prolongation of the vector field to $\mathrm{J}^n(M,p)$, whose coefficients are given by the well-known prolongation formula

$$\varphi_\alpha^J = D_J\Big(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi^i\Big) + \sum_{i=1}^{p} u_{J,i}^\alpha \xi^i, \tag{4}$$

obtained by repeatedly applying the total derivatives $D_i = D_{x^i}$, $i = 1, \dots, p$, to its characteristic. Observe that each φ_α^J is a certain linear function of the derivatives $\xi_A^i = \partial^{\#A}\xi^i/\partial z^A$, $\varphi_A^\alpha = \partial^{\#A}\varphi^\alpha/\partial z^A$, of the vector field coefficients with respect to all variables $z = (x,u) = (x^1, \dots, x^p, u^1, \dots, u^q)$ with coefficients that are polynomials of the derivative coordinates u_K^β.

A vector field $\mathbf{v} \in \mathcal{X}(M)$ is an *infinitesimal symmetry* of the system of differential equations (1) if and only if it satisfies the infinitesimal invariance condition

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \quad \text{on} \quad S_\Delta \qquad \text{for all} \qquad \nu = 1, 2, \dots, k. \tag{5}$$

When expanded out, this typically forms an overdetermined system of homogeneous linear partial differential equations for the coefficients ξ^i, φ^α of the vector field (2). We let

$$\mathcal{L}(\,\dots\,, x^i, \,\dots\,, u^\alpha, \,\dots\,, \xi_A^i, \,\dots\,, \varphi_A^\alpha, \,\dots\,) = 0 \tag{6}$$

denote the completion of the system of *infinitesimal determining equations*, which includes the original determining equations along with all equations obtained by repeated differentiation.

The solution space $\mathfrak{g} \subset \mathcal{X}(M)$ to the infinitesimal determining equations (6) is the Lie algebra of infinitesimal symmetries of the system (1), and can be either finite- or infinite-dimensional. In [4], we developed new algorithms for directly determining the structure of the symmetry algebra $\mathfrak{g}$ that completely avoided integration of the determining equations. The goal of the present paper is to develop analogous computational algorithms for studying the structure of its differential invariant algebra $\mathcal{I}(\mathcal{G})$.

2.1. **The KdV equation.** Our running example will be the celebrated Korteweg–deVries (KdV) equation, [1],

$$u_t + u_{xxx} + u u_x = 0. \tag{7}$$

The total space $M = \mathbb{R}^3$ has coordinates (t,x,u), and its solutions $u = f(t,x)$ define $p = 2$-dimensional submanifolds of M. The prolongation of a vector field

$$\mathbf{v} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \varphi(t,x,u)\frac{\partial}{\partial u}$$

on M to $\mathrm{J}^n(M,2)$ has the form

$$\mathbf{v}^{(n)} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \sum_{\#J\geq 0} \varphi^J \frac{\partial}{\partial u_J}\,,$$

whose coefficients are given by the well-known explicit formulas, [23],

$$\begin{aligned}
\varphi^t &= \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u, \\
\varphi^x &= \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u, \\
&\ \vdots
\end{aligned} \tag{8}$$

The vector field $\mathbf{v}$ is an infinitesimal symmetry of the KdV equation if and only if

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + uu_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0 \qquad \text{whenever} \qquad u_t + u_{xxx} + uu_x = 0.$$

Substituting the prolongation formulas (8), and equating the coefficients of the independent derivative monomials to zero, leads to the infinitesimal determining equations which together with their differential consequences reduce to the system

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0, \qquad \varphi = \xi_t - \tfrac{2}{3}u\tau_t, \qquad \varphi_u = -\tfrac{2}{3}\tau_t = -2\xi_x, \tag{9}$$

while all the derivatives of the components of order two or higher vanish. The general solution

$$\tau = c_1 + 3c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2c_4 u,$$

defines the four-dimensional KdV symmetry algebra with the basis given by

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = t\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3t\partial_t + x\partial_x - 2u\partial_u. \tag{10}$$

3. Structure of Lie Pseudo-groups

Each vector field in the symmetry algebra $\mathfrak{g}$ generates a one-parameter local transformation group. These combine to form the (connected component of) the symmetry pseudo-group $\mathcal{G}$ of the system, which forms a sub-pseudo-group of the pseudo-group $\mathcal{D} = \mathcal{D}(M)$ of all local diffeomorphisms of the total space M. Let us briefly discuss the structure and geometry of the diffeomorphism and symmetry pseudo-groups, referring the reader to [4, 27] for details.

For $0 \le n \le \infty$, let $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M,M)$ be the subbundle $\mathcal{D}^{(n)} \to M$ consisting of the n^{th} order jets, $\mathrm{j}_n\psi$, of local diffeomorphisms $\psi\colon M \to M$. Local coordinates $(x,u,X^{(n)},U^{(n)})$ on $\mathcal{D}^{(n)}$ consist of the source (base) coordinates x^i,u^α on M, the corresponding target coordinates[5] X^i,U^α, along with their derivatives X^i_A,U^α_A, $1 \le \#A \le n$, with respect to the source coordinates. We view the jet coordinates X^i_A,U^α_A as representing the group parameters of the diffeomorphism pseudo-group $\mathcal{D}$.

The local coordinate expressions for the prolonged action of a local diffeomorphism of M on the submanifold jet bundle $\mathrm{J}^n(M,p)$ are obtained by implicit differentiation. In view of the chain rule, this action only depends on n^{th} order derivatives of the diffeomorphism at the base point, and so factors through $\mathcal{D}^{(n)}$. To formalize the process we introduce the *lifted horizontal coframe*

$$\mathrm{d}_H X^i = \sum_{j=1}^p (D_j X^i)\mathrm{d}x^j = \sum_{j=1}^p \left(X^i_{x^j} + \sum_{\alpha=1}^q u^\alpha_j X^i_{u^\alpha} \right) \mathrm{d}x^j, \qquad i = 1,2,\ldots,p, \tag{11}$$

where d_H denotes the horizontal differential. Here $X^i_{x^j}, X^i_{u^\alpha}$ are first order jet coordinates on $\mathcal{D}^{(1)}$, while u^α_j are first order jet coordinates on $\mathrm{J}^1(M,p)$. Thus, strictly speaking, the lifted horizontal coframe consists of p one-forms on the bundles $\mathcal{E}^{(n)} \to \mathrm{J}^n(M,p)$ obtained by forming the pull-back bundle of $\mathcal{D}^{(n)} \to M$ under the usual jet projection $\pi^n\colon \mathrm{J}^n(M,p) \to M$. Coordinates on $\mathcal{E}^{(n)}$

[5]Throughout, we adopt Cartan's convention that source coordinates are denoted with lower case letters, while target coordinates of diffeomorphisms and their jets are denoted by the corresponding upper case letters.

consist of the submanifold jet coordinates x^i, u^α_J along with the diffeomorphism jet coordinates (or group parameters) X^i_A, U^α_A.

The dual *lifted total differential operators*, denoted $D_{X^1}, \dots, D_{X^p}$, are defined so that

$$\mathrm{d}_H F = \sum_{j=1}^{p} (D_{X^i} F)\, \mathrm{d}_H X^i \qquad \text{for any function} \quad F\colon \mathcal{E}^{(n)} \to \mathbb{R}. \tag{12}$$

The prolonged action of a diffeomorphism jet $(X^{(n)}, U^{(n)}) \in \mathcal{D}^{(n)}$ maps the submanifold jet $(x, u^{(n)}) \in \mathrm{J}^n(M,p)$ to the target jet $(X, \widehat{U}^{(n)}) \in \mathrm{J}^n(M,p)$, whose components[6]

$$\widehat{U}^\alpha_J = D_{X^{j_1}} \cdots D_{X^{j_k}} U^\alpha, \qquad 0 \le k = \#J \le n, \quad \alpha = 1, \dots q, \tag{13}$$

are obtained by repeatedly applying the lifted total differential operators to the target dependent variables $U^\alpha = \widehat{U}^\alpha$.

Warning: In these formulas, as in (11), the total derivatives $D_i = D_{x^i}$ act on both the submanifold jet coordinates u^α_J and the diffeomorphism jet coordinates X^i_A, U^α_A in a natural manner. See [27, 28] for full details.

The symmetry group $\mathcal{G}$ forms a sub-pseudo-group of the diffeomorphism pseudo-group $\mathcal{D}$, and hence its n^{th} order jets determine a subbundle[7] $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$. When $n < \infty$, we let r_n be the fiber dimension of the subbundle $\mathcal{G}^{(n)}$, which can be identified with the pseudo-group dimension at order n. Clearly

$$0 \le r_0 \le r_1 \le r_2 \le \ \cdots \ . \tag{14}$$

In the finite-dimensional case when the pseudo-group $\mathcal{G}$ represents the (local) action of a Lie group G, the fiber dimensions stabilize: $r_n = r \le \dim G$ for $n \gg 0$. On the other hand, for infinite-dimensional pseudo-group actions, the fiber dimensions continue to increase without bound as $n \to \infty$. Local coordinates on $\mathcal{G}^{(n)}$ consist of the source coordinates x^i, u^α on M along with r_n group parameters $\lambda^{(n)} = (\lambda_1, \dots, \lambda_{r_n})$ that parametrize the fibers.

The prolonged action of the pseudo-group $\mathcal{G}$ on the submanifold jets $\mathrm{J}^n(M,p)$ is then given by restricting the prolonged diffeomorphism action (13) to $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$. Or, once a parametrization of the pseudo-group subbundle is specified, one can directly apply the induced lifted differentiation operators as in (13).

3.1. **The KdV symmetry pseudo-group.** When $M = \mathbb{R}^3$ has coordinates (t, x, u), the induced coordinates on the diffeomorphism jet bundle $\mathcal{D}^{(n)}$ are denoted by

$$(t, x, u, T, X, U, T_t, T_x, T_u, X_t, X_x, X_u, U_t, U_x, U_u, T_{tt}, T_{tx}, T_{xx}, T_{tu}, T_{xu}, T_{uu}, X_{tt}, X_{tx}, X_{xx}, \dots).$$

By integrating the infinitesimal symmetries (10), we recover the action of the KdV symmetry group $\mathcal{G}_{KdV}$ on M, which can be obtained by composing the flows of the symmetry algebra basis and is given by

$$\begin{aligned}(T, X, U) &= \exp(\lambda_4 \mathbf{v}_4) \circ \exp(\lambda_3 \mathbf{v}_3) \circ \exp(\lambda_2 \mathbf{v}_2) \circ \exp(\lambda_1 \mathbf{v}_1)(t, x, u) \\ &= \big(e^{3\lambda_4}(t + \lambda_1),\ e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2),\ e^{-2\lambda_4}(u + \lambda_3)\big),\end{aligned} \tag{15}$$

[6] We place hats over the transformed submanifold jet coordinates $\widehat{U}^\alpha_J$ to avoid confusing them with the diffeomorphism jet coordinates U^α_A.

[7] As always, we are assuming regularity of the symmetry pseudo-group.

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the group parameters. A parametrization of the subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is obtained by repeatedly differentiating T, X, U with respect to t, x, u, which yields the expressions

$$\begin{gathered} T = e^{3\lambda_4}(t+\lambda_1), \qquad X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2), \qquad U = e^{-2\lambda_4}(u+\lambda_3), \\ T_t = e^{3\lambda_4}, \;\; T_x = 0, \;\; T_u = 0, \;\; X_t = \lambda_3 e^{\lambda_4}, \;\; X_x = e^{\lambda_4}, \;\; X_u = 0, \;\; U_t = 0, \;\; U_x = 0, \;\; U_u = e^{-2\lambda_4}, \\ T_{tt} = 0, \;\; T_{tx} = 0, \;\; T_{xx} = 0, \;\; T_{tu} = 0, \;\; T_{xu} = 0, \;\; T_{uu} = 0, \;\; X_{tt} = 0, \;\; X_{tx} = 0, \;\; X_{xx} = 0, \;\; \ldots\;, \end{gathered} \tag{16}$$

implying that the fiber dimension of $\mathcal{G}^{(n)}$ is $r_n = 4 = \dim \mathcal{G}$ for all $n \geq 1$.

The lifted horizontal coframe, when restricted to $\mathcal{G}$, is

$$\begin{aligned} \mathrm{d}_H T &= (T_t + u_t T_u)\mathrm{d}t + (T_x + u_x T_u)\mathrm{d}x = e^{3\lambda_4}\mathrm{d}t, \\ \mathrm{d}_H X &= (X_t + u_t X_u)\mathrm{d}t + (X_x + u_x X_u)\mathrm{d}x = \lambda_3 e^{\lambda_4}\mathrm{d}t + e^{\lambda_4}\mathrm{d}x, \end{aligned} \tag{17}$$

with dual lifted total derivative operators

$$D_T = e^{-3\lambda_4} D_t - \lambda_3 e^{-3\lambda_4} D_x, \qquad D_X = e^{-\lambda_4} D_x, \tag{18}$$

where now D_t, D_x are the usual total derivative operators on $\mathrm{J}^\infty(M,2)$. A repeated application of these to $\widehat{U} = U = e^{-2\lambda_4}(u+\lambda_3)$, as in (13), produces the explicit formulas for prolonged action of $\mathcal{G}$ on the submanifold jet space $\mathrm{J}^n(M,2)$. Specifically, we have

$$\begin{aligned} &T = e^{3\lambda_4}(t+\lambda_1), \qquad X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2), \qquad \widehat{U} = U = e^{-2\lambda_4}(u+\lambda_3), \\ &\widehat{U}_T = D_T \widehat{U} = e^{-5\lambda_4}(u_t - \lambda_3 u_x), \qquad \widehat{U}_X = D_X \widehat{U} = e^{-3\lambda_4} u_x, \\ &\widehat{U}_{TT} = D_T^2 \widehat{U} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}), \qquad \widehat{U}_{TX} = D_X D_T \widehat{U} = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}), \\ &\widehat{U}_{XX} = D_X^2 \widehat{U} = e^{-4\lambda_4} u_{xx}, \qquad \widehat{U}_{TTT} = D_T^3 \widehat{U} = e^{-11\lambda_4}(u_{ttt} - 3\lambda_3 u_{ttx} + 3\lambda_3^2 u_{txx} - \lambda_3^3 u_{xxx}), \\ &\widehat{U}_{TTX} = D_X D_T^2 \widehat{U} = e^{-9\lambda_4}(u_{ttx} - 2\lambda_3 u_{txx} + \lambda_3^2 u_{xxx}), \\ &\widehat{U}_{TXX} = D_X^2 D_T \widehat{U} = e^{-7\lambda_4}(u_{txx} - \lambda_3 u_{xxx}), \qquad \widehat{U}_{XXX} = D_X^3 \widehat{U} = e^{-5\lambda_4} u_{xxx}, \quad \ldots\;. \end{aligned} \tag{19}$$

4. Moving Frames and Invariantization

In the finite-dimensional theory, [10], a moving frame is defined to be an equivariant map from (an open subset of) the jet bundle $\mathrm{J}^n(M,p)$ back to the Lie group G. In the more general context of pseudo-groups, [27, 28], the role of the group is played by the bundles (or, more accurately, groupoids) $\mathcal{G}^{(n)} \to M$. Let $\mathcal{H}^{(n)} \longrightarrow \mathrm{J}^n(M,p)$ be the pull-back of $\mathcal{G}^{(n)}$ along the usual jet projection $\pi^n \colon \mathrm{J}^n(M,p) \to M$, which, assuming regularity, forms a subbundle $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$. Local coordinates on $\mathcal{H}^{(n)}$ have the form $(x, u^{(n)}, \lambda^{(n)})$, where $(x, u^{(n)})$ are jet coordinates on $\mathrm{J}^n(M,p)$ while the fiber coordinates $\lambda^{(n)}$ are the pseudo-group parameters of order $\leq n$. Since $\mathcal{G}$ acts on $\mathrm{J}^n(M,p)$ by prolongation, and on $\mathcal{G}^{(n)}$ through right jet multiplication, $\mathcal{G}$ also acts on $\mathcal{H}^{(n)}$. The key definition was first proposed in [28]:

Definition 4.1. An n^{th} order *moving frame* for a pseudo-group $\mathcal{G}$ acting on p-dimensional submanifolds $N \subset M$ is a locally $\mathcal{G}$-equivariant section $\rho^{(n)} \colon \mathrm{J}^n(M,p) \to \mathcal{H}^{(n)}$ of the bundle $\mathcal{H}^{(n)} \to \mathrm{J}^n(M,p)$.

As in the finite-dimensional version, necessary and sufficient conditions for the existence of a moving frame are that the action be locally free and regular, [28]:

Theorem 4.1. *A locally equivariant moving frame exists in a neighborhood of a jet* $(x, u^{(n)}) \subset \mathrm{J}^n(M,p)$ *if and only if the* $\mathcal{G}$*-orbits near* $(x, u^{(n)})$ *form a regular foliation having* r_n*-dimensional leaves.*

Remark: The local freeness condition requires that, at order n, the dimension of the intersection of the pseudo-group orbits with the jet fibers $\mathrm{J}^n(M,p) \mid_z$ is the same as the fiber dimension of $\mathcal{G}^{(n)}$. In the case of finite-dimensional group actions, local freeness in the usual sense (discrete isotropy) implies local freeness as a pseudo-group, but not conversely, [28]. In practice, local freeness does not need to be checked a priori, but is a consequence of the successful solution to the moving frame normalization equations.

A practical way of constructing a moving frame $\rho^{(n)}$ is through *normalization* based on the choice of a cross-section to the $\mathcal{G}$-orbits. The computational algorithm is as follows:

(i) First, explicitly write out the local coordinate formulas (13) for the prolonged pseudo-group action on $\mathrm{J}^n(M,p)$ in terms of the jet coordinates $(x, u^{(n)})$ and the r_n independent pseudo-group parameters $\lambda^{(n)}$,

$$(X, \widehat{U}^{(n)}) = P^{(n)}(x, u^{(n)}, \lambda^{(n)}). \tag{20}$$

(ii) Set r_n of the coordinate functions (20) to constants,

$$P_\kappa(x, u^{(n)}, \lambda^{(n)}) = c_\kappa, \qquad \kappa = 1, 2, \ldots, r_n, \tag{21}$$

suitably chosen so as to form a cross-section[8] to the pseudo-group orbits.

(iii) Solve the *normalization equations* (21) for the independent group parameters

$$\lambda^{(n)} = h^{(n)}(x, u^{(n)}) \tag{22}$$

in terms of the submanifold jet coordinates. The induced moving frame section $\rho^{(n)} \colon \mathrm{J}^n(M,p) \to \mathcal{H}^{(n)}$ is given by

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, h^{(n)}(x, u^{(n)})). \tag{23}$$

From here on, we assume that the pseudo-group acts *eventually locally freely*, which means that it acts (locally) freely on an open subset of $\mathrm{J}^n(M,p)$ for all sufficiently large $n \geq n^\star$. According to [26], all finite-dimensional Lie groups (more correctly, all those that act locally effectively on subsets) act eventually locally freely. For infinite-dimensional pseudo-group, it can be proved, [29], that local freeness at order n automatically implies local freeness at all higher orders; the minimal such n will be called the *order of freeness* of the pseudo-group. Pseudo-groups that act eventually freely admit an *infinite order moving frame*: a hierarchy of mutually compatible moving frames. In practice, compatibility is assured by retaining all lower order cross-section normalizations when proceeding to the next higher order. See [28] for details, as well as a Taylor series version of the algorithm that performs all moving frame normalizations simultaneously.

4.1. **A moving frame for the KdV equation.** As noted earlier, the KdV symmetry group has dimension 4. Let us choose the cross-section to the $\mathcal{G}$-orbits in $\mathrm{J}^n(M,2)$, $n \geq 1$, defined by the four normalization equations

$$\begin{aligned} T &= e^{3\lambda_4}(t + \lambda_1) = 0, & \widehat{U} &= e^{-2\lambda_4}(u + \lambda_3) = 0, \\ X &= e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0, & \widehat{U}_T &= e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1. \end{aligned} \tag{24}$$

[8]Thus, we restrict our attention here, for simplicity, to coordinate cross-sections. Moving frames based on general cross-sections can be treated by adapting the methods presented by Mansfield, [18].

On the subset[9] $\mathcal{V} = \{u_t + uu_x > 0\}$, these can be solved for the group parameters

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + uu_x), \tag{25}$$

thereby prescribing the moving frame $\rho^{(n)} \colon \mathrm{J}^n(M,2) \to \mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ for $n \geq 1$. Namely, by substituting into (16), $\rho^{(n)}$ maps the point $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \dots) \in \mathrm{J}^n(M,2)$ to the pseudogroup jet in $\mathcal{H}^{(n)}$ with fiber coordinates

$$\begin{gathered} T = 0, \quad X = 0, \quad U = 0, \quad T_t = 1, \quad T_x = 0, \quad T_u = 0, \quad X_t = -u(u_t + uu_x)^{1/5}, \\ X_x = (u_t + uu_x)^{1/5}, \quad X_u = 0, \quad U_t = 0, \quad U_x = 0, \quad U_u = (u_t + uu_x)^{-2/5}, \\ T_{tt} = 0,\ T_{tx} = 0,\ T_{xx} = 0,\ X_{tt} = 0,\ T_{tu} = 0,\ T_{xu} = 0,\ T_{uu} = 0,\ X_{tx} = 0,\ X_{xx} = 0,\ \dots . \end{gathered} \tag{26}$$

By Theorem 4.1, the existence of a moving frame implies that the action of $\mathcal{G}$ is locally free and regular on the subset $\mathcal{V} = \{u_t + uu_x > 0\} \subset \mathrm{J}^n(M,2)$ for all $n \geq 1$. In practice, the existence of moving frames can be verified through direct (and successful) implementation of the normalization procedure, rather than checking the condition of local freeness and regularity of the action *a priori*.

Once a moving frame is fixed, the induced *invariantization process* ι associates to each object on $\mathrm{J}^n(M,p)$ — differential function, differential form, differential operator, etc. — a uniquely prescribed invariant counterpart with the property that they coincide when restricted to the cross-section. In local coordinates, this is accomplished by writing out the transformed version of the object, and then replacing all occurrences of the pseudo-group parameters by their moving frame expressions (22). In particular, invariantizing the n^{th} order jet coordinates $(x, u^{(n)})$ leads to the normalized differential invariants

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J). \tag{27}$$

These naturally split into two classes: Those that correspond to the r_n coordinate functions used in the normalization equations (21) are equal to the corresponding normalization constants c_κ, and are known as the *phantom differential invariants*. The remaining $s_n = \dim \mathrm{J}^n(M,p) - r_n$ differential functions form a complete system of functionally independent differential invariants, in the sense that any differential invariant of order $\leq n$ can be locally uniquely written as a function of the non-phantom differential invariants (27).

Theorem 4.2. *If the pseudo-group $\mathcal{G}$ acts locally freely and regularly on an open subset of $\mathrm{J}^n(M,p)$ for $n \gg 0$, then the non-phantom normalized differential invariants* (27) *of all orders $n \geq 0$ are functionally independent and generate the differential invariant algebra $\mathcal{I}(\mathcal{G})$.*

Invariantization is clearly an algebra morphism, so

$$\iota\big(\Phi(F_1, \dots, F_k)\big) = \Phi\big(\iota(F_1), \dots, \iota(F_k)\big)$$

for any function Φ of the differential functions $F_1, \dots, F_k$. Moreover, it defines a projection, meaning that $\iota \circ \iota = \iota$; see [10, 28]. In particular, ι does not affect differential invariants, which implies the elementary, but extremely powerful Replacement Theorem, [10]:

Theorem 4.3. *If $I(x, u^{(n)}) = I(\dots, x^i, \dots, u^\alpha_J, \dots)$ is any differential invariant, then*

$$I(x, u^{(n)}) = \iota\big(I(x, u^{(n)})\big) = I(\dots, H^i, \dots, I^\alpha_J, \dots)$$

[9]One can define alternative moving frames that include points where $u_t + uu_x = 0$ by employing different cross-sections. For brevity, in this paper we only deal with this particular choice of moving frame.

on the domain of definition of the moving frame. Similarly, any invariant system of differential equations $\Delta(x, u^{(n)}) = 0$ *can be rewritten*[10] *in terms of the normalized differential invariants by invariantization*:

$$\iota\big(\Delta(x, u^{(n)})\big) = \Delta(\ldots, H^i, \ldots, I^\alpha_J, \ldots) = 0.$$

The alternative, classical method for generating higher order differential invariants is by invariant differentiation. A basis for the *invariant differential operators* $\mathcal{D}_1, \ldots, \mathcal{D}_p$ can be obtained by invariantizing the total differential operators $D_1, \ldots, D_p$. More explicitly, we let

$$\omega^i = (\rho^{(n)})^* \mathrm{d}_H X^i, \qquad i = 1, 2, \ldots, p, \tag{28}$$

be the *contact-invariant*[11] *horizontal coframe* obtained by pulling back the lifted horizontal coframe (11) via the moving frame. In practice, the one-forms (28) are found by substituting for the pseudo-group parameters in the lifted horizontal coframe (11) in accordance with the moving frame formulas (22). The invariant differential operators are the dual total differential operators, defined so that

$$\mathrm{d}_H F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \omega^i \qquad \text{for any differential function} \qquad F\colon \mathrm{J}^n(M, p) \to \mathbb{R}. \tag{29}$$

They can be obtained directly by replacing the pseudo-group parameters in the lifted total differential operators D_{X^i} by their moving frame expressions.

4.2. **Differential invariants for the KdV equation.** For the KdV symmetry group, the differential invariants are obtained by invariantizing the jet coordinates $t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots$, which is equivalent to substituting the moving frame expressions (25) into the prolonged action formulas (19). The constant phantom differential invariants

$$H^1 = \iota(t) = 0, \qquad H^2 = \iota(x) = 0, \qquad I_0 = \iota(u) = 0, \qquad I_{10} = \iota(u_t) = 1, \tag{30}$$

result from our particular choice of normalization (24). The invariantizing the remaining coordinate functions yields a complete system of functionally independent normalized differential invariants:

$$\begin{aligned}
&I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}, \\
&I_{11} = \iota(u_{tx}) = \frac{u_{tx} + u u_{xx}}{(u_t + u u_x)^{6/5}}, \qquad I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + u u_x)^{4/5}}, \\
&I_{30} = \iota(u_{ttt}) = \frac{u_{ttt} + 3u u_{ttx} + 3u^2 u_{txx} + u^3 u_{xxx}}{(u_t + u u_x)^{11/5}}, \qquad I_{21} = \iota(u_{ttx}) = \frac{u_{ttx} + 2u u_{txx} + u^2 u_{xxx}}{(u_t + u u_x)^{9/5}}, \\
&I_{12} = \iota(u_{txx}) = \frac{u_{txx} + u u_{xxx}}{(u_t + u u_x)^{7/5}}, \qquad I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + u u_x}, \qquad \ldots .
\end{aligned} \tag{31}$$

The Replacement Theorem 4.3 allows us to immediately rewrite the KdV equation in terms of the differential invariants by applying the invariantization process to it:

$$0 = \iota(u_t + u u_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + u u_x + u_{xxx}}{u_t + u u_x}.$$

[10] The invariantized system may include an additional multiplier.

[11] These one-forms are invariant if the pseudo-group acts projectably, but only invariant modulo contact forms in general, cf. [12, 28]. A familiar example is the arc length form $\omega = ds$ in Euclidean curve geometry, which is only contact-invariant under general Euclidean motions, [24].

Note the appearance of a nonzero multiplier indicating that the KdV equation is initially defined by a relative differential invariant, [24]. The invariant horizontal one-forms

$$\omega^1 = (u_t + uu_x)^{3/5}\mathrm{d}t, \qquad \omega^2 = -u(u_t + uu_x)^{1/5}\mathrm{d}t + (u_t + uu_x)^{1/5}\mathrm{d}x, \tag{32}$$

are obtained by substituting (25) into the lifted horizontal coframe (17), while the corresponding invariant differential operators

$$\mathcal{D}_1 = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x, \qquad \mathcal{D}_2 = (u_t + uu_x)^{-1/5} D_x, \tag{33}$$

can be found either by using duality (29), or by directly substituting the moving frame expressions (25) into the lifted total derivative operators (18). Higher order differential invariants can then be constructed by repeatedly applying the invariant differential operators to the lower order differential invariants, and hence can be expressed in terms of the normalized differential invariants. For example,

$$\mathcal{D}_1 I_{01} = -\tfrac{3}{5} I_{01}^2 + I_{11} - \tfrac{3}{5} I_{01} I_{20}, \qquad \mathcal{D}_2 I_{01} = -\tfrac{3}{5} I_{01}^3 + I_{02} - \tfrac{3}{5} I_{01} I_{11},$$

as can be checked by a somewhat tedious explicit calculation. In the next section, we will develop an algorithm for constructing these recurrence formulas in a much simpler, direct fashion.

5. The Algebra of Differential Invariants

Unlike the normalized differential invariants obtained from Theorem 4.2, the *differentiated invariants* are typically not functionally independent. Thus, it behooves us to establish the *recurrence formulas* relating the normalized and differentiated invariants, which will, in turn, enable us to write down a finite generating system of differential invariants as well as a complete system of *syzygies* or functional dependencies among the differentiated invariants. The required recurrence formulas rely on the Maurer–Cartan forms for the pseudo-group, and so we begin by briefly reviewing their construction, as developed in [4, 27].

5.1. **The Maurer–Cartan forms.** First, the Maurer–Cartan forms for the diffeomorphism pseudo-group $\mathcal{D}$ are explicitly realized as the right-invariant contact forms on the infinite jet bundle $\mathcal{D}^{(\infty)}$. A basis is labeled by the fiber coordinates X^i_A, U^α_A on $\mathcal{D}^{(\infty)}$, and we use the symbols

$$\chi^i_A, \quad \mu^\alpha_A, \qquad \text{for} \qquad i = 1, \ldots p, \quad \alpha = 1, \ldots, q, \quad \#A \geq 0, \tag{34}$$

to denote the corresponding basis Maurer–Cartan forms. Their explicit formulas, along with the complete system of diffeomorphism structure equations, will not be required here, but can be found in [4, 27] .

The Maurer–Cartan forms for a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms[12] (34) to the subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$. Of course, the resulting differential forms are no longer (pointwise) linearly independent. But remarkably, the complete system of linear dependencies among the restricted forms can be immediately described in terms of the infinitesimal determining equations for the pseudo-group.

Theorem 5.1. *The restricted Maurer–Cartan forms satisfy the* lifted determining equations

$$\mathcal{L}(\,\ldots, X^i, \ldots, U^\alpha, \ldots, \chi^i_A, \ldots, \mu^\alpha_A, \ldots) = 0 \tag{35}$$

[12] To avoid unnecessary clutter, we will retain the same notation for the restricted forms.

that are obtained by applying the following replacement rules to the infinitesimal determining equations (6):

$$x^i \longmapsto X^i, \qquad u^\alpha \longmapsto U^\alpha, \qquad \xi^i_A \longmapsto \chi^i_A, \qquad \varphi^\alpha_A \longmapsto \mu^\alpha_A, \qquad \textit{for all} \qquad i,\ \alpha,\ A. \tag{36}$$

As discussed in [27] (see also [4]), the structure equations for the pseudo-group $\mathcal{G}^{(\infty)}$ can simply be obtained by imposing the dependencies (35) on the structure equations of the diffeomorphism pseudo-group.

In the construction of recurrence formulas, the most important forms are not the Maurer–Cartan forms per se, but rather, their pull-backs under the moving frame map. In what follows, we will only need the horizontal components of the resulting invariantized forms, as specified by the splitting of coordinates on M into independent and dependent variables, [24, 28].

Definition 5.1. Given a moving frame $\rho^{(n)}\colon \mathrm{J}^n(M,p) \to \mathcal{H}^{(n)}$, we define the *invariantized Maurer–Cartan forms* to be the horizontal components of the pull-backs

$$\beta^i_A = \pi_H[\,(\rho^{(n)})^* \chi^i_A\,], \qquad \zeta^\alpha_A = \pi_H[\,(\rho^{(n)})^* \mu^\alpha_A\,]. \tag{37}$$

Remark 5.1.1. In general, the pull-backs $(\rho^{(n)})^* \chi^i_A$ and $(\rho^{(n)})^* \mu^\alpha_A$ are one-forms on $\mathrm{J}^n(M,p)$ with non-trivial vertical or contact components. Only the horizontal components are required in the analysis of the algebraic structure of differential invariants. The contact components *are* important in the study of invariant variational problems and the invariant variational bicomplex, [12], and will be the focus of future research.

Applying the moving frame pull-back map to (35) and then extracting the horizontal components of the resulting linear system, we deduce the corresponding dependencies among the invariantized Maurer–Cartan forms.

Theorem 5.2. *The invariantized Maurer–Cartan forms satisfy the* invariantized determining equations

$$\mathcal{L}(\,\ldots\,, H^i,\,\ldots\,, I^\alpha,\,\ldots\,, \beta^i_A,\,\ldots\,, \zeta^\alpha_A,\,\ldots\,) = 0. \tag{38}$$

We next extend the invariantization process to include, besides differential functions and forms, the derivatives (jet coordinates) of vector field coefficients[13] by setting

$$\iota(\xi^i_A) = \beta^i_A, \qquad \iota(\varphi^\alpha_A) = \zeta^\alpha_A. \tag{39}$$

The *invariantization* of any linear differential function[14]

$$\sum_{i,A} F^i_A(x,u^{(n)})\xi^i_A + \sum_{\alpha,A} F^\alpha_A(x,u^{(n)})\varphi^\alpha_A,$$

on the space of vector fields $\mathcal{X}(M)$ is the corresponding invariant linear combination

$$\sum_{i,A} F^i_A(H,I^{(n)})\beta^i_A + \sum_{\alpha,A} F^\alpha_A(H,I^{(n)})\zeta^\alpha_A, \tag{40}$$

of invariantized Maurer–Cartan forms. In other words, to invariantize, we replace jet coordinates x^i, u^α_J by the corresponding differential invariants H^i, I^α_J, while derivatives of the vector field

[13] Each derivative $\xi^i_A, \varphi^\alpha_A$ serves to define a linear function on the space of vector fields $\mathcal{X}(M)$, and so should properly be viewed as a differential form. Thus, the fact that its invariantization is another differential form should not come as a complete surprise.

[14] All sums are finite.

coefficient are replaced by the corresponding invariantized Maurer–Cartan forms for the pseudo-group. In particular, applying the invariantization process ι to the infinitesimal determining equations (6) yields the linear dependencies (38) among the invariantized Maurer–Cartan forms.

5.2. **Maurer–Cartan forms for the KdV symmetry group.** Let us apply our constructions to the KdV symmetry group. Its Maurer–Cartan forms are obtained by restricting the diffeomorphism Maurer–Cartan forms to the pseudo-group subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$. Let ν_A, χ_A, μ_A be the diffeomorphism Maurer–Cartan forms (34) corresponding to the target jet coordinates T_A, X_A, U_A. According to Theorem 5.1, the restricted forms satisfy the linear equations

$$\begin{aligned} &\nu_X = \nu_U = \chi_U = \mu_T = \mu_X = 0, \qquad \mu - \chi_T + \tfrac{2}{3}\, U \nu_T = 0, \\ &\mu_U = -\tfrac{2}{3}\nu_T = -2\chi_X, \qquad \nu_{TT} = \nu_{TX} = \cdots = 0, \end{aligned} \tag{41}$$

obtained from the determining equations (9) by using the replacement rules (36). From these equations we see that the forms ν, χ, μ, ν_T form a basis for the Maurer–Cartan forms for the four-dimensional symmetry group $\mathcal{G}$ of the KdV equation. In [4], this fact was used to establish the structure of the KdV symmetry group directly without having to solve the determining equations.

We now pull back the Maurer–Cartan forms by our moving frame map. The resulting (horizontal) invariantized Maurer–Cartan forms are denoted by

$$\iota(\tau_A) = \alpha_A, \qquad \iota(\xi_A) = \beta_A, \qquad \iota(\varphi_A) = \gamma_A. \tag{42}$$

They are subject to the equations obtained by invariantization of the determining equations (9), and so, in view of the normalizations (30),

$$\begin{aligned} &\alpha_X = \alpha_U = \beta_U = \gamma_T = \gamma_X = 0, \qquad \gamma - \beta_T = 0, \\ &\gamma_U = -\tfrac{2}{3}\alpha_T = -2\beta_X, \qquad \alpha_{TT} = \alpha_{TX} = \cdots = 0. \end{aligned} \tag{43}$$

As with the lifted forms, we can use these linear relations to write all of the invariantized Maurer–Cartan forms as linear combinations of

$$\alpha = \iota(\tau), \qquad \beta = \iota(\xi), \qquad \gamma = \iota(\varphi), \qquad \zeta = \alpha_T = \iota(\tau_t). \tag{44}$$

5.3. **The Recurrence Formulas.** According to the prolongation formula (4), the coefficients φ_α^J of a prolonged vector field are certain well-prescribed linear combinations of the derivatives ξ_A^i, φ_A^α, $\#A \leq \#J$, of its original coefficients. Let

$$\psi_\alpha^J = \iota(\varphi_\alpha^J) \tag{45}$$

denote their invariantizations, which, in accordance with the general procedure (40), are linear combinations of invariantized Maurer–Cartan forms $\beta_A^i, \zeta_A^\alpha$ defined in (39) whose coefficients are differential invariants; in fact, they are certain universal polynomial functions of the basic normalized differential invariants I_J^α. These particular invariant differential forms provide the crucial correction terms in the recurrence relations for the differentiated invariants. See [28] for a proof of this key result.

Theorem 5.3. *The* recurrence formulas *for the normalized differential invariants* (27) *are*

$$\mathrm{d}_H H^j = \sum_{i=1}^p \bigl(\mathcal{D}_i H^j\bigr)\omega^i = \omega^i + \beta^i, \qquad \mathrm{d}_H I_J^\alpha = \sum_{i=1}^p \bigl(\mathcal{D}_i I_J^\alpha\bigr)\omega^i = \sum_{i=1}^p I_{J,i}^\alpha\, \omega^i + \psi_\alpha^J. \tag{46}$$

The recurrence formulas (46) split into two types: First, whenever H^j or I_J^α is a phantom (constant) differential invariant, its differential is identically zero, and so the left hand side of the phantom recurrence equation in (46) vanishes. Under the assumption that the pseudo-group acts locally freely at order n, the resulting equations can be solved for all the independent invariantized Maurer–Cartan forms of order $\# A \leq n$; see [29] for the general proof. We then substitute the resulting expressions for the invariantized Maurer–Cartan forms into the remaining non-phantom recurrence equations in (46) to produce the recurrence formulas. Identifying the induced coefficients of the invariant horizontal coframe $\omega^1, \ldots, \omega^p$ results in a complete system of recurrence formulas relating the differentiated and normalized invariants.

Thus, the basic recurrence formulas have the form

$$\mathcal{D}_i H^j = \delta_i^j + \hat{M}_i^j, \qquad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha, \tag{47}$$

where δ_j^i is the usual Kronecker delta, and the *correction terms* $\hat{M}_i^j, M_{J,i}^\alpha$ are fixed by the preceding algorithm. Iterating, we establish the general *recurrence formulas*

$$\mathcal{D}_K I_J^\alpha = I_{J,K}^\alpha + M_{J,K}^\alpha, \tag{48}$$

valid for any multi-indices J, K. In computations, the *correction terms* $M_{J,K}^\alpha$ are rewritten in terms of the generating differential invariants.

The most striking fact is that the preceding algorithm establishes the recurrence formulas, without any need to explicitly compute the Maurer–Cartan forms or their pull-backs in advance, nor the explicit formulas for the differential invariants and invariant differential forms, nor the infinitesimal generators or symmetry group transformations. Once the cross-section normalizations have been chosen, the algorithm is entirely based on the standard prolongation formula, and the resulting infinitesimal determining equations for the symmetry group!

With the recurrence formulas (47, 48) in hand, the generating set of differential invariants and the syzygies can, at least in relatively simple examples, be found by inspection along the same lines as in the finite-dimensional version presented in [10]. A more sophisticated approach relies on the additional algebraic structure underlying the differential invariant algebra revealed in [29], which we now briefly summarize.

Let $\mathbb{R}[x]$ be the ring of real-valued polynomials $p(x) = \sum_J c_J x^J$ in the independent variables $x^1, \ldots, x^p$. Let $\mathbb{R}[x;u]$ be the $\mathbb{R}[x]$ module consisting of polynomials $q(x,u) = \sum_J c_{J,\alpha} x^J u^\alpha$ which are linear in the dependent variables $u^1, \ldots, u^q$. By Dickson's Lemma, [5], any *monomial submodule* $\mathcal{J} \subset \mathbb{R}[x;u]$ is generated by finite number of monomials $x^{J_1} u^{\alpha_1}, \ldots, x^{J_k} u^{\alpha_k}$. We call a subspace $\mathcal{J} \subset \mathbb{R}[x;u]$ an *eventual monomial module* of *order* n if it is spanned by monomials, and its "high degree" component $\mathcal{J}_{>n}$, that is spanned by all monomials $x^J u^\alpha$ of degree $\# J > n$ in $\mathcal{J}$, forms a module. A *generating set* for an eventual monomial ideal of order n is given by a Gröbner basis for $\mathcal{J}_{>n}$ along with all monomials $x^I u^\beta \in \mathcal{J}$ of degree $\# I \leq n$.

Given a moving frame (of infinite order) based on compatible coordinate cross-sections, we identify each non-phantom normalized differential invariant I_J^α with the monomial $x^J u^\alpha$. We let $\mathcal{M}(\mathcal{G}) \subset \mathbb{R}[x;u]$ be the subspace spanned by these non-phantom monomials. An infinite order moving frame is called *algebraic* of order n if $\mathcal{M}(\mathcal{G})$ is an eventual monomial module of order n. Moving frames for finite-dimensional Lie group actions are always algebraic; indeed, if the moving frame has order n, then $\mathcal{M}(\mathcal{G})_{>n}$ contains *all* monomials of degree $> n$, and so is trivially a module. Under mild regularity assumptions, it can be proved, [29], that if an infinite-dimensional pseudo-group action is eventually free of order n, then it admits an algebraic moving frame of order n.

The following results will be established in [29]. For simplicity, we shall assume that the pseudo-group acts transitively on the independent variables, and that the cross-section has been chosen so that all $H^i = \iota(x^i) = c_i$, $i = 1, \ldots, p$, are phantom differential invariants. (Including cases when some of the independent variables lead to non-phantom differential invariants requires only technical modifications of the results.) We now state a constructive version of Tresse's Basis Theorem.

Theorem 5.4. *Suppose $\mathcal{G}$ acts eventually freely at order n, and let $\rho^{(\infty)} \colon \mathrm{J}^\infty(M,p) \to \mathcal{H}^{(\infty)}$ be an algebraic moving frame. Then, the non-phantom differential invariants I^α_J corresponding to the generators of its order n eventual monomial module $\mathcal{M}(\mathcal{G})$ generate its differential invariant algebra $\mathcal{I}(\mathcal{G})$.*

Furthermore, in [29], we apply Gröbner basis methods to analyse the algebraic structure of syzygies amongst the generating system constructed in Theorem 5.4. As in the finite-dimensional theory, [10], under suitable regularity assumptions, the generating syzygies fall into two classes. The first one consist of syzygies of the form

$$\mathcal{D}_K I^\alpha_J = c^\alpha_{JK} + M^\alpha_{J,K}, \tag{49}$$

where I^α_J is a generating differential invariant and $I^\alpha_{JK} = c^\alpha_{JK}$ is a phantom differential invariant, while the second one consists of all equations of the form

$$\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}, \tag{50}$$

where I^α_{LK} and I^α_{LJ} are generating differential invariants, the multi-indices $K \cap J = \emptyset$ are disjoint and non-zero, and L is an arbitrary multi-index. Note that the first type of syzygy (49) only arises when I^α_J has order $\le n$, and usually don't occur. All other syzygies amongst the generating differential invariants are invariant linear combinations of the invariant derivatives of the generating syzygies.

Fine details of the algorithm are illustrated in the course of the following examples.

5.4. **Recurrence formulas for the KdV equation.** In the case of the KdV equation, the prolongation of the general infinitesimal symmetry generator

$$\mathbf{v} = (c_1 + 3c_4 t)\partial_t + (c_2 + c_3 t + c_4 x)\partial_x + (c_3 - 2c_4 u)\partial_u$$

has

$$\varphi^{jk} = -j\, c_3 u_{t^{j-1}x^{k+1}} - (3j+k+2)\, c_4 u_{t^j x^k}, \qquad j+k \ge 1, \tag{51}$$

as the coefficient of $\partial/\partial u_{t^j x^k}$. Identifying $c_3 = \xi_t$, $c_4 = \frac{1}{3}\tau_t$, the corresponding invariantized forms are

$$\begin{gathered} \alpha = \iota(\tau), \qquad \beta = \iota(\xi), \qquad \psi = \iota(\varphi) = \gamma, \\ \psi^{jk} = -j\, I_{j-1,k+1}\, \gamma - \frac{3j+k+2}{3}\, I_{j,k}\, \zeta, \qquad j+k \ge 1. \end{gathered} \tag{52}$$

Thus, according to (46),

$$
\begin{aligned}
0 = \mathrm{d}_H H^1 &= \omega^1 + \alpha, \\
0 = \mathrm{d}_H H^2 &= \omega^2 + \beta, \\
0 = \mathrm{d}_H I_{00} &= I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \gamma, \\
0 = \mathrm{d}_H I_{10} &= I_{20}\omega^1 + I_{11}\omega^2 + \psi^T = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\gamma - \tfrac{5}{3}\,\zeta, \\
\mathrm{d}_H I_{01} &= I_{11}\omega^1 + I_{02}\omega^2 + \psi^X = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\zeta, \\
\mathrm{d}_H I_{20} &= I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\gamma + \psi^{TT} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\gamma - \tfrac{8}{3}I_{20}\zeta, \\
\mathrm{d}_H I_{11} &= I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\gamma + \psi^{TX} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\gamma - 2I_{11}\zeta, \\
\mathrm{d}_H I_{02} &= I_{12}\omega^1 + I_{03}\omega^2 + \psi^{XX} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\zeta, \\
&\ \ \vdots
\end{aligned}
\tag{53}
$$

The left-hand-sides of the first four recurrence formulas in (53) are all zero since they are the differentials of the phantom invariants (30). Thus we can solve those phantom recurrence equations to establish the explicit formulas for the independent invariantized Maurer–Cartan forms in terms of the invariant horizontal coframe:

$$
\alpha = -\omega^1, \qquad \beta = -\omega^2, \qquad \gamma = -\omega^1 - I_{01}\omega^2, \qquad \zeta = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2. \tag{54}
$$

Substituting these results into the recurrence formulas for the differentials

$$
\mathrm{d}_H I = (\mathcal{D}_1 I)\omega^1 + (\mathcal{D}_2 I)\omega^2
$$

of non-phantom invariants, and equating the coefficients of ω^1, ω^2 on both sides yields the complete collection of recurrence formulas for the differentiated invariants:

$$
\begin{aligned}
\mathcal{D}_1 I_{01} &= I_{11} - \tfrac{3}{5}I_{01}^2 - \tfrac{3}{5}I_{01}I_{20}, & \mathcal{D}_2 I_{01} &= I_{02} - \tfrac{3}{5}I_{01}^3 - \tfrac{3}{5}I_{01}I_{11}, \\
\mathcal{D}_1 I_{20} &= I_{30} + 2I_{11} - \tfrac{8}{5}I_{01}I_{20} - \tfrac{8}{5}I_{20}^2, & \mathcal{D}_2 I_{20} &= I_{21} + 2I_{01}I_{11} - \tfrac{8}{5}I_{01}^2 I_{20} - \tfrac{8}{5}I_{11}I_{20}, \\
\mathcal{D}_1 I_{11} &= I_{21} + I_{02} - \tfrac{6}{5}I_{01}I_{11} - \tfrac{6}{5}I_{11}I_{20}, & \mathcal{D}_2 I_{11} &= I_{12} + I_{01}I_{02} - \tfrac{6}{5}I_{01}^2 I_{11} - \tfrac{6}{5}I_{11}^2, \\
\mathcal{D}_1 I_{02} &= I_{12} - \tfrac{4}{5}I_{01}I_{02} - \tfrac{4}{5}I_{02}I_{20}, & \mathcal{D}_2 I_{02} &= I_{03} - \tfrac{4}{5}I_{01}^2 I_{02} - \tfrac{4}{5}I_{02}I_{11},
\end{aligned}
\tag{55}
$$

and so on.

In general, the expressions (52) yield the recurrence formulas

$$
\begin{aligned}
\mathcal{D}_1 I_{j,k} &= I_{j+1,k} - \frac{3j+k+2}{5}(I_{20} + I_{01})I_{j,k} + j\,I_{j-1,k+1}, \\
\mathcal{D}_2 I_{j,k} &= I_{j,k+1} - \frac{3j+k+2}{5}(I_{01}^2 + I_{11})I_{j,k} + j\,I_{01}I_{j-1,k+1},
\end{aligned}
\qquad \text{for all} \quad i,j \geq 0, \tag{56}
$$

for the normalized differential invariants, where, by definition, $I_{-1,k} = 0$. As a consequence, we conclude that every normalized differential invariant can be obtained from the two fundamental differential invariants

$$
I_{01} = \frac{u_x}{(u_t + uu_x)^{3/5}}, \qquad\qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}, \tag{57}
$$

by invariant differentiation, and hence I_{01} and I_{20} generate the KdV differential invariant algebra $\mathcal{I}(\mathcal{G}_{KdV})$. This is in accordance with Theorem 5.4, since the module corresponding to the non-phantom differential invariants induced by our choice of cross-section is generated by the monomials $xu \sim I_{01}$ and $t^2 u \sim I_{20}$. Finally, according to (49, 50), there is one fundamental syzygy, namely,

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5} I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5} I_{20} + \tfrac{19}{5} I_{01}\right) \mathcal{D}_1 I_{01} - \mathcal{D}_2 I_{01} - \tfrac{6}{25} I_{01} I_{20}^2 - \tfrac{7}{25} I_{01}^2 I_{20} + \tfrac{24}{25} I_{01}^3 = 0.$$

5.5. **The algebra of differential invariants for the KP equation.** In this example, we will show how to obtain the structure of the algebra of differential invariants, including a minimal set of generators and a complete list of basic syzygies, for the symmetry pseudo-group $\mathcal{G}_{KP}$ of the KP equation

$$u_{tx} + \tfrac{3}{2} u u_{xx} + \tfrac{3}{2} u_x^2 + \tfrac{1}{4} u_{xxxx} + \tfrac{3}{4}\,\epsilon\, u_{yy} = 0, \qquad \epsilon = \pm 1, \tag{58}$$

without having to establish its (prolonged) action in advance. Earlier work on its symmetry group and differential invariants can be found in [6, 7, 8, 14, 15, 16, 17]. The underlying total space is $M = \mathbb{R}^4$ with coordinates (t, x, y, u). Applying the standard Lie algorithm, [23], we find that a vector field

$$\mathbf{v} = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \eta(t,x,y,u)\frac{\partial}{\partial y} + \varphi(t,x,y,u)\frac{\partial}{\partial u}$$

is an infinitesimal symmetry of the KP equation if and only if its coefficients satisfy the infinitesimal symmetry determining equations

$$\begin{gathered} \tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \xi_x - \tfrac{1}{3}\tau_t = 0, \quad \xi_y + \tfrac{2}{3}\,\epsilon\,\eta_t = 0, \quad \xi_u = 0, \\ \eta_x = 0, \quad \eta_y - \tfrac{2}{3}\tau_t = 0, \quad \eta_u = 0, \quad \varphi - \tfrac{2}{3}\,\xi_t + \tfrac{2}{3}\,u\tau_t = 0, \end{gathered} \tag{59}$$

along with all their differential consequences.

Our actual choice of cross-section that defines the moving frame will be deferred until we acquire some familiarity with the structure of the recurrence formulas. First, invariantization of the determining equations (59) implies the complete system of linear dependencies among the invariantized Maurer–Cartan forms

$$\alpha_{ijkl} = \iota(\tau_{ijkl}), \qquad \beta_{ijkl} = \iota(\xi_{ijkl}), \qquad \gamma_{ijkl} = \iota(\eta_{ijkl}), \qquad \zeta_{ijkl} = \iota(\varphi_{ijkl}),$$

namely,

$$\begin{gathered} \alpha_X = 0, \quad \alpha_Y = 0, \quad \alpha_U = 0, \quad \beta_X = \frac{1}{3}\alpha_T, \quad \beta_Y = -\frac{2}{3}\epsilon\,\gamma_T, \quad \beta_U = 0 \\ \gamma_X = 0, \quad \gamma_Y = \tfrac{2}{3}\alpha_T, \quad \gamma_U = 0, \quad \zeta = \tfrac{2}{3}\beta_T - \tfrac{2}{3} I_{000}\alpha_T, \end{gathered} \tag{60}$$

and so on. Here we denote the corresponding normalized differential invariants by

$$H^1 = \iota(t), \qquad H^2 = \iota(x), \qquad H^3 = \iota(y), \qquad I_{ijk} = \iota(u_{ijk}),$$

some of which will be phantom, i.e., constant, once the moving frame is fixed. As in [4], a basis of the invariantized Maurer–Cartan forms is obtained from the involutive completion of the lifted determining equations (60) and, for example, is seen to be given by the forms

$$\alpha_{T^n}, \quad \beta_{T^n}, \quad \gamma_{T^n}, \quad n \geq 0. \tag{61}$$

The remaining lifted invariant forms can now easily be expressed in terms of the basis forms (61). We have

$$\begin{aligned}
&\alpha_{T^nX^pY^qU^r} = 0, \qquad \text{if } (p,q,r) \neq (0,0,0);\\
&\beta_{T^nX} = \tfrac{1}{3}\alpha_{T^{n+1}}, \quad \beta_{T^nY} = -\tfrac{2}{3}\epsilon\,\gamma_{T^{n+1}}, \quad \beta_{T^nYY} = -\tfrac{4}{9}\epsilon\,\beta_{T^{n+2}}, \quad \gamma_{T^nY} = \tfrac{1}{3}\alpha_{T^{n+1}},\\
&\beta_{T^nX^pY^qU^r} = 0, \quad \gamma_{T^nX^pY^qU^r} = 0, \qquad \text{for all other choices of } (p,q,r);\\
&\zeta_{T^n} = \tfrac{2}{3}\beta_{T^{n+1}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s00}\alpha_{T^{n-s+1}}, \quad \zeta_{T^nX} = \tfrac{2}{9}\alpha_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s10}\alpha_{T^{n-s+1}},\\
&\zeta_{T^nY} = -\tfrac{4}{9}\epsilon\,\gamma_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s01}\alpha_{T^{n-s+1}}, \quad \zeta_{T^nYY} = -\tfrac{4}{27}\epsilon\,\alpha_{T^{n+3}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s02}\alpha_{T^{n-s+1}},\\
&\zeta_{T^nX^pY^q} = -\tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{spq}\alpha_{T^{n-s+1}}, \qquad \text{for all other choices of } (p,q),\\
&\zeta_{T^nU} = -\tfrac{2}{3}\alpha_{T^{n+1}}, \quad \zeta_{T^nX^pY^qU^r} = 0, \qquad \text{if } r \geq 2.
\end{aligned} \tag{62}$$

Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ be the invariant differential operators dual to the invariantized horizontal coframe

$$\omega^1 = \iota(dt), \qquad \omega^2 = \iota(dx), \qquad \omega^3 = \iota(dy). \tag{63}$$

As above, the explicit formulas are not required at the moment.

It follows from (60) that the correction terms ψ_α^J in equation (45) for the KP symmetry algebra are precisely the coefficients of the invariantization of the vector field obtained by first prolonging the vector field

$$\mathbf{w} = \tau(t)\frac{\partial}{\partial t} + \xi(t,x,y)\frac{\partial}{\partial x} + \eta(t,y)\frac{\partial}{\partial y} + \left(-\tfrac{2}{3}u\tau_t(t) + \tfrac{2}{3}\xi_t(t,x,y)\right)\frac{\partial}{\partial u} \tag{64}$$

and then applying the relations

$$\xi_x = \tfrac{1}{3}\tau_t, \quad \xi_y = -\tfrac{2}{3}\epsilon\,\eta_t, \quad \eta_y = \tfrac{2}{3}\tau_t \tag{65}$$

and their differential consequences to express the resulting coefficient functions solely in terms of the repeated t-derivatives of τ, ξ and η. This yields the expression

$$\begin{aligned}
\psi_{pqr} = {}& \tfrac{2}{9}\delta_{q1}\delta_{r0}\alpha_{T^{p+2}} - \tfrac{4}{9}\delta_{q0}\delta_{r1}\epsilon\,\gamma_{T^{p+2}} - \tfrac{8}{27}\delta_{q0}\delta_{r2}\epsilon\,\alpha_{T^{p+3}}\\
&- \sum_{s=0}^{p}\binom{p}{s}\left(\frac{2+q+2r}{3} + \frac{p-s}{s+1}\right) I_{p-s,q,r}\alpha_{T^{s+1}} + \tfrac{2}{9}\epsilon\, r(r-1)\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-2}\alpha_{T^{s+2}}\\
&- \sum_{s=1}^{p}\binom{p}{s} I_{p-s,q+1,r}\beta_{T^s} - \sum_{s=1}^{p}\binom{p}{s} I_{p-s,q,r+1}\gamma_{T^s} + \tfrac{2}{3}\epsilon\, r\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-1}\gamma_{T^{s+1}}
\end{aligned} \tag{66}$$

for the correction terms $\psi_{pqr} = \psi_{T^pX^qY^r}$, where δ_{ij} denotes the Kronecker delta.

With (66), equations (46) directly yield the recurrence formulas

$$
\begin{aligned}
&\mathrm{d}_H H^1 = \omega^t + \alpha, \qquad \mathrm{d}_H H^2 = \omega^x + \beta, \qquad \mathrm{d}_H H^3 = \omega^y + \gamma,\\
&\mathrm{d}_H I_{000} = I_{100}\omega^t + I_{010}\omega^x + I_{001}\omega^y - \tfrac{2}{3} I_{000}\alpha_T + \tfrac{2}{3}\beta_T,\\
&\mathrm{d}_H I_{100} = I_{200}\omega^t + I_{110}\omega^x + I_{101}\omega^y - \tfrac{5}{3} I_{100}\alpha_T - \tfrac{2}{3} I_{000}\alpha_{TT} - I_{010}\beta_T + \tfrac{2}{3}\beta_{TT} - I_{001}\gamma_T,\\
&\mathrm{d}_H I_{010} = I_{110}\omega^t + I_{020}\omega^x + I_{011}\omega^y - I_{010}\alpha_T + \tfrac{2}{9}\alpha_{TT},\\
&\mathrm{d}_H I_{001} = I_{101}\omega^t + I_{011}\omega^x + I_{002}\omega^y - \tfrac{4}{3} I_{001}\alpha_T + \tfrac{2}{3}\,\epsilon\, I_{010}\gamma_T - \tfrac{4}{9}\,\epsilon\,\gamma_{TT},\\
&\mathrm{d}_H I_{200} = I_{300}\omega^t + I_{210}\omega^x + I_{201}\omega^y - \tfrac{8}{3} I_{200}\alpha_T - \tfrac{7}{3} I_{100}\alpha_{TT} - \tfrac{2}{3} I_{000}\alpha_{TTT}\\
&\qquad\qquad - 2I_{110}\beta_T - I_{010}\beta_{TT} + \tfrac{2}{3}\beta_{TTT} - 2I_{101}\gamma_T - I_{001}\gamma_{TT},\\
&\mathrm{d}_H I_{110} = I_{210}\omega^t + I_{120}\omega^x + I_{111}\omega^y - 2I_{110}\alpha_T - I_{010}\alpha_{TT} + \tfrac{2}{9}\alpha_{TTT} - I_{020}\beta_T - I_{011}\gamma_T,\\
&\mathrm{d}_H I_{101} = I_{201}\omega^t + I_{111}\omega^x + I_{102}\omega^y - \tfrac{7}{3} I_{101}\alpha_T - \tfrac{4}{3} I_{001}\alpha_{TT} - I_{011}\beta_T\\
&\qquad\qquad + (\tfrac{2}{3}\,\epsilon\, I_{110} - I_{002})\gamma_T + \tfrac{2}{3}\,\epsilon\, I_{010}\gamma_{TT} - \tfrac{4}{9}\,\epsilon\,\gamma_{TTT},\\
&\mathrm{d}_H I_{020} = I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y - \tfrac{4}{3} I_{020}\alpha_T,\\
&\mathrm{d}_H I_{011} = I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y - \tfrac{5}{3} I_{011}\alpha_T + \tfrac{2}{3}\,\epsilon\, I_{020}\gamma_T,\\
&\mathrm{d}_H I_{002} = I_{102}\omega^t + I_{012}\omega^x + I_{003}\omega^y - 2I_{002}\alpha_T + \tfrac{4}{9}\,\epsilon\, I_{010}\alpha_{TT} - \tfrac{8}{27}\,\epsilon\,\alpha_{TTT} - \tfrac{4}{3}\,\epsilon\, I_{011}\gamma_T,\\
&\qquad\vdots
\end{aligned}
\tag{67}
$$

In general, a specification of normalization equations defines a valid cross-section to the pseudo-group orbits if and only if the resulting phantom recurrence equations (67) can be solved for the basis (61) of invariantized Maurer–Cartan forms. For this, we choose the normalizations

$$
\begin{array}{llllll}
H^1 \longmapsto 0, & H^2 \longmapsto 0, & H^3 \longmapsto 0, & I_{000} \longmapsto 0, & I_{100} \longmapsto 0, & I_{010} \longmapsto 0,\\
I_{001} \longmapsto 0, & I_{200} \longmapsto 0, & I_{101} \longmapsto 0, & I_{020} \longmapsto 1, & I_{011} \longmapsto 0, & I_{002} \longmapsto 0,\\
I_{i,0,0} \longmapsto 0, & I_{i-1,0,1} \longmapsto 0, & I_{i-2,0,2} \longmapsto 0, & \text{for all } \ i \geq 3, & &
\end{array}
\tag{68}
$$

which, when substituted into equations (67), yield the expressions

$$
\begin{aligned}
&\alpha = -\omega^t, \qquad \beta = -\omega^x, \qquad \gamma = -\omega^y,\\
&\alpha_T = \tfrac{3}{4}(I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y), \quad \alpha_{TT} = \tfrac{9}{2}(I_{110}\omega^t + \omega^x),\\
&\alpha_{TTT} = \tfrac{27}{8}\,\epsilon\,(I_{012}\omega^x + I_{003}\omega^y), \quad \ldots \quad ;\\
&\beta_T = 0, \quad \beta_{TT} = -\tfrac{3}{2} I_{110}\omega^x, \quad \beta_{TTT} = -\tfrac{3}{2} I_{210}\omega^x, \quad \ldots \quad ;\\
&\gamma_T = -\tfrac{3}{2}\epsilon\,(I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y), \quad \gamma_{TT} = 0,\\
&\gamma_{TTT} = \tfrac{9}{4}\epsilon\,(-I_{110}I_{111}\omega^t + (I_{111} - I_{110}I_{021})\omega^x - I_{110}I_{012}\omega^y), \quad \ldots
\end{aligned}
\tag{69}
$$

for the basic invariant forms. The higher order invariantized Maurer-Cartan forms can be recursively determined from the equations

$$\begin{aligned}
\alpha_{T^{p+3}} &= \tfrac{27}{8}\epsilon\,(I_{p12}\omega^x + I_{p03}\omega^y) + \tfrac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-2,1,0}\alpha_{T^{s+2}} - \tfrac{27}{8}\epsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,1,2}\beta_{T^s} \\
&\qquad + \tfrac{9}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,1}\gamma_{T^{s+1}} - \tfrac{27}{8}\epsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,0,3}\gamma_{T^s}, \\
\beta_{T^{p+1}} &= -\tfrac{3}{2} I_{p10}\omega^x + \tfrac{3}{2}\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,0}\beta_{T^s}, \\
\gamma_{T^{p+2}} &= \tfrac{9}{4}\epsilon\, I_{p11}\omega^x - \tfrac{9}{4}\epsilon\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,1}\beta_{T^s} + \tfrac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,0}\gamma_{T^{s+1}}.
\end{aligned} \tag{70}$$

Next we substitute expressions (69) for the invariantized Maurer-Cartan forms into the equations for the non-phantom variables in (67) to derive the recurrence formulas between the differentiated and normalized invariants

$$\begin{aligned}
&\mathcal{D}_1 I_{110} = I_{210} - \tfrac{3}{2} I_{110} I_{120}, \quad \mathcal{D}_2 I_{110} = I_{120} - \tfrac{3}{2} I_{110} I_{030} + \tfrac{3}{4}\epsilon\, I_{012}, \\
&\mathcal{D}_3 I_{110} = I_{111} - \tfrac{3}{2} I_{110} I_{021} + \tfrac{3}{4}\epsilon\, I_{003}, \\
&\mathcal{D}_1 I_{210} = I_{310} - \tfrac{9}{4} I_{210} I_{120} + \tfrac{3}{2}\epsilon\, I_{111}^2 + \tfrac{9}{8} I_{111} I_{003} + 12 I_{110}^2, \\
&\mathcal{D}_2 I_{210} = I_{220} - \tfrac{9}{4} I_{210} I_{030} + \tfrac{3}{4}\epsilon\, I_{112} + \tfrac{3}{2}\epsilon\, I_{111} I_{021} + \tfrac{9}{8} I_{003} I_{021} + \tfrac{27}{2} I_{110}, \\
&\mathcal{D}_3 I_{210} = I_{211} - \tfrac{9}{4} I_{210} I_{021} + \tfrac{3}{2}\epsilon\, I_{111} I_{012} + \tfrac{3}{4}\epsilon\, I_{103} + \tfrac{9}{8} I_{012} I_{003}, \\
&\mathcal{D}_1 I_{120} = I_{220} + \tfrac{3}{2}\epsilon\, I_{111} I_{021} - \tfrac{7}{4} I_{120}^2 + 6 I_{110}, \\
&\mathcal{D}_2 I_{120} = I_{130} - \tfrac{7}{4} I_{120} I_{030} + \tfrac{3}{2}\epsilon\, I_{021}^2 + 6, \\
&\mathcal{D}_3 I_{120} = I_{121} - \tfrac{7}{4} I_{120} I_{021} + \tfrac{3}{2}\epsilon\, I_{021} I_{012}, \\
&\mathcal{D}_1 I_{111} = I_{211} - \big(3 I_{120} - \tfrac{3}{2}\epsilon\, I_{012}\big) I_{111}, \\
&\mathcal{D}_2 I_{111} = I_{121} - \big(I_{120} - \tfrac{3}{2}\epsilon\, I_{012}\big) I_{021} - 2 I_{111} I_{030}, \\
&\mathcal{D}_3 I_{111} = I_{112} - \big(I_{120} - \tfrac{3}{2}\epsilon\, I_{012}\big) I_{012} - 2 I_{111} I_{021}, \\
&\mathcal{D}_1 I_{030} = I_{130} - \tfrac{5}{4} I_{030} I_{120}, \quad \mathcal{D}_2 I_{030} = I_{040} - \tfrac{5}{4} I_{030}^2, \quad \mathcal{D}_3 I_{030} = I_{031} - \tfrac{5}{4} I_{030} I_{021}, \\
&\mathcal{D}_1 I_{021} = I_{121} - I_{030} I_{111} - \tfrac{3}{2} I_{120} I_{021}, \\
&\mathcal{D}_2 I_{021} = I_{031} - \tfrac{5}{2} I_{030} I_{021}, \\
&\mathcal{D}_3 I_{021} = I_{022} - I_{030} I_{012} - \tfrac{3}{2} I_{021}^2, \\
&\mathcal{D}_1 I_{012} = I_{112} - 2 I_{021} I_{111} - \tfrac{7}{4} I_{120} I_{012} - 2 I_{110}, \\
&\mathcal{D}_2 I_{012} = I_{022} - 2 I_{021}^2 - \tfrac{7}{4} I_{030} I_{012} - 2, \\
&\mathcal{D}_3 I_{012} = I_{013} - \tfrac{15}{4} I_{021} I_{012}, \\
&\mathcal{D}_1 I_{003} = I_{103} - 3 I_{012} I_{111} - 2 I_{003} I_{120}, \quad \mathcal{D}_2 I_{003} = I_{013} - 3 I_{012} I_{021} - 2 I_{003} I_{030}, \\
&\mathcal{D}_3 I_{003} = I_{004} - 3 I_{012}^2 - 2 I_{003} I_{021}, \qquad \ldots .
\end{aligned} \tag{71}$$

By a repeated application of the operators $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, these formulas can be used to express higher order normalized invariants in terms of the lower order invariants. As an example, the first four equations in (71) yield the expressions

$$\begin{aligned} I_{310} &= \mathcal{D}_1^2 I_{110} + (\tfrac{3}{2} I_{120} + \tfrac{9}{4}\mathcal{D}_2 I_{110} + \tfrac{27}{8} I_{110} I_{030} - \tfrac{27}{16}\epsilon\, I_{012})\mathcal{D}_1 I_{110} + \tfrac{27}{8} I_{110} I_{120} \mathcal{D}_2 I_{110} \\ &\quad - \tfrac{3}{2}\epsilon\,(\mathcal{D}_3 I_{110})^2 - \big(\tfrac{9}{2}\epsilon\, I_{110} I_{021} - \tfrac{9}{8} I_{003}\big)\mathcal{D}_3 I_{110} + \tfrac{3}{2} I_{110} \mathcal{D}_1 I_{120} \\ &\quad + \big(\tfrac{81}{16} I_{120} I_{030} - \tfrac{27}{8}\epsilon\, I_{021}^2\big) I_{110}^2 - \tfrac{81}{32}\epsilon\, I_{110} I_{120} I_{012} + \tfrac{27}{16} I_{110} I_{021} I_{003} - 12 I_{110}^2. \end{aligned} \tag{72}$$

Moreover, in light of the results in [29], we derive the following fundamental syzygies amongst the basic differential invariants I_{110}, I_{030}, I_{021}, I_{012}, I_{003}:

$$\begin{aligned} &\mathcal{D}_3 I_{012} - \mathcal{D}_2 I_{003} + \tfrac{3}{4} I_{012} I_{021} - 2 I_{030} I_{003} = 0, \\ &\mathcal{D}_2 I_{021} - \mathcal{D}_3 I_{030} + \tfrac{5}{4} I_{021} I_{030} = 0, \\ &\mathcal{D}_3 I_{021} - \mathcal{D}_2 I_{012} - \tfrac{1}{2} I_{021}^2 - \tfrac{3}{4} I_{012} I_{030} - 2\epsilon = 0, \\ &\mathcal{D}_2\mathcal{D}_2 I_{110} - \mathcal{D}_1 I_{030} - \tfrac{3}{4}\epsilon\,\mathcal{D}_3 I_{021} + \tfrac{3}{2} I_{110}\mathcal{D}_2 I_{030} + 2 I_{030}\mathcal{D}_2 I_{110} \\ &\qquad\qquad - \tfrac{9}{8}\epsilon\, I_{021}^2 + \tfrac{3}{16}\epsilon\, I_{030} I_{012} + \tfrac{3}{4} I_{030}^2 I_{110} - \tfrac{9}{2} = 0, \\ &\mathcal{D}_2\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{021} + I_{030}\mathcal{D}_3 I_{110} + I_{021}\mathcal{D}_2 I_{110} + \tfrac{3}{2} I_{110}\mathcal{D}_3 I_{030} \\ &\qquad\qquad - \tfrac{3}{4}\epsilon\,\mathcal{D}_2 I_{003} - \tfrac{9}{8} I_{110} I_{030} I_{021} - \tfrac{3}{4}\epsilon\, I_{030} I_{003} - \tfrac{9}{8}\epsilon\, I_{021} I_{012} = 0, \\ &\mathcal{D}_2\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{021} + I_{030}\mathcal{D}_3 I_{110} + I_{021}\mathcal{D}_2 I_{110} + \tfrac{3}{2} I_{110}\mathcal{D}_2 I_{021} \\ &\qquad\qquad + \tfrac{3}{4} I_{110} I_{030} I_{021} - \tfrac{3}{4}\epsilon\,\mathcal{D}_2 I_{003} - \tfrac{9}{8}\epsilon\, I_{021} I_{012} - \tfrac{3}{4}\epsilon\, I_{030} I_{003} = 0, \\ &\mathcal{D}_3\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{012} + \tfrac{3}{2} I_{021}\mathcal{D}_3 I_{110} - \tfrac{3}{4} I_{012}\mathcal{D}_2 I_{110} \\ &\qquad\qquad + \big(\tfrac{3}{2}\mathcal{D}_2 I_{012} + \tfrac{3}{4} I_{021}^2 + \epsilon\big) I_{110} - \tfrac{3}{4}\epsilon\,\mathcal{D}_3 I_{003} - \tfrac{15}{16}\epsilon\, I_{012}^2 = 0, \\ &\mathcal{D}_3\mathcal{D}_3\mathcal{D}_3 I_{110} - \mathcal{D}_1\mathcal{D}_2 I_{003} - 2 I_{030}\mathcal{D}_1 I_{003} + 3 I_{021}\mathcal{D}_1 I_{012} - 2 I_{003}\mathcal{D}_1 I_{030} \\ &\qquad\qquad + \big(2\mathcal{D}_3 I_{021} - \tfrac{7}{4} I_{021}^2\big)\mathcal{D}_3 I_{110} + \big(\tfrac{21}{16} I_{012} I_{021} - \tfrac{5}{4}\mathcal{D}_3 I_{012}\big)\mathcal{D}_2 I_{110} \\ &\qquad\qquad + \big(\tfrac{3}{2}\mathcal{D}_3\mathcal{D}_3 I_{021} - \tfrac{15}{4} I_{021}\mathcal{D}_3 I_{021} - \tfrac{15}{8}\mathcal{D}_3 I_{012} I_{030} + \tfrac{63}{32} I_{021} I_{012} I_{030} \\ &\qquad\qquad + \tfrac{3}{4} I_{021}^3 + 6\epsilon\, I_{021}\big) I_{110} - \tfrac{3}{4}\epsilon\,\mathcal{D}_3\mathcal{D}_3 I_{003} + \tfrac{9}{8}\epsilon\, I_{021}\mathcal{D}_3 I_{003} \\ &\qquad\qquad - \tfrac{57}{16}\epsilon\, I_{012}\mathcal{D}_3 I_{012} + \tfrac{3}{4}\epsilon\, I_{003}\mathcal{D}_3 I_{021} - \tfrac{3}{8}\epsilon\, I_{021}^2 I_{003} - \tfrac{3}{2} I_{003} + \tfrac{9}{64}\epsilon\, I_{021} I_{012}^2 = 0. \end{aligned} \tag{73}$$

These allow us to reduce the number of generating differential invariants:

Theorem 5.5. *The differential invariants I_{110}, I_{021}, I_{003} form a generating set for the algebra $\mathcal{I}(\mathcal{G}_{KP})$ of differential invariants for the KP symmetry pseudo-group.*

Computations indicate that I_{110}, I_{021}, I_{003} form, in fact, a minimal generating set. However, a few details remain to be overcome.

After some work, the standard algorithm [23] for constructing a group action from the infinitesimal generators yields the finite KP symmetry transformations which are given by

$$\begin{aligned} &T = F(t), \\ &X = x F'(t)^{1/3} - \tfrac{2}{9}\epsilon\, y^2 F'(t)^{-2/3} F''(t) - \tfrac{2}{3}\epsilon\, y F'(t)^{-1/3} H'(t) + G(t), \\ &Y = y F'(t)^{2/3} + H(t), \\ &U = u F'(t)^{-2/3} + \tfrac{2}{9} x F'(t)^{-5/3} F''(t) - \tfrac{4}{27} y^2 \big(\epsilon\, F'(t)^{-5/3} F'''(t) + \tfrac{4}{3} F'(t)^{-8/3} F''(t)^2\big) \\ &\qquad + \tfrac{4}{9}\epsilon\, y \big(F'(t)^{-7/3} F''(t) h'(t) - F'(t)^{-4/3} H''(t)\big) + \tfrac{2}{9}\epsilon\, F'(t)^{-2} H'(t)^2 + \tfrac{2}{3} F'(t)^{-1} G'(t), \end{aligned} \tag{74}$$

where $F(t)$ is an arbitrary smooth, invertible function and $G(t)$, $H(t)$ are arbitrary smooth functions; see also [16]. Thus the prolonged action of the KP symmetry algebra on submanifold jets can be obtained by applying the differential operators

$$\begin{aligned} D_X &= F'(t)^{-1/3} D_x, \qquad D_Y = F'(t)^{-2/3} D_y + \epsilon\,(\tfrac{4}{9} y F'(t)^{-5/3} F''(t) + \tfrac{2}{3} F'(t)^{-4/3} G'(t)) D_x, \\ D_T &= F'(t)^{-1} D_t + \big(-\tfrac{1}{3} x F'(t)^{-2} F''(t) + \epsilon\, y^2 (\tfrac{2}{9} F'(t)^{-2}\ F'''(t) - \tfrac{4}{9} F'(t)^{-3} F''(t)^2) \\ &\qquad + \epsilon\, y (\tfrac{2}{3} F'(t)^{-5/3} G''(t) - \tfrac{10}{9} F'(t)^{-8/3} F''(t) G'(t)) - F'(t)^{-4/3} H'(t) \\ &\qquad - \tfrac{2}{3} \epsilon\, F'(t)^{-7/3} G'(t)^2)\big) D_x + (-\tfrac{2}{3} y F'(t)^{-2} F''(t) - F'(t)^{-5/3} G'(t)) D_y, \end{aligned} \tag{75}$$

to U in (74). Now normalizations (68) yield the expressions

$$\begin{aligned} I_{110} &= u_{xx}^{-3/2} \big(u_{tx} + \tfrac{3}{2} u u_{xx} + \tfrac{3}{2} u_x^2 + \tfrac{3}{4} \epsilon\, u_{yy}\big), \\ I_{030} &= u_{xx}^{-5/4} u_{xxx}, \\ I_{021} &= u_{xx}^{-5/2} (u_{xx} u_{xxy} - u_{xy} u_{xxx}), \\ I_{012} &= u_{xx}^{-15/4} (u_{xx}^2 u_{xyy} - 2 u_{xx} u_{xy} u_{xxy} - 2\epsilon\, u_x u_{xx}^3 + u_{xy}^2 u_{xxx}), \\ I_{003} &= u_{xx}^{-5} (u_{xx}^3 u_{yyy} - 3 u_{xx}^2 u_{xy} u_{xyy} + 3 u_{xx} u_{xy}^2 u_{xxy} - u_{xy}^3 u_{xxx}), \end{aligned} \tag{76}$$

for the basic differential invariants for the KP symmetry algebra as well as the expressions

$$\begin{aligned} \mathcal{D}_1 &= u_{xx}^{-3/4} D_t + \tfrac{3}{4} u_{xx}^{-11/4} (2 u u_{xx}^2 - \epsilon\, u_{xy}^2) D_x + \tfrac{3}{2} \epsilon\, u_{xy} u_{xx}^{-7/4} D_y, \\ \mathcal{D}_2 &= u_{xx}^{-1/4} D_x, \\ \mathcal{D}_3 &= -u_{xx}^{-3/2} u_{xy} D_x + u_{xx}^{-1/2} D_y, \end{aligned} \tag{77}$$

for the invariant differential operators.

Additionally, by applying the invariantization map as in Theorem 4.3, we see that KP equation (58) can be written in terms of the normalized differential invariants as

$$I_{110} + \tfrac{1}{4}\, I_{040} = u_{xx}^{-3/2} (u_{tx} + \tfrac{3}{2} u_x^2 + \tfrac{3}{4} \epsilon\, u_{yy} + \tfrac{3}{2} u u_{xx}) + \tfrac{1}{4} u_{xx}^{-3/2} u_{xxxx} = 0. \tag{78}$$

The KP symmetry algebra is known to possess a Kac–Moody–Virasoro structure. It would be an interesting problem, which now can be systematically studied by our methods, to investigate to what extent the Lie algebra structure of a symmetry algebra determines the structure of its differential invariant algebra.

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