STRONG STABILITY OF PDE SEMIGROUPS VIA A RESOLVENT CRITERION OF Y. TOMILOV

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Strong Stability of PDE Semigroups via a Resolvent Criterion of Y. Tomilov

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Abstract

We give a rather general procedure by which one may invoke a recently derived operator theoretic result in [15], so as to obtain strong stability of those C_0-semigroups which model partial differential equations in Hilbert space. This procedure is illustrated here by means of two concrete PDE examples. The novelty of adopting this new strong stability technique is that one does not need to have an explicit representation of the resolvent.

1 Introduction

The intent of this note is to demonstrate the applicability of recently derived sufficiency criteria of Y. Tomilov, for establishing the strong stability of C_0-semigroups (see [15]). In particular, we discuss here the implementation of these abstract results in the context of ascertaining strong decays for solutions of partial differential equations (PDE's) under the influence of inserted dissipation. We were first made aware of these stability results in [15], at the March 2005 AMS Sectional Meeting (in Bowling Green, Kentucky), wherein Y. Tomilov himself presented these and newer results in the "Special Session on Semigroups of Operators and Applications". At the time of this meeting, Y. Tomilov was not aware of any concrete PDE examples by which strong stability could be inferred by the use of his criteria. The present paper is written to demonstrate that, aside from the intrinsic worth of the results in [15], and their implications in operator theory, they can also be effectively applied to general PDE models in order to show their strong stability.

Throughout, we dwell on the Hilbert space formulation of the said stability criteria in [15]: Let $\mathbf{H}$ be a Hilbert space, and linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ be the generator of a bounded C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$. Recall that this semigroup is said to be *strongly stable* if for all $x \in \mathbf{H}$, one has

$$\lim_{t\to\infty} e^{\mathcal{A}t}x = 0.$$

In this connection, we have the following:

Theorem 1 *(see Theorem 3.4 of [15]; see also Theorem 8.4 of [3]) Let $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ be a bounded C_0-semigroup in a Hilbert space $\mathbf{H}$ with generator $\mathcal{A}$. If exists a dense set $M \subset \mathbf{H}$ such that,*

$$\lim_{\alpha\to 0^+} \sqrt{\alpha}\mathcal{R}(\alpha + i\beta; \mathcal{A})x = 0 \quad \textit{for every } x \in M \textit{ and every } \beta \in \mathbb{R},$$

then the semigroup is strongly stable.

We note that by virtue of this result, the strong stability of a semigroup depends upon the behavior of the resolvent $\mathcal{R}(\lambda;\mathcal{A})$, as (positive) $\mathrm{Re}\,\lambda$ tends to zero. This formulation of asymptotic stability, in terms of the resolvent behavior of the generator, stands in contrast to the very well-known results of [1] and [10], in which strong stability is determined by the *spectrum* of the generator, as it appears on the imaginary axis.

We will give, in the next two sections, two concrete PDE examples for which one can use the Theorem 1 to ascertain the strong decay of each respective PDE. The first example will involve the wave equation on a bounded domain, with boundary dissipative (velocity) feedback inserted in the Neumann boundary condition. The feedback, as it appears here, is locally supported. It is well-known that unless the feedback control region is a "large enough" portion of the boundary (Γ_0 below), then the PDE will *not* be exponentially stable by means of such feedback. Lacking this uniform stability for a general geometrical configuration, it makes sense to consider the question of asymptotic decay. The problem of strong decay for boundary-damped waves has been studied for decades (see [13], [14], [17], [7],[16]); the classical theme adopted for this problem (and generally for all purely hyperbolic PDE's under control) is to exploit the underlying *weak stability* of the associated semigroup and the *compactness* of the resolvent. These ingredients will allow the use of well-established stability tools, such as the Lasalle Invariance principle or Nagy-Foias decompositions. We here, on the other hand, will proceed in an altogether different manner. In fact, the relatively simple form of the boundary-damped wave model will serve as an appropriate setting so as to demonstrate the algorithm by which the resolvent criterion Theorem 1 is to be used. One will see that this methodology uses no information on the compactness properties of the resolvent; nor do we particularly even need any explicit resolvent representation. These details are given below in Section 2.

Subsequently, by way of showing the generality of the algorithm outlined in Section 2 for the Neumann boundary-damped wave equation, we demonstrate in Section 3 how the resolvent criterion Theorem 1 may be used to infer the asymptotic decay of a PDE model which describes a 2-D structural acoustic interaction (see; e.g., [11] and [5]). It is also wellknown that the solutions of this PDE model will not decay exponetially (see [2]), and so a natural follow-up question is whether the structural dissipation, which is inherent in the model, gives rise to strongly decaying dynamics. What distinguishes this structural acoustics PDE from more classical (hyperbolic) models, such as the wave equation, is the fact that the resolvent corresponding to its generator is *not* compact; this circumstance is not unexpected inasmuch as this PDE constitutes a coupling of distinct dynamics, with this coupling being accomplished via boundary (unbounded) interfaces (see [9] for another nontrivial example of an acoustic-flow generator with noncompact resolvent). This lack of compactness precludes the use of Nagy-Foias theory or the Lasalle Invariance Principle in order to determine strong stability of this model. Instead, the asymptotic decay of the 2-D structural acoustic dynamics in Section 3 was originally shown in [2], by an invocation of the aforesaid spectral criterion condition in [1] and [10]. Alternatively, in Section 3, we demonstrate how one can systematically proceed so as deduce strong stability of the structural acoustics PDE, by verifying the resolvent limit in Theorem 1.

We should conclude our introductory remarks by pointing out that possibly one advantage of using the resolvent criterion of Tomilov, vis-à-vis the the earlier approaches–e.g., Lasalle Invariance Principles, Nagy-Foias or the spectral criterion in [1]–is that in working to show the pointwise limit in Theorem 1, one does not actually need the explicit form of the resolvent; one only needs to know the action of the resolvent. Since the algorithm we employ below seems fairly general, we believe that Theorem 1 can be effectively used to determine stability properties of other, much more complicated PDE's, such as those which model systems of higher-dimensional elasticity and thermoelasticity, shallow shells, multi-layer plates, fluid-structure interactions; etc. Given that the resolvents of such models are extremely difficult to write down, in the main, it might in fact seem preferable to adopt a methodology, such as that involving Theorem 1, which is independent of explicit resolvent representations.

2 The Wave Equation with Neumann Boundary Damping

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set ($n \geq 2$), with C^2 boundary $\Gamma = \Gamma_1 \cup \Gamma_0$, and with $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 \neq \emptyset$. We will also assume that $\Gamma_0 \neq \emptyset$ (this is for convenience only). On this geometry, we consider the following PDE in variable $z(t,x)$:

$$\begin{cases} z_{tt}(t,x) = \Delta z(t,x) \;\text{ in } (0,T)\times\Omega \\ \\ z(t,\xi)|_{\Gamma_1} = 0 \;\text{ on } (0,T)\times\Gamma_1 \\ \left.\dfrac{\partial z(t,\xi)}{\partial \nu}\right|_{\Gamma_0} = -\chi_S(\xi) z_t(t,x) \;\text{ on } (0,T)\times\Gamma_0 \\ \\ [z(0), z_t(0)] = [\phi_0, \phi_1] \in H^1_{\Gamma_1}(\Omega)\times L^2(\Omega), \end{cases} \tag{1}$$

where, above, $H^1_{\Gamma_1}(\Omega) = \left\{ f \in H^1(\Omega) : f|_{\Gamma_1} = 0 \right\}$. Also, given a set $S \subseteq \Gamma_0$ of nonzero measure, χ_S denotes its characteristic function. That is,

$$\chi_S(\xi) = \begin{cases} 1, & \text{if } \xi \in S, \\ 0, & \text{if } \xi \notin S \end{cases}$$

The semigroup formulation for (1) is quite well-known (see e.g., [16]): in fact, on Hilbert space

$$\mathbf{H} \equiv H^1_{\Gamma_1}(\Omega)\times L^2(\Omega), \tag{2}$$

we define $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ by

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} 0 & I \\ -A & -AN\chi_S N^* A \end{bmatrix}; \\ D(\mathcal{A}) &= \left\{ [f_0, f_1] \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) : f_0 + N\chi_S N^* A f_1 \in D(A) \right\}. \end{aligned} \tag{3}$$

Above: (i) the positive definite, self-adjoint operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$Af = -\Delta f, \;\text{ with }\; D(A) = \left\{ f \in H^2(\Omega) : f|_{\Gamma_1} = 0 \;\text{ and }\; \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_0} = 0 \right\}. \tag{4}$$

By the characterization of the fractional powers in [4], we have then $D(A^{\frac{1}{2}}) = H^1_{\Gamma_1}(\Omega)$.

Also, the Neumann map $N : L^2(\Gamma_0) \to L^2(\Omega)$ is given by

$$Ng = h \iff \Delta h = 0, \;\text{ with }\; h|_{\Gamma_1} = 0 \;\text{ and }\; \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_0} = g. \tag{5}$$

By elliptic theory, one has that $N \in \mathcal{L}(L^2(\Gamma_0), D(A^{\frac{1}{2}}))$ (conservatively; see [8]). Consequently, $AN \in \mathcal{L}(L^2(\Gamma_0), [D(A^{\frac{1}{2}})]')$. By the use of Green's Theorem, one can show that $N^*A \in \mathcal{L}(D(A^{\frac{1}{2}}), L^2(\Gamma_0))$ has the explicit characterization:

$$N^* A f = f|_{\Gamma_0}, \;\text{ for all } f \in D(A^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega). \tag{6}$$

By the Lumer-Phillips Theorem, one can show directly that $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ generates a contraction C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$. Thus, given initial data $[\phi_0, \phi_1] \in \mathbf{H}$, the solution of (1) may be given as

$$\begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \in C([0,T]; \mathbf{H}).$$

With the dynamics $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ in hand, we are out to show the following:

Theorem 2 *The resolvent of the generator $\mathcal{A}$, defined in (3), satisfies the following limit:*

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\left\|\mathcal{R}(\alpha+i\beta;\mathcal{A})\begin{bmatrix} f_0\\ f_1\end{bmatrix}\right\|_{\mathbf{H}}=0 \ \text{ for every } [f_0,f_1]\in\mathbf{H} \ \text{ and every } \beta\in\mathbb{R}. \tag{7}$$

In other words, by Theorem 1, the semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is strongly stable.

In the course of proof below, we will see that the case $\beta=0$ can be dealt with rather easily (as there will then be no coupling between real and complex solution parts). Accordingly, we assume throughout that $\beta\neq 0$.

2.1 The Proof of Theorem 2

With $\lambda=\alpha+i\beta$, we look at the equation

$$(\lambda I-\mathcal{A})\begin{bmatrix} z_0\\ z_1\end{bmatrix}=\begin{bmatrix} f_0\\ f_1\end{bmatrix}\in D(A^{\frac{1}{2}})\times L^2(\Omega),$$

where, of course,

$$\begin{bmatrix} z_0\\ z_1\end{bmatrix}=\mathcal{R}(\lambda;\mathcal{A})\begin{bmatrix} f_0\\ f_1\end{bmatrix}.$$

Component-wise, we have then

$$\begin{aligned}\lambda z_0 - z_1 &= f_0;\\ \lambda z_1 + Az_0 + AN\chi_S N^*Az_1 &= f_1.\end{aligned}\tag{8}$$

This gives the single equation,

$$\lambda^2 z_0 + Az_0 + \lambda AN\chi_S N^*Az_0 = f_1+\lambda f_0 + AN\chi_S N^*Af_0.$$

Decomposing this equation into its real and imaginary parts, with $z_0=\operatorname{Re} z_0+i\operatorname{Im} z_0$, we have then

$$\begin{aligned}&(\alpha^2-\beta^2)\operatorname{Re} z_0 + A\operatorname{Re} z_0 - 2\alpha\beta\operatorname{Im} z_0 + \alpha AN\chi_S N^*A\operatorname{Re} z_0 - \beta AN\chi_S N^*A\operatorname{Im} z_0 = \operatorname{Re} F_\alpha \ \text{ in } [D(A^{\frac{1}{2}})]'\\ &(\alpha^2-\beta^2)\operatorname{Im} z_0 + A\operatorname{Im} z_0 + 2\alpha\beta\operatorname{Re} z_0 + \alpha AN\chi_S N^*A\operatorname{Im} z_0 + \beta AN\chi_S N^*A\operatorname{Re} z_0 = \operatorname{Im} F_\alpha \ \text{ in } [D(A^{\frac{1}{2}})]',\end{aligned}\tag{9}$$

where

$$F_\alpha \equiv \operatorname{Re} f_1+\alpha\operatorname{Re} f_0-\beta\operatorname{Im} f_0+AN\chi_S N^*A\operatorname{Re} f_0+i\left(\operatorname{Im} f_1+\beta\operatorname{Re} f_0+\alpha\operatorname{Im} f_0+AN\chi_S N^*A\operatorname{Im} f_0\right).\tag{10}$$

Step 1 (A priori bounds for the damping mechanism) To start, we multiply the first equation in (9) by $-\alpha\operatorname{Im} z_0$, and the second by $\alpha\operatorname{Re} z_0$. Integrating both subsequent relations and adding yields

$$\begin{aligned}&\alpha\beta\left(\|N^*A\operatorname{Re} z_0\|^2_{L^2(S)}+\|N^*A\operatorname{Im} z_0\|^2_{L^2(S)}\right)+2\alpha^2\beta\left(\|\operatorname{Re} z_0\|^2_{L^2(\Omega)}+\|\operatorname{Im} z_0\|^2_{L^2(\Omega)}\right)\\ =\;&(\operatorname{Im} F_\alpha,\alpha\operatorname{Re} z_0)_{L^2(\Omega)}-(\operatorname{Re} F_\alpha,\alpha\operatorname{Im} z_0)_{L^2(\Omega)}.\end{aligned}\tag{11}$$

Estimating the right hand side of this relation, we have then

$$\begin{aligned}&\alpha\beta\left(\|N^*A\operatorname{Re} z_0\|^2_{L^2(S)}+\|N^*A\operatorname{Im} z_0\|^2_{L^2(S)}\right)+2\alpha^2\beta\left(\|\operatorname{Re} z_0\|^2_{L^2(\Omega)}+\|\operatorname{Im} z_0\|^2_{L^2(\Omega)}\right)\\ \leq\;& C_\beta\,\|[f_0,f_1]\|^2_{\mathbf{H}},\end{aligned}\tag{12}$$

where the constant C_β is independent of $0<\alpha<M$. In arriving at this inequality, we used the following basic contraction semigroup estimate: *Given Banach space X*, if $\mathcal{A}:D(\mathcal{A})\subset X\to X$ *is*

the infinitesimal generator of a contraction semigroup, then for all $\lambda = \alpha + i\beta$, with $\alpha > 0$, we have the estimate

$$\|\mathcal{R}(\lambda;\mathcal{A})\|_X \leq \frac{1}{\alpha} \tag{13}$$

(see p. 11 of [12]).

Step 2 (a priori bounds in a lower topology) We will use the estimate (12) to show the following:

Lemma 3 *For* $\alpha > 0$*, the solution* $[\sqrt{\alpha}\,\mathrm{Re}\, z_0, \sqrt{\alpha}\,\mathrm{Im}\, z_0]$ *obeys the following estimate:*

$$\left\| \left[\sqrt{\alpha}\,\mathrm{Re}\, z_0, \sqrt{\alpha}\,\mathrm{Im}\, z_0\right] \right\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_\beta \left\| [f_0, f_1] \right\|_{\mathbf{H}}, \tag{14}$$

where the constant C *is independent of* $0 < \alpha \leq M$ *(say).*

Proof of Lemma 3:

Case 1. β^2 *is in the resolvent set of* $A : D(A) \to L^2(\Omega)$. In this case, we multiply both equations of (9) by $\sqrt{\alpha}\mathcal{R}(\beta^2; A)$, so as to have

$$\begin{aligned}
\sqrt{\alpha}\,\mathrm{Re}\, z_0 &= \sqrt{\alpha}\mathcal{R}(\beta^2; A)\left(\alpha^2\,\mathrm{Re}\, z_0 - 2\alpha\beta\,\mathrm{Im}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \beta A N \chi_S N^* A\,\mathrm{Im}\, z_0 - \mathrm{Re}\, F_\alpha\right);\\
\sqrt{\alpha}\,\mathrm{Im}\, z_0 &= \sqrt{\alpha}\mathcal{R}(\beta^2; A)\left(\alpha^2\,\mathrm{Im}\, z_0 + 2\alpha\beta\,\mathrm{Re}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Im}\, z_0 + \beta A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \mathrm{Im}\, F_\alpha\right),
\end{aligned}$$

whence we obtain the estimate (14), upon an invocation of (12).

Case 2. β^2 *is an eigenvalue of* A. Here, we will make use of a classic operator theoretic estimate: Namely, there is a positive constant M^* such that one has the inequality,

$$\|\mathcal{R}(-s; A)\|_{L^2(\Omega)} \leq \frac{M^*}{s}, \quad \text{for all } -s \in \rho(A), \ s > 0 \tag{15}$$

(see; e.g., p. 115 of [6]). Now, since the eigenvalues of A are at most countable (as A^{-1} is compact), then there is surely a parameter δ, with $0 < \delta < \frac{\beta^2}{1+M^*}$ (where constant M^* is as in (15)) and with $\beta^2 - \delta$ being in the resolvent set $\rho(A)$.

From (9), we then have

$$\begin{aligned}
(\beta^2 - \delta - A)\,\mathrm{Re}\, z_0 + \delta\,\mathrm{Re}\, z_0 &= \alpha^2\,\mathrm{Re}\, z_0 - 2\alpha\beta\,\mathrm{Im}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \beta A N \chi_S N^* A\,\mathrm{Im}\, z_0 - \mathrm{Re}\, F_\alpha\\
(\beta^2 - \delta - A)\,\mathrm{Im}\, z_0 + \delta\,\mathrm{Im}\, z_0 &= \alpha^2\,\mathrm{Im}\, z_0 + 2\alpha\beta\,\mathrm{Re}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Im}\, z_0 + \beta A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \mathrm{Im}\, F_\alpha,
\end{aligned}$$

or

$$\begin{aligned}
& (I + \delta\mathcal{R}(\beta^2 - \delta; A))\,\mathrm{Re}\, z_0\\
=\ & \mathcal{R}(\beta^2 - \delta; A)\left(\alpha^2\,\mathrm{Re}\, z_0 - 2\alpha\beta\,\mathrm{Im}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \beta A N \chi_S N^* A\,\mathrm{Im}\, z_0 - \mathrm{Re}\, F_\alpha\right);\\
\\
& (I + \delta\mathcal{R}(\beta^2 - \delta; A))\,\mathrm{Im}\, z_0\\
=\ & \mathcal{R}(\beta^2 - \delta; A)\left(\alpha^2\,\mathrm{Im}\, z_0 + 2\alpha\beta\,\mathrm{Re}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Im}\, z_0 + \beta A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \mathrm{Im}\, F_\alpha\right).
\end{aligned}$$

Since, from the estimate (15) and the choice of parameter δ,

$$\left\|\delta\mathcal{R}(\beta^2 - \delta; A)\right\|_{\mathcal{L}(L^2(\Omega))} < \frac{\delta M^*}{\beta^2 - \delta} < 1,$$

we have then

$$\begin{aligned}
& \sqrt{\alpha}\,\mathrm{Re}\, z_0 = \sqrt{\alpha}\left[I + \delta\mathcal{R}(\beta^2 - \delta; A)\right]^{-1}\mathcal{R}(\beta^2 - \delta; A)\left(\alpha^2\,\mathrm{Re}\, z_0 - 2\alpha\beta\,\mathrm{Im}\, z_0\right)\\
& +\sqrt{\alpha}\left[I + \delta\mathcal{R}(\beta^2 - \delta; A)\right]^{-1}\mathcal{R}(\beta^2 - \delta; A)\left(\alpha A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \beta A N \chi_S N^* A\,\mathrm{Im}\, z_0 - \mathrm{Re}\, F_\alpha\right);\\
\\
& \sqrt{\alpha}\,\mathrm{Im}\, z_0 = \sqrt{\alpha}\left[I + \delta\mathcal{R}(\beta^2 - \delta; A)\right]^{-1}\mathcal{R}(\beta^2 - \delta; A)\left(\alpha^2\,\mathrm{Im}\, z_0 + 2\alpha\beta\,\mathrm{Re}\, z_0\right)\\
& +\sqrt{\alpha}\left[I + \delta\mathcal{R}(\beta^2 - \delta; A)\right]^{-1}\mathcal{R}(\beta^2 - \delta; A)\left(\alpha A N \chi_S N^* A\,\mathrm{Im}\, z_0 + \beta A N \chi_S N^* A\,\mathrm{Re}\, z_0 - \mathrm{Im}\, F_\alpha\right),
\end{aligned}$$

from which the estimate (14) follows, upon another usage of (12). $\Box$

Step 3 (a priori bounds in finite energy topology)

Proposition 4 *For $\alpha > 0$, the solution $[\sqrt{\alpha}\,\mathrm{Re}\, z_0, \sqrt{\alpha}\,\mathrm{Im}\, z_0]$ obeys the following estimate:*

$$\left\| \left[\sqrt{\alpha} A^{\frac{1}{2}}\,\mathrm{Re}\, z_0, \sqrt{\alpha} A^{\frac{1}{2}}\,\mathrm{Im}\, z_0 \right] \right\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_\beta \left\| [f_0, f_1] \right\|_{\mathbf{H}}, \tag{16}$$

where the constant C is independent of $0 < \alpha \leq M$ (say).

Proof of Proposition 4: We multiply the first equation in (9) by $\alpha\,\mathrm{Re}\, z_0$, and the second by $\alpha\,\mathrm{Im}\, z_0$. Integrating these relations and adding gives

$$\begin{aligned} & \alpha \left(\left\| A^{\frac{1}{2}}\,\mathrm{Re}\, z_0 \right\|^2_{L^2(\Omega)} + \left\| A^{\frac{1}{2}}\,\mathrm{Im}\, z_0 \right\|^2_{L^2(\Omega)} \right) + \alpha^2 \left(\left\| N^* A\,\mathrm{Re}\, z_0 \right\|^2_{L^2(S)} + \left\| N^* A\,\mathrm{Im}\, z_0 \right\|^2_{L^2(S)} \right) \\ = \; & \alpha(\beta^2 - \alpha^2) \left(\left\| \mathrm{Re}\, z_0 \right\|^2_{L^2(\Omega)} + \left\| \mathrm{Im}\, z_0 \right\|^2_{L^2(\Omega)} \right) + (\mathrm{Re}\, F_\alpha, \alpha\,\mathrm{Re}\, z_0)_{L^2(\Omega)} + (\mathrm{Im}\, F_\alpha, \alpha\,\mathrm{Im}\, z_0)_{L^2(\Omega)}. \end{aligned}$$

Invoking the estimate in Lemma 3 now yields the desired conclusion. $\Box$

Step 4 (Conclusion of the proof of Theorem 2: show that the weak convergence implied in (16) can be improved to strong convergence) From Proposition 4,

$$\begin{aligned} \lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Re}\, z_0 &= z_r^* \text{ (say) } \textit{weakly} \text{ in } D(A^{\frac{1}{2}}); \\ \lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Im}\, z_0 &= z_i^* \text{ (say) } \textit{weakly} \text{ in } D(A^{\frac{1}{2}}). \end{aligned} \tag{17}$$

In consequence, since $A^{-\frac{1}{2}} \in \mathcal{L}(L^2(\Omega))$ is compact, we have further

$$\begin{aligned} \lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Re}\, z_0 &= z_r^* \;\; \textit{strongly} \text{ in } L^2(\Omega); \\ \lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Im}\, z_0 &= z_i^* \;\; \textit{strongly} \text{ in } L^2(\Omega). \end{aligned} \tag{18}$$

Moreover, since $N^* A f = f|_{\Gamma_0}$ for $f \in D(A^{\frac{1}{2}})$, and the inclusion $H^{\frac{1}{2}}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is compact, and as $N \in \mathcal{L}(L^2(\Gamma_0), D(A^{\frac{1}{2}}))$, we then have that

$$\begin{aligned} \lim_{\alpha\to 0^+} \sqrt{\alpha} A^{\frac{1}{2}} N \chi_S N^* A\,\mathrm{Re}\, z_0 &= A^{\frac{1}{2}} N \chi_S \; z_r^*|_{\Gamma_0} \; \textit{strongly} \text{ in } L^2(\Omega); \\ \lim_{\alpha\to 0^+} \sqrt{\alpha} A^{\frac{1}{2}} N \chi_S N^* A\,\mathrm{Im}\, z_0 &= A^{\frac{1}{2}} N \chi_S \; z_i^*|_{\Gamma_0} \; \textit{strongly} \text{ in } L^2(\Omega). \end{aligned} \tag{19}$$

Furthermore, from the estimate in (12), we also have for $0 < \alpha \leq M$,

$$\begin{aligned} & \left\| \sqrt{\alpha} \left(\alpha^2\,\mathrm{Re}\, z_0 - 2\alpha\beta\,\mathrm{Im}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Re}\, z_0 \right) \right\|_{[D(A^{\frac{1}{2}})]'} \\ \leq \; & C_\beta \left(\sqrt{\alpha}\alpha \left\| \mathrm{Re}\, z_0 \right\|_{L^2(\Omega)} + \sqrt{\alpha}\alpha \left\| \mathrm{Im}\, z_0 \right\|_{L^2(\Omega)} + \alpha\sqrt{\alpha} \left\| N^* A\,\mathrm{Re}\, z_0 \right\|_{L^2(S)} \right) \\ \leq \; & \sqrt{\alpha} C_\beta \left\| [f_0, f_1] \right\|_{\mathbf{H}}; \end{aligned}$$

$$\begin{aligned} & \left\| \sqrt{\alpha} \left(\alpha^2\,\mathrm{Im}\, z_0 + 2\alpha\beta\,\mathrm{Re}\, z_0 + \alpha A N \chi_S N^* A\,\mathrm{Im}\, z_0 \right) \right\|_{[D(A^{\frac{1}{2}})]'} \\ \leq \; & C_\beta \left(\sqrt{\alpha}\alpha \left\| \mathrm{Im}\, z_0 \right\|_{L^2(\Omega)} + \sqrt{\alpha}\alpha \left\| \mathrm{Re}\, z_0 \right\|_{L^2(\Omega)} + \alpha\sqrt{\alpha} \left\| N^* A\,\mathrm{Im}\, z_0 \right\|_{L^2(S)} \right) \\ \leq \; & \sqrt{\alpha} C_\beta \left\| [f_0, f_1] \right\|_{\mathbf{H}}. \end{aligned}$$

Consequently,

$$\begin{aligned}\lim_{\alpha\to 0^+}\sqrt{\alpha}\left(\alpha^2\operatorname{Re}z_0-2\alpha\beta\operatorname{Im}z_0+\alpha AN\chi_S N^*A\operatorname{Re}z_0\right)&=0\ \ \textit{strongly in } [D(A^{\frac12})]';\\ \lim_{\alpha\to 0^+}\sqrt{\alpha}\left(\alpha^2\operatorname{Im}z_0+2\alpha\beta\operatorname{Re}z_0+\alpha AN\chi_S N^*A\operatorname{Im}z_0\right)&=0\ \ \textit{strongly in } [D(A^{\frac12})]'.\end{aligned}\tag{20}$$

In addition, we apply the operator $\sqrt{\alpha}A^{-\frac12}$ to both equations in (9). This gives relation

$$\begin{aligned}\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0&=-\sqrt{\alpha}A^{-\frac12}\left((\alpha^2-\beta^2)\operatorname{Re}z_0-2\alpha\beta\operatorname{Im}z_0+\alpha AN\chi_S N^*A\operatorname{Re}z_0\right.\\&\qquad\left.-\beta AN\chi_S N^*A\operatorname{Im}z_0-\operatorname{Re}F_\alpha\right)\\ \sqrt{\alpha}A^{\frac12}\operatorname{Im}z_0&=-\sqrt{\alpha}A^{-\frac12}\left((\alpha^2-\beta^2)\operatorname{Im}z_0+2\alpha\beta\operatorname{Re}z_0+\alpha AN\chi_S N^*A\operatorname{Im}z_0\right.\\&\qquad\left.+\beta AN\chi_S N^*A\operatorname{Re}z_0-\operatorname{Im}F_\alpha\right).\end{aligned}\tag{21}$$

Now,

$$\begin{aligned}&\left\|\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right\|^2_{L^2(\Omega)}\\ =\ &\left(\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}-\left(A^{\frac12}z_r^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}\\ =\ &-\left(\sqrt{\alpha}\left[(\alpha^2-\beta^2)\operatorname{Re}z_0-2\alpha\beta\operatorname{Im}z_0+\alpha AN\chi_S N^*A\operatorname{Re}z_0\right],\sqrt{\alpha}\operatorname{Re}z_0-z_r^*\right)_{L^2(\Omega)}\\ &+\left(\sqrt{\alpha}\left[\beta AN\chi_S N^*A\operatorname{Im}z_0+\operatorname{Re}F_\alpha\right],\sqrt{\alpha}\operatorname{Re}z_0-z_r^*\right)_{L^2(\Omega)}-\left(A^{\frac12}z_r^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)},\end{aligned}$$

after using (21). Thus, using the strong convergences posted in (18), (19), (20), and the weak convergence posted in (17), we have

$$\begin{aligned}&\lim_{\alpha\to 0^+}\left\|\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right\|^2_{L^2(\Omega)}\\ =\ &\lim_{\alpha\to 0^+}\left\{\beta\left(\beta\sqrt{\alpha}A^{-\frac12}\operatorname{Re}z_0-\beta A^{-\frac12}z_r^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}\right.\\ &+\beta\left(\sqrt{\alpha}A^{\frac12}N\chi_S N^*A\operatorname{Im}z_0-A^{\frac12}N\chi_S N^*Az_i^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}\\ &+\left(\beta^2A^{-\frac12}z_r^*+\beta A^{\frac12}N\chi_S N^*Az_i^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}\\ &-\left(\sqrt{\alpha}\left(\alpha^2\operatorname{Re}z_0-2\alpha\beta\operatorname{Im}z_0+\alpha AN\chi_S N^*A\operatorname{Re}z_0-\operatorname{Re}F_\alpha\right),\sqrt{\alpha}\operatorname{Re}z_0-z_r^*\right)_{L^2(\Omega)}\\ &\left.-\left(A^{\frac12}z_r^*,\sqrt{\alpha}A^{\frac12}\operatorname{Re}z_0-A^{\frac12}z_r^*\right)_{L^2(\Omega)}\right\}\\ =\ &0.\end{aligned}\tag{22}$$

We can repeat the very same argument for the difference $\sqrt{\alpha}A^{\frac12}\operatorname{Im}z_0-A^{\frac12}z_i^*$. In short, we have

$$\begin{aligned}\lim_{\alpha\to 0^+}\sqrt{\alpha}\operatorname{Re}z_0&=z_r^*\ \ \textit{strongly in } D(A^{\frac12});\\ \lim_{\alpha\to 0^+}\sqrt{\alpha}\operatorname{Im}z_0&=z_i^*\ \ \textit{strongly in } D(A^{\frac12}).\end{aligned}\tag{23}$$

Taking moreover, the limit of both sides of (21), and using the strong convergences in (23), (18), (19) and (20), we have that the limits z_r^* and z_i^* satisfy the coupled equations

$$\begin{aligned}A^{\frac12}z_r^*&=A^{-\frac12}\beta^2z_r^*+\beta A^{\frac12}N\chi_S N^*Az_i^*;\\ A^{\frac12}z_i^*&=A^{-\frac12}\beta^2z_i^*-\beta A^{\frac12}N\chi_S N^*Az_r^*,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}(\beta^2 - A)z_r^* &= -\beta A N \chi_S N^* A z_i^*;\\ (\beta^2 - A)z_i^* &= \beta A N \chi_S N^* A z_r^*.\end{aligned} \tag{24}$$

Therewith, we can multiply the first equation by $-z_i^*$, the second equation by z_r^*, and integrate, so as to have

$$\|N^* A z_i^*\|^2_{L^2(S)} + \|N^* A z_r^*\|^2_{L^2(S)} = 0. \tag{25}$$

This essentially concludes the proof: In fact, the relations (25) and (24) imply that z_r^* solves $(\beta^2 - A)z_r^* = 0$, with $\frac{\partial z_r^*}{\partial \nu}\Big|_{\Gamma_0} = 0$, $z_r^*|_{\Gamma_1} = 0$ and $z_r^*|_S = 0$ (where again, $S \subseteq \Gamma_0$, and $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 \neq \emptyset$). By elliptic theory, then necessarily $\lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Re}\, z_0 = z_r^* = 0$. In the same way, $\lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Im}\, z_0 = z_i^* = 0$. Combining these conclusions with the first relation in (8) now verifies the limit in (7). The proof of Theorem 2 is complete.

3 The Structural Acoustics Model

Again, $\Omega \subset \mathbb{R}^n$ is a bounded open set, with C^2 boundary $\Gamma = \Gamma_1 \cup \Gamma_0$, and with $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 \neq \emptyset$. We will also assume that boundary portion Γ_0 is nonempty and *flat*. On this geometry, we consider the following PDE system in variables $[z(t), z_t(t), v(t), v_t(t)]$:

$$\begin{aligned}&\begin{cases} z_{tt} = \Delta z \ \text{ in } (0,T)\times\Omega \\ z|_{\Gamma_1} = 0 \ \text{ on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu}\Big|_{\Gamma_0} = v_t \ \text{ on } (0,T)\times\Gamma_0 \\ [z(0), z_t(0)] = [\phi_0, \phi_1] \in D(A^{\frac12}) \times L^2(\Omega) \end{cases}\\ &\begin{cases} v_{tt} = -\mathring{\mathbf{A}} v - \mathring{\mathbf{A}}^{\eta} v_t - z_t|_{\Gamma_0} \ \text{ on } (0,T)\times\Gamma_0 \\ [v(0), v_t(0)] = [\psi_0, \psi_1] \in D(\mathring{\mathbf{A}}^{\frac12}) \times L^2(\Gamma_0) \end{cases}.\end{aligned} \tag{26}$$

Here, the operator $\mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}^{\frac12}) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ is any positive definite, self-adjoint operator on $L^2(\Gamma_0)$ (and so its fractional powers are well-defined). Also, the non-negative parameter η is in the range $0 \leq \eta \leq 1$. Like the wave equation of Section 2, this PDE enjoys a semigroup formulation (see [2]). In fact, using the operator theoretic definitions given in (4) and (5), and with Hilbert space $\mathbf{H}$ now defined as

$$\mathbf{H} \equiv D(A^{\frac12}) \times L^2(\Omega) \times D(\mathring{\mathbf{A}}^{\frac12}) \times L^2(\Gamma_0),$$

we have that the PDE system (26) can be associated with the following generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$:

$$\begin{aligned}\mathcal{A} &= \begin{bmatrix} 0 & I & 0 & 0 \\ -A & 0 & 0 & AN \\ 0 & 0 & 0 & I \\ 0 & -N^*A & -\mathring{\mathbf{A}} & -\mathring{\mathbf{A}}^{\eta} \end{bmatrix};\\ D(\mathcal{A}) &= \Big\{ [\phi_0, \phi_1, \psi_0, \psi_1] \in \Big[D(A^{\frac12})\Big]^2 \times \Big[D(\mathring{\mathbf{A}}^{\frac12})\Big]^2 : \\ &\qquad -\phi_0 + N\psi_1 \in D(A) \ \text{ and } \ \mathring{\mathbf{A}}^{1-\eta}\psi_0 + \psi_1 \in D(\mathring{\mathbf{A}}^{\eta}) \Big\}\end{aligned} \tag{27}$$

(where again, $0 \leq \eta \leq 1$). It is shown in [2] that $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ generates a C_0-semigroup of contractions $\{e^{\mathcal{A}t}\}_{t\geq 0}$. Thus given initial data $[\vec{\phi}_0, \vec{\psi}_0] \equiv [\phi_0, \phi_1, \psi_0, \psi_1] \in \mathbf{H}$, the solution of (26) is given by

$$\begin{bmatrix} \vec{z}(t) \\ \vec{v}(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \vec{\phi}_0 \\ \vec{\psi}_0 \end{bmatrix} \in C([0,T]; \mathbf{H}).$$

The task here is to show the following:

Theorem 5 *The resolvent of the generator $\mathcal{A}$, defined in (27), satisfies the following limit:*

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \mathcal{R}(\alpha + i\beta; \mathcal{A}) \begin{bmatrix} f_0 \\ f_1 \\ g_0 \\ g_1 \end{bmatrix} = 0, \text{ for every } [f_0, f_1, g_0, g_1] \in \mathbf{H} \text{ and every } \beta \in \mathbb{R}. \tag{28}$$

In other words, by Theorem 1, the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is strongly stable.

As in the proof of Theorem 2, we take $\beta \neq 0$ below, as the case $\beta = 0$ is easy.

3.1 Proof of Theorem 5

Given $[f_0, f_1, g_0, g_1] \in \mathbf{H}$ where $\beta \in \mathbb{R} \setminus \{0\}$, we consider

$$\begin{bmatrix} z_0 \\ z_1 \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} z_0(\lambda) \\ z_1(\lambda) \\ v_0(\lambda) \\ v_1(\lambda) \end{bmatrix} = \mathcal{R}(\lambda; \mathcal{A}) \begin{bmatrix} f_0 \\ f_1 \\ g_0 \\ g_1 \end{bmatrix}, \quad \text{where } \lambda = \alpha + i\beta. \tag{29}$$

This gives

$$\begin{aligned} \lambda z_0 - z_1 &= f_0; \\ \lambda z_1 + A z_0 - ANv_1 &= f_1; \\ \lambda v_0 - v_1 &= g_0; \\ \lambda v_1 + N^* A z_1 + \mathring{\mathbf{A}} v_0 + \mathring{\mathbf{A}}^\eta v_1 &= g_1, \end{aligned} \tag{30}$$

or

$$\begin{aligned} \lambda^2 z_0 + A z_0 - \lambda ANv_0 &= f_1 + \lambda f_0 - ANg_0; \\ \lambda^2 v_0 + \mathring{\mathbf{A}} v_0 + \lambda \mathring{\mathbf{A}}^\eta v_0 + \lambda N^* A z_0 &= g_1 + \lambda g_0 + N^* A f_0 + \mathring{\mathbf{A}}^\eta g_0. \end{aligned}$$

Distinguishing between real and imaginary parts (with $z_0 = \operatorname{Re} z_0 + i \operatorname{Im} z_0$ and $v_0 = \operatorname{Re} v_0 + i \operatorname{Im} v_0$), we have then the following coupled system:

$$(\alpha^2 - \beta^2) \operatorname{Re} z_0 + A \operatorname{Re} z_0 - 2\alpha\beta \operatorname{Im} z_0 - \alpha AN \operatorname{Re} v_0 + \beta AN \operatorname{Im} v_0 = \operatorname{Re} F_\alpha; \tag{31}$$

$$(\alpha^2 - \beta^2) \operatorname{Im} z_0 + A \operatorname{Im} z_0 + 2\alpha\beta \operatorname{Re} z_0 - \beta AN \operatorname{Re} v_0 - \alpha AN \operatorname{Im} v_0 = \operatorname{Im} F_\alpha; \tag{32}$$

$$(\alpha^2 - \beta^2) \operatorname{Re} v_0 + \mathring{\mathbf{A}} \operatorname{Re} v_0 - 2\alpha\beta \operatorname{Im} v_0 + \alpha N^* A \operatorname{Re} z_0 - \beta N^* A \operatorname{Im} z_0 + \alpha \mathring{\mathbf{A}}^\eta \operatorname{Re} v_0 - \beta \mathring{\mathbf{A}}^\eta \operatorname{Im} v_0 = \operatorname{Re} G_\alpha; \tag{33}$$

$$(\alpha^2 - \beta^2) \operatorname{Im} v_0 + \mathring{\mathbf{A}} \operatorname{Im} v_0 + 2\alpha\beta \operatorname{Re} v_0 + \alpha N^* A \operatorname{Im} z_0 + \beta N^* A \operatorname{Re} z_0 + \alpha \mathring{\mathbf{A}}^\eta \operatorname{Im} v_0 + \beta \mathring{\mathbf{A}}^\eta \operatorname{Re} v_0 = \operatorname{Im} G_\alpha, \tag{34}$$

where

$$\begin{aligned} F_\alpha &= \operatorname{Re} f_1 + \alpha \operatorname{Re} f_0 - \beta \operatorname{Im} f_0 - AN \operatorname{Re} g_0 + i \left(\operatorname{Im} f_1 + \beta \operatorname{Re} f_0 + \alpha \operatorname{Im} f_0 - AN \operatorname{Im} g_0 \right); \\ G_\alpha &= \operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0 + N^* A \operatorname{Re} f_0 + \mathring{\mathbf{A}}^\eta \operatorname{Re} g_0 \\ &\quad + i \left(\operatorname{Im} g_1 + \beta \operatorname{Re} g_0 + \alpha \operatorname{Im} g_0 + N^* A \operatorname{Im} f_0 + \mathring{\mathbf{A}}^\eta \operatorname{Im} g_0 \right). \end{aligned} \tag{35}$$

We now: multiply (31) by $-\alpha \operatorname{Im} z_0$; multiply (32) by $\alpha \operatorname{Re} z_0$; multiply (33) by $-\alpha \operatorname{Im} v_0$; multiply (34) by $\alpha \operatorname{Re} v_0$. Integrating and adding the four consequent relations give now

$$\begin{aligned}
& \alpha\beta \left(\left\| \mathring{\mathbf{A}}^{\frac{\eta}{2}} \operatorname{Re} v_0 \right\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{\eta}{2}} \operatorname{Im} v_0 \right\|^2_{L^2(\Gamma_0)} \right) \\
& +2\alpha^2\beta \left(\|\operatorname{Re} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Im} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Re} v_0\|^2_{L^2(\Gamma_0)} + \|\operatorname{Im} v_0\|^2_{L^2(\Gamma_0)} \right) \\
= \; & 2\alpha^2 \left[(\operatorname{Im} v_0, N^* A \operatorname{Re} z_0)_{L^2(\Gamma_0)} - (\operatorname{Re} v_0, N^* A \operatorname{Im} z_0)_{L^2(\Gamma_0)} \right] - \alpha (\operatorname{Re} F_\alpha, \operatorname{Im} z_0)_{L^2(\Omega)} \\
& +\alpha \left[(\operatorname{Im} F_\alpha, \operatorname{Re} z_0)_{L^2(\Omega)} - (\operatorname{Re} G_\alpha, \operatorname{Im} v_0)_{L^2(\Gamma_0)} + (\operatorname{Im} G_\alpha, \operatorname{Re} v_0)_{L^2(\Gamma_0)} \right]. \qquad (36)
\end{aligned}$$

Applying the basic contraction semigroup inequality in (13), we obtain, for $0 < \alpha \leq M$,

$$\begin{aligned}
& \alpha \left(\left\| \mathring{\mathbf{A}}^{\frac{\eta}{2}} \operatorname{Re} v_0 \right\|^2_{L^2(\Gamma_0)} + \left\| \mathring{\mathbf{A}}^{\frac{\eta}{2}} \operatorname{Im} v_0 \right\|^2_{L^2(\Gamma_0)} \right) \\
& +2\alpha^2 \left(\|\operatorname{Re} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Im} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Re} v_0\|^2_{L^2(\Gamma_0)} + \|\operatorname{Im} v_0\|^2_{L^2(\Gamma_0)} \right) \\
\leq \; & C_\beta \, \|[f_0, f_1, g_0, g_1]\|^2_{\mathbf{H}}. \qquad (37)
\end{aligned}$$

Proposition 6 *The component* $[\operatorname{Re} z_0, \operatorname{Im} z_0]$ *of the relation (29) obeys the estimate*

$$\left\| \left[\sqrt{\alpha} \operatorname{Re} z_0, \sqrt{\alpha} \operatorname{Im} z_0 \right] \right\|_{L^2(\Omega) \times L^2(\Omega)} \leq C_\beta \, \|[f_0, f_1, g_0, g_1]\|_{\mathbf{H}}, \qquad (38)$$

where C_β *may depend on* β, *but not on* α, $0 < \alpha \leq M$.

Proof of Lemma 6: From (31) and (32), we have

$$\begin{aligned}
(\beta^2 - A) \operatorname{Re} z_0 &= \alpha^2 \operatorname{Re} z_0 - 2\alpha\beta \operatorname{Im} z_0 - \alpha A N \operatorname{Re} v_0 + \beta A N \operatorname{Im} v_0 - \operatorname{Re} F_\alpha; \\
(\beta^2 - A) \operatorname{Im} z_0 &= \alpha^2 \operatorname{Im} z_0 + 2\alpha\beta \operatorname{Re} z_0 - \beta A N \operatorname{Re} v_0 - \alpha A N \operatorname{Im} v_0 - \operatorname{Im} F_\alpha. \qquad (39)
\end{aligned}$$

If $\beta^2 \in \rho(A)$, we have then,

$$\begin{aligned}
\operatorname{Re} z_0 &= \mathcal{R}(\beta^2; A) \left[\alpha^2 \operatorname{Re} z_0 - 2\alpha\beta \operatorname{Im} z_0 - \alpha A N \operatorname{Re} v_0 + \beta A N \operatorname{Im} v_0 - \operatorname{Re} F_\alpha \right]; \\
\operatorname{Im} z_0 &= \mathcal{R}(\beta^2; A) \left[\alpha^2 \operatorname{Im} z_0 + 2\alpha\beta \operatorname{Re} z_0 - \beta A N \operatorname{Re} v_0 - \alpha A N \operatorname{Im} v_0 - \operatorname{Im} F_\alpha \right].
\end{aligned}$$

Multiplying both sides of these expressions by $\sqrt{\alpha}$, and employing the estimate in (37), we obtain the estimate in (38).

On the other hand, if β^2 is an eigenvalue of A, then as noted in the proof of Lemma 3 above, there exists a δ, $0 < \delta < \frac{\beta^2}{1+M^*}$ (where again positive constant M^* is that which appears in (15)), such that $\beta^2 - \delta \in \rho(A)$, and $I + \delta \mathcal{R}(\beta^2 - \delta; A) \in \mathcal{L}(L^2(\Omega))$ is boundedly invertible. Applying this operator to (39) (similar to what was done in Lemma 3) we obtain

$$\begin{aligned}
\sqrt{\alpha} \operatorname{Re} z_0 &= \sqrt{\alpha} \left[I + \delta \mathcal{R}(\beta^2 - \delta; A) \right]^{-1} \mathcal{R}(\beta^2; A) \left[\alpha^2 \operatorname{Re} z_0 - 2\alpha\beta \operatorname{Im} z_0 - \alpha A N \operatorname{Re} v_0 + \beta A N \operatorname{Im} v_0 \right] \\
& \quad - \sqrt{\alpha} \left[I + \delta \mathcal{R}(\beta^2 - \delta; A) \right]^{-1} \mathcal{R}(\beta^2; A) \operatorname{Re} F_\alpha; \\
\sqrt{\alpha} \operatorname{Im} z_0 &= \sqrt{\alpha} \left[I + \delta \mathcal{R}(\beta^2 - \delta; A) \right]^{-1} \mathcal{R}(\beta^2; A) \left[\alpha^2 \operatorname{Im} z_0 + 2\alpha\beta \operatorname{Re} z_0 - \beta A N \operatorname{Re} v_0 - \alpha A N \operatorname{Im} v_0 \right] \\
& \quad - \sqrt{\alpha} \left[I + \delta \mathcal{R}(\beta^2 - \delta; A) \right]^{-1} \mathcal{R}(\beta^2; A) \operatorname{Im} F_\alpha. \qquad (40)
\end{aligned}$$

Estimating the right hand sides of this relation, by the use of (37), gives the estimate in (38). $\square$

We now use the estimates in (37) and (38) to obtain the weak convergence of $[\sqrt{\alpha}z_0, \sqrt{\alpha}v_0]$: Multiplying the first equation (31) by $\alpha \operatorname{Re} z_0$; (32) by $\alpha \operatorname{Im} z_0$; (33) by $\alpha \operatorname{Re} v_0$; (34) by $\alpha \operatorname{Im} v_0$; integrating each and adding the consequent relations, we obtain

$$\begin{aligned}
&\alpha\left(\left\|A^{\frac{1}{2}}\operatorname{Re} z_0\right\|^2_{L^2(\Omega)} + \left\|A^{\frac{1}{2}}\operatorname{Im} z_0\right\|^2_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Re} v_0\right\|^2_{L^2(\Gamma_0)} + \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Im} v_0\right\|^2_{L^2(\Gamma_0)}\right)\\
&\alpha^2\left(\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Re} v_0\right\|^2_{L^2(\Gamma_0)} + \left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Im} v_0\right\|^2_{L^2(\Gamma_0)}\right)\\
=\;& 2\alpha\beta\left[(\operatorname{Re} v_0, N^*A\operatorname{Im} z_0)_{L^2(\Gamma_0)} - (\operatorname{Im} v_0, N^*A\operatorname{Re} z_0)_{L^2(\Gamma_0)}\right]\\
&+\alpha(\beta^2-\alpha^2)\left(\|\operatorname{Re} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Im} z_0\|^2_{L^2(\Omega)} + \|\operatorname{Re} v_0\|^2_{L^2(\Gamma_0)} + \|\operatorname{Im} v_0\|^2_{L^2(\Gamma_0)}\right)\\
&+\alpha\left[(\operatorname{Re} F_\alpha, \operatorname{Re} z_0)_{L^2(\Omega)} + (\operatorname{Im} F_\alpha, \operatorname{Im} z_0)_{L^2(\Omega)} + (\operatorname{Re} G_\alpha, \operatorname{Re} v_0)_{L^2(\Gamma_0)} + (\operatorname{Im} G_\alpha, \operatorname{Im} v_0)_{L^2(\Gamma_0)}\right].
\end{aligned} \tag{41}$$

Using the Sobolev Trace Theorem and the inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, we can estimate the first term of the right hand side by

$$\begin{aligned}
&2\alpha\beta\left|(\operatorname{Im} v_0, N^*A\operatorname{Re} z_0)_{L^2(\Gamma_0)} - (\operatorname{Re} v_0, N^*A\operatorname{Im} z_0)_{L^2(\Gamma_0)}\right|\\
\le\;& \alpha C_\beta\left(\|\operatorname{Im} v_0\|_{L^2(\Gamma_0)}\left\|A^{\frac{1}{2}}\operatorname{Re} z_0\right\|_{L^2(\Omega)} + \|\operatorname{Re} v_0\|_{L^2(\Gamma_0)}\left\|A^{\frac{1}{2}}\operatorname{Im} z_0\right\|_{L^2(\Omega)}\right)\\
\le\;& \alpha C_\beta\left(\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Im} v_0\right\|_{L^2(\Gamma_0)}\left\|A^{\frac{1}{2}}\operatorname{Re} z_0\right\|_{L^2(\Omega)} + \left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Re} v_0\right\|_{L^2(\Gamma_0)}\left\|A^{\frac{1}{2}}\operatorname{Im} z_0\right\|_{L^2(\Omega)}\right)\\
\le\;& \frac{\alpha}{2}\left(\left\|A^{\frac{1}{2}}\operatorname{Re} z_0\right\|^2_{L^2(\Omega)} + \left\|A^{\frac{1}{2}}\operatorname{Im} z_0\right\|^2_{L^2(\Omega)}\right) + \alpha\frac{C_\beta^2}{2}\left(\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Im} v_0\right\|^2_{L^2(\Gamma_0)} + \left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Re} v_0\right\|^2_{L^2(\Gamma_0)}\right)\\
\le\;& \frac{\alpha}{2}\left(\left\|A^{\frac{1}{2}}\operatorname{Re} z_0\right\|^2_{L^2(\Omega)} + \left\|A^{\frac{1}{2}}\operatorname{Im} z_0\right\|^2_{L^2(\Omega)}\right) + C_\beta\|[f_0, f_1, g_0, g_1]\|^2_{\mathbf{H}},
\end{aligned} \tag{42}$$

after using the estimate (37). Applying now (42) to the first term on the right hand side of (41), and also the estimates (38) and (37) to handle the other terms thereof, we have now the following uniform bound:

Proposition 7 *For $0 < \alpha \le M$, the component $[\sqrt{\alpha}z_0, \sqrt{\alpha}v_0]$ of the relation (29) satisfies the estimate*

$$\begin{aligned}
&\left\|\left[\sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Re} z_0, \sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Im} z_0\right]\right\|_{[L^2(\Omega)]^2} + \left\|\left[\sqrt{\alpha}\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Re} v_0, \sqrt{\alpha}\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Im} v_0\right]\right\|_{[L^2(\Gamma_0)]^2}\\
\le\;& C_\beta\|[f_0, f_1, g_0, g_1]\|_{\mathbf{H}}.
\end{aligned} \tag{43}$$

Conclusion of the proof of Theorem 5: From Proposition 7, we have the existence of $[z_r^*, z_i^*] \in [D(A^{\frac{1}{2}})]^2$, $[v_r^*, v_i^*] \in [D(\mathring{\mathbf{A}}^{\frac{1}{2}})]^2$ such that

$$\begin{aligned}
\sqrt{\alpha}\operatorname{Re} z_0(\alpha) &\rightharpoonup z_r^*,\ \sqrt{\alpha}\operatorname{Im} z_0(\alpha) \rightharpoonup z_i^* \text{ weakly in } D(A^{\frac{1}{2}});\\
\sqrt{\alpha}\operatorname{Re} v_0(\alpha) &\rightharpoonup v_r^*,\ \sqrt{\alpha}\operatorname{Im} v_0(\alpha) \rightharpoonup v_i^* \text{ weakly in } D(\mathring{\mathbf{A}}^{\frac{1}{2}}).
\end{aligned} \tag{44}$$

In addition, since $A^{-\frac{1}{2}} \in \mathcal{L}(L^2(\Omega))$ and $\mathring{\mathbf{A}}^{-\frac{1}{2}} \in \mathcal{L}(L^2(\Gamma_0))$ are compact operators, these and (44) yield that

$$\begin{aligned}
\sqrt{\alpha}\operatorname{Re} z_0(\alpha) &\to z_r^*,\ \sqrt{\alpha}\operatorname{Im} z_0(\alpha) \to z_i^* \text{ strongly in } L^2(\Omega);\\
\sqrt{\alpha}\operatorname{Re} v_0(\alpha) &\to v_r^*,\ \sqrt{\alpha}\operatorname{Im} v_0(\alpha) \to v_i^* \text{ strongly in } L^2(\Gamma_0).
\end{aligned} \tag{45}$$

With these convergences, the estimate in (37), and the relations (31)-(32), we can proceed just as we did at the end of Theorem 2 (cf. the limit in (22)), so as to have the *strong* convergences

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Re} z_0(\alpha)=A^{\frac{1}{2}}z_r^*;\ \lim_{\alpha\to 0^+}\sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Im} z_0(\alpha)=A^{\frac{1}{2}}z_i^* \text{ in } L^2(\Omega). \tag{46}$$

Moreover, the $D(A^{\frac{1}{2}})$-strong limits $[z_r^*, z_i^*]$ and limits $[v_r^*, v_i^*]$ (strong in $L^2(\Gamma_0)$, but at present weak in $D(\mathring{\mathbf{A}}^{\frac{1}{2}})$) satisfy the following coupled system (from (31)-(34)):

$$\begin{aligned}
(\beta^2-A)z_r^* &= \beta ANv_i^* \text{ in } [D(A^{\frac{1}{2}})]'; && (47)\\
(\beta^2-A)z_i^* &= -\beta ANv_r^* \text{ in } [D(A^{\frac{1}{2}})]'; && (48)\\
(\beta^2\mathring{\mathbf{A}}^{-1}-I)v_r^* &= -\beta\mathring{\mathbf{A}}^{-1}N^*Az_i^*-\beta\mathring{\mathbf{A}}^{\eta-1}v_i^* \text{ in } L^2(\Gamma_0); && (49)\\
(\beta^2\mathring{\mathbf{A}}^{-1}-I)v_i^* &= \beta\mathring{\mathbf{A}}^{-1}N^*Az_r^*+\beta\mathring{\mathbf{A}}^{\eta-1}v_r^* \text{ in } L^2(\Gamma_0). && (50)
\end{aligned}$$

Multiplying the first of these equations by $-z_i^*$, the second by z_r^*, the third by $-\mathring{\mathbf{A}}v_i^*$ and the fourth by $\mathring{\mathbf{A}}v_r^*$, integrating and adding the consequent relations, we have

$$\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}v_r^*\right\|^2_{L^2(\Gamma_0)}+\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}v_i^*\right\|^2_{L^2(\Gamma_0)}=0. \tag{51}$$

Thus, from (47), (50) and (51), we see that z_r^* satisfies $(\beta^2-A)z_r^*=0$ with $\left.\frac{\partial z_r^*}{\partial\nu}\right|_{\Gamma_0}=0$ and $z_r^*|_\Gamma=0$; consequently by elliptic theory, $z_r^*=0$. Likewise $z_i^*=0$. These limits, combined with (46), give then

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Re} z_0(\alpha)=0 \text{ and } \lim_{\alpha\to 0^+}\sqrt{\alpha}A^{\frac{1}{2}}\operatorname{Im} z_0(\alpha)=0 \text{ in } L^2(\Omega). \tag{52}$$

Moreover, (45) and (51) give

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\operatorname{Re} v_0(\alpha)=0 \text{ and } \lim_{\alpha\to 0^+}\sqrt{\alpha}\operatorname{Im} v_0(\alpha)=0 \text{ in } L^2(\Gamma_0). \tag{53}$$

With this convergence in mind, we multiply the equation (33) by $\alpha\operatorname{Re} v_0$ and (34) by $\alpha\operatorname{Im} v_0$. Integrating and adding the subsequent relations give now

$$\begin{aligned}
&\alpha\left(\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Re} v_0\right\|^2_{L^2(\Gamma_0)}+\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Im} v_0\right\|^2_{L^2(\Gamma_0)}\right)+\alpha^3\left(\|\operatorname{Re} v_0\|^2_{L^2(\Gamma_0)}+\|\operatorname{Im} v_0\|^2_{L^2(\Gamma_0)}\right)\\
&+\alpha^2\left(\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Re} v_0\right\|^2_{L^2(\Gamma_0)}+\left\|\mathring{\mathbf{A}}^{\frac{\eta}{2}}\operatorname{Im} v_0\right\|^2_{L^2(\Gamma_0)}\right)\\
=\;&\alpha\beta^2\left(\|\operatorname{Re} v_0\|^2_{L^2(\Gamma_0)}+\|\operatorname{Im} v_0\|^2_{L^2(\Gamma_0)}\right)-\alpha^2\left[(N^*A\operatorname{Re} z_0,\operatorname{Re} v_0)_{L^2(\Gamma_0)}+(N^*A\operatorname{Im} z_0,\operatorname{Im} v_0)_{L^2(\Gamma_0)}\right]\\
&+\alpha\beta\left[(N^*A\operatorname{Im} z_0,\operatorname{Re} v_0)_{L^2(\Gamma_0)}-(N^*A\operatorname{Re} z_0,\operatorname{Im} v_0)_{L^2(\Gamma_0)}\right]\\
&+\alpha\left[(\operatorname{Re} G_\alpha z,\operatorname{Re} v_0)_{L^2(\Gamma_0)}+(\operatorname{Im} G_\alpha,\operatorname{Im} v_0)_{L^2(\Gamma_0)}\right].
\end{aligned}$$

Invoking the limits in (52) and (53) to estimate the right hand side of this expression, we have finally the *strong* convergences,

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Re} v_0(\alpha)=0 \text{ and } \lim_{\alpha\to 0^+}\sqrt{\alpha}\mathring{\mathbf{A}}^{\frac{1}{2}}\operatorname{Im} v_0(\alpha)=0. \tag{54}$$

The convergences in (52) and (54), and the first and third relations in (30), now complete the proof of Theorem 5.

References

[1] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one parameter semigroups*, Trans. Amer. Math. Soc. **306** No. 8 (1988), pp. 837-852.

[2] G. Avalos and I. Lasiecka, *The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system*, Semigroup Forum, Vol. 57 (1998), pp. 278-292.

[3] R. Chill and Y. Tomilov, *Stability of operator semigroups: ideas and results*, preprint (2005).

[4] P. Grisvard, *Caracterization de qualques espaces d'interpolation*, Arch. Rational Mechanics and Analysis **25** (1967), pp. 40-63.

[5] M. S. Howe, *Acoustics of fluid-structure interaction*, Cambridge University Press (1998).

[6] S. G. Kreĭn, "Linear Differential Equations in Banach Space, American Mathematical Society, Providence, Rhode Island (1971).

[7] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Differential Equations **50** (1983), pp. 163-182.

[8] J. L. Lions and Magenes, "Nonhomogeneous Boundary-Value Problems and Applications", Springer-Verlag, Berlin, Germany (1972).

[9] W. Littman and B. Liu, "On the spectral properties and stabilization of acoustic flow", IMA, University of Minnesota, *IMA Preprint Series #1436*, (November, 1996).

[10] Y. I. Lyubich and V. Q. Phong, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Matematica, LXXXVII (1988), pp. 37-42.

[11] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York (1968).

[12] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, New York (1983).

[13] J. Quinn and D. L. Russell, *Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping*, Proc. Roy. Soc. Edinburgh Sect. A **77** (1977), pp. 97-127.

[14] M. Slemrod, *Stabilization of boundary control systems*, J. Differential Equations **22** (1976), pp. 402-415.

[15] Y. Tomilov, *A resolvent approach to stability of operator semigroups*, J. Operator Theory **46** (2001), pp. 63-98.

[16] R. Triggiani, *Wave equation on a bounded domain with boundary dissipation: an operator approach*, Journal of Mathematical Analysis and Applications **137** (1989), pp. 438-461.

[17] J. Zabczyk, *Stabilization of boundary control systems*, in "International Symposium on Systems Optimization and Analysis, December 1978", Lecture Notes in Control and Information Sciences (A. Bensoussan and J. L. Lions, Eds.), Vol. 14, Springer-Verlag, New York/Berlin.