THE NEW SOLITARY WAVE SOLUTION FOR NONLINEAR WAVE, CKGZ, GDS, DS AND GZ EQUATIONS

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The new solitary wave solution for the nonlinear wave, CKGZ, GDS, DS and GZ equations

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ABSTRACT

A simple transformation technique is used to reduce the nonlinear wave equation, the coupled Klein-Gordon-Zakharov (CKGZ) equations, the generalized Davey Stewartson (GDS) equations, the Davey Stewartson (DS) equations, the generalized Zakharov (GZ) equations to the elliptic-like equation. Then, their new solutions are derived using a property of the reciprocal Weierstrass elliptic function. By using the relationship between the Weierstrass elliptic functions and the Jacobian elliptic functions, new Jacobi elliptic function solutions and degenerate solutions in terms of solitary wave solutions to this class of NLPDEs have been obtained.

KEYWORDS: elliptic-like equation, nonlinear wave equation, coupled Klein-Gordon-Zakharov equations, generalized Davey-Stewartson equations, Davey-Stewartson equations, generalized Zakharov equations, Weierstrass elliptic functions, Jacobian elliptic functions solitary waves solutions.

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1 INTRODUCTION

In the recent paper [1], by using a property of the reciprocal Weierstrass elliptic function, solutions for the new Hamiltonian amplitude equation were reported. In that work, Krishnan and Peng expressed subsequently the Weierstrass elliptic function in terms of Jacobi elliptic function. The solitary wave solution which is obtained as limit of the Jacobi elliptic function as the modulus m tends to 1, was also obtained. These authors gave an account of the representation of the solutions of higher order ordinary differential equations in terms of Weierstrass elliptic functions and of the relation between the Weierstrass elliptic functions and the Jacobian elliptic functions [2].

In the present paper, we will show how the same technique can be useful for obtaining new exact solutions to some class of nonlinear partial differential equations (NLPDEs) which can be reduced to a simple elliptic-like equation. By the method proposed in [1], we have successfully found the solutions of the nonlinear wave equation [3], the coupled Klein-Gordon-Zakharov (CKGZ) equations [4], the generalized Davey-Stewartson (GDS) equations [5], the Davey-Stewartson (DS) equations [6], the generalized Zakharov (GZ) equations [7]. Very recently, the above mentioned equations were solved by us using the extended F expansion method [8]. To our knowledge, the method proposed by Krishnan and Peng [1] actually allows one to obtain new exact travelling wave solutions for single equation. The present work is motivated by the desire: firstly to continue using the method in [1] for solving single equations, secondly, to use the above mentioned method for solving some coupled systems of two different equations in terms of two unknowns u and v. However, to seek the exact solutions of these coupled systems of equations, we could not use this method directly. For this aim, we will introduce a delicate implicit assumed relation between the two unknowns $u = f(\phi)$ and $v = g(\phi)$ in order to decouple the system; then a suitable ansatz for ϕ , combined with this implicit assumed relation between u and v, will solve the problem.

The paper is organized as follows: In section 2, first we briefly give the steps of the method and apply the method to solve the elliptic-like equation. In section 3, by using the results obtained in section 2, the corresponding solutions of the nonlinear wave equation, CKGZ equations, GDS equations, DS equations, GZ equations can be obtained. The last section is devoted to the conclusion.

2 Property of Weierstrass elliptic function and solutions of elliptic-like equation

Following the same line as in [1)] , the method simply proceeds as follows: Considering a given NLPDE with independent variables $x = (x_1, x_2, x_3, ..., x_l, t)$, and dependent variable ϕ, the unknowns $\phi(x_1, x_2, x_3, ..., x_l, t)$ are solutions of the NODE obtained by the travelling wave reduction $\phi(x_1, x_2, x_3, ..., x_l, t) \rightarrow \phi(\xi = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + ... + \lambda_l x_l - \omega t)$. Thus, we get a NODE of order $2k$ for $\phi(\xi)$ as follows

$$\frac{d^{2k}\phi}{d\xi^{2k}} = f(\phi; r+1) \qquad (\mathbf{1})$$

where $f(\phi; r+1)$ is an degree polynomial in ϕ. To determine $\phi(\xi)$ explicitly, we take the following three steps.

Step 1.

Supposing that $\phi(\xi)$ is as follows

$$\phi(\xi) = \lambda Q^{2s}(\xi) + \mu, \qquad (\mathbf{2})$$

is a solution of (1), where λ and μ are arbitrary constants to be determined later and $Q^{2s}(\xi)$ is the $(2s)^{th}$ derivative of the reciprocal elliptic $Q(\xi) = 1/[\wp(\xi)]$, $\wp(\xi)$ being the Weierstrass elliptic function.

Step 2.

Therefore, for (2) to be a solution of (1), the following relation should be satisfied

$$2k - r = 2rs, \qquad (\mathbf{3})$$

where the necessary condition $2k \geq r$ is required to assume a solution in the form (2). But this is not a sufficient condition for the existence of periodic wave solution in the form (2).

Step 3.

The Weierstrass elliptic function $\wp(\xi; g_2, g_3)$ with the invariants g_2 and g_3 verify

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad (\mathbf{4})$$

where

$$g_2^3 - 27g_3^2 > 0. \qquad (\mathbf{5})$$

The Weierstrass elliptic function $\wp(\xi; g_2, g_3)$ is related to the Jacobian elliptic function $ns(\xi; m)$ as follows

$$\wp(\xi; g_2, g_3) = e_3 + (e_1 - e_3)ns^2(\sqrt{e_1 - e_3}\xi, \sqrt{m}), \qquad (\mathbf{6})$$

where the modulus m is

$$m = \frac{e_2 - e_3}{e_1 - e_3}, \quad (\mathbf{7})$$

and $e_1, \quad e_2, e_3$ satisfy

$$4z^3 - g_2 z - g_3 = 0, \quad (\mathbf{8})$$

with

$$e_1 > e_2 > e_3. \quad (\mathbf{9})$$

To apply the above method now, let us choose the following elliptic-like equation

$$A\phi^{''}(\xi) + B\phi(\xi) + C\phi^3(\xi) = 0, \quad (\mathbf{10})$$

where A, B and C are arbitrary constants.
Therefore, for (2) to be a solution of (10), we have, $r = 2$ and $k = 1$ so that $s = 0$.
So as requires by the first step of this method, the solution is [1)]

$$\phi(\xi) = \frac{\lambda}{\wp(\xi)} + \mu, \quad (\mathbf{11})$$

where λ and μ are constants to be determined later.
After substituting (11) in (10), equating the coefficients of $\wp(\xi)$ to zero and solving the four obtained equations, the solutions are given by [1)]

$$\phi(\xi) = \pm\frac{\sqrt{-\frac{B^3}{27A^2C}}}{\wp(\xi)} \pm \sqrt{-\frac{B}{3C}}, \quad (\mathbf{12})$$

with

$$g_2 = \frac{2C}{A}\lambda\mu, \quad (\mathbf{13})$$

$$g_3 = -\frac{B^3}{54A^3}. \quad (\mathbf{14})$$

Taking into account relations (13) and (14) into Eq.(5) leads to the following constraint condition

$$6(2^{2/3})AC\lambda\mu > B^2. \quad (\mathbf{15})$$

Owing to Eqs.(5) and (12), it is necessary that g_2 and BC are always positive and negative respectively.

Suppose that $AB > 0$ then $g_3 < 0$, and if $AC < 0$ then $\lambda\mu < 0$; or $AB < 0$ then $g_3 > 0$, and if $AC > 0$ then $\lambda\mu > 0$, thus Eq.(10) admits solutions (12)-(15).
The periodic wave solution in terms of Jacobian elliptic function is

$$\phi(\xi) = \pm\frac{\sqrt{-\frac{B^3}{27A^2C}}}{e_3 + (e_1 - e_3)ns^2(\sqrt{e_1 - e_3}\xi, \sqrt{m})} \pm \sqrt{-\frac{B}{3C}}. \quad (\mathbf{16})$$

In the limiting case when $m \longrightarrow 1$, $sn(\xi) \longrightarrow\ tanh(\xi)$ and $e_1 \longrightarrow e_2$.
Using the fact that $e_1 + e_2 + e_3 = 0$, one gets $e_3 = -2e_1$.
So the solitary wave solution is obtained as

$$\phi(\xi) = \pm\frac{\sqrt{-\frac{B^3}{27A^2C}}(1 - sech^2(\sqrt{3e_1}\xi))}{e_1(1 + 2sech^2(\sqrt{3e_1}\xi))} \pm \sqrt{-\frac{B}{3C}}, \quad (\mathbf{17})$$

it requires $e_1 > 0$.

3 Exact solutions of some class of NLPDEs

In this section, by using the results obtained in the preceding section, we will construct the corresponding solutions of the nonlinear wave equation, CKGZ equations, GDS equations, DS equations, GZ equations.

3.1 nonlinear wave equation

Consider the nonlinear wave equation in Rev. [3)]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \quad (\mathbf{18})$$

where α, β and γ are constants. Eq.(18) contains some particular important equations such as Duffing, Klein-Gordon and Landau-Ginzburg-Higgs equation. We assume that Eq.(18) has exact solution in the form

$$u(x,t) = \phi(\xi), \quad \xi = px - \omega t. \quad (\mathbf{19})$$

Substituting Eq.(19) into Eq.(18), we have

$$A\phi^{''}(\xi) + B\phi(\xi) + C\phi^3(\xi) = 0. \quad (\mathbf{20})$$

Eq.(20) coincides with Eq.(10) where A, B and C are defined by

$$A = (\omega^2 + \alpha p^2), \quad B = \beta, \quad C = \gamma \tag{21}$$

Then the solutions of (18) are

$$u = \phi(\xi). \tag{22}$$

$\phi(\xi)$ is given by relations (11)-(17) and $\xi = px - \omega t$, $A,\ B,\ C$ are defined by (21).

3.2 Coupled Klein-Gordon-Zakharov equations

The coupled nonlinear Klein-Gordon-Zakarov equations [4)] read

$$\begin{aligned} u_{tt} - c_0^2 \nabla^2 u + f_0^2 u + \delta uv = 0, \\ v_{tt} - c_0^2 \nabla^2 v - \beta \nabla^2 |u|^2 = 0. \end{aligned} \tag{23}$$

We seek its following wave packet solution

$$u(x,y,z,t) = \phi(\xi) e^{i(kx+ly+nz-\Omega t)}, \ \ v = v(\xi), \ \ \xi = px + qy + rz - \omega t \tag{24}$$

where both $\phi(\xi)$ and $v(\xi)$ are real functions. Substituting Eq.(24) into Eq.(23) yields

$$(\omega^2 - c_0^2\ \mathbf{P^2})\ \phi^{''}(\xi) + 2i(\omega\Omega - c_0^2\ \mathbf{K.P})\ \phi^{'}(\xi) - (\omega^2 - c_0^2 \mathbf{K^2}\ - f_0^2)\phi(\xi) + \delta v(\xi)\phi(\xi) = 0$$

$$(\omega^2 - c_0^2\ \mathbf{P^2})\ v^{''}(\xi) - \beta\ \mathbf{P^2}\ (\phi^2(\xi))^{''} = 0, \tag{25}$$

where

$$\mathbf{K}{=}(k,l,n), \quad \mathbf{K^2}{=}k^2 + l^2 + n^2, \quad \mathbf{P}{=}(p,q,r),$$
$$\mathbf{P^2} = p^2 + q^2 + r^2, \quad \mathbf{K.P}{=}kp + lq + nr. \tag{26}$$

If we take

$$\omega\Omega = c_0^2 \mathbf{K.P}, \tag{27}$$

then (25) is reduced to

$$(\omega^2 - \mathbf{P^2}\ c_0^2)\phi^{''}(\xi) - (\omega^2 - \mathbf{K^2}\ c_0^2 - f_0^2)\phi(\xi) + \delta v(\xi)\phi(\xi) = 0, \tag{28.a}$$
$$(\omega^2 - \mathbf{P^2}\ c_0^2) v^{''}(\xi) - \beta\ \mathbf{P^2}\ (\phi^2(\xi))^{''} = 0. \tag{28.b}$$

Integrating (28.b) twice with respect to ξ, we get

$$v(\xi) = \frac{c}{\omega^2 - c_0^2\mathbf{P^2}} + \frac{\beta\mathbf{P^2}}{\omega^2 - c_0^2\mathbf{P^2}}\phi^2(\xi), \qquad (\mathbf{29})$$

where c is an integration constant. Substituting (29) into (28.a) the obtained equation can be expressed as Eq.(10), while the parameters $A,\ B$ and C are defined by

$$A = (\omega^2 - c_0^2\ \mathbf{P^2}\ )^2, \quad B = [(\omega^2 - c_0^2\ \mathbf{P^2}\ )(-\omega^2 + c_0^2\ \mathbf{K^2}\ c_0^2 + f_0^2) + \delta c],$$

$$C = \delta\beta\mathbf{P^2}. \qquad (\mathbf{30})$$

Then solutions of Eq.(23) are defined as follows

$$u(x,y,z,t) = \phi(\xi)e^{i(kx+ly+nz-\Omega t)},$$

$$v(x,y,z,t) = \frac{c}{\omega^2 - c_0^2\mathbf{P^2}} + \frac{\beta\mathbf{P^2}}{\omega^2 - c_0^2\mathbf{P^2}}\phi^2(\xi), \qquad (\mathbf{31})$$

$$\Omega = \frac{c_0^2\mathbf{K}.\mathbf{P}}{\omega}, \qquad (\mathbf{32})$$

where $\phi(\xi)$ appearing in these solutions is given by relations (11)-(17) and $A,\ B$ and C are defined by (30), $\xi = px + qy + rz - \omega t$

3.3 GDS, DS and GZ equations

We consider a class of NLPDEs with constant coefficients [5)]

$$iu_t + \nu(u_{xx} + D_1 u_{yy}) + E_1|u|^2 u + C_1 uv = 0, \qquad (\mathbf{33}.\mathbf{a})$$

$$D_2 v_{tt} + (v_{xx} - E_2 u_{yy}) + C_2(|u|^2)_{xx} = 0, \qquad (\mathbf{33}.\mathbf{b})$$

where $\nu,\ D_i,\ E_i,\ C_i$ (i=1,2) are real constants and $\nu \neq 0,\ D_1 \neq 0,\ C_1 \neq 0,\ C_2 \neq 0$.
Eqs.(33.a), (33.b) are a class of physically important equations. In fact, if one takes

$$\nu = \tfrac{1}{2}\kappa^2, \quad D_1 = 2\nu, \quad E_1 = \alpha, \quad C_1 = -1,$$
$$D_2 = 0, \quad E_2 = D_1, \quad C_2 = -2\alpha, \quad \kappa^2 = \pm 1, \tag{34}$$

then Eqs.(33.a), (33.b) represent the Davey-Stewartson (DS) equations [6)]

$$iu_t + \frac{1}{2}\kappa^2(u_{xx} + \kappa^2 u_{yy}) + \alpha|u|^2 u - uv = 0, \qquad (\mathbf{35}.\mathbf{a})$$

$$v_{xx} - \kappa^2 v_{yy} - 2\alpha(|u|^2)_{xx} = 0. \qquad (\mathbf{35}.\mathbf{b})$$

If one takes

$$v = v(x,t) \ i.e. \ v_y = 0, \ \nu = 1, \ D_1 = 0, \ E_1 = -2\sigma,$$
$$E_2 = -1, \quad C_2 = -1, \quad C_1 = 2, \tag{36}$$

then Eqs.(33.a) and (33.b) become generalized Zakharov (GZ) equations [7)]

$$iu_t + u_{xx} - 2\sigma|u|^2 u + 2uv = 0, \qquad (\mathbf{37}.\mathbf{a})$$

$$v_{tt} - v_{xx} + (|u|^2)_{xx} = 0. \qquad (\mathbf{37}.\mathbf{b})$$

Since u is a complex function, we assume that

$$u(x,y,t) = \phi(\xi)e^{i(kx+ly-\Omega t)}, \quad v(x,y,t) = v(\xi), \ \xi = px+qy-\omega t \tag{38}$$

where both $\phi(\xi)$ and $v(\xi)$ are real functions, k, l, p, q, Ω and ω are constants to be determined later. Substituting Eq.(38) into Eqs.(33.a) and (33.b), we have the following ODE for $\phi(\xi)$ and $v(\xi)$

$$\nu(p^2 + D_1 q^2)\phi''(\xi) + [\Omega - \nu(k^2 + D_1 l^2)]\phi(\xi) + E_1\phi^3(\xi) + i[-\omega +$$
$$2\nu(kp + D_1 lq)]\phi'(\xi) + C_1\phi(\xi)v(\xi) = 0, \qquad (\mathbf{39}.\mathbf{a})$$

$$(D_2\omega^2 + p^2 - E_2 q^2)v''(\xi) + C_2 p^2(\phi^2(\xi))'' = 0. \qquad (\mathbf{39}.\mathbf{b})$$

If we set

$$\omega = 2\nu(kp + D_1 lq), \tag{40}$$

Then (39.a) reduces to

$$\nu(p^2 + D_1 q^2)\phi''(\xi) + [\Omega - \nu(k^2 + D_1 l^2)]\phi(\xi) + E_1\phi^3(\xi) + C_1\phi(\xi)v(\xi) = 0, \tag{41}$$

Integrating (39.b) twice, we get

$$v(\xi) = \frac{c}{D_2\omega^2 + p^2 - E_2q^2} - \frac{C_2p^2}{D_2\omega^2 + p^2 - E_2q^2}\phi^2(\xi), \qquad \textbf{(42)}$$

where c is an integration constant. Substituting (42) into (41) yields

$$\nu(p^2+D_1q^2)(D_2\omega^2+p^2-E_2q^2)\phi^{''}(\xi)+[C_1c-(D_2\omega^2+p^2-E_2q^2)(\Omega-\nu(k^2+D_1l^2))]\phi(\xi)+[E_1(D_2\omega^2+p^2-E_2q^2)-C_1C_2p^2]\phi^3(\xi) = 0, \qquad \textbf{(43)}$$

Eq.(43) can be written as the elliptic-like equation (10), while $A,\ B$ and C are given by the following equation,

$$A = \nu(p^2+D_1q^2)(D_2\omega^2+p^2-E_2q^2), \quad B = C_1c-(D_2\omega^2+p^2-E_2q^2)(\Omega-\nu(k^2+D_1l^2)),$$

$$C = E_1(D_2\omega^2 + p^2 - E_2q^2) - C_1C_2p^2. \qquad \textbf{(44)}$$

Then the solutions of Eqs.(33) are

$$u(x, y, t) = \phi(\xi)e^{i(kx+ly-\Omega t)},$$

$$v(x, y, t) = \frac{c}{D_2\omega^2 + p^2 - E_2q^2} - \frac{C_2p^2}{D_2\omega^2 + p^2 - E_2q^2}\phi^2(\xi), \qquad \textbf{(45)}$$

$$\omega = 2\nu(kp + D_1lq). \qquad \textbf{(46)}$$

The expression $\phi(\xi)$ appearing in these solutions is given by relations (11)-(17) and $A,\ B$ and C are defined by (44) and $\xi = px + qy - \omega t$.
We may obtain from (35) that

$$\omega = \kappa^2(kp + \kappa^2lq), \qquad \textbf{(47)}$$

$$v(x, y, t) = \frac{c}{p^2 - \kappa^2q^2} + \frac{2\alpha p^2}{p^2 - \kappa^2q^2}\phi^2(\xi), \qquad \textbf{(48)}$$

$$u(x, y, t) = \phi(\xi)e^{i(kx+ly-\Omega t)},$$

where $\phi(\xi)$ satisfies the elliptic-like equation (10) with $A,\ B$ and C are defined as follows

$$A = \kappa^2(p^2+\kappa^2 q^2)(\kappa^2 q^2 - p^2), \quad B = 2c + (p^2-\kappa^2 q^2)(-2\Omega + \kappa^2(k^2+\kappa^2 l^2))$$

$$B = 2\alpha\kappa^2(p^2+\kappa^2 q^2). \qquad (\mathbf{49})$$

The expression $\phi(\xi)$ appearing in these solutions is given by relations (11)-(17) and $A,\ B$ and C are defined by (49) and $\xi = px + qy - \omega t$.
Then from (37) we have that

$$\omega = 2kp, \qquad (\mathbf{50})$$

$$v(x,t) = \frac{c}{p^2-\omega^2} + \frac{p^2}{p^2-\omega^2}\phi^2(\xi), \qquad (\mathbf{51})$$

$$u(x,t) = \phi(\xi)e^{i(kx-\Omega t)},$$

where $\phi(\xi)$ satisfies Eq.(10), while $A,\ B$ and C are defined as follows

$$A = p^2(p^2-\omega^2), \quad B = 2c - (p^2-\omega^2)(\Omega - k^2))$$

$$C = 2(p^2 - \sigma(p^2-\omega^2)). \qquad (\mathbf{52})$$

The expression $\phi(\xi)$ appearing in these solutions is given by relations (11)-(17) and $A,\ B$ and C are defined by (52) and $\xi = px - \omega t$.

4 Conclusion

In this paper, by using the relationship between the Weierstrass elliptic functions and the Jacobian elliptic functions, new solutions to a class of NLPDEs have been obtained. This class of NLPDEs is characterized by the fact that it can be reduced through a simple transformation to the elliptic-like equation $A\phi''(\xi) + B\phi(\xi) + C\phi(\xi)^3 = 0$. Thus, the method has proved its efficiency to the nonlinear wave equation, the coupled Klein-Gordon-Zakharov equations, the generalized Davey Stewartson (GDS) equations, the Davey Stewartson (DS) equations, the generalized Zakharov (GZ) equations.

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