ON EXPLICIT EXACT SOLUTIONS FOR THE LIÉNARD EQUATION AND ITS APPLICATION TO THE COMPLEX GINZBURG-LANDAU EQUATION WITH HIGHER-ORDER TERMS

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0N Explicit Exact Solutions for the Liénard Equation and its Application to the Complex Ginzburg-Landau Equation with Higher-Order Terms

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ABSTRACT

Feng [Phys. Lett. A 293 (2002)] obtained a kind of explicit exact solutions to the one form of the Liénard equation, and applied these results to find some explicit exact solitary wave solutions to some nonlinear partial differential equations. In this work, exact solutions of the complex Ginzburg-Landau equation with higher-order terms are obtained by seeking solutions of another more generalized Liénard equation.

1 INTRODUCTION

Consider the Liénard equation

$x''+f(x)x'+g(x)=e(t),$ (1)

where f(x)x' is the damping force, g(x) is the restoring force and e(t) is the external force. Eq.(1) which is a generalization of the damped pendulum equation or a damped spring-mass system, is also used as nonlinear models in many physically significant fields when taking different choice for f(x), g(x) and e(t). One can see that when $f(x)=\varepsilon(x^2-1)$, $g(x)=x$, and e(t)=0, Eq.(1) leads to the Van der Pol equation and served as a nonlinear model of electronic oscillation [1,2]. Moreover, some nonlinear partial differential equations (NLPDEs) which arise from various physical phenomena can be transformed to Eq.(1). During the past years, quite a good number of some papers published where concerned with Eq.(1). Some authors were concerned with Eq.(1) with the conditions f(x)=0, and g(x) is a polynomial of order 5, and the results to Eq.(1) were applied to several NLPDEs.

Then an exact solution to the following special case of Eq.(1)

$a''(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0,$ (2)

where l, m, n are real coefficients, was obtained by Behera and Khare in 1980 in terms of Weierstrass function [3]. Exact solitary wave solutions were presented by Kong in 1996 by a direct method [4], a more general solution of Eq.(2) have been presented by Feng in 2002 [5]. The purpose of this work is to study another special case of Eq.(1) for general solutions by a direct method. Consider the following choice of

$f(x)=r+s\,x^2, \qquad g(x)=l\,x+m\,x^3+n\,x^5, \qquad e(t)=0.$ (3)

We can get the following remarkable ordinary differential equation (ODE)

$a''(\xi)+r\,a'(\xi)+s\,a^2(\xi)\,a'(\xi)+l\,a(\xi)+m\,a^3(\xi)+n\,a^5(\xi)=0,$ (4)

where r, s, l, m and n are real coefficients, $l\times m\times n\neq 0$

By means of the general solutions of Eq.(4), travelling solitary wave solutions of the complex Ginzburg-Landau equation with higher-order terms are presented in section 2. Section 3 gives a brief conclusion.

2 Exact solutions of Eq.(4) and its applica-

tion to CGLE with higher-order terms

Let

$$a(\xi) = \sqrt{\phi(\xi)}, \tag{5}$$

then by Eq.(4) we can get the following equation satisfied by $\phi(\xi)$

$$2\phi''(\xi)\phi(\xi) - \phi'^2(\xi) + 2r\phi'(\xi)\phi(\xi) + 2s\phi'(\xi)\phi^2(\xi) + 4l\ \phi^2(\xi) + 4m\phi^3(\xi) + 4n\phi^4(\xi) = 0 \tag{6}$$

Next, we suppose the following transformation

$$\phi(\xi) = \frac{G(\xi)}{F(\xi)}, \tag{7}$$

where G(ξ) and F(ξ) are real functions. Using Eq.(7), we obtain the following quadrilinear equation in F and G.

$$2GF[G_{\xi\xi}F - GF_{\xi\xi}] + [-4GF_\xi + 2rGF + 2sG^2](G_\xi F - GF_\xi) - (G_\xi F - GF_\xi)^2 + 4lG^2F^2 + 4mG^3F + 4nG^4 = 0. \tag{8}$$

Further, we assume that

$$G(\xi) = ce^{2k(\xi+\xi_0)}, \tag{9.a}$$

$$F(\xi) = d + ge^{2k(\xi+\xi_0)}, \tag{9.b}$$

where d, g and ξ_0 are real constants.

Substituting Eqs.(9.a) and (9.b) into Eq.(8), and collecting the coefficients of different powers of e to zero, we obtain the following equations:

$$e^{4k(\xi+\xi\_0)}: \quad 4d^2c^2[k^2 + rk + l] = 0, \tag{10.a}$$

$$e^{6k(\xi+\xi_0)}: \quad dc^2[(-2k^2 + rk + 2l)g + c(sk + m)] = 0, \tag{10.b}$$

$$e^{8k(\xi+\xi_0)}: \quad lg^2 + mgc + nc^2 = 0. \tag{10.c}$$

Then solving Eqs.(10.a), (10.b) and (10.c), we obtain the following results

$$g = \frac{c}{2l}[-m \pm \sqrt{m^2 - 4nl}], \quad k = \frac{1}{2}[-r \pm \sqrt{r^2 - 4l}], \quad s = \frac{2(m \mp 2\sqrt{m^2-4nl})}{-r \pm \sqrt{r^2-4l},} - \frac{3r}{2l}[-m \pm \sqrt{m^2 - 4nl}], \tag{11}$$

with $m^2 - 4nl > 0$, $r^2 - 4l > 0$, c and d are arbitrary constants.

Substituting Eqs.(11) and (9) into Eq.(7), we have the following solution

$$\phi_1(\xi) = \frac{ce^{(-r\pm\ \sqrt{r^2-4l})(\xi+\xi_0)}}{d + \frac{c}{2l}[-m \pm\ \sqrt{m^2-4nl}]e^{(-r\pm\ \sqrt{r^2-4l})(\xi+\xi_0)}}. \tag{12}$$

under the constraint condition

$$s = \frac{2(m \mp 2\ \sqrt{m^2-4nl})}{-r \pm \sqrt{r^2-4l}} - \frac{3r}{2l}[-m \pm\ \sqrt{m^2 - 4nl}] \tag{13}$$

Setting d=g in (12), and making use of equality $\frac{e^{(2kx)}}{e^{(2kx)}+1} = \frac{1}{2}[\tanh(kx)+1]$ we obtain

$$\phi_2(\xi) = \frac{l}{-m\pm\sqrt{m^2-4nl}}[1 + tanh[\frac{1}{2}(-r \pm \sqrt{r^2 - 4l})(\xi + \xi_0)]]. \qquad (14)$$

Therefore, from (12) and (13), we obtain

Theorem 1.

i) Suppose that $s = \frac{2(m\mp2\sqrt{m^2-4nl})}{-r\pm\ \sqrt{r^2-4l}} - \frac{3r}{2l}[-m\pm\sqrt{m^2 - 4nl})]$, $m^2-4nl > 0$, $r^2-4l > 0$, $\frac{1}{l}[-m\pm\sqrt{m^2 - 4nl})] >0$ and c>0, d>0 (c<0, d<0), then Eq.(4) admits an exact solution

$$a_1(\xi) = \pm\left[\frac{ce^{\{(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)\}}}{d+\frac{c}{2l}[-m\pm\ \sqrt{m^2-4nl}]e^{\{(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)\}}}\right]^{(\frac{1}{2})} \qquad (15)$$

ii) In particular,

(a) for $d = \frac{c}{2l}[-m+ \sqrt{m^2 - 4nl}]$, $s = \frac{2(m-2\sqrt{m^2-4nl})}{-r\pm\ \sqrt{r^2-4l}} - \frac{3r}{2l}[-m+\sqrt{m^2 - 4nl}]$, if m>0, l>0, n<0, and $r^2 - 4l \geq 0$ or m<0, l<0, n<0 and $m^2 - 4nl \geq 0$ then Eq.(4) admits the following exact solution

$$a_2(\xi) = \pm\{\frac{l}{-m+\sqrt{m^2-4nl}}[1+\tanh[\frac{1}{2}(-r\pm\sqrt{r^2 - 4l})(\xi+\xi_0)]]\}^{(\frac{1}{2})}. \qquad (16)$$

(b) for d=$\frac{c}{2l}[-m-\sqrt{m^2 - 4nl}]$, $s = \frac{2(m+2\sqrt{m^2-4nl})}{-r\pm\sqrt{r^2-4l}} - \frac{3r}{2l}[-m-\sqrt{m^2 - 4nl}]$, if m>0, l<0, n<0, then Eq.(4) admits the following exact solution

$$a_3(\xi) = \pm\{\frac{l}{-m-\sqrt{m^2-4nl}}[1+\tanh[\frac{1}{2}(-r\pm\sqrt{r^2 - 4l})(\xi+\xi_0)]]\}^{(\frac{1}{2})}. \qquad (17)$$

(c) for $d = \frac{c}{2l}[-m\pm\sqrt{m^2 - 4nl}]$, $s = \frac{2(m\mp2\sqrt{m^2-4nl}}{-r\pm\sqrt{r^2-4l}} - \frac{3r}{2l}[-m\pm\sqrt{m^2 - 4nl}]$, if m>0, l<0, n<0, and $m^2 - 4nl \geq 0$ or m<0, l>0, n<0 and $r^2 - 4l \geq 0$ or m<0, l>0, n>0, $m^2 - 4nl \geq 0$ and $r^2 - 4l \geq 0$, then Eq.(4) admits the following exact solution

$$a_4(\xi) = \pm\{\frac{l}{-m\pm\sqrt{m^2-4nl}}[1+\tanh[\frac{1}{2}(-r\pm\sqrt{r^2 - 4l})(\xi+\xi_0)]]\}^{(\frac{1}{2})}. \qquad (18)$$

When r=0, then Eq.(4) is reduced to

$$a''(\xi) + sa^2(\xi)a'(\xi) + la(\xi) + ma^3(\xi) + na^5(\xi) = 0, \qquad (19)$$

Taking into account Eqs.(5)-(10.c) for r=0, and proceeding as in the previous manner, we obtain the following results

$$g = \frac{c}{2l}[-m\pm\sqrt{m^2 - 4nl}], \quad k = \pm\sqrt{-l}, \quad s = \frac{m\mp2\sqrt{m^2-4nl}}{\pm\sqrt{-l}}. \qquad (20)$$

We obtain exact solutions to Eq.(19) as follows:

$$\phi_3(\xi) = \frac{ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}}{d+\frac{c}{2l}[-m\pm\sqrt{m^2-4nl}]e^{\{2\sqrt{-l}(\xi+\xi_0)\}}}. \qquad (21.a)$$

$$\phi_4(\xi) = \frac{ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{d+\frac{c}{2l}[-m\pm\ \sqrt{m^2-4nl}]e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}. \qquad (21.b)$$

under the constraint conditions $s = \frac{(m\mp2\sqrt{m^2-4nl})}{\sqrt{-l}}$ and $s = \frac{(m\mp2\sqrt{m^2-4nl})}{-\sqrt{-l}}$ respectively.

Setting $d = \frac{c}{2l}[-m \pm \sqrt{m^2-4nl}]$, in Eqs.(21.a) and (21.b), we obtain

$$\phi_5(\xi) = \frac{l}{-m\pm\sqrt{m^2-4nl}}[1+\tanh[\sqrt{-l}(\xi+\xi_0)]]. \qquad (22.a)$$

and

$$\phi_6(\xi) = \frac{l}{-m\pm\sqrt{m^2-4nl}}[1-\tanh[\sqrt{-l}(\xi+\xi_0)]]. \qquad (22.b)$$

respectively.

Theorem 2.

Suppose that l<0, $m^2-4nl \geq 0$, $\frac{1}{2l}[-m\pm\sqrt{m^2-4nl}] > 0$, and c>0, d>0 (c<0, d<0), then Eq.(19) admits an exact solution

$$a_5(\xi) = \pm\left[\frac{ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}}{d+\frac{c}{2l}[-m\pm\ \sqrt{m^2-4nl}]e^{\{2\sqrt{-l}(\xi+\xi_0)\}}}\right]^{(\frac{1}{2})}, \qquad (23.a)$$

$$a_6(\xi) = \pm\left[\frac{ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{d+\frac{c}{2l}[-m\pm\ \sqrt{m^2-4nl}]e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}\right]^{(\frac{1}{2})}, \qquad (23.b)$$

with the constraint conditions $s = \frac{(m\ \mp2\ \sqrt{m^2-4nl})}{\sqrt{-l},}$ and $s = \frac{(m\mp2\ \sqrt{m^2-4nl})}{-\sqrt{-l}}$ respectively.

If setting $d = \frac{c}{2l}[-m\pm\sqrt{m^2-4nl}]$, in Eqs.(23.a) and (23.b), then Eq.(19) admits the particular solutions

$$a_7(\xi)=\pm\{\frac{l}{-m\pm\sqrt{m^2-4nl}}[1+\tanh[\sqrt{-l}(\xi+\xi_0)]]\}^{(\frac{1}{2})}, \qquad (24.a)$$

and

$$a_8=\pm\{\frac{l}{-m\pm\sqrt{m^2-4nl}}[1-\tanh[\sqrt{-l}(\xi+\xi_0)]]\}^{(\frac{1}{2})}, \qquad (24.b)$$

When s=0, then Eq.(4) is reduced to

$$a''(\xi)+ra'(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0, \qquad (25)$$

Taking into account Eqs.(5)-(10.c) for s=0, and proceeding as in the previous manner, we obtain the following results

$$g = \frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})}c, \qquad k = \frac{1}{2}[-r\pm\sqrt{r^2-4l}], \qquad (26.a)$$

$$n = 3m^2\frac{3l+r(-r\pm\sqrt{r^2-4l})}{2l(16l-9r^2)+3r(8l-3r^2)(-r\pm\sqrt{r^2-4l})}. \qquad (26.b)$$

Then, the following exact solution is obtained

$$\phi_7(\xi) = \frac{ce^{\{(-r\pm\ \sqrt{r^2-4l})(\xi+\xi\_0)\}}}{d+\frac{(-2m)c}{8l-3r)(-r\pm\sqrt{r^2-4l})}e^{\{(-r\pm\ \sqrt{r^2-4l})(\xi+\xi\_0)\}}}, \qquad (27)$$

with the constraint condition (26.b) and for $r^2-4l \geq 0$.
Setting $d = \frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})}c$, we get

$$\phi_8(\xi) = -\frac{8l+3r(-r\pm\sqrt{r^2-4l})}{4m}[1+\tanh[\tfrac{1}{2}(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)]](28)$$

with the constraint condition (26.b).
Therefore, substituting (27) and (28) into (5), we can get the following in

Theorem 3.

Suppose i) that $n = 3m^2\frac{3l+r(-r\pm\sqrt{r^2-4l})}{2l(16l-9r^2)+3r(8l-3r^2)(-r\pm\sqrt{r^2-4l})}$, $r^2-4l \geq 0$, $\frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})} > 0$, and c>0, d>0 (c<0, d<0), then Eq.(25) admits the following exact solution

$$a_9(\xi) = \pm\left[\frac{ce^{\{(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)\}}}{d+\frac{(-2m)c}{(8l-3r)(-r\pm\sqrt{r^2-4l})}e^{\{(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)\}}}\right]^{(\frac{1}{2})}, \qquad (29)$$

where c and d are arbitrary constants.
ii) In particular, if $d = \frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})}c$, $n = 3m^2\frac{3l+r(-r\pm\sqrt{r^2-4l})}{2l(16l-9r^2)+3r(8l-3r^2)(-r\pm\sqrt{r^2-4l})}$,
$-\frac{8l+3r(-r\pm\sqrt{r^2-4l})}{4m} > 0$, $r^2-4l \geq 0$, then Eq.(25) admits exact solution

$$a_{10}(\xi) = \pm\{-\frac{8l+3r(-r\pm\sqrt{r^2-4l})}{4m}[1+\tanh[\tfrac{1}{2}(-r\pm\sqrt{r^2-4l})(\xi+\xi_0)]]\}^{(\frac{1}{2})}(30)$$

When r=s=0, then Eq.(4) is reduced to Eq.(2) taking into account Eqs.(5)-(10.c) for r=s=0 and proceeding in the same manner as above, we have the following results:

$$g = -\frac{m}{4l}c, \quad k = \pm\sqrt{-l}, \quad \frac{m^2}{4} - \frac{4nl}{3} = 0, \qquad (31)$$

with l<0, c and d as arbitrary constants.
Substituting (31) into (9.a), (9.b) and (7), we have the following solutions of Eq.(2):

$$\phi_9(\xi) = \frac{ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}}{d-\frac{m}{4l}ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}}. \qquad (32.a)$$

$$\phi_{10}(\xi) = \frac{ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{d-\frac{m}{4l}ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}. \qquad (32.b)$$

Setting $d = -\frac{m}{4l}c$, we obtain

$$\phi_{11}(\xi) = -\tfrac{2l}{m}[1+\tanh[\sqrt{-l}(\xi+\xi_0)]]. \tag{33.a}$$

and

$$\phi_{12}(\xi) = -\tfrac{2l}{m}[1-\tanh[\sqrt{-l}(\xi+\xi_0)]]. \tag{33.b}$$

Solutions (32) and (32) are the solutions found by Feng [5].

Other solutions are obtained by assuming the solutions of the form

$$G = be^{k(\xi+\xi_0)}, \qquad F = d+fe^{k(\xi+\xi_0)}+ge^{2k(\xi+\xi_0)}. \tag{34}$$

Hence, if substitute the proposed solutions (34) into Eq.(8) for r=s=0 and collecting the coefficients of different powers of e to zero, we obtain the following equations:

$$\begin{aligned}
&e^{2k(\xi+\xi_0)}: \quad b^2d^2(k^2+4l) = 0, && (35.a)\\
&e^{3k(\xi+\xi_0)}: \quad 2b^2d(-k^2f+2mb+4lf) = 0, && (35.b)\\
&e^{4k(\xi+\xi_0)}: \quad 2b^2[-5dgk^2+2l(f^2+2dg)+2mbf+2nb^2] = 0, && (35.c)\\
&e^{5k(\xi+\xi_0)}: \quad 2b^2g(-k^2f+2mb+4lf) = 0, && (35.d)\\
&e^{6k(\xi+\xi_0)}: \quad b^2g^2(k^2+4l) = 0. && (35.e)
\end{aligned}$$

Solving these equations for $bdgf \neq 0$ yields

$$g = \tfrac{3m^2-16nl}{12m^2}\tfrac{1}{d}f^2, \quad b = -\tfrac{4l}{m}f, \quad k = \pm 2\sqrt{-l}, \tag{36}$$

with $l<0$, f and d are arbitrary constants. Substituting (36) into (34) and (7), we can get after some arrangements the following solutions:

$$\phi_{13}(\xi) = \frac{-\frac{4l}{m}fde^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{\left(de^{\{-2\sqrt{-l}(\xi+\xi_0)\}}+f\sqrt{\frac{3m^2-16nl}{12m^2}}\right)^2+df\left(1-2\sqrt{\frac{3m^2-16nl}{12m^2}}\right)e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}, \tag{37.a}$$

$$\phi_{14}(\xi) = \frac{-\frac{4l}{m}fde^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{\left(d+f\sqrt{\frac{3m^2-16nl}{12m^2}}e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}\right)^2+df\left(1-2\sqrt{\frac{3m^2-16nl}{12m^2}}\right)e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}. \tag{37.b}$$

with $l<0$, $m^2-16nl>0$, d and f are arbitrary constants.

Setting $f=m$, and $d=\sqrt{\frac{3m^2-16nl}{12m^2}}$ in (37.a) and (37.b), we see that these solutions are reduced to the same solution as follows:

$$\phi_{15}(\xi) = \frac{4\sqrt{\frac{3m^2-16nl}{12m^2}}sech^2\sqrt{-l}(\xi+\xi_0)}{2+\left(-1+\frac{\sqrt{3}m}{\sqrt{3m^2-16nl}}\right)sech^2\sqrt{-l}(\xi+\xi_0)}. \tag{38}$$

It is important to say that solution (37.a) in the special case of f=m, and solution (38) obtained for $f=m$ and $d=\sqrt{\frac{3m^2-16nl}{12m^2}}$ in Eqs.(37.a) and (37.b) were found by Feng [5].

Therefore by (32.a), (32.b), (33.a), (33.b), (37.a), (37.b) and (38), we can get

Theorem 4.

Suppose that (i) $l < 0, n \geq 0,\ m > 0,$, then Eq.(2) admits exact solutions

$$a_{11}(\xi) = \pm \left[\frac{-\frac{4l}{m} fde^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{\left(de^{\{-2\sqrt{-l}(\xi+\xi_0)\}} + f\sqrt{\frac{3m^2-16nl}{12m^2}}\right)^2 + df\left(1-2\sqrt{\frac{3m^2-16nl}{12m^2}}\right)e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}} \right]^{1/2}, (39.a)$$

$$a_{12}(\xi) = \pm \left[\frac{-\frac{4l}{m} fde^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{\left(d + f\sqrt{\frac{3m^2-16nl}{12m^2}}e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}\right)^2 + df\left(1-2\sqrt{\frac{3m^2-16nl}{12m^2}}\right)e^{\{-2\sqrt{-l}(\xi+\xi_0)\}}} \right]^{1/2}, (39.b)$$

where f and d are arbitrary positive constants.

In particular if $f = m$ and $d = \sqrt{\frac{3m^2-16nl}{12m^2}},\ l < 0,\ n \geq 0,\ m > 0$, then Eq.(2) admits exact solutions

$$a_{13}(\xi) = \pm \left[\frac{4\sqrt{\frac{3m^2-16nl}{12m^2}}\, sech^2\sqrt{-l}(\xi+\xi_0)}{2 + \left(-1 + \frac{\sqrt{3}m}{\sqrt{3m^2-16nl}}\right) sech^2\sqrt{-l}(\xi+\xi_0)} \right]^{1/2}, \qquad (40)$$

that (ii) $l < 0,\ m > 0$ and $3\mathrm{m}^2 - 16nl = 0$, c and d are arbitrary positive constants, then Eq.(2) admits exact solutions

$$a_{14}(\xi) = \pm \left[\frac{ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}}{d - \frac{m}{4l}ce^{\{2\sqrt{-l}(\xi+\xi_0)\}}} \right]^{1/2}, \qquad (41.a)$$

$$a_{15}(\xi) = \pm \left[\frac{ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}}{d - \frac{m}{4l}ce^{\{-2\sqrt{-l}(\xi+\xi_0)\}}} \right]^{1/2}, \qquad (41.b)$$

in particular if $d = -\frac{m}{4l}c$, then solutions of Eq.(2) are reduced to

$$a_{16}(\xi) = \pm\sqrt{-\frac{2l}{m}(1 + \tanh\sqrt{-l}(\xi + \xi_0)}, \qquad (42.a)$$

$$a_{17}(\xi) = \pm\sqrt{-\frac{2l}{m}(1 - \tanh\sqrt{-l}(\xi + \xi_0)} \qquad (43.b)$$

We can apply theorems I, II, III and IV to seeking travelling solitary wave solutions to some NLDEs. It is well-known that the theory of solitary wave solutions plays an important crucial role in mathematical physics and represent especially realistic physical phenomena in physical experiments and nature.

For example, consider the generalized Ginzburg-Landau equation for complex function A(x,t) in one-dimension space has the following form [6,7]

$$A_t = \varepsilon A + (b_1 + ic_1)A_{xx} - (b_3 - ic_3)|A|^2A - (b_5 - ic_5)|A|^4A + (p_r + ip_i)(|A|^2A)_x + (q_r + iq_i)(|A|^2)_x A \quad (44)$$

where b, c, p, and q are real constants. If the last two terms on the right-hand side are neglected, Eq.(44) reduce to cubic-quintic complex Ginzburg-Landau, whose dynamical behaviors have extensively been investigated [8-12]. However, there is little corresponding study in the presence of the higher-order terms $(p_r + ip_i)(|A|^2A)_x$ and $(q_r + iq_i)(|A|^2)_xA$. Noting that the model parameters are generally dependent on selected physical systems. For propagation of nonlinear light pulses in optical systems, A(z,t) is the complex envelope of electric field, t is the normalized propagation distance, and x is the retarded time. $\varepsilon > 0$ (<0) represents linear gain (loss), c_1 is group velocity dispersion (GVD), c_3 is nonlinear Kerr effect, b_1 describes the effect of spectral limitation due to gain bandwith-limited amplification and (or) spectral filtering, b_3 accounts for nonlinear gain [and (or) absorption] processes, b_5 and c_5 describe the saturable effects of nonlinear gain [and (or) absorption] and nonlinear refractive index, p_r is the nonlinear dispersion term, q_r and q_i are nonlinear gradient terms which result from the time-retarded induced Raman process usually, p_i and q_r are neglected in optical transmission systems because they are much more smaller than p_r and q_i. Analytical solutions of Eq.(44) were found [13, 7, 14].

Exact analytical solutions of Eq.(44) can be found by seeking solutions in the form

$$A(x,t) = a(\xi)e^{i[\psi(\xi)+Kx-\Omega t]}, \quad (45)$$

where a and ψ are real functions of $\xi = x - vt$, v is the velocity of propagative wave solution. Substituting (45) into (44), we obtain an equation for two coupled functions a and ψ Separating real and imaginary parts, we get the following set of two ODE's:

$$-b_1a_{\xi\xi} + (2Kc_1 - v)a_\xi + b_1\psi_\xi^2 a + 2Kb_1a\psi_\xi + c_1\psi_{\xi\xi}a + 2c_1\psi_\xi a_\xi + p_i\psi_\xi a^3 + (b_1K^2 - \varepsilon)a + (b_3 + p_iK)a^3 + b_5a^5 - (3p_r + 2q_r)a^2a\_\xi = 0, \quad (46.a)$$

$$c_1a_{\xi\xi} + 2Kb_1a_\xi - c_1\psi_\xi^2 a + (v - 2Kc_1a\psi_\xi + b_1\psi_{\xi\xi}a + 2b_1\psi_\xi a_\xi + p_r\psi_\xi a^3 + (\Omega - c_1K^2)a + (c_3 + p_rK)a^3 + c_5a^5 - (3p_i + 2q_i)a^2a\_\xi = 0, \quad (46.b)$$

Let

$$\psi'(\xi) = D + Ba^2(\xi), \quad (47)$$

where D, and B are constants. Substituting (47) into (46), then yields

$$-b_1a_{\xi\xi}+(2Kc_1-v+2c_1D)a_\xi+(4c_1-3p_r-2q_r)a^2a_\xi+(b_1D^2+2Kb_1D+b_1K^2-\varepsilon)a+(2b_1DB+2Kb_1B+p_iD+b_3+p_iK)a^3+(b_1B^2+p_iB+b_5)a^5=0, \quad (48.a)$$

$$c_1a_{\xi\xi}+(2Kb_1+2b_1D)a_\xi+(4b_1+3p_i+2q_i)a^2a_\xi+(-c_1D^2+(v-2Kc_1)D-c_1K^2+\Omega)a+(-2c_1DB+(v-2Kc_1)B+p_rD+c_3+p_rK)a^3+(-c_1B^2+p_rB+c_5)a^5=0, \quad (48.b)$$

Case 1

In order to make the left-hand side of (48.a) equal to zero identically, we set

$$b_1=0,\ \varepsilon=0,\ B=-\tfrac{b_5}{p_i},\ c_1=-\tfrac{p_i(3p_r+2q_r)}{4b_5},\ D=-K-\tfrac{b_3}{p_i},\ v=-\tfrac{2b_3}{p_i}c_1,\ v=\tfrac{b_3}{2b_5}(3p_r+2q_r), \quad (49)$$

Using (49), (48.b) becomes

$$a''(\xi)+sa^2(\xi)a'(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0, \quad (50)$$

by (50), taking

$$s=\tfrac{3p_i+2q_i}{c_1},\ r=0,\ l=[(K+\tfrac{b_3}{p_i})^2-K^2+\tfrac{\Omega}{c_1}],\ \ m=\tfrac{1}{c_1}(p_rK+c_3)-\tfrac{1}{c_1}(p_r+\tfrac{c_1b_5}{p_i})(K+\tfrac{b_3}{p_i}),\quad n=\tfrac{c_5}{c_1}-\tfrac{b_5}{p_ic_1}(p_r+\tfrac{b_5}{p_i}c_1), \quad (51)$$

in (19), according to theorem II, we have

Theorem 5.

I)Suppose that l<0, $m^2-4nl\geq 0$, $\frac{1}{2l}[-m\pm\sqrt{m^2-4nl}]>0$, and c>0, d>0 (c<0, d<0), then Eq.(50) admits an exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_5(\xi)$, $a_6(\xi)$ in (23.a) and (23.b) and s, l, m and n in (23.a) and (23.b) are the same as (51).

II) Suppose that $d=\frac{c}{2l}[-m\pm\sqrt{m^2-4nl}]$, l<0, $m^2-4nl\geq 0$, $\frac{1}{2l}[-m\pm\sqrt{m^2-4nl}]>0$, then Eq.(50) admits exact solutions in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_7(\xi)$, $a_8(\xi)$ in (24.a) and (24.b) and s, l, m and n in (24.a) and (24.b) are the same as (51).

Case 2

Another case is obtained if we make the left-hand side of (48.b) equal zero identically, then we have

$c_1=0,\ B=-\frac{c_5}{p_r},\ b_1=\frac{p_r(3p_i+2q_r)}{4c_5},\ D=-K,\ v=\frac{p_rc_3}{c_5},\ \ \Omega=K\frac{p_rc_3}{c_5},$ (52)

utilizing (39), (35.a) becomes

$$a''(\xi)+ra'(\xi)+sa^2(\xi)a'(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0, \tag{53}$$

where

$$r=\frac{c_3p_r}{b_1c_5},\ s=\frac{3p_r+2q_r}{b_1},\ l=\frac{\varepsilon}{b_1},\ m=-\frac{b_3}{b_1},\ n=\frac{c_5}{p_r^2b_1}(p_ip_r-c_5b_1)-\frac{b_5}{b_1}. \tag{54}$$

Case 3

If taking into account relation (52) with $s=0$ i.e $q_r=-\frac{3p_r}{2}$, then we obtain the following equation

$$a''(\xi)+ra'(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0. \tag{55}$$

Letting

$$r=\frac{c_3p_r}{b_1c_5},\quad l=\frac{\varepsilon}{b_1},\ m=-\frac{b_3}{b_1},\ \ n=\frac{c_5}{p_r^2b_1}(p_ip_r-c_5b_1)-\frac{b_5}{b_1}, \tag{56}$$

and using theorem 3, we obtain

Theorem 6.

Suppose i) that $n=3m^2\frac{3l+r(-r\pm\sqrt{r^2-4l})}{2l(16l-9r^2)+3r(8l-3r^2)(-r\pm\sqrt{r^2-4l})},\ r^2-4l\geq 0,$ $\frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})}>0,$ and c>0, d>0 (c<0, d<0), then Eq.(25) admits exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_9(\xi)$ in (29) and r, l, m and n in (29) are the same as (56).

ii) In particular, if $d=\frac{-2m}{8l+3r(-r\pm\sqrt{r^2-4l})}c,$

$n=3m^2\frac{3l+r(-r\pm\sqrt{r^2-4l})}{2l(16l-9r^2)+3r(8l-3r^2)(-r\pm\sqrt{r^2-4l})},\ -\frac{8l+3r(-r\pm\sqrt{r^2-4l})}{4m}>0,\ r^2-4l\geq 0,$ then Eq.(25) admits exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_{10}(\xi)$ in (30) and r, l, m and n in (30) are the same as (56).

Case 4

If we impose equality between the coefficients in the left-hand side of (46.a) and those in the left-hand side of (46.b), then we obtain the following relations:

$b_1=-c_1=\frac{(p_i-p_r)(3p_r+2q_r+3p_i+2q_i)}{8(b_5-c_5)},\quad D=\frac{2(b_3-c_3)}{5p_r+2q_r+p_i+2q_i}-K,\quad v=-\frac{(p_i-p_r)(3p_r+2q_r+3p_i+2q_i)(b_3-c_3)}{(5p_r+2q_r+p_i+2q_i)(b_5-c_5)},$

$$B=\frac{b_5-c_5}{p_i-p_r},\ \Omega=-\varepsilon\frac{(p_i-p_r)(3p_r+2q_r+3p_i+2q_i)(b_3-c_3)}{(5p_r+2q_r+p_i+2q_i)(b_5-c_5)}\left[\frac{2(b_3-c_3)}{5p_r+2q_r+p_i+2q_i}-K\right]. \tag{57}$$

Taking into account Eq.(57) into Eq.(48.a), we have

$$a''(\xi)+ra'(\xi)+sa^2(\xi)a'(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0, \tag{58}$$

letting

$$r=\frac{4(b_3-c_3)}{5p_r+2q_r+p_i+2q_i},\ s=4\frac{(3p_r+2q_r+3p_i+2q_i)(b_5-c_5)}{(5p_r+2q_r+p_i+2q_i)(p_r-p_i)},$$
$$l=\left[\frac{(p_i-p_r)(3p_r+2q_r+3p_i+2q_i)(b_3-c_3)^2}{2(5p_r+2q_r+p_i+2q_i)^2(b_5-c_5)}-\varepsilon\right]\left[\frac{8(b_5-c_5)}{(p_r-p_i)(3p_r+2q_r+3p_i+2q_i)}\right],$$
$$m=\left[b_3+\frac{(3p_r+2q_r+2q_i-p_i)(c_3-b_3)}{2(5p_r+2q_r+p_i+2q_i)}\right]\left[\frac{8(b_5-c_5)}{(p_r-p_i)(3p_r+2q_r+3p_i+2q_i)}\right],$$
$$n=\left[b_5+\frac{(b_5-c_5)(3p_r+2q_r+2q_i-5p_i)}{8(p_r-p_i)}\right]\left[\frac{8(b_5-c_5)}{(p_r-p_i)(3p_r+2q_r+3p_i+2q_i)}\right]. \tag{59}$$

Relations (53), and (58) are of the form of the Liénard equation (4) with r, s, l, m and n given by (54), and (59) respectively. Therefore, making use of Theorem I, we obtain

Theorem 7.

i) Suppose that $s=\frac{2(m\ \mp2\sqrt{m^2-4nl})}{-r\pm\sqrt{r^2-4l}}-\frac{3r}{2l}[-m\pm\sqrt{m^2-4nl}]$, $m^2-4nl>0$, $r^2-4l>0$, $\frac{1}{l}[-m\pm\ \sqrt{m^2-4nl}]$ and c>0, d>0 (c<0, d<0), then Eqs.(53) and (58) then admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_1(\xi)$ in (15) and r, s, l, m and n in (15) are the same as (54). and (59).

ii) In particular,

(a) for $d=\frac{c}{2l}[-m+\sqrt{m^2-4nl}]$, $s=\frac{2(m-2\sqrt{m^2-4nl})}{-r\pm\sqrt{r^2-4l}}-\frac{3r}{2l}[-m+\sqrt{m^2-4nl}]$, if m>0, l>0, n<0, and $r^2-4l\geq0$ or m<0, l<0, n<0 and $m^2-4nl\geq0$ then Eq.(53) and (58) admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_2(\xi)$ in (16) and r, l, m and n in (16) are the same as (54). and (59)

(b) for d=$\frac{c}{2l}[-m-\sqrt{m^2-4nl}]$, $s=\frac{2(m+2\sqrt{m^2-4nl})}{-r\pm\sqrt{r^2-4l}}-\frac{3r}{2l}[-m-\sqrt{m^2-4nl}]$, if m>0, l<0, n<0, then Eq.(53) and (58) admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_3(\xi)$ in (17) and r, l, m and n in (17) are the same as (54). and (59)

(c) for $d=\frac{c}{2l}[-m\pm\sqrt{m^2-4nl}]$, $s=\frac{2(m\mp\sqrt{m^2-4nl})}{-r\pm\sqrt{r^2-4l}}-\frac{3r}{2l}[-m\pm\sqrt{m^2-4nl}]$, if m>0, l<0, n<0, and $m^2-4nl\geq0$ or m<0, l>0, n<0 and $r^2-4l\geq0$ or m<0, l>0, n>0, $m^2-4nl\geq0$ and $r^2-4l\geq0$, then Eq.(53) and (58) admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_4(\xi)$ in (18) and r, l, m and n in (18) are the same as (54). and (59)

Case 5

In order to make the left-hand side of (46.a) equal to zero identically, we set

$$b_1=0,\ \varepsilon=0,\ B=-\frac{b_5}{p_i},\ c_1=-\frac{p_i(3p_r+2q_r)}{4b_5},\ D=-K-\frac{b_3}{p_i},\ v=-\frac{2b_3}{p_i}c_1,$$

$v=\frac{b_3}{2b_5}(3p_r+2q_r), \quad q_i=-\frac{3}{2}p_i$ (60)
Using (60), (46.b) becomes

$$a''(\xi)+la(\xi)+ma^3(\xi)+na^5(\xi)=0, \quad (61)$$

by (61), taking

$$s=0,\ r=0,\ l=[(K+\frac{b_3}{p_i})^2-K^2+\frac{\Omega}{c_1}], \quad m=\frac{1}{c_1}(p_rK+c_3)-\frac{1}{c_1}(p_r+\frac{c_1b_5}{p_i})(K+\frac{b_3}{p_i}), \quad n=\frac{c_5}{c_1}-\frac{b_5}{p_ic_1}(p_r+\frac{b_5}{p_i}c_1), \quad (62)$$

in (19), according to theorem IV, we have

Theorem 8.

(i) Suppose that $l<0,\ n\geq 0,\ m>0$, and $d>0,\ f>0$ then Eqs.(61) and (62) then admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_{11}(\xi)$ and $a_{12}(\xi)$ in (39.a) and (39.b) respectively and l, m and n in (2) are the same as (62).

In particular, for $f=m,\ d=\sqrt{\frac{3m^2-16nl}{12m^2}},\ l<0,\ n\geq 0,\ m>0$ then Eq.(61) and (62) admit exact solutions in the form (32), where $\psi(\xi)$ is expressed as (34), $a(\xi)$ is described as $a_{13}(\xi)$ in (40) and l, m and n in (2) are the same as (62).

(ii) suppose that $l<0,\ n\geq 0,\ m>0$, and $c>0,\ d>0$ then Eq(61) and (62) then admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_{13}(\xi)$ and $a_{14}(\xi)$ in (41.a) and (41.b) respectively and l, m and n in (2) are the same as (62).

In particular if d=-$\frac{m}{4l}c$, then Eq(61) and (62) admit exact solution in the form (32), where $\psi(\xi)$ is expressed as (47), $a(\xi)$ is described as $a_{15}(\xi)$ and $a_{16}(\xi)$ in (42.a) and (42.b) respectively and l, m and n in (2) are the same as (62).

3 Case of degeneracy of Eq.(19)

If D=B=0 and if $\phi=0$, then Eq.(46.a) and (46.b) are reduced to

$$-b_1a_{\xi\xi}+(2Kc_1-v)a_\xi+(b_1K^2-\varepsilon)a+(b_3+p_iK)a^3+b_5a^5-(3p_r+2q_r)a^2a_\xi=0, \quad (63.a)$$

$$c_1a_{\xi\xi}+2Kb_1a_\xi+(\Omega-c_1K^2)a+(c_3+p_rK)a^3+c_5a^5+(3p_i+2q_i)a^2a_\xi=0, \quad (63.b)$$

Imposing equality between the coefficients in the left-hand side of (63.a) and those in (63.b), we have the following relation

$$c_1=-b_1,\ v=-4Kb_1,\ \Omega=-\varepsilon,\ K=\frac{b_3-c_3}{p_r-p_i},$$

$b_5 = c_5, q_r = -\frac{3p_r}{2}, q_i = -\frac{3p_r+2q_r+3p_i}{2}.$ (64)

Taking into account (64) in (63.a), we obtain

$$a''(\xi) + ra'(\xi) + la(\xi) + ma^3(\xi) + na^5(\xi) = 0, \quad (65)$$

where

$$r = -2K,\ l = \frac{\varepsilon}{b_1} - \frac{(b_3-c_3)^2}{(p_r-p_i)^2},\ m = \frac{p_i(b_3-c_3)}{b_1(p_i-p_r)},\ n = -\frac{b_5}{b_1}, \quad (66)$$

then theorem 6 can be used for $\psi = 0$ to determine the solution of Eq.(65).

Another case is obtained when $q_r \neq -\frac{3p_r}{2}$, then we have

$$c_1 = -b_1, v = -4Kb_1, \Omega = -\varepsilon, K = \frac{b_3-c_3}{p_r-p_i}, b_5 = c_5, q_i = -\frac{3p_r+2q_r+3p_i}{2}. \quad (67)$$

Taking into account (67) in (63.a), we obtain

$$a''(\xi) + ra'(\xi) + sa^2(\xi)a'(\xi) + la(\xi) + ma^3(\xi) + na^5(\xi) = 0, \quad (68)$$

where

$$r = -2K,\ s = \frac{3p_r+2q_r}{b_1}\ l = \frac{\varepsilon}{b_1} - \frac{(b_3-c_3)^2}{(p_r-p_i)^2},\ m = \frac{p_i(b_3-c_3)}{b_1(p_i-p_r)},\ n = -\frac{b_5}{b_1}. \quad (69)$$

Then Theorem 7 is used for $\psi = 0$ to determine the exact solution of Eq.(68).

Another case is obtained by taking the left-hand side of Eq.(63.a) equal to zero identically, thus we have the following relations

$$b_1 = 0,\ \varepsilon = 0,\ b_5 = 0,\ q_r = -\frac{3}{2}p_r,\ q_i = -\frac{3}{2}p_i,\ K = -\frac{b_3}{p_i},\ v = 2Kc_1. \quad (70)$$

Then Eq.(63.a) is reduced to

$$a''(\xi) + la(\xi) + ma^3(\xi) + na^5(\xi) = 0, \quad (71)$$

where

$$l = \frac{\Omega}{c_1} - \frac{b_3^2}{c_1 p_i^2},\quad m = \frac{c_3}{c_1} - \frac{b_3 p_r}{c_1 p_i},\quad n = \frac{c_5}{c_1}, \quad (72)$$

Then Theorem 8 is used for $\psi = 0$ to determine the exact solution of Eq.(71).

4 Conclusion

In this work, the explicit exact solitary wave solutions in the form $a(\xi)e^{i[\psi(\xi)+Kx-\Omega t]}$, $\xi = x - vt$ for the complex Ginzburg-Landau equation with higher-order terms have been derived by seeking solutions for a new class of Liénard equation. The method used can also be easily extended to the nonlinear Schroedinger equation with higher-order terms.

5 References

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