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ABSTRACT

In the present paper defining fractional derivative operator as an extension of power series method, we have obtained new results in which H and G-functions have been expressed in terms of Gauss hypergeometric functions.

1. Introduction-

Wright [13] introduced the concept of coefficient of power series having exponential singularities, and discussed various representations with there suitability for studying the special functions. Lavoie et al [6] have discussed the Cauchy Pochhammer complex contour type integrals of the properties of fractional derivatives. Kalla [2] has studied the fractional calculus with generalized H-function and Galue et al [3] have evaluated the fractional integral involving hypergeometric functions. Ghitany [4] has worked for the negative binomial distribution on the hypergeometric functions.

We are making an attempt function to evaluate more generalized functions from among special functions, the H-function & G-functions, by the technique of the simplest representations of power series including the fractional derivatives of the two functions $e^{az}\, and\; z^{p}$.

2. Fractional Derivative for G-function and H-function in term of Gauss hypergeometric functions

Hence we shall prove the result

$$\mathrm{H}_{1,3}^{1,1}\left[\frac{z^2}{4}\left|\begin{matrix}\left(\frac{\alpha+v+1}{2},1\right)\\ \left(\frac{\alpha+v+1}{2},1\right),\left(\frac{\alpha+v}{2},1\right),\left(\frac{\alpha-v}{2},1\right)\end{matrix}\right.\right]$$

$$=\frac{2}{\sqrt{p}}z^{v+1}\Gamma\left[\begin{matrix}\alpha+v+2\\ v+3/2,v+2\end{matrix}\right]{}_1F_0\left(1+\alpha;-;\tfrac{-4}{z}\right)\,{}_1F_2\left(1;\tfrac{3}{2},v+\tfrac{3}{2};-\tfrac{z^2}{4}\right) \tag{2.1}$$

Proof –

Making use of the value of $H_{1,3}^{1,1}$ to terms of Hermite function and then converting the same is terms of Gauss hypergeometric functions, we have following expression for left hand side of above result

$$\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{a+v+1}\sum_{n=0}^{\infty}\frac{(1)_n}{\left(\frac{3}{2}\right)_n\left(v+\frac{3}{2}\right)_n n!}\left(-\frac{z^2}{4}\right)^n$$

$$=\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{a+v+1}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma\left(3/2\right)\Gamma\left(v+3/2\right)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(v+\frac{3}{2}+n\right)n!}\left(-\frac{z^2}{4}\right)^n, \tag{2.2}$$

which by the application of fractional derivative yields

$$\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}D_z^a\left(\frac{z}{2}\right)^{a+v+1}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+3/2\right)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(v+\frac{3}{2}+n\right)n!}\left(-\frac{z^2}{4}\right)^n$$

$$=\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}\sum_{m,n=0}\binom{\alpha}{m}D_z^{a-m}\left(\frac{z}{2}\right)^{a+v+1}D_z^m\frac{(1)_n}{n!\left(\frac{3}{2}\right)_n\left(v+\frac{3}{2}\right)_n}\left(-\frac{z^2}{4}\right)$$

Application of the result $D_z^a z^p=\frac{p!}{\Gamma(p-\alpha+1)}z^{p-a}$ can transform above expression as

$$\frac{z^{v+1}}{\Gamma\left(\tfrac{3}{2}\right)\Gamma\left(v+\tfrac{3}{2}\right)}\sum_{m,n=0}^{\infty}\frac{(1+\alpha)_m\,\Gamma(\alpha+v+2)\Gamma(1+n)\Gamma\left(\tfrac{3}{2}+m\right)\Gamma\left(v+\tfrac{3}{2}\right)}{m!\,\Gamma(m+v+2)\Gamma\left(\tfrac{3}{2}+n\right)\Gamma\left(v+\tfrac{3}{2}+n\right)n!}\left(-\frac{z^2}{4}\right)^{n-m} z^m$$

which after simplification generates the result (2.1).

Now we shall prove the result

$$H_{2,4}^{1,2}\left[z^2\left|\begin{matrix}\left(\frac{\lambda}{2}\right),\left(\frac{\lambda+1}{2},1\right)\\ \left(\frac{\lambda+m+v}{2},1\right),\left(\frac{\lambda-m+v}{2},1\right),\left(\frac{\lambda+m-v}{2},1\right)\left(\frac{\lambda-m-v}{2},1\right)\end{matrix}\right.\right]$$

$$=\sqrt{\pi}\,z^{\lambda}\left(\frac{z}{2}\right)^{v-\alpha+m}\Gamma\left[\begin{matrix}1+v+m\\ 1+v,1+m,1+v+m-\alpha\end{matrix}\right]$$

$$ {}_1F_0\left(1+\alpha;-;-\tfrac{2}{z}\right)\ {}_0F_2\left(-;1+v,1+m;\tfrac{z^4}{16}\right) \qquad (2.3)$$

Proof –

Converting the result $H_{2,4}^{1,2}$ in form of the product of Bessel's function $J_m(z)\ and\ J_v(z)$ and then expanding with the help of power series form

$$\sum_{r=0}^{\infty}\frac{(1)^r}{r!}\frac{\left(\frac{z}{2}\right)^{v+2r}}{\Gamma(v+r+1)},$$

and use of Gauss hypergeometric function, we can have

$$\Gamma\left(\tfrac{1}{2}\right)2^{\lambda}\,\frac{\left(\frac{z}{2}\right)^{v+m}}{\Gamma(v+1)\Gamma(m+1)}\sum_{n=0}^{\infty}\frac{1}{n!(v+1)_n(m+1)_n}\left(-\frac{z^2}{4}\right)^{2n}$$

Application of fractional calculus converts above expression as

$$\frac{\left(\sqrt{p}\right)\,2^{l}}{\Gamma(v+1)\Gamma(m+1)}\sum_{m,n=0}^{\infty}\frac{\Gamma(1+a+m)\Gamma(1+v+m)}{m!\,\Gamma(1+a)_n(v+1)_n(m+1)_n}$$

$$\frac{\Gamma(2n+1)}{n!\,\Gamma(2n-m+1)\Gamma(1+v+m-a+m)}\left(\frac{-z^2}{4}\right)^{2n-m}\left(\frac{z}{2}\right)^{v+m-a+m}$$

which on simplification we produces result (2.3).

We shall prove the result

$$H_{1,3}^{3,1}\left[\frac{z^2}{4}\left|\begin{matrix}\left(\frac{\boldsymbol{a}+\boldsymbol{m}+1}{2},1\right)\\ \left(\frac{\boldsymbol{a}+\boldsymbol{m}+1}{2},1\right)\left(\frac{\boldsymbol{a}+v}{2},1\right)\left(\frac{\boldsymbol{a}-v}{2},1\right)\end{matrix}\right.\right]$$

$$\times H_{2,2}^{1,2}\left[z\left|\begin{matrix}(1+\boldsymbol{l}-\boldsymbol{a},1)(1+\boldsymbol{l}-\boldsymbol{b},1)\\ (\boldsymbol{l},1)(1+\boldsymbol{l}-\boldsymbol{g},1)\end{matrix}\right.\right]$$

$$=\frac{2z^{l+1-d}\left(\frac{z}{2}\right)^{a+m}}{(m-v+1)(m+v+1)}\Gamma\left[\begin{matrix}\boldsymbol{a},\boldsymbol{b},\frac{1-m+v}{2},\frac{1-m-v}{2}\\ \boldsymbol{g}\end{matrix}\right]$$

$$\times\,{}_3F_3\left(\boldsymbol{a},\boldsymbol{b},1;\boldsymbol{g},\frac{\boldsymbol{g}-v+3}{2},\frac{\boldsymbol{m}+v+3}{2};\frac{-z^3}{4}\right) \qquad (2.4)$$

Proof-

Expanding the value of $H_{1,3}^{3,1}$ and in form of the Lommel's function and also expand $H_{2,2}^{1,2}$ thus we have obtain the left hand side of (2.4) as

$$\left(\frac{z}{2}\right)^{a} 2^{1-m}\,\Gamma\left(\frac{1-m+v}{2}\right)\Gamma\left(\frac{1-m-v}{2}\right)\frac{\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{b})}{\Gamma(g)} z^{l}$$
$$\times\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_n(\boldsymbol{b})_n}{(g)_n\, n!}(-z)^n S_{m,v}(z)$$

We know that

$$S_{m,v}(z)=\frac{z^{m+1}}{(m-v+1)(m+v+1)}\,{}_1F_2\left(1;\frac{m-v+3}{2},\frac{m+v+3}{2},\frac{z^2}{4}\right) \tag{2.5}$$

Putting the value of (2.5) in the (2.4), we have

$$\sum_{n=0}^{\infty}\Gamma\left(\frac{1-m+v}{2}\right)\Gamma\left(\frac{1-m-v}{2}\right)\frac{\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{b})}{\Gamma(g)(m-v+1)(m+v+1)n!}\left(\frac{z}{2}\right)^{a} 2^{1-m}$$
$$\times\frac{(\boldsymbol{a})_n(\boldsymbol{b})_n(1)_n}{(g)_n\left(\frac{m-v+3}{2}\right)_n\left(\frac{m+v+3}{2}\right)_n}(-z)^n z^{l+m+1}\left(\frac{z^2}{4}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_n(\boldsymbol{b})_n(1)_n\,\Gamma\left(\frac{1-m+v}{2}\right)\Gamma\left(\frac{1-m-v}{2}\right)\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{b})}{(g)_n\left(\frac{m-v+3}{2}\right)_n\left(\frac{m+v+3}{2}\right)_n n!\,\Gamma(g)(m-v+1)(m+v+1)}$$
$$\times(-1)^n z^{3n+l+m+1+a}\, 2^{1-m-a-2n} \tag{2.6}$$

application of fractional calculus yields

$$\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_n(\boldsymbol{b})_n(1)_n\,\Gamma\left(\frac{1-m+v}{2}\right)\Gamma\left(\frac{1-m-v}{2}\right)\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{b})(-1)^n}{(g)_n\left(\frac{m-v+3}{2}\right)_n\left(\frac{m+v+3}{2}\right)_n n!\,\Gamma(g)(m-v+1)(m+v+1)} z^{1-m-a-2n} D_z^d z^{3n+l+a+m+1}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(1)_n\Gamma\left(\frac{1-m+\nu}{2}\right)\Gamma\left(\frac{1-m-\nu}{2}\right)\Gamma(\alpha)\Gamma(\beta)(-1)^n}{(\gamma)_n\left(\frac{m-\nu+3}{2}\right)_n\left(\frac{m+\nu+3}{2}\right)_n n!\,\Gamma(\gamma)(m-\nu+1)(m+\nu+1)}$$

$$\times 2^{1-m-\alpha-2n}\frac{\Gamma(3n+\lambda+\alpha+m+2)}{\Gamma(3n+\lambda+\alpha+m+2-\delta)}z^{3n+\lambda+\alpha+m+1-\delta}$$

Finally applying the result of Gauss hypergeometric function we can generate the result (2.4).

Applying the same procedure as adopted for H-function, we can have following result for G-function

$$G_{04}^{20}\left(4^{-4}x^4\middle|\tfrac{1}{4}\nu+\tfrac{1}{4}m,\tfrac{1}{4}\nu+\tfrac{1}{4}m+\tfrac{1}{2},\tfrac{1}{4}m-\tfrac{1}{4}\nu,\tfrac{1}{2}+\tfrac{1}{4}m-\tfrac{1}{4}\nu\right)$$

$$=\frac{\Gamma(1+\alpha)}{2^{2m+\nu}}z^{m+\nu-\alpha}\;{}_2F_1\left[\begin{matrix}1+\alpha,1+m\\1+m+\nu\end{matrix};\tfrac{-4}{z}\right]. \qquad (2.7)$$

Now we shall prove

$$G_{24}^{12}\left[z^2\middle|\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}s,\tfrac{1}{2}s\\ \tfrac{1}{2}(m+\nu+s),\tfrac{1}{2}(\nu+s-m),\tfrac{1}{2}(m+s-\nu),\tfrac{1}{2}(s-m-\nu)\end{matrix}\right]$$

$$G_{04}^{10}\left[\frac{z^4}{64}\middle|\tfrac{1}{4}m+\tfrac{1}{2}\nu,\tfrac{1}{4}m-\tfrac{1}{2}\nu,\tfrac{1}{4}m,\tfrac{1}{4}m+\tfrac{1}{2}\right]$$

$$=(-1)^{3n}\frac{z^{s+2m+3\nu-\alpha}}{2^{5m/2+3\nu}}\Gamma\left[\begin{matrix}1+\nu+\alpha\\1+\nu,1+\nu,1+m,1+\nu,1+\nu\end{matrix}\right]$$

$${}_1F_2\left[\begin{matrix}1+\nu\\1+s+2m+3\nu\end{matrix};\frac{256}{z^7}\right]{}_0F_3\left[\begin{matrix}-\\1+\alpha,1+s,1+\beta\end{matrix};\frac{z^8}{256}\right] \qquad (2.8)$$

Proof-

Converting the values of G_{24}^{12} *and* G_{04}^{10} in the product form of Bessel's functions and modified Bessel's function of first kind, we have

$$\frac{z^{s+m}}{2^{3m/2}}\frac{\left(\frac{z}{2}\right)^{m+3v}}{\Gamma(m+1)(\Gamma(v+1))^{3}}\sum_{n=0}^{\infty}\frac{1}{(m+1)_{n}(v+1)_{n}(v+1)_{n}(v+1)_{n}}(-1)^{3n}\left(\frac{z^{2}}{4}\right)^{4n}$$

$$=\frac{z^{s+2m+3v}}{2^{5m/2+3v}\Gamma(m+1)(\Gamma(v+1))^{3}}\sum_{n=0}^{\infty}\frac{(-1)^{3n}}{(v+1)_{n}(m+1)_{n}(v+1)_{n}(v+1)_{n}}\left(\frac{z^{2}}{4}\right)^{4n} \quad (2.9)$$

Where $(m+1)_{n}=\dfrac{\Gamma(m+1+n)}{\Gamma(m+1)}$

Use of fractional derivatives can result in derivation of

$$\frac{1}{2^{5m/2+3v}\Gamma(m+1)(\Gamma(v+1))^{3}}\sum_{n=0}^{\infty}\frac{(-1)^{3n}}{(v+1)_{n}(m+1)_{n}(v+1)_{n}(v+1)_{n}}D_{z}^{a}Z^{s+2m+3v}\left(\frac{z^{8}}{256}\right)^{n}$$

$$=\frac{1}{2^{5m/2+3v}\Gamma(m+1)(\Gamma(v+1))^{3}}$$

$$\times\sum_{n=0}^{\infty}\frac{(-1)^{3n}}{(v+1)_{n}(m+1)_{n}(v+1)_{n}(v+1)_{n}}\sum_{m=0}^{\infty}\binom{a}{m}D_{z}^{a-m}Z^{s+2m+3v}D_{z}^{m}\left(\frac{z^{8}}{256}\right)^{n}$$

(2.10)

Which can generate the result (2.8) by using the result of power function and simplification.

3. The following list (without proof) depicts our results obtained for H and G-function in the form of hypergeometric function

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{z^{2}}{4}\middle|(0,1),(\tfrac{1}{2},1)\right]=\frac{z^{-a}}{2\sqrt{p}\,\Gamma(1-a)}\left\{{}_{1}F_{1}(1;1-a;iz)+{}_{1}F_{1}(1;1-a;-iz)\right\} \quad (3.1)$$

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{z^2}{4}\middle|\left(\tfrac{1}{2},1\right),(0,1)\right]$$

$$= \frac{z^{-a}}{2i\sqrt{\boldsymbol{p}}\,\Gamma(1-\boldsymbol{a})}\left\{{}_1F_1(1;1-\boldsymbol{a};\quad iz)-{}_1F_1(1;1-\boldsymbol{a};\quad -iz)\right\} \tag{3.2}$$

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{-z^2}{4}\middle|\left(\tfrac{1}{2},1\right),(0,1)\right]$$

$$= \frac{z^{-a}}{2i\sqrt{\boldsymbol{p}}\,\Gamma(1-\boldsymbol{a})}\left\{{}_1F_1(1;1-\boldsymbol{a};\quad -az)-{}_1F_1(1;1-\boldsymbol{a};\quad az)\right\} \tag{3.3}$$

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{-z^2}{4}\middle|(0,1),\left(\tfrac{1}{2},1\right)\right]$$

$$= \frac{z^{-a}}{2\sqrt{\boldsymbol{p}}\,\Gamma(1-\boldsymbol{a})}\left\{{}_1F_1(1;1-\boldsymbol{a};\quad az)+{}_1F_1(1;1-\boldsymbol{a};\quad -az)\right\} \tag{3.4}$$

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{z^2}{4}\middle|\left(\tfrac{1}{4},1\right),\left(\tfrac{-1}{4},1\right)\right]$$

$$= \frac{n!\,z^{-n}}{\sqrt{2i}}\,{}_1F_1\left(\tfrac{1}{2};\tfrac{3}{2};\quad z\right)\left\{{}_1F_1(1;1-n;\quad iz)-{}_1F_1(1;1-n;\quad -iz)\right\} \tag{3.5}$$

$$\mathrm{H}_{2,2}^{1,0}\left[z\middle|\begin{matrix}(1,1),(1,1)\\(1,1),(0,1)\end{matrix}\right] = z\;{}_2F_1(1,1;\ 2;\quad z) \tag{3.6}$$

$$\mathrm{H}_{2,2}^{1,2}\left[z^2\middle|\begin{matrix}(1,1),\left(\tfrac{1}{2},1\right)\\\left(\tfrac{1}{2},1\right),(0,1)\end{matrix}\right] = 2z\;{}_2F_1\left(\tfrac{1}{2},1;\tfrac{3}{2};\quad -z^2\right) \tag{3.7}$$

$$\mathrm{H}_{2,2}^{1,2}\left[-z^2\middle|\begin{matrix}\left(\tfrac{1}{2},1\right),\left(\tfrac{1}{2},1\right)\\(0,1),\left(\tfrac{-1}{2},1\right)\end{matrix}\right] == 2z\;{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};\quad z^2\right) \tag{3.8}$$

$$\mathrm{H}_{1,1}^{1,0}\left[z\middle|\begin{matrix}(\boldsymbol{a}+\boldsymbol{b}+1,1)\\(\boldsymbol{a},1)\end{matrix}\right] = \frac{{}_2F_1(1,-\boldsymbol{a};1;\quad 1-z)\times{}_2F_1(1,-\boldsymbol{b};1;\quad z)}{\Gamma(1+\boldsymbol{b})} \tag{3.9}$$

$$\mathrm{H}_{0,2}^{1,0}\left[\frac{z^2}{4}\left|\left(\tfrac{a+n}{2},1\right),\left(\tfrac{a-n}{2},1\right)\right.\right]=\left(\frac{z}{2}\right)^{a+n}\frac{{}_0F_1\left(n+1;\tfrac{-1}{4}z^2\right)}{\Gamma(n+1)} \tag{3.10}$$

$$\mathrm{H}_{1,2}^{2,0}\left[2z\left|\begin{matrix}\left(\tfrac{1}{2}+a,1\right)\\ (a+n,1),(a-n,1)\end{matrix}\right.\right]=\sqrt{p}\,2^{a-1}z^{a}\,\Phi(a,a;-z)\times{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;z^2\right) \tag{3.11}$$

$$G_{02}^{10}\left[\frac{1}{4}x^2\left|\tfrac{1}{2}n+\tfrac{1}{2}m,\tfrac{1}{2}m-\tfrac{1}{2}n\right.\right]=\left(\frac{x}{2}\right)^{m+n}\frac{{}_0F_1\left(n+1;\tfrac{-1}{4}x^2\right)}{\Gamma(n+1)} \tag{3.12}$$

$$G_{13}^{11}\left(\tfrac{1}{4}x^2\left|\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}n+\tfrac{1}{2}m\\ \tfrac{1}{2}+\tfrac{1}{2}n+\tfrac{1}{2}m,\ \tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{1}{2}m+\tfrac{1}{2}n\end{matrix}\right.\right)$$

$$=\left(\frac{x}{2}\right)^{m+n+1}\frac{{}_1F_2\left(1,\tfrac{3}{2},n+\tfrac{3}{2};\tfrac{-x^2}{4}\right)}{\tfrac{1}{2}\sqrt{p}\,\Gamma\left(v+\tfrac{3}{2}\right)} \tag{3.13}$$

$$G_{02}^{10}(x|a,b)=\frac{x^{a/\left(a^2-b^2\right)}}{\Gamma(a-b+1)}\,{}_0F_1(a-b+1;-x) \tag{3.14}$$

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