FRACTIONAL DERIVATIVE OPERATOR INVOLVING PRODUCTS OF SPECIAL FUNCTIONS AND GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The present paper aims at the derivation of certain fractional calculus formulae for Fox's H- function by the application of fractional calculus formulae involving a general class of polynomials.

1.INTRODUCTION

Raina [5] obtained a fractional derivative formula for the function z^{p} using generalized Gauss theorem, while Ross [7] obtained the fractional integral formula for the function $(\alpha z+\beta)^{a}$ using series expansion method. Kalla et al [4] has derived the fractional integral transformation by orthogonal polynomials. Ali et al [1] generated the expansion of the Laguerre Polynomials and Soni & Singh [12] obtained the fractional derivative formulae involving the product of a general class of polynomials.

The present work is an attempt in the direction of obtaining fractional calculus formula by utilizing series expression method, introduced by Srivastava [9]. The name general class of polynomials, itself indicates the importance of the results, because we can derive a number of fractional calculus formulae for various classical orthogonal polynomials.

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2.FRACTIONAL DERIVATIVES

In this section we shell prove following three fractional calculus formulae.

1

$$D_z^{\alpha}\left[z^{k}(z+\xi)^{-\lambda} S_n^{m}\left(z^{p}(z+\xi)^{-\sigma}\right)\right]$$

$$=\xi^{-\lambda} z^{k-\alpha}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j\,!}A_{n,j}\Gamma\begin{bmatrix}\rho j+k+1\\ \rho j+k-\alpha+1\end{bmatrix}$$

$$\times\left(\frac{z^{\rho}}{\xi^{\sigma}}\right)^{j}{}_2F_1\begin{bmatrix}\lambda+\sigma j,\, k+\rho j+1; & -\dfrac{z}{\xi}\\ k+\rho j-\alpha+1; & \end{bmatrix}\text{---------}(2.1)$$

$$\textit{valid for}\ \min\,(k,\lambda,\rho,\sigma)>0,\left|z/\xi\right|<1\,\textit{and}\ \operatorname{Re}(k+\rho j-\mu+1)>0.$$

Proof –

For the proof of this result we shall utilize following definition introduced by Srivastava [9] or general class of polynomials

$$S_n^m(z)=\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j\,!}\,A_{n,j}z^j \text{ ------------ } (2.2),$$

where m is an arbitrary positive integer and the coefficient $A_{n,j}(n,\ j\geq 0)$ are arbitrary constants, real or complex.

Expressing the general class of polynomials $S_n^m(z)$ occurring on its left hand side in the series form given (2.2), the left hand side of (2.1) (say $\oplus$) takes the following form-

$$\oplus = D_z^{\alpha}\left[z^k(z+\xi)^{-\lambda}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}z^{\rho j}(z+\xi)^{-\sigma j}\right]$$

Using the following form of the binomial theorem

$$[z+\xi]^{-\lambda}=\xi^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m\,!}\left(\frac{-z}{\xi}\right)^m \text{ ---------- } (2.3),$$

in the above expression, we have

$$\oplus=\xi^{-\lambda}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}\xi^{-\sigma j}\sum_{m=0}^{\infty}\frac{(\lambda+\sigma j)_m}{m!}D_z^{\alpha}z^{k+\rho j}\left(\frac{-z}{\xi}\right)^m$$

Use of the fractional derivative of product of two functions, and after simplification we can deduce the result (2.1).

We can also express equation (2.1) in following form

$$D_z^{\alpha}\left(z^k(z+\xi)^{-\lambda}S_n^m\left(z^p(z+\xi)^{-\sigma}\right)S_p^q(z^{\delta})\right)$$

$$==\xi^{-\lambda}z^{k-\alpha}\sum_{j=0}^{[n/m]}\sum_{i=0}^{[p/q]}\frac{(-n)_{mj}(-P)_{qi}}{i\,!\,j\,!}A_{n,j}A_{p,i}$$

$$\times\Gamma\left[\begin{matrix}\rho j+\delta i+k+1\\ \rho j+\delta i+k-\alpha+1\end{matrix}\right]z^{\rho j+\delta i}\xi^{-\sigma j}\times {}_2F_1\left[\begin{matrix}\lambda+\sigma j,k+\rho j+\delta i+1; & -z\\ k+\rho j+\delta i-\alpha+1; & \xi\end{matrix}\right] \text{ ------- } (2.4)$$

valid for $\min(k,\lambda,\rho,\sigma,\delta)>0,\left|z/\xi\right|<1,$

$\mathrm{Re}(k+\rho j+\delta i-\alpha+1)>0,$ *where* ${}_2F_1(*)$ *is the well known Gauss functions.*

2.

$$D_z^{\alpha}\{z^kS_n^m z^{\rho}(z+\xi)^{-\sigma}H_{P,Q}^{M,N}(z^k)\}$$

$$=(-1)^{\alpha}z^{k-\alpha}\xi^{\alpha}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}\left(\frac{z^{\rho}}{\xi^{\sigma}}\right)^j\Gamma(1+\alpha)\,{}_2F_1\left(\begin{matrix}\sigma_j,1+\beta\\ 1-\alpha\end{matrix};-\frac{z}{\xi}\right){}_1F_0\left(1+\alpha;;-\tfrac{1}{\xi}\right)$$

$$H_{P+1,Q+1}^{M,N+1}\left[z^k\left|\begin{matrix}(-k-\rho_j,k),(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,Q},(-\rho_j-k+n,k\end{matrix}\right.\right] \text{ ---------------- } (2.5)$$

Proof –

Expending $S_n^m(z^\rho(z+\xi)^{-\sigma})$ by polynomial (2.2) and then expending it in terms of binomial theorem, we shall have

$$\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}\xi^{-\sigma j}A_{n,j}D_z^\alpha\left\{\sum_{m=0}^{\infty}\frac{(\sigma j)_m}{m!}\left(\frac{-z}{\xi}\right)^m z^{k+\rho j}\mathrm{H}_{P,Q}^{M,N}(z^k)\right\},$$

which yields

$$\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}\xi^{-\sigma j}A_{n,j}\sum_{m,n=0}^{\infty}\frac{(\sigma j)_m}{m!}\binom{\alpha}{n}D_z^{\alpha-n}\left(-\frac{z}{\xi}\right)^m D_z^n\left(z^{k+\rho j}\mathrm{H}_{P,Q}^{M,N}(z^k)\right)$$

and following result by the application of fractional product rule

$$\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}\xi^{-\sigma j}A_{n,j}\sum_{m,n=0}^{\infty}\frac{(\sigma j)_m}{m!}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)\,n!}\frac{\Gamma(1+m)}{\Gamma(1+m-\alpha+n)}$$

$$\left(-\frac{z}{\xi}\right)^{m-\alpha+n}z^{k+\rho j-n}\sum_{l=0}^{\infty}\binom{n}{l}\frac{1}{\Gamma(l-n+1)}S_n^{(k+\rho j)}(z),$$

$$where\quad S_n^{(k+\rho j)}(z)=\mathrm{H}_{P+1,Q+1}^{M,N+1}\left[z^k\left|\begin{matrix}(-k-\rho j,k),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-k-\rho j+n,k)\end{matrix}\right.\right]$$

on mere simplification above result can be expressed as (2.5).

3.

$$D_z^\alpha H_{P,Q}^{M,N}\left[z^k\left|\begin{matrix}(a_j,\alpha_j)_{1.P}\\(b_j,\beta_j)_{1.Q}\end{matrix}\right.\right]H_{1,1}^{1,0}\left[z\left|\begin{matrix}(\alpha+\beta+1,1)\\(\alpha,,1)\end{matrix}\right.\right]\mathrm{H}_{U,V}^{u,v}\left[z^\rho\left|\begin{matrix}(e_j,E_j)_{1.U}\\(f_j,F_j)_{1.V}\end{matrix}\right.\right]H_{1,1}^{1,1}\left[-z\left|\begin{matrix}(1-\alpha,1)\\(0,1)\end{matrix}\right.\right]$$

$$=(-1)^{-\alpha}\Gamma\left[\begin{matrix}2\alpha\\1+\alpha+\beta\end{matrix}\right]F\left[\begin{matrix}\alpha,\beta,1+\alpha+\beta\\1+\alpha-\beta\end{matrix};-z^2\right]F(\alpha,\beta\;;-;-1)$$

$$\mathrm{H}_{P+1,Q+1}^{M,N+1}\left[z^k\left|\begin{matrix}(-n+\alpha,k),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-\alpha,k)\end{matrix}\right.\right]\mathrm{H}_{U+1,V+1}^{u,v+1}\left[z^h\left|\begin{matrix}(0,h),(e_j,E_j)_{1,U}\\(f_j,F_j)_{1,V},(l,h)\end{matrix}\right.\right]------(2.6)$$

Proof –

We observe that H_{11}^{10} can be expended in the exponential form in while H_{11}^{11} when exposed to binomial theorem(2.3), we shall have the value of the expression given in left hand side, say $\oplus$ as-

$$\oplus=D_z^\alpha\frac{\Gamma(\alpha)}{\Gamma(\beta+1)}\sum_{n=0}^{\infty}\binom{\beta}{n}\binom{\alpha}{n}z^{n+\alpha}(-z)^n H_{P,Q}^{M,N}\left[z^k\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]H_{U,V}^{u.v}\left[z^\rho\left|\begin{matrix}(e_j,E_j)_{1,U}\\(f_j,F_j)_{1,V}\end{matrix}\right.\right]$$

$$=\frac{\Gamma(\alpha)}{\Gamma(\beta+1)}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)n!}D_z^\alpha z^{n+\varepsilon}(-z)^n$$

$$H_{P,Q}^{M,N}\left[z^k\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]H_{U,V}^{u,v}\left[z^h\left|\begin{matrix}(e_j,E_j)_{1,U}\\(f_j,F_j)_{1,V}\end{matrix}\right.\right]$$

which by the application of the fractional product rule yields

$$\oplus=\frac{\Gamma(\alpha)}{\Gamma(\beta+1)}\sum_{n=0}^{\infty}\frac{(\beta)_n(\alpha)_n}{n!}\sum_{r=0}^{\infty}\binom{\alpha}{r}D_z^{\alpha-r}(-z)^n$$

$$D_z^r z^{n+\alpha}H_{P,Q}^{M,N}\left[z^k\left|\begin{matrix}(a_j,\alpha_j)_{1.P}\\(b_j,\beta_j)_{1.Q}\end{matrix}\right.\right]H_{U,V}^{u,v}\left[z^{\rho}\left|\begin{matrix}(e_j,E_j)_{1.U}\\(f_j,F_j)_{1.V}\end{matrix}\right.\right]$$

$$=\frac{\Gamma(\alpha)}{\Gamma(\beta+1)}\sum_{m,n,l,r=0}^{\infty}\frac{(\alpha)_n(\beta)_n\Gamma(\alpha+r)\Gamma(n+1)}{n!r!\Gamma(\alpha)\Gamma(n-\alpha+r+1)}(-z)^{n-\alpha+r}z^{n+\alpha-r}$$

$$\binom{r}{m}\binom{r-m}{m}\binom{m}{l}\frac{1}{\Gamma(1-r+m+n)\Gamma(1-m+l)}$$

$$S_n^{(n+\alpha)}(z)H_{U+1,V+1}^{u.v+1}\left[z^h\left|\begin{matrix}(0,1),(e_j,E_j)_{1,U}\\(f_j,F_j)_{1,V},(l,h)\end{matrix}\right.\right]$$

where $S_n^{(n+\alpha)}(z)=H_{P+1,Q+1}^{M,N+1}\left[z^k\left|\begin{matrix}(-n+\alpha,k),(a_j,\alpha_j)_{1.P}\\(b_j,\beta_j)_{1.Q,}(-\alpha,,k)\end{matrix}\right.\right]$

The terms $\binom{r}{m}\binom{r-m}{m}\binom{m}{l}\frac{1}{\Gamma(1-r+m+n)\Gamma(1-m+l)}$ *involving Gamma functions can be* transformed is the form of hypergeometric function

$$\frac{\Gamma(2\alpha)}{\Gamma(1+\alpha+\beta)}\sum_{r,n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\alpha)_r(\beta)_r(1+\alpha+\beta)_n}{n!r!(1+\alpha-\beta)_n}(-1)^r(-z^2)^n$$

$$S_n^{(n+\alpha)}(z)H_{U+1,V+1}^{u.v+1}\left[z^h\left|\begin{matrix}(0,1),(e_j,E_j)_{1,U}\\(f_j,F_j)_{1,V},(l,h)\end{matrix}\right.\right]$$

so as to have the result (2.6) on simplification.

3. SPECIAL CASES

As the special cases of our main results, if we take $\sigma=0$ and $\lambda=0$, we deduce the results given earlier by Saigo et al [8].

If we take $\sigma=0$, then for formula (2.1), we have

$$D_z^{\alpha}\left[z^k(z+\xi)^{-\lambda}S_n^m z^{\rho}\right]=\xi^{-\lambda}z^{k-\alpha}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}\quad A_{n,j}$$

$$\times\Gamma\begin{bmatrix}\rho j+k+1\\ \rho j+k-\alpha+1\end{bmatrix}z^{\rho j}\ {}_2F_1\begin{bmatrix}\lambda,k+\rho j+1; & \frac{-z}{\xi}\\ k+\rho j-\alpha+1; & \end{bmatrix}\text{---------}(3.1)$$

while (2.4) reduces to

$$D_z^{\alpha}\left[z^k(z+\xi)^{-\lambda}\, S_n^m\left(z^p\right)S_p^q z^{\delta}\right]$$

$$=\xi^{-\lambda} z^{k-\alpha}\sum_{j=0}^{[n/m]}\sum_{i=0}^{[p/q]}\frac{(-n)_{mj}(-p)_{qi}}{i\,!\,j\,!}A_{n,j}A_{p,i}$$

$$\times\Gamma\begin{bmatrix}\rho j+\delta\, i+k+1\\ \rho j+\delta\, i+k-\alpha+1\end{bmatrix}z^{\rho\, j+\delta\, i}\;{}_2F_1\begin{bmatrix}\lambda,k+\rho j+\delta\, i+1; & \frac{-z}{\xi}\\ k+\rho j+\delta\, i-\alpha+1; & \end{bmatrix}----------(3.2)$$

If we take $\lambda=0$ in (3.1) and (3.2), these formulae reduce to

$$D_z^{\alpha}\left[z^k\, S_n^m\left(z^{\rho}\right)\right]=z^{k-\alpha}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j\,!}A_{n,j}\times\Gamma\begin{bmatrix}\rho j+k+1\\ \rho j+k-\alpha+1\end{bmatrix}z^{\rho\, j}----------(3.3)$$

$$and\; D_z^{\alpha}\left[z^k\, S_n^m\left(z^{\rho}\right)S_p^q\left(z^{\delta}\right)\right]=z^{k-\alpha}\sum_{j=0}^{[n/m]}\sum_{i=0}^{[p/q]}\frac{(-n)_{mj}(-p)_{qi}}{i!\;j!}A_{n,j}\,A_{p,i}$$

$$\times\Gamma\begin{bmatrix}\rho j+\delta i+k+1\\ \rho j+\delta i+k-\alpha+1\end{bmatrix}z^{\rho\, j+\delta\, i}-----------(3.4)$$

Out of these two formulae (3.3) and (3.4) on specialization of $A_{n,j}$ reduces to a very special type of polynomial representation which was used by Vyas et al [13].

By putting $m=\rho=1,\quad k=\alpha,\lambda=0$ *and*

$A_{n,j}=\dfrac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+j)}$ in (3.1) we shall have the result given by Raina [5].

Substituted $\rho=\delta=m=q=1$

$k=\alpha$ and

$$A_{n,j}=\frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+j)\,n!}\;and\;A_{p,i}=\frac{\Gamma(1+p+\beta)}{\Gamma(1+i+\beta)\,p!}$$

in (3.4) produced the Kample de Feriet function and $L_p^{(\beta)}$.

Setting $m=\rho=\sigma=1$

$$k=\alpha\;and\;A_{n.j}=\frac{(1+\alpha+\beta+n)_j(1+\alpha)_n}{(1+\alpha)_j\;n!}\,in\,(2.1)\,we\;can\;obtain\;results$$

involving Jacobi Polynomial $P_n^{(\alpha,\beta)}$.

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