NEW RESULTS FOR H-FUNCTION AND G-FUNCTION BY THE APPLICATION OF FRACTIONAL CALCULUS

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ABSTRACT

In the present study an attempt is being made to obtain numerically evaluable new results for Fox's H-function and Meijer's G-function by the application of Leibniz rule product technique. the discussed process has been testified by establishment of known results for hypergeometric functions of unit argument.

1. INTRODUCTION

The Fractional derivative, extension of the familiar derivative $\left[\frac{d^n}{dz^n} f(z)\right]$ to non integral values of n, is of immense utility in finding the solution of ordinary, Partial and integral equation, as well as in other contexts.

Fractional derivatives for conventional functions were introduced long before the systematic development of the generalized function. Indeed one of the more creditable aspects of the new generalized functional theory would seem to be the fractional derivatives as well as hypergeometric function.

Osler [6] has discussed the generalization of the Leibniz rule for the derivative of the product of two functions and has used Leibniz rule to compute the value of hypergeometric function of unit argument is terms of gamma function. Raimond [15] expressed the fractional powers of the integration operator identified with their traditionalcounter parts.

Samko. et. Al. [3] has compiled differential aspects of fractional integrals and derivatives in their text book. Ali & Kalla [1] have dealt with application of fractional calculus to the solution of general terms of differential equations. Ali &Tuan [4] have studied the application of basic hypergeometric series to stable analytic continuation. Monreal. et. Al. [2] have worked for the evaluation of special functions using continuous fraction method.

We are making an attempt in the direction of evaluation of generalized functions like G- function and H-function by finding expressions for them in the form of Gamma function. The validity of the product of Leibniz rule technique has been demonstrated in the beginning of the study so as to venture with greater confidence in the direction of obtaining new results more generalized functions, by it application in finding expressions for Gauss hypergeometric functions of unit argument.

2. GENERAL FORMULATION

Osler [6] has used the technique of Nekrassov [12] is order to yield

a definition of fractional differentiation by generalizing the Cauchy integral formula.

$$D_z^N f(z) = \frac{n!}{2\pi i} \oint_c f(t)(t-z)^{N-1} dt \text{ ------------------------------------ } (2.1)$$

He has computed the fractional derivative of z^p as.

$$D_z^N z^P = \frac{p!z^{P-N}}{(N-P)!} \text{ --- } (2.2)$$

Following form of Leibniz rule for the derivative of product of two functions, from elementary calculus, has been also utilized

$$D_z^{\alpha} uv = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_z^{\alpha-n} u \, D_z^{n} v \text{ -- } (2.3)$$

3.VALUES OF HYP FUNCTIONS BY THE APPICATION OF PRODUCT OF LEIBNITZ RULE

We know that

$$= \sum_{n=0}^{\infty} \frac{(a)_n (1-a)_n \left(\frac{1}{2}\right)^n}{(c)_n} \text{ --------------------- } \quad (3.1)$$

where $(a)_n \;=\; a(a+1)(a+2)-----(a+n-1)$

Substituting $u(z) = z^{c-a+1}$, $v(z) = z^{a-1}$ and α = -a in (2.3), we obtain

$$D_z^{-a} z^{c} = \sum_{n=0}^{\infty} \frac{\Gamma(1-a) D_z^{-a+n} z^{c-a+1} D_z^{n} z^{a-1}}{\Gamma(1-a-n)\, n!}$$

which assumes following formula by the application of (2.2)

$$\frac{\Gamma(c+1)}{\Gamma(c+a+1)} \;=\; \sum_{n=0}^{\infty} \frac{\Gamma(1-a)\Gamma(c-a+2)\Gamma(a)}{\Gamma(1-a-n)\Gamma(c+n+2)\Gamma(a-n) \quad n!}$$

Simplification of above result finally yields following result

$$\frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{c}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{c}{2}-\frac{a}{2}\right)} \;=\; {}_2F_1\left(a,1-a;c;\tfrac{1}{2}\right)$$

In order to prove $\;{}_2F_1(a,b;b-a+1;-1) \;=\; \dfrac{\Gamma(b-a+1)\Gamma\left(\frac{b}{2}+1\right)}{\Gamma(b+1)\Gamma\left(\frac{b}{2}-a+1\right)}$ ---------------------- (3.2)

we shall set $u(z) = z^{1-a}, v(z) = z^{b}\, and \alpha = a$ and shall proceed on the similar lines.

Following result

$${}_4F_3\left[\begin{array}{ll} a, 1+\frac{a}{2}, b, c; & -1 \\ \frac{a}{2}, 1+a-b, 1+a-c; & \end{array}\right]$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b+c)} \text{ ---------------- } (3.3)$$

can be obtained by taking u(z) = $z^{b-c-1}, v(z) = z^{1-2b}$ $and \alpha = -a$ in equation (2.3)so as to have application of (2.2) generates

$$\frac{\Gamma(1-b-c)}{\Gamma(1+a-b-c)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-a)\Gamma(b-c)\Gamma(2-2b)}{\Gamma(1-n-a)\Gamma(a+b-c+n)\Gamma(2-2b-n) \quad n!}$$

whose simplified form shall be (3.3)

The result

$$F_D^{(n)}(a, b_1 + ----- + b_n; c; 1) = \frac{\Gamma(c)\Gamma\left(c-a-\sum_{r=1}^{n} b_r\right)}{\Gamma(c-a)\Gamma\left(c-\sum_{r-1}^{n} b_r\right)} \text{ -------------- } (3.4)$$

can be obtained by setting u(z) = $z^{c-a-1}, v(z) = z^{-(b_1+b_2+---+b_n)}$ and α = -a and proceeding on the similar lines.

Similarly the following result of Lauricella function

$$F_A^{(n)}(a, b_1 --- b_n; c_1 ---- c_n; 1)$$

$$= \frac{\Gamma\prod_{r=1}^{n} c_r \; \Gamma\prod_{r=1}^{n}(c_r - b_r) - a}{\Gamma\prod_{r=1}^{n} c_r - a \quad \Gamma\prod_{r=1}^{n}(c_r - b_r)} \text{ ------------------- } (3.5)$$

can obtained by assuming

u(z) = $z^{(c_1 ----- c_n - a - 1)}, v(z) = z^{-(b_1 --- b_n)}, \alpha = -a$

and proceeding on the similar lines.

4.RESULTS FOR G-FUNCTIONS AND H-FUNCTIONS IN TERMS OF GAMMA FUNCTION BY THE APPLICATION OF FRACTIONAL CALCULUS.

Hence we shall prove the result

$$H_{2,2}^{1,2}\left[Z\left| \begin{matrix} (1-\alpha,1), & (1-\beta,1) \\ (0,1), & (1-\gamma,1) \end{matrix}\right.\right] = \frac{\Gamma(\alpha)\,\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}(-z)^n \text{ ------------- } (4.1)$$

Proof:-

We shall take the known value of H- function and shall convert it in such form so that our technique could be applicable. Then on substituting $u(z) = z^{\gamma-\alpha-1}, v(z) = z^{-\beta}\, and \alpha = -\alpha$ in (2.3), we obtain

$$D_z^{-\alpha} Z^{\gamma-\beta-\alpha-1} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha)\, D_z^{-\alpha-n}\, z^{\gamma-\alpha-1}\, D_z^{n} Z^{-\beta}}{\Gamma(1-\alpha-n)\quad n!}$$

which assumes following formula by the applicable of equation (2.2)

$$\frac{\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\beta)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha)\,\Gamma(\gamma-\alpha)\,\Gamma(1-\beta)}{\Gamma(1-\alpha-n)\,\Gamma(\gamma+n)\,\Gamma(1-\beta-n)\quad n!}$$

on simplification we can be produce result (4.1)

for the proof of the result

$$H_{2,2}^{1,2}\left[Z\left| \begin{matrix} (\alpha,1), & (\beta,1) \\ (\gamma,1), & (\delta,1) \end{matrix}\right.\right]$$

$$= \frac{\Gamma(\gamma-\alpha+1)\Gamma(\gamma-\beta+1)\Gamma(\alpha+\beta-\gamma-\delta-1)(-1)^n Z^{n+\gamma}}{\Gamma(\alpha-\delta)\Gamma(\beta-\delta)} \text{------------}(4.2)$$

Proof :-

Taking following value of the function

$$H_{2,2}^{1,2}\left[Z\left| \begin{matrix} (\alpha,1),(\beta,1) \\ (\gamma,1),(\delta,1) \end{matrix}\right.\right]$$

$$= \frac{\Gamma(\gamma-\alpha+1)\Gamma(\gamma-\beta+1)z^{\gamma}}{\Gamma(\gamma-\delta+1)}\ {}_2F_1(\gamma-\alpha+1,\gamma-\beta+1;\gamma-\delta+1;\quad -z)$$

and substituting u(z) = $z^{\alpha-\delta-1}, v(z) = z^{\beta-\gamma-1}, \alpha = \alpha-\gamma-1$ in (2.3),we can obtain

application of (2.2) to above result can yield

$$\frac{\Gamma(\alpha+\beta-\gamma-\delta-1)}{\Gamma(\beta-\delta)} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha-\gamma)\Gamma(\alpha-\delta)\Gamma(\beta-\gamma)}{\Gamma(\alpha-\gamma-n)\Gamma(\gamma-\delta+1+n)\Gamma(\beta-\gamma-n)}$$

and finally (4.2).

We can obtain following results for H- function in the same manner.

$$\mathrm{H}_{2,2}^{2,2}\left[Z\left|\begin{matrix}(1+\lambda-\alpha,1),(1+\lambda-\beta,1)\\(\lambda,1),(\gamma-\alpha-\beta+\lambda,1)\end{matrix}\right.\right] = \Gamma[\alpha,\ \beta,\ \gamma-\beta-\alpha;\quad (1-z)^n z^{\lambda}]$$

$$\mathrm{H}_{2,2}^{2,0}\left[Z\left|\begin{matrix}(\alpha-1,1),(\beta-1,1)\\(\gamma-1,1),(\delta-1,1)\end{matrix}\right.\right] = Z^{\delta}(1-z)^{\alpha+\beta-\delta-\gamma-1+n}\frac{\Gamma(\gamma-\delta)}{\Gamma(\beta-\delta)\Gamma(\alpha-\delta)}$$

Following results for G- functions have been compiled by as

$$G_{22}^{12}\left(Z\left|\begin{matrix}-c,-d\\a-1,-b\end{matrix}\right.\right) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b-c-a-d)(-1)^n Z^{n+a-1}}{\Gamma(b-d)\Gamma(b-c)}$$

$$G_{44}^{14}\left[z\left|\begin{matrix}-a,-b,-c,-d\\-1,-e,-f,-l\end{matrix}\right.\right) = \Gamma\left[\begin{matrix}-a-d,a,b,c,d & \\ & (-1)^n z^{n-1}\\ c-a,c-d,e,f,l & \end{matrix}\right]$$

$$G_{22}^{12}\left[Z\left|\begin{matrix}-a,-b\\-1,-c\end{matrix}\right.\right) = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}(-1)^n z^{n-1}$$

REFERENCES

[1] I.Ali and S.L.Kalla. (1996) An application of fractional calculus to the solution of a general class of differential equations, Appl.Math. Comput., 77.pp.137-152.

[2] L.Monreal,P.Fernaadez de Cordoba, J.A. Monsorru, and S.Abrahum Ibrahim. (2003) Special Functions evaluation using continued Fractions methods, Applications in Physics and

Engineering,AIP Conference Proceedings vol 661(1) pp. 267-268.

[3] S.G.Samko; A.A. Kilbas; and O.I. Marichev. (1993) Fractional Integrals and Derivatives.Y Verdon Switzerland ; Gordon and Breach. pp. 21-22.

[4] Ismail Ali and Vu Kim Tuan. (2001) Application of basic hypergeometric series to stable analytic continuation, Journal of computational and Applied Mathematics Applications, vol. 120, 118, pp.193-202.

.

[5] I.J.Zucker and G.S.Joyce. (2003) Special values of the Hypergeometric series, Math Pro 213-222.

[6] T.J.Osler. (1971) Leibniz rule for fractional derivatives. SAIM J. Appl. Math., vol 18, 658-674.

[7] S.T.Tu, D.K.Chyan and H.M.Srivastava.(2000) some families of ordinary and partial fractional differintegaral equations, Integral Transform, Spec. funct.9.

[8] B- Al-Saqabi & V . Kiryakiva (1997) Transmutation methods for solving Erdelyi-Kober fractional differintegral equations, J. Math. Analysis & Appl-S , 211, 347-364.

[9] M.Abramowitz, and I.A.Stegun., “Hypergeometric functions”. Ch. 15 in Hand book, formulas.Graphs,and Mathematical Tables. 9^{th} Printing.

New York: Dover, pp. 555-556.

[10] G.Arfken (1985) ”Hypergeometric functions” § 13.5 in Mathematical methods for physicists, 3^{rd} pp.748-752.

[11] V- Kiryakova (1997) All the special functions are fractional differintegrals of elementary function,J. Physics A : Math. & General, vol 30, No 14 , 5085-5103.

[12] P.A.Nekrassov. (1888) General differentaion, (in Russin).Math. sb., vol 14. pp. 45-168.

[13] A. Carpinteri, F. Mainardi ed., (1997)Fractals and Fractional calculus in continuum mechanics,springer.

[14] W.Grecksch and v.v. Anh. (2000) Approximation of stochastic evalution equation and application toequation with fractional power of infitesimal operators. Random operators stochastic, 8: 27-38.

[15] A.Struble. Raimond (1985) Fractional calculus in the operator field of generalized functions. North Carolina State University, Raleignh, North Carolina.

[16] V.S.Kiryakova(1988) A generalized fractional calculus and integral transforms. Proc. Conf. GFCA. Dubrovnik, (plenum Pub.co. New York).205-217.

[17] V. Kiryakova(1995) Generalized fractional calculus. Special functions and integral transforms. Transform Methods & special functions: Sofia ’94 (Pro. Internate. workshop). SCTP-

Singapore, 123-149.