EVALUATION OF G-FUNCTION BY MULTIPLICATION AND DIVISION TECHNIQUES OF CONTINUED FRACTIONS

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IMA Preprint Series # 1962

(February 2004)

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EVALUATION OF G-FUNCTION BY MULTIPLICATION AND DIVISION TECHNIQUES OF CONTINUED FRACTIONS

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ABSTRACT

In present paper an attempt is being made to develop algorithm for the evaluation of G-function are wider ranges with the help of multiplication and division techniques of continued fractions. Application of qd scheme according RITZ method have been also dealt with for computation purpose.

1. INTRODUCTION

Elementary arithmetic operation on continued fraction in simple case through computing have been developed by Henery Baker [1] Peter Henrici [3] has presented qd scheme and RITZ method for continued fractions. L. Lorentzen [4] has discussed continued fraction and their applicatioins. W. Gautshi [2] has developed continued fraction approximation of the modified

Bessel function. We [5] in our earlier study, have tabulated H-functions for values of parameters over certain ranges. Wider range of tables can be had by the use of multiplication and division techniques on continued fraction as generalized G-function is expressible as the product of several other special functions.

2. GENERAL FORMULATION

The G-function can be defined as

$$G_{p,q}^{m,n}(x) = X.[w(u,v)] \tag{2.1}$$

where X is a function of x.

Let us assume that one continued fraction be

$$u = \frac{a_1}{1+}\frac{a_2 x}{1+}\frac{a_3 x}{1+}\frac{a_4 x}{1+}\frac{a_5 x}{1+}.................. \tag{2.2a}$$

and another be

$$v = \frac{a'_1}{1+}\frac{a'_2\, x}{1+}\frac{a'_3 x}{1+}\frac{a'_4\, x}{1+}\frac{a'_5\, x}{1+}................ \tag{2.2b}$$

Then w(u, v) shall be a function in the terms of u, v which are in expressible as u.v or u/v.

For computing the G-function through multiplication or division of continued fraction as

$$w(u,v) = \left\{ a\left(\frac{a_1}{1+} \frac{a_2 x}{1+} \frac{a_3 x}{1+} ------ \right) . \left(\frac{a'_1}{1+} \frac{a'_2\, x}{1+} \frac{a'_3\, x}{1+} ----- \right) + b\left(\frac{a_1}{1+} \frac{a_2 x}{1+} \right.\right.$$

$$\left.\left. \frac{a_3 x}{1+} ----- \right) + c\left(\frac{a'_1}{1+} \frac{a'_2\, x}{1+} \frac{a'_3\, x}{1+} ------ \right) + d \right\} \Big/ \left\{ e\left(\frac{a_1}{1+} \frac{a_2 x}{1+} \right.\right.$$

$$\left. \frac{a_3 x}{1+} ----- \right) . \left(\frac{a'_1}{1+} \frac{a'_2\, x}{1+} \frac{a'_3\, x}{1+} ----- \right) + f\left(\frac{a_1}{1+} \frac{a_2 x}{1+} \frac{a_3 x}{1+} ----- \right)$$

$$\left. + g\left(\frac{a'_1}{1+} \frac{a'_2\, x}{1+} \frac{a'_3\, x}{1+} ------- \right) + h \right\} \tag{2.3}$$

we prefer to choose initial values of integer's variable of 'a' through 'h' as

u.v = (1 0 0 0)/(0 0 0 1) (2.4)

u/v = (0 1 0 0)/(0 0 1 0) (2.5)

The process of inputing term u, v and out putting term of 'w' will reduce to replacing the eight integers 'a' through 'h' with linear combination of each other.

For numerical evaluation of G-function through continued fraction we are using descending method and the approximants algorithm which alternately furnish upper and lower bounds for the accurate value of the fraction as discussed in the book of Henirici [3] for logarithm function.

3. COMPUTATIONAL ALGORITHM

In this section we have defined G-function by multiplication of continued fraction and have also used qd scheme according RITZ method.

Defining the function $G_{1,3}^{2,1}\left(x \middle| \begin{matrix} 1/2 \\ a,0,-a \end{matrix}\right)$ as

$$\frac{2^{\frac{1}{2}+a}.\pi.x^{\frac{a}{2}}}{\Gamma(a+1).e^{x}}[w(u,v)] \tag{3.1}$$

and making use of the results (2.1) and (2.2) of general formulation, we shall have.

$$u = \frac{x^{-\frac{1}{2}}}{1+}\frac{a_2 x^{\frac{1}{2}}}{1+}\frac{a_3 x^{\frac{1}{2}}}{1+}\frac{a_4 x^{\frac{1}{2}}}{1+}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.2a}$$

$$v = \frac{x^{-\frac{3}{2}}}{1+}\frac{a'_2}{(x^{\frac{1}{2}})+}\frac{a'_3}{1+}\frac{a'_4}{(x^{\frac{1}{2}})+}\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.2b}$$

and then using qd scheme as

$$\begin{aligned}
a_2 &= -\frac{1}{1} \\
a_{2k+1} &= \frac{2a+1}{4(a+k)} \\
a_{2k+2} &= -\frac{1}{2(a+k)} \\
a'_2 &= \frac{1}{2}\left(\frac{1}{2}+b\right)\left(\frac{1}{2}-b\right) \\
a'_{2k+1} &= \frac{k}{4(2k-1)} \\
a'_{2k+2} &= \frac{\left(\frac{1}{2}+b+k\right)^2}{4k(2k+1)}
\end{aligned} \tag{3.3}$$

k = 1, 2, 3, …………..

we can tabulate the functions by making the substitutions.

x = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 0, 1, 2

in the form of table (Appendix – I)

Defining the function $G_{2,4}^{2,2}\left(x\,\middle|\,\begin{matrix} a, a+\frac{1}{2} \\ b, c, 2a-c, 2a-b \end{matrix}\right)$ as

$$\frac{\pi.2^{2a-b-c+\frac{1}{2}}x^{\frac{b+c}{2}}}{e^{x}\Gamma.(1+b+c-2a)}[w(u,v)] \tag{3.4}$$

and making use of the result (2.1) and (2.2) of general formulation, we shall have.

$$\left.\begin{aligned} u &= \frac{x^{-\frac{1}{2}}}{1+}\,\frac{a_2 x^{\frac{1}{2}}}{1+}\,\frac{a_3 x^{\frac{1}{2}}}{1+}\,\frac{a_4 x^{\frac{1}{2}}}{1+}\,\ldots\ldots\ldots\ldots \\ v &= \frac{x^{-\frac{3}{2}}}{1+}\,\frac{a'_2}{(x^{\frac{1}{2}})+}\,\frac{a'_3}{1+}\,\frac{a'_4}{x^{\frac{1}{2}}}\,\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{3.5}$$

Utilization of qd scheme yields

$$a_2 = -\frac{1}{1}$$

$$a_{2k+1} = \frac{b+c-2a+\frac{1}{2}}{2(b+c-2a+k)}$$

$$a_{2k+2} = -\frac{1}{2(b+c-2a+k)}$$

$$a'_2 = \frac{1}{2}\left(\frac{1}{2}+b-c\right)\left(\frac{1}{2}-b+c\right)$$

$$a'_{2k+1} = \frac{k}{4(2k-1)}$$

$$a'_{2k+2} = \frac{\left(\frac{1}{2} + b - c + k\right)^2}{4k(1+2k)}$$

$$\tag{3.6}$$

k = 1, 2, 3………..

Thus we can tabulate the functions by making the substitutions

x = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 1

b = 1, 2, 3

c = 1

in the form of table (Appendix – II)

For the function $G_{1,3}^{3,0}\left(x \,\middle|\, \begin{matrix} a + \frac{1}{2} \\ a+b, a-b, a \end{matrix} \right)$ defined as

$$\frac{\pi^{\frac{1}{2}} x^a}{e^{2\sqrt{x}}}[w(u,v)] \tag{3.7}$$

The use of the results (2.1) and (2.2) of general formulation procreates.

$$u = v = \frac{x^{-\frac{3}{2}}}{1+}\,\frac{a_2}{(x^{1/2})+1+}\,\frac{a_3}{(x^{1/2})+1+}\,\frac{a_4}{}\,\frac{a^5}{}\ldots\ldots\ldots\ldots \tag{3.8}$$

and then using qd scheme as

$$a_2 = \frac{1}{2}\left(\frac{1}{2}+b\right)\left(\frac{1}{2}-b\right)$$

$$a_{2k+1} = \frac{k}{4(2k-1)} \tag{3.9}$$

$$a_{2k+2} = \frac{\left(b+k+\frac{1}{2}\right)^2}{4k(2k+1)}$$

k = 1, 2, 3 ……….

we can tabulate the functions by making the substitutions

x = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0,

11.0, 12.0, 13.0, 14.0, 15.0

a = 1/2, 1, 3/2, 2

b = 0, 1, 2

in the form of table (Appendix – III)

Defining the function $G_{2,4}^{4,0}\left(x \,\middle|\, \begin{matrix} a, a+b \\ a+b, a+c, a-c, a-b \end{matrix}\right)$ as

$$\frac{\pi^{\frac{1}{2}} x^a}{e^x}[z(u,v)]\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.10}$$

and making use of the results (2.1) and (2.2) of general formulation, we shall have

$$u = \frac{x^{-\frac{3}{2}}}{1+}\frac{a_2}{(x^{\frac{1}{2}})+}\frac{a_3}{1+}\frac{a_4}{(x^{\frac{1}{2}})+}\ldots\ldots\ldots\ldots$$

$$\tag{3.11}$$

$$u = \frac{x^{-\frac{3}{2}}}{1+}\frac{a'_2}{(x^{\frac{1}{2}})+}\frac{a'_3}{1+}\frac{a'_4}{x^{\frac{1}{2}}}\ldots\ldots\ldots$$

and then using qd scheme as

$$
\begin{aligned}
a_2 &= \frac{1}{2}\left(\frac{1}{2}+b+c\right)\left(\frac{1}{2}-b-c\right) \\
a_{2k+1} &= \frac{k}{4(2k-1)} \\
a_{2k+2} &= -\frac{\left(\frac{1}{2}+b+c+k\right)^2}{4k(2k+1)} \\
a'_2 &= \frac{1}{2}\left(\frac{1}{2}+b-c\right)\left(\frac{1}{2}-b+c\right) \\
a'_{2k+1} &= \frac{k}{4(2k-1)} \\
a'_{2k+2} &= \frac{\left(\frac{1}{2}+b-c+k\right)^2}{4k(2k+1)}
\end{aligned} \tag{3.12}
$$

k = 1, 2, 3 ………………..

we can tabulate the function by making the substitutions

x= 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0, 12.0, 13.0, 14.0, 15.0

a = 1/2, 1, 2, 3

b = 1/2, 1, 3/2

c = 1/2, 1

in the form of table (Appendix – IV)

Define the function $G_{1,3}^{2,1}\left(x^2 \,\middle|\, \begin{matrix} \frac{1}{2}+a \\ a+b, a, a-b \end{matrix} \right)$ similarly as

$$\frac{\pi . x^{2a+b}}{2^{b-\frac{1}{2}} e^{2x}}[w(u,v)] \tag{3.13}$$

so as have the continued fraction as

$$u = \frac{x^{-2}}{1+}\frac{a_2 \frac{1}{x}}{1+}\frac{a_3 \frac{1}{x}}{1+}\frac{a_4 \frac{1}{x}}{1+}\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(3.14)$$

$$u = \frac{1}{1+}\frac{a'_2 x}{1+}\frac{a'_3 x}{1+}\frac{a'_4 x}{1+}\ldots\ldots\ldots\ldots\ldots\ldots$$

Use of qd scheme yields

$$a_2 = \frac{1-4b^2}{8}$$

$$a_{2k+1} = \frac{k}{4(2k-1)}$$

$$a_{2k+2} = -\frac{\left(b+k+\frac{1}{2}\right)^2}{4k(1+2k)}$$

$$a'_2 = -\frac{1}{1}$$

$$a'_{2k+1} = \frac{\left(b+\frac{1}{2}\right)}{2(k+b)}$$

$$a'_{2k+2} = -\frac{1}{4(b+k)}$$

$$(3.15)$$

k = 1, 2, 3………………..

Now we can tabulate the functions by making the substitutions

x = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0

a = 1/2, 1

b = 1, 2

in the form of table (Appendix – V)

APPENDIX- I

a . x	0	1	2
0.2	4.1113	1.5801	0.8649
0.4	3.0104	1.4491	0.8417
0.6	2.6377	1.3848	0.8280
0.8	2.3382	1.3226	0.8134
1.0	1.8899	1.2061	0.7820
2.0	1.3014	1.0079	0.7157
3.0	1.0937	0.8880	0.6665
4.0	0.9205	0.7888	0.6198
5.0	0.8321	0.7322	0.5905
6.0	0.7591	0.6822	0.5629
7.0	0.6979	0.6377	0.5369
8.0	0.6459	0.5982	0.5124
9.0	0.6011	0.5628	0.4896
10.0	0.5810	0.5465	0.4788

APPENDIX - II

a, b, c X	a=1 b=1 c=1	a=1 b=2 c=1	a=1 b=3 c=1
0.2	0.8222	0.3160	0.1729
0.4	1.2042	0.5796	0.3366
0.6	1.5826	0.8309	0.4968
0.8	1.8706	1.0581	0.6507
1.0	1.8899	1.2061	0.7820
2.0	2.6839	2.0159	1.4314
3.0	3.2812	2.6641	1.9995
4.0	3.6821	3.1552	2.4794
5.0	4.1607	3.6612	2.9529
6.0	4.5549	4.0932	3.3775
7.0	4.8855	4.4645	3.7583
8.0	5.1672	4.7858	4.0999
9.0	5.4106	5.0658	4.4069
10.0	5.8108	5.4659	4.7881

APPENDIX - III

a, b X	a=1/2 b=1	a =3/2 b=1	a=2 b=1
0.2	(1) 0.4711	(0) 0.9422	(0) 0.4214
0.4	(1) 0.1211	(0) 0.4844	(0) 0.3063
0.6	(0) 0.9639	(0) 0.5783	(0) 0.4480
0.8	(0) 0.7493	(0) 0.5994	(0) 0.5362
1.0	(0) 0.4645	(0) 0.4645	(0) 0.4087
2.0	(0) 0.1573	(0) 0.3146	(0) 0.4645
3.0	(-1)+ 0.8564	(0) 0.2569	(0) 0.4450
4.0	(-1)+ 0.4413	(0) 0.1765	(0) 0.3531
5.0	(-1)+ 0.2936	(0) 0.1468	(0) 0.3283
6.0	(-1)+ 0.1937	(0) 0.1162	(0) 0.2846
7.0	(-1)+ 0.1163	(-1)+ 0.8141	(0) 0.2154
8.0	(-2)+ 0.9521	(-1)+ 0.7616	(0) 0.1886
9.0	(-2)+ 0.5457	(-1)+ 0.4911	(0) 0.1473
10.0	(-2)+ 0.4529	(-1)+ 0.4529	(0) 0.1432
11.0	(-2)+ 0.2957	(-1)+ 0.3252	(0) 0.1079
12.0	(-2)+ 0.2442	(-1)+ 0.2930	(0) 0.1015
13.0	(-2)+ 0.1593	(-1)+ 0.2070	(-1)+ 0.7470
14.0	(-2)+ 0.1311	(-1)+ 0.1835	(-1)+ 0.6871
15.0	(-2)+ 0.1077	(-1)+ 0.1615	(-1)+ 0.6261

CONTINUE OF APPENDIX - III

a, b X	a=1 b=0		a =1 b=1		a=2 b=2	
0.2	(0)	0.2802	(0)	1.0765	(0)	6.5366
0.4	(0)	0.2727	(0)	0.7658	(0)	4.7377
0.6	(0)	0.2952	(0)	0.7465	(0)	5.4436
0.8	(0)	0.2884	(0)	0.6701	(0)	5.3402
1.0	(0)	0.1999	(0)	0.4086	(0)	2.9780
2.0	(0)	0.1339	(0)	0.2058	(0)	2.2234
4.0	(-1)+	0.5852	(-1)+	0.8826	(0)	1.1620
6.0	(-1)+	0.3336	(-1)+	0.4743	(0)	0.7957
8.0	(-1)+	0.1732	(-1)+	0.2357	(0)	0.4658
10.0	(-2)+	0.3426	(-1)+	0.1432	(0)	0.3281

APPENDIX - IV

a, b, c X	a=1/2 b=1/2 c=1/2		a=1 b=1/2 c=1/2		a=1 b=3/2 c=1/2	
0.2	(0)	1.0580	(0)	0.4731	(2)	0.1499
0.4	(0)	0.7227	(0)	0.4571	(1)	0.3010
0.6	(0)	0.2540	(0)	0.1968	(1)	0.2602
0.8	(0)	0.4916	(0)	0.4397	(1)	0.1586
1.0	(0)	0.7771	(0)	0.7771	(1)	0.5017
2.0	(0)	0.1105	(0)	0.1563	(0)	0.5081
3.0	(-1)+	0.6770	(0)	0.1172	(0)	0.2917
4.0	(-1)+	0.3594	(-1)+	0.7188	(0)	0.1601
5.0	(-1)+	0.2429	(-1)+	0.5433	(0)	0.1140
6.0	(-1)+	0.1440	(-1)+	0.3527	(-1)+	0.5477
7.0	(-1)+	0.1079	(-1)+	0.2856	(-1)+	0.5445
8.0	(-2)+	0.7146	(-1)+	0.2021	(-1)+	0.3705
9.0	(-2)+	0.4721	(-1)+	0.1416	(-1)+	0.2507
10.0	(-2)+	0.3935	(-1)+	0.1244	(-1)+	0.2168
11.0	(-2)+	0.2589	(-2)+	0.8587	(-1)+	0.1453
12.0	(-2)+	0.2145	(-2)+	0.7431	(-1)+	0.1240
13.0	(-2)+	0.1409	(-2)+	0.5080	(-2)+	0.8272
14.0	(-2)+	0.1163	(-2)+	0.4351	(-2)+	0.7004
15.0	(-3)+	0.9583	(-2)+	0.3711	(-2)+	0.5908

CONTINUE OF APPENDIX -IV

X \ a, b, c	A=1 b=1 c=1	a=2 b=1 c=1	a=3 b=1 c=1
0.2	(0) 1.3657	(0) 0.2731	(-1)+ 0.5462
0.4	(0) 0.9117	(0) 0.3646	(0) 0.1458
0.6	(0) 0.7540	(0) 0.4524	(0) 0.2714
0.8	(0) 0.6261	(0) 0.5008	(0) 0.4007
1.0	(0) 0.4352	(0) 0.4352	(0) 0.4352
2.0	(0) 0.2133	(0) 0.4266	(0) 0.8532
3.0	(0) 0.1253	(0) 0.3759	(0) 1.1277
4.0	(-1)+ 0.7356	(0) 0.2942	(0) 1.1769
5.0	(-1)+ 0.5150	(0) 0.2575	(0) 1.2875
6.0	(-1)+ 0.3602	(0) 0.2161	(0) 1.2967
7.0	(-1)+ 0.2516	(0) 0.1761	(0) 1.2328
8.0	(-1)+ 0.2385	(0) 0.1908	(0) 1.5264
9.0	(-1)+ 0.1223	(0) 0.1100	(0) 0.9906
10.0	(-1)+ 0.1021	(0) 0.1021	(0) 1.0210

APPENDIX -V

a, b X	a=1/2 b=1		a=1 b=1		a=1/2 b=2	
0.1	(0)	0.1749	(-1)+	0.1749	(-1)+	0.8849
0.2	(0)	0.3403	(-1)+	0.6806	(0)	0.1761
0.3	(0)	0.4930	(0)	0.1479	(0)	0.2621
0.4	(0)	0.6320	(0)	0.2528	(0)	0.3459
0.5	(0)	0.7573	(0)	0.3786	(0)	0.4270
0.6	(0)	0.8689	(0)	0.5213	(0)	0.5050
0.7	(0)	0.9693	(0)	0.6785	(0)	0.5796
0.8	(0)	1.0581	(0)	0.8464	(0)	0.6507
0.9	(0)	1.1366	(0)	1.0229	(0)	0.7182
1.0	(0)	1.2061	(0)	1.2061	(0)	0.7820
2.0	(0)	1.5776	(0)	3.1552	(0)	1.2397
3.0	(0)	1.6886	(0)	5.0658	(0)	1.4689
4.0	(0)	1.7278	(0)	6.9112	(0)	1.5849
5.0	(0)	1.7449	(0)	8.7245	(0)	1.6475
6.0	(0)	1.7537	(1)	0.1052	(0)	1.6840
7.0	(0)	1.7589	(1)	0.1231	(0)	1.7068
8.0	(0)	1.7622	(1)	0.1409	(0)	1.7219
9.0	(0)	1.7645	(1)	0.1588	(0)	1.7324
10.0	(0)	1.7660	(1)	0.1766	(0)	1.7400

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