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By

Guozhen Lu

and

Biao Ou

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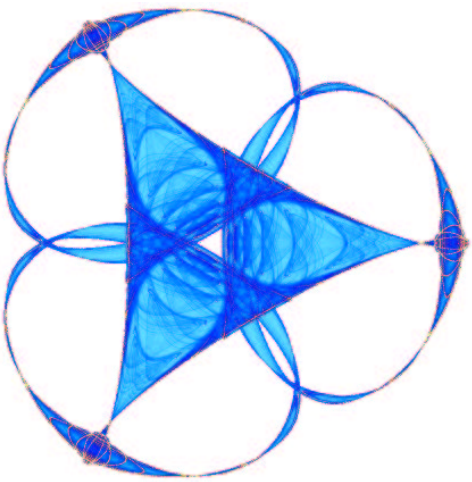

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

A Poincaré Inequality on R^n and Its Application to Potential Fluid Flows in Space

Guozhen Lu
Wayne State University

Biao Ou
University of Toledo

Abstract Consider a function $u(x)$ in the standard localized Sobolev space $W^{1,p}_{loc}(R^n)$ where $n \geq 2$, $1 \leq p < n$. Suppose that the gradient of $u(x)$ is globally L^p integrable; i.e., $\int_{R^n} |\nabla u|^p dx$ is finite. We prove a Poincaré inequality for $u(x)$ over the entire space R^n. Using this inequality we prove that the function subtracting a certain constant is in the space $W^{1,p}_0(R^n)$, which is the completion of $C_0^\infty(R^n)$ functions under the norm $||\phi|| = (\int_{R^n} |\nabla \phi|^p dx)^{1/p}$ where $\phi \in C_0^\infty(R^n)$. As a result, we come to know the best constant and the optimizing functions for the Poincaré inequality on R^n.

We then prove a similar inequality for functions whose higher order derivatives are L^p integrable on R^n.

Next we study functions whose gradients are L^p integrable on an exterior domain of R^n and apply the results to another proof of an existence theorem for irrotational and incompressible flows around a body in space.

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In the study of fluid flows around a body in space, the notion of disturbance with a finite energy arises. This leads us to consider functions on an entire Euclidean space R^n ($n \geq 2$) whose gradients are L^p integrable for some p satisfying $1 \leq p < n$.

Let $u(x)$ be such a function. That is, $u(x)$ satisfies

$$u(x) \in W^{1,p}_{loc}(R^n) \text{ and } \int_{R^n} |\nabla u|^p dx < \infty. \tag{0.1}$$

Here the function is in $W^{1,p}_{loc}(R^n)$ means that it is in the standard Sobolev space on each finite ball. We refer to Evans-Gariepy [EG], Gilbarg-Trudinger [GT], Lieb-Loss [LL], Stein [S], and Ziemer [Z] for the background material about the Sobolev spaces.

Assume $1 \leq p < n$. In Theorem 1.1 we prove the following Poincaré inequality for functions satisfying (0.1):

$$(\int_{R^n} |u(x) - (u)_\infty|^{np/(n-p)} dx)^{(n-p)/(np)} \leq c_{p,n} (\int_{R^n} |\nabla u|^p dx)^{1/p}.$$

Here $c_{p,n}$ is a positive constant depending on p, n only, and $(u)_\infty$ is the limit of $(u)_R$, the average value of u on the ball B_R centered at the origin and with radius R, as R approaches infinity. In Theorem 2.1 we show that it follows from this inequality that the linear space consisting of functions satisfying (0.1) is a complete Banach space under the norm

$$||u|| = (\int_{R^n} |\nabla u|^p dx + |(u)_\infty|^p)^{1/p}.$$

It also follows from the inequality and a probably known result, our Proposition 2.2, that for any $\epsilon > 0$ there exists a $C_0^\infty(R^n)$ function ϕ such that

$$(\int_{R^n} |u(x) - (u)_\infty - \phi|^{np/(n-p)} dx)^{(n-p)/(np)} + (\int_{R^n} |\nabla u - \nabla \phi|^p dx)^{1/p} < \epsilon.$$

See Theorem 2.3.

In section 3, we make three remarks on these results. The first is about the best constant $c_{p,n}$ and the optimizing functions for the inequality. To our best knowledge, there has been no complete result on the best constant and the optimizing functions for the Poincaré inequality on a finite domain. Yet we know everything about the Poincaré inequality on R^n thanks to Proposition 2.2 and the well known results for the Sobolev inequalities.

The second remark is about the breakdown of our results if $p \geq n$. This is not unexpected, nevertheless. For Sobolev functions in $W^{1,p}$ on a bounded domain with $p \geq n$, we already know that the Poincaré inequality is not suitable. In place of it we have the John-Nirenberg inequality, Moser-Trudinger inequality, Morrey's estimate, and so on (cf. the references we gave earlier and the paper of Ball [B]).

The third remark is about an alternative perspective to look at our results, especially for the case of $n \geq 3$, $p = 2$.

It is not hard to extend our results to functions whose higher order derivatives are integrable. We do that in Section 4.

In section 5 we study functions whose gradients are L^p integrable on an exterior domain of R^n outside a bounded domain Ω with a reasonably smooth boundary. Particularly, we have a similar Poincaré inequality following a direct and elementary proof on a Gagliado-Nirenberg-Sobolev inequality on an exterior domain, see (5.2) and (5.3).

In section 6 we apply what we obtain in section 5 to derive another proof of an existence theorem for irrotational and incompressible flows around a body in space. We especially mention that the linear space $U^{1,2}(\Omega^C)$ consisting of functions on Ω^C with a square integrable gradient is a complete Hilbert space under certain norm if and only if the dimension n of the Euclidean space is larger than or equal to three. Thus our proof works for space flows only.

A function in $U^{1,2}(\Omega^C)$ represents a potential function whose gradient describes the disturbance of a non-uniform vector field from a uniform vector field. The square integral of the gradient can be viewed as part of the disturbance energy. It is this consideration that has motivated our work in this paper. In the previous work of Dong and Ou [DO] on an existence theorem for subsonic potential flows around a body in space, attention was directed at potential functions that have a finite disturbance energy and is *approachable by finitely disturbed potential functions.* To a large extent, the purpose of this paper is to show that a potential function in $U^{1,2}(\Omega^C)$, after subtracting a proper constant, is indeed approachable by finitely disturbed potential functions. See section 6 for more detail.

1 The Poincaré Inequality on R^n $(n \geq 2)$

Let $u(x)$ be a function satisfying (0.1). Let $(u)_R$ denote the average value of u on the ball B_R; that is,

$$(u)_R \equiv \frac{1}{|B_R|} \int_{B_R} u dx.$$

We first establish the following theorem for $u(x)$.

Theorem 1.1 *As R approaches infinity, $(u)_R$ approaches a finite limit $(u)_\infty$; moreover, there exists a constant $c_{p,n}$, depending on p, n but not on u, such that*

$$||u - (u)_\infty||_{np/(n-p)} \leq c_{p,n}||\nabla u||_p. \tag{1.1}$$

Proof: Let ω_n be the area of the unit sphere in R^n, and let (r, ω) be the polar coordinate of $x \in R^n$. Instead of working on $(u)_R$ directly, we consider the average value of u on the sphere ∂B_R, which equals

$$\frac{1}{\omega_n}\int_{\partial B_1} u(R, \omega)dS_\omega \tag{1.2}$$

where dS_ω is the surface measure on the unit sphere. Suppose $0 < R_1 < R_2$. In case $1 < p < n$ we have

$$\begin{aligned}
&|\frac{1}{\omega_n}\int_{\partial B_1} u(R_1, \omega)dS_\omega - \frac{1}{\omega_n}\int_{\partial B_1} u(R_2, \omega)dS_\omega| \\
\leq\ & \frac{1}{\omega_n}\int_{R_1}^{R_2}\int_{\partial B_1} |u_r(r, \omega)|drdS_\omega = \frac{1}{\omega_n}\int_{B_{R_2}\backslash B_{R_1}} \frac{|u_r(r, \omega)|}{r^{n-1}}dx \\
\leq\ & \frac{1}{\omega_n}(\int_{B_{R_2}\backslash B_{R_1}} |u_r(r, \omega)|^p dx)^{1/p}(\int_{B_{R_2}\backslash B_{R_1}} \frac{1}{r^{q(n-1)}}dx)^{1/q} \\
& \text{(where } q = p/(p-1)) \\
\leq\ & c_{p,n}(\int_{B_{R_2}\backslash B_{R_1}} |\nabla u|^p dx)^{1/p}(\frac{1}{R_1^{(q-1)(n-1)-1}} - \frac{1}{R_2^{(q-1)(n-1)-1}})^{1/q}.
\end{aligned}$$

This difference approaches zero as R_1, R_2 approach infinity, because $(q-1)(n-1) - 1 = (n-1)/(p-1) - 1 > 0$. In case $p = 1$ the calculation is a little simpler, and we have

$$\begin{aligned}
&|\frac{1}{\omega_n}\int_{\partial B_1} u(R_1, \omega)dS_\omega - \frac{1}{\omega_n}\int_{\partial B_1} u(R_2, \omega)dS_\omega| \\
\leq\ & \frac{1}{\omega_n}\int_{R_1}^{R_2}\int_{\partial B_1} |u_r(r, \omega)|drd\omega = \frac{1}{\omega_n}\int_{B_{R_2}\backslash B_{R_1}} \frac{|u_r(r, \omega)|}{r^{n-1}}dx \\
\leq\ & \frac{1}{\omega_n R_1^{n-1}}\int_{B_{R_2}\backslash B_{R_1}} |\nabla u|dx.
\end{aligned}$$

Again the difference approaches zero as R_1, R_2 approach infinity.

Thus the average value of u on the sphere ∂B_R, (1.2), has a finite limit as R approaches infinity. It follows that, as a weighted average of (1.2),

$$(u)_R = \frac{n}{R^n}\int_0^R (\frac{1}{\omega_n}\int_{\partial B_1} u(r,\omega)dS_\omega) r^{n-1}dr \to (u)_\infty$$

for some finite limit $(u)_\infty$ as $R \to \infty$.

Now we recall the standard Poincaré inequality

$$(\int_{B_R} |u(x) - (u)_R|^{np/(n-p)}dx)^{(n-p)/(np)} \leq c_{p,n}(\int_{B_R} |\nabla u|^p dx)^{1/p}$$

for some constant $c_{p,n}$. The fact that $c_{p,n}$ does not depend on R is due to the inequality's invariance of a scaling transform. That is, if in the inequality we replace R by R_1 and $u(x)$ by $u_1(x) = u(xR/R_1)$, the constant is not changed. We refer to [EG, p. 141] for a proof of the Poincaré inequality on a bounded ball. In case $1 < p < n$, a proof of the Poincaré inequality on a bounded domain follows from a pointwise representation formula of fractional integral type together with the Hardy-Littlewood-Sobolev inequalities (cf. [S]). The case $p = 1$ follows from a truncation argument and a weak type estimate of fractional integral (see e.g. [FLW]).

For any $0 < R_1 < R_2$, we have

$$\begin{aligned}
&(\int_{B_{R_1}} |u(x) - (u)_{R_2}|^{np/(n-p)}dx)^{(n-p)/(np)} \\
\leq\ &(\int_{B_{R_2}} |u(x) - (u)_{R_2}|^{np/(n-p)}dx)^{(n-p)/(np)} \leq c_{p,n}(\int_{B_{R_2}} |\nabla u|^p dx)^{1/p}.
\end{aligned}$$

Letting $R_2 \to \infty$ in the inequality above, we obtain

$$(\int_{B_{R_1}} |u(x) - (u)_\infty|^{np/(n-p)}dx)^{(n-p)/(np)} \leq c_{p,n}(\int_{R^n} |\nabla u|^p dx)^{1/p}.$$

We come to (1.1) after sending $R_1 \to \infty$.

2 The complete Banach space $U^{1,p}(R^n)$

Let $U^{1,p}(R^n)$ be the linear space consisting of functions satisfying (0.1). As the first application of the Poincaré inequality we have

Theorem 2.1 *The linear space $U^{1,p}(R^n)$ consisting of functions satisfying (0.1) is a complete Banach space with the norm*

$$||u|| \equiv (\int_{R^n} |\nabla u|^p dx + |(u)_\infty|^p)^{1/p} \tag{2.1}$$

where $(u)_\infty$ is the limit whose existence is ensured by Theorem 1.1.

Proof: We need only to prove the completeness. Let $\{u^i\}$ be a Cauchy sequence. Let $w^i = u^i - (u^i)_\infty,\ i = 1, 2, \ldots$. By Theorem 1.1,

$$||w^i - w^j||_{np/(n-p)} = ||(u^i - u^j) - (u^i - u^j)_\infty||_{np/(n-p)} \leq c_{p,n}||\nabla(u^i - u^j)||_p.$$

Note also that

$$||\nabla w^i - \nabla w^j||_p = ||\nabla u^i - \nabla u^j||_p.$$

Thus the sequence of w_i has a limit w such that w is in $L_{np/(n-p)}$, ∇w is in L_p, and

$$||w^i - w||_{np/(n-p)} + ||\nabla w^i - \nabla w||_p \to 0$$

as i approaches infinity. For each i we know $(w^i)_\infty = 0$. For the limit w, it is also true that

$$(w)_\infty = 0, \tag{2.2}$$

because

$$\begin{aligned} |(w)_R| &\leq (\frac{1}{|B_R|}\int_{B_R} |w|^{np/(n-p)}dx)^{(n-p)/(np)} \\ &\quad \text{(by the Hölder inequality)} \\ &\leq c_{p,n}R^{-(n-p)/p}||w||_{np/(n-p)} \to 0 \end{aligned}$$

as R approaches infinity. Let

$$u = w + \lim_{i\to\infty}(u^i)_\infty.$$

Then u is in $U^{1,p}(R^n)$ and is the limit of the sequence of u^i in $U^{1,p}(R^n)$ with the norm (2.1). The completeness is now proved.

Q.E.D

The Sobolev space $W_0^{1,p}(R^n)$ is a closed subspace of $U^{1,p}(R^n)$ that contains the $C_0^\infty(R^n)$ functions as a dense set. We will have more discussions on the definitions of this space. Next we prove that the codimension of this subspace in $U^{1,p}(R^n)$ is one. It amounts to showing that for every u in $U^{1,p}(R^n)$, the difference $w = u-(u)_\infty$ is in $W_0^{1,p}(R^n)$. We present the following proposition with a simple proof.

Proposition 2.2 *Suppose w is in $W_{loc}^{1,p}(R^n)$ and satisfies*

$$||w||_{np/(n-p)} + ||\nabla w||_p < \infty. \tag{2.3}$$

Then for any $\epsilon > 0$ there is a $C_0^\infty(R^n)$ function $\phi(x)$ such that

$$||w-\phi||_{np/(n-p)} + ||\nabla w - \nabla\phi||_p < \epsilon. \tag{2.4}$$

Proof: Let $R>0$ and let $\psi_R(x)$ be a $C_0^\infty(R^n)$ function satisfying

$$\begin{cases} \psi_R(x) &=& 1 & \text{if } |x|\le R, \\ \psi_R(x) &=& 0 & \text{if } |x|\ge 2R, \\ |\psi_R(x)| &\le& 1 & \text{for all } x, \\ |\nabla\psi_R(x)| &\le& 2/R & \text{for all } x. \end{cases}$$

In fact we can first choose a specific function $\psi_1(x)$ and let $\psi_R(x) = \psi_1(x/R)$.

Notice that

$$\begin{aligned} &\int_{R^n} |w\nabla\psi_R|^p dx \le \frac{2^p}{R^p}\int_{B_{2R}\setminus B_R} |w|^p dx \\ &= c_{p,n}R^{n-p}(\frac{1}{|B_{2R}\setminus B_R|}\int_{B_{2R}\setminus B_R}|w|^p dx) \\ &\le c_{p,n}R^{n-p}(\frac{1}{|B_{2R}\setminus B_R|}\int_{B_{2R}\setminus B_R}|w|^{np/(n-p)}dx)^{(n-p)/n} \\ &\le c_{p,n}(\int_{B_{2R}\setminus B_R}|w|^{np/(n-p)}dx)^{(n-p)/n}. \end{aligned}$$

Thus

$$\begin{aligned} &||w-w\psi_R||_{np/(n-p)} + ||\nabla w - \nabla(w\psi_R)||_p \\ &\le ||w(1-\psi_R)||_{np/(n-p)} + ||\nabla w(1-\psi_R)||_p + ||w\nabla\psi_R||_p \\ &\le (\int_{B_R^C}|w|^{np/(n-p)}dx)^{(n-p)/(np)} + (\int_{B_R^C}|\nabla w|^p dx)^{1/p} \\ &\quad + c_{p,n}(\int_{B_{2R}\setminus B_R}|w|^{np/(n-p)}dx)^{(n-p)/(np)} \to 0 \text{ as } R\to\infty. \end{aligned}$$

Apparently, we can choose R so large that

$$||w - w\psi_R||_{np/(n-p)} + ||\nabla w - \nabla(w\psi_R)||_p < \epsilon/2.$$

Next, since $w\psi_R$ has a bounded support, we can find a $C_0^\infty(R^n)$ function $\phi(x)$ such that

$$||w\psi_R - \phi||_{np/(n-p)} + ||\nabla(w\psi_R) - \nabla\phi||_p < \epsilon/2.$$

Then this ϕ satisfies (2.4). The proposition is proved.

Q.E.D

The following remarks are in order. First of all, by Theorem 1.1, for every u in $U^{1,p}(R^n)$, the function $w = u - (u)_\infty$ satisfies the condition of Proposition 2.2. Thus for every $\epsilon > 0$ there is a $C_0^\infty(R^n)$ function $\phi(x)$ satisfying $||\nabla w - \nabla\phi||_p < \epsilon$. Moreover, since it is apparent that $(\phi)_\infty = 0$, the norm of $w - \phi$ as defined in (2.1) is less than ϵ. Thus w is in the Sobolev space $W_0^{1,p}(R^n)$ by our definition.

Second, we note that the definition of the Sobolev space $W_0^{1,p}(R^n)$ does not have to depend on the definition of $U^{1,p}(R^n)$. Indeed, we can define $W_0^{1,p}(R^n)$ as the linear space consisting of functions satisfying (2.3), yet the norm of w is defined to be $||w|| = ||\nabla w||_p$. It follows from Proposition 2.2 and the Gagliado-Nirenberg-Sobolev inequality (cf. [EG], [GT], and [S])

$$||\phi||_{np/(n-p)} \leq c_{p,n}||\nabla\phi||_p, \quad \phi \in C_0^\infty(R^n) \tag{2.5}$$

that w also satisfies

$$||w||_{np/(n-p)} \leq c_{p,n}||\nabla w||_p.$$

It should be easy to show that $W_0^{1,p}(R^n)$ defined this way is the same complete Banach space.

Third, the definition of $W_0^{1,p}(R^n)$ does not have to be inseparable from Proposition 2.2. We can define this same $W_0^{1,p}(R^n)$ as the linear space consisting of functions w satisfying (2.3) and that for every $\epsilon > 0$ there is a $C_0^\infty(R^n)$ function ϕ satisfying (2.4). It is not difficult to show that $W_0^{1,p}(R^n)$, defined in this more primitive way, is a complete Banach space. A good consequence of Theorem 1.1 and Proposition 2.2 is that all these different definitions for $W_0^{1,p}(R^n)$ are equivalent.

In short, $W_0^{1,p}(R^n)$ is the completion of $C_0^\infty(R^n)$ functions under the norm $(\int_{R^n} |\nabla\phi|^p dx)^{1/p}$, $\phi \in W_0^{1,p}(R^n)$. We caution the reader that $W_0^{1,p}(R^n)$ defined in these different but equivalent ways is different from, and in fact larger than, the Banach space by the completion of $C_0^\infty(R^n)$ functions under the norm $(\int_{R^n}(|\nabla\phi|^p + |\phi|^p)dx)^{1/p}$.

We summarize our discussions in the following theorem.

Theorem 2.3 *The Sobolev space $W_0^{1,p}(R^n)$ is a subspace of $U^{1,p}(R^n)$ with co-dimension one. Equivalently, for any function $u(x) \in U^{1,p}(R^n)$ satisfying (0.1) and for any $\epsilon > 0$, there exists a $C_0^\infty(R^n)$ function ϕ such that*

$$(\int_{R^n} |u(x) - (u)_\infty - \phi|^{np/(n-p)} dx)^{(n-p)/(np)} + (\int_{R^n} |\nabla u - \nabla\phi|^p dx)^{1/p} < \epsilon.$$

It is notable that on the Euclidean space R^n with a dimension $n \geq 3$, the linear spaces $U^{1,2}(R^n), W_0^{1,2}(R^n)$ are complete Hilbert spaces with the inner products respectively

$$\begin{aligned} < u_1, u_2 > &= \int_{R^n} \nabla u_1 \cdot \nabla u_2 dx + (u_1)_\infty (u_2)_\infty \quad u_1,\, u_2 \in U^{1,2}(R^n); \\ < w_1, w_2 > &= \int_{R^n} \nabla w_1 \cdot \nabla w_2 dx \quad w_1,\, w_2 \in W_0^{1,2}(R^n). \end{aligned}$$

3 Three remarks

We make three remarks about our results. The first remark is about the best possible constant $c_{p,n}$ and the optimizing functions for the inequality (1.1). To our best knowledge, the problem of finding the best possible constant for the Poincaré inequality on a bounded domain, even a ball, is not solved completely at present. In the special case of $L^2 \to L^2$ Poincaré inequality, it is known that the best constant is equivalent to the lower bound of the least positive eigenvalue of the Neumann problem (see also a result of best constants for $L^1 \to L^1$ Poincaré inequality on rectangles in [LW]). However, for the Poincaré inequality on R^n, (1.1), all is known! Indeed, by Theorem 2.3, $u-(u)_\infty$ is in the Sobolev space $W_0^{1,p}(R^n)$; therefore, the inequality (1.1) is essentially the Sobolev inequality. The problem for the Sobolev inequality was solved in the well-known work of Talenti [T] and Aubin [A] in the case of $1 < p < n$ and the work of Fleming-Rishel [FR] in the case of $p = 1$. We also refer to the works of P.L. Lions on the theory of concentrated compactness, cf. [Li]. In the special case of $p = 2, n \geq 3$ there is extensive work, using the method of moving planes, on the semilinear elliptic pde satisfied by an optimizing function. By these results, we know the following.

In the case of $1 < p < n$ the optimizing functions for (1.1) are, after a translation, of form

$$c_1 + c_2(a + b|x|^{p/(p-1)})^{1-n/p}$$

where c_1 is any constant, c_2 any non-zero constant, and a, b any positive constants. The best constant for $c_{p,n}$ has a specific elementary expression in terms of the Gamma function. In the case of $p = 1$, the best constant equals a constant from the isoperimetric inequality in geometry, and no function in $U^{1,1}(R^n)$ attains the best constant. However, every optimizing sequence of functions, after a translation and a subtraction of a proper constant for each function, converges to the characteristic function of a ball in a weak sense in the space of functions with a bounded variation in R^n. It is notable that the best constant for $c_{p,n}$ is a continuous function of p on $[1, n)$.

We mentioned earlier that the best constant for the Poincaré inequality on a ball is not known. It follows from our remark that this best constant is not less than the corresponding best constant for the Sobolev inequality.

Our next remark is about the assumption $1 \leq p < n$ in Theorems 1.1 and 2.1. If $p \geq n$, then these theorems are not valid. In case $p = n$, consider

$$u(x) = \begin{cases} \ln \ln |x| & \text{if } |x| \geq e; \\ 0 & \text{if } |x| < e. \end{cases}$$

In case $p > n$, let α satisfy $0 < \alpha < 1 - n/p$ and consider

$$u(x) = \begin{cases} |x|^\alpha & \text{if } |x| \geq 1; \\ 1 & \text{if } |x| < 1. \end{cases}$$

It is elementary to verify that in the respective cases u is in $W^{1,p}_{loc}(R^n)$ and $||\nabla u||_p < \infty$, yet $(u)_\infty = \infty$. Thus Theorem 1.1 is no longer valid if $p \geq n$.

As for Theorem 2.1, we can construct a sequence of functions $\{u^i(x)\}$ such that $u^i \in W^{1,p}_{loc}(R^n)$ for each i and $||\nabla u^i||_p \to 0$ as $i \to \infty$, yet the sequence of the functions has no limit. In case $p = n$, let

$$u^i(x) = \begin{cases} 0 & \text{if } |x| \geq i; \\ \ln \ln i - \ln \ln |x| & \text{if } i > |x| \geq e; \\ \ln \ln i & \text{if } |x| < e. \end{cases}$$

In case $p > n$, let α satisfy $0 < \alpha < 1 - n/p$ and let

$$u^i(x) = \begin{cases} 0 & \text{if } |x| \geq i; \\ i^\alpha - |x|^\alpha & \text{if } i > |x| \geq 1; \\ i^\alpha - 1 & \text{if } |x| < 1. \end{cases}$$

Note that in the construction above each $u^i(x)$ even has a bounded support. It is easy to verify that the sequence has no limit and therefore Theorem 2.1 is not valid if $p \geq n$.

One more remark we would like to make is that there is an alternative way to look at Theorem 1.1 and Proposition 2.2 in special cases. Suppose u satisfies (0.1). Consider the minimization problem

$$\min_{w \in W_0^{1,p}(R^n)} \int_{R^n} |\nabla u - \nabla w|^p dx.$$

If $1 < p < n$, by the standard Hilbert method (cf. [KS]), this problem has a unique minimizer. (For this statement we need the uniform convexity of the L^p space for $p > 1$. We also recall that by the discussion on the definition of $W_0^{1,p}(R^n)$ in the previous section the space $W_0^{1,p}(R^n)$ can be defined without the knowledge of Theorem 1.1 and Proposition 2.2. It's not hard to see that every minimizing sequence of w converges strongly to a unique minimizer.)

Let w be the minimizer of the variational problem. It satisfies the Euler-Lagrange equation

$$\mathrm{div}(|\nabla(u - w)|^{p-2} \nabla(u - w)\,) = 0.$$

In case $p = 2$ and $n \geq 3$, the function $u - w$ is harmonic with $\int_{R^n} |\nabla u - \nabla w|^2 dx < \infty$. Applying the mean value theorem for harmonic functions to each component of $\nabla u - \nabla w$, we can easily show that $\nabla u - \nabla w$ is identically zero and therefore $u - w$ is a constant. In this way, we can come to the conclusions of Theorem 1.1 and Proposition 2.2 in the special case of $p = 2, n \geq 3$ from a different approach, except that we would have no knowledge about the constant $u - w$.

Of course our Theorem 1.1 and Proposition 2.2 tell us that the minimizer for the minimization problem for all p satisfying $1 \leq p < n$ is $u - (u)_\infty$.

4 Inequalities for functions with higher order derivatives

In this section we consider $n \geq 3$. Let k be an integer satisfying $2 \leq k < n$, and let p satisfy $1 \leq p < n/k$. Let $u(x)$ be a function on R^n satisfying

$$u \in W^{k,p}_{loc}(R^n) \text{ and } ||D^k u||_p < \infty. \tag{4.1}$$

In the inequality above, we mean that all the $k-$th derivatives, all the $D^\beta u$ with $|\beta| = |(\beta_1, \ldots, \beta_n)| = |\beta_1| + \cdots + |\beta_n| = k$, are in L_p.

We extend Theorem 1.1 to the following theorem.

Theorem 4.1 *Suppose u satisfies (4.1). Then there exists a unique polynomial $P(x)$ of order $k-1$ such that*

$$||u - P||_{np/(n-kp)} \leq c_{p,n} ||D^k u||_p, \tag{4.2}$$

where $||D^k u||_p$ denotes the sum of the L_p norm of all the $k-$th derivatives of u.

Proof: Let

$$p_i = \frac{np}{n - ip}, \quad i = 0, 1, \ldots, k. \tag{4.3}$$

Note that the p_i satisfy

$$1 \leq p = p_0 < p_1 \cdots < p_{k-1} < n \tag{4.4}$$

and

$$p_{i+1} = \frac{np_i}{n - p_i}, \quad i = 0, 1, \ldots, (k-1). \tag{4.5}$$

Applying Theorem 1.1 to $D^{k-1}u$, any $(k-1)-$th derivative of u, we know that $(D^{k-1}u)_\infty$ exists and

$$||D^{k-1}u - (D^{k-1}u)_\infty||_{p_1} \leq c_{p,n} ||D^k u||_p.$$

Define

$$u_1 = u - \sum_{|\beta| = k-1} \frac{1}{\beta!} (D^\beta u)_\infty x^\beta.$$

Then u_1 satisfies

$$||D^{k-1}u_1||_{p_1} \le c_{p,n}||D^k u||_p$$

We next define $u_2, \dots, u_k$ inductively. Assume $u_i (1 \le i \le k)$ is defined and satisfies

$$||D^{k-i}u_i||_{p_i} \le c_{p,n}||D^k u||_p. \tag{4.6}$$

Applying Theorem 1.1 to $D^{k-i-1}u_i$, we know that $(D^{k-i-1}u_i)_\infty$ exists and define

$$u_{i+1} = u_i - \sum_{|\beta|=k-i-1} \frac{1}{\beta!}(D^\beta u_i)_\infty x^\beta. \tag{4.7}$$

Then u_{i+1} satisfies

$$D^{k-i-1}u_{i+1} = D^{k-i-1}u_i - (D^{k-i-1}u_i)_\infty$$

and thus

$$||D^{k-i-1}u_{i+1}||_{p_{i+1}} \le c_{p,n}||D^{k-i}u_i||_{p_i} \le c_{p,n}||D^k u||_p.$$

Comparing this last inequality with (4.6), we see that the inductive definitions of $u_2, \dots, u_k$ are successful. By (4.7), we know $u_k = u - P$ where P is a polynomial of order $k-1$. Then the inequality (4.6) with $i = k$ gives the inequality (4.2) in the theorem.

Q.E.D

Next we extend Theorem 2.1.

Theorem 4.2 *Let $U^{k,p}(R^n)$ be the function space consisting of all functions satisfying (4.1). For each element $u(x)$, let P be the polynomial of order $k-1$ from $u(x)$ in Theorem 4.1 and let the P_β be the coefficients of the polynomial. Then $U^{k,p}(R^n)$ is a complete Banach space with the norm*

$$||u|| = (\int_{R^n} |D^k u|^p dx + \sum_{|\beta|\le k-1} |P_\beta|^p)^{1/p}.$$

Proof: Again we need only to prove the completeness. Before the proof, let's note that for an element u, the difference $w = u - P$ satisfies, for each $i = 1, \dots, k$, $D^{k-i}w = D^{k-i}u_i$ where the u_i are defined in the proof of the previous theorem. By (4.6), we have

$$||D^{k-i}w||_{p_i} \le c_{p,n}||D^k w||_p, \ i = 1, \dots, k. \tag{4.8}$$

Now let $\{u^i\}$ be a Cauchy sequence in $U^{k,p}(R^n)$. Consider the sequence of $w^i = u^i - P^i$. Then by (4.8), a limit w exists and satisfies (4.8). Let P be the limit of the sequence of P^i. We know that $u = w + P$ is the limit of the sequence of u^k. The completeness simply follows.

Q.E.D

Similar to Proposition 2.2, we can prove that if w satisfies (4.8) then for any $\epsilon > 0$ there exists a $C_0^\infty(R^n)$ function ϕ such that

$$\sum_{i=1}^{k} ||D^{k-i}w - D^{k-i}\phi||_{p_i} + ||D^k w - D^k\phi||_p < \epsilon.$$

We need only to modify the proof there slightly to prove this statement. Also, we can define $W_0^{k,p}(R^n)$ as the closed subspace of $U^{k,p}(R^n)$ which contains the $C_0^\infty(R^n)$ functions as a dense subset. The codimension of this subspace equals the dimension of the linear space consisting of all polynomials of order less than k.

5 Inequalities for functions on an exterior domain

Let Ω be a bounded domain in R^n such that the boundary $\partial\Omega$ is a Lipschitz continuous surface and the exterior domain Ω^C is connected. We come back to the assumption $1 \le p < n$. Consider a function on Ω^C satisfying

$$u \in W_{loc}^{1,p}(\Omega^C) \text{ and } \int_{\Omega^C} |\nabla u|^p dx < \infty. \tag{5.1}$$

We use $c_{p,n}(\Omega)$ to denote a constant that depends on p, n, and Ω.

We first state a proposition on $C_0^\infty(R^n)$ functions.

Proposition 5.1 *There exists a constant $c_{p,n}(\Omega)$ such that for any $\phi \in C_0^\infty(R^n)$,*

$$\left(\int_{\Omega^C} |\phi|^{np/(n-p)}dx\right)^{(n-p)/(np)} \le c_{p,n}(\Omega)\left(\int_{\Omega^C} |\nabla\phi|^p dx\right)^{1/p}. \tag{5.2}$$

Assume this proposition for the moment. We move on to the following two theorems similar to Theorems 1.1 and 2.1.

Theorem 5.2 *For each $u(x)$ satisfying (5.1), there exists a number $(u)_\infty$ such that*

$$\int_{\Omega^C} |u-(u)_\infty|^{np/(n-p)}dx)^{(n-p)/(np)} \le c_{p,n}(\Omega)(\int_{\Omega^C} |\nabla u|^p dx)^{1/p}. \quad (5.3)$$

Theorem 5.3 *The linear space $U^{1,p}(\Omega^C)$ consisting of functions satisfying (5.1) is a complete Banach space with the norm*

$$||u|| = (\int_{\Omega^C} |\nabla u|^p dx + |(u)_\infty|^p)^{1/p}.$$

Proof of Theorems 5.2 and 5.3: For every u satisfying (5.1) there is an extension of u to a function $\tilde{u}$ on R^n satisfying (0.1) and $\tilde{u} = u$ on Ω^C. Applying Theorem 1.1 to $\tilde{u}$, we have the existence of $(\tilde{u})_\infty$ such that

$$||\tilde{u}-(\tilde{u})_\infty||_{np/(n-p)} < \infty \text{ and } ||\nabla\tilde{u}||_p < \infty. \quad (5.4)$$

Note that $(\tilde{u})_\infty$ does not depend on the choice of the extension because Ω is assumed to be a bounded domain. Thus we can let $(u)_\infty$ be equal to $(\tilde{u})_\infty$ and then

$$(\int_{\Omega^C} |u-(u)_\infty|^{np/(n-p)}dx)^{(n-p)/(np)} \le c_{p,n}||\tilde{u}-(\tilde{u})_\infty||_{np/(n-p)} < \infty.$$

This is, of course, short of (5.3). Next, we apply Proposition 2.2 to $\tilde{u}-(\tilde{u})_\infty$. It follows that for any $\epsilon > 0$, there exists a $C_0^\infty(R^n)$ function ϕ such that

$$||\tilde{u}-(\tilde{u})_\infty-\phi||_{np/(n-p)} + ||\nabla\tilde{u}-\nabla\phi||_p < \epsilon.$$

This inequality implies that

$$(\int_{\Omega^C} |u-(u)_\infty-\phi|^{np/(n-p)}dx)^{(n-p)/(np)} + (\int_{\Omega^C} |\nabla u-\nabla\phi|^p dx)^{1/p} < \epsilon.$$

Using Proposition 5.1 for ϕ, we know that (5.3) is satisfied by taking a sequence of $C_0^\infty(R^n)$ functions that converges to $u-(u)_\infty$ in the sense above.

The proof of Theorem 5.3 is similar to that of Theorem 2.1.

Q.E.D

Now we come back to Proposition 5.1.

We remark that the inequality (5.2) is essentially the Gagliado-Nirenberg-Sobolev inequality on an exterior domain. We present a proof using the standard extension theorem for Sobolev functions on a domain with a Lipschitz continuous boundary (cf. [EG] and [S]). Here we pursue a direct approach, in contrast to the contradiction argument that is often used in proving similar inequalities.

We refer to the works of Jones [J] and Chua [C] for more general extension theorems for Sobolev functions. Their results might be used to further generalize the inequality (5.2).

We break the proof into five steps.

Proof of Proposition 5.1:

Step one: Let $R \geq 0$. We prove that there is a constant $c_{p,n}$, independent on R, such that

$$\left(\int_{B_R^C} |\phi|^{np/(n-p)} dx\right)^{(n-p)/(np)} \leq c_{p,n}\left(\int_{B_R^C} |\nabla \phi|^p dx\right)^{1/p} \tag{5.5}$$

for any $\phi \in C_0^\infty(R^n)$. Note that when $R = 0$, this equality is the standard G-N-S inequality (2.5). For $R > 0$, we define

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } |x| \geq R; \\ \phi(R^2 x/|x|^2) & \text{if } |x| < R. \end{cases}$$

This $\tilde{\phi}$ is not a $C_0^\infty(R^n)$ function anymore but is still in $W_0^{1,p}(R^n)$. Thus $\tilde{\phi}$ still satisfies the inequality (2.5). Moreover, upon the change of variables $x = R^2 y/|y|^2$,

$$\begin{aligned} \int_{B_R} |\nabla \tilde{\phi}|^p dx &= \int_{B_R^C} |\nabla \phi(y)|^p \left(\frac{|y|}{R}\right)^{2p} \left(\frac{R}{|y|}\right)^{2n} dy \\ &= \int_{B_R^C} |\nabla \phi(y)|^p \left(\frac{R}{|y|}\right)^{2n-2p} dy \leq \int_{B_R^C} |\nabla \phi(y)|^p dy. \end{aligned}$$

In the calculation above, we have used that

$$\frac{\partial x_i}{\partial y_j} = R^2 \frac{|y|^2 \delta_{ij} - 2y_i y_j}{|y|^4}$$

and that the matrix

$$\left(\frac{|y|^2 \delta_{ij} - 2y_i y_j}{|y|^2}\right)_{n\times n}$$

for each $y \neq 0$ represents a reflection in R^n and has eigenvalues $-1, 1, \ldots, 1$.

In a short form, the previous inequality gives

$$\int_{B_R} |\nabla \tilde{\phi}|^p dx \leq \int_{B_R^C} |\nabla \phi|^p dx.$$

Then we have

$$\begin{aligned}
& (\int_{B_R^C} |\phi|^{np/(n-p)} dx)^{(n-p)/(np)} \leq (\int_{R^n} |\tilde{\phi}|^{np/(n-p)} dx)^{(n-p)/(np)} \\
\leq \;& c_{p,n}(\int_{R^n} |\nabla \tilde{\phi}|^p dx)^{1/p} \\
\leq \;& c_{p,n}((\int_{B_R} |\nabla \tilde{\phi}|^p dx)^{1/p} + (\int_{B_R^C} |\nabla \phi|^p dx)^{1/p}) \\
\leq \;& c_{p,n}(\int_{B_R^C} |\nabla \phi|^p dx)^{1/p}.
\end{aligned}$$

That is, (5.5) is valid. Note that if Ω is a ball, then (5.2) is simply the same as (5.5).

Step two: We choose a constant $R_0 > 0$ such that the closure $\bar{\Omega}$ is contained in the ball B_{R_0}. Suppose $\xi(x)$ is a smooth function on the closed set $\Omega^C \cap \bar{B}_{2R_0}$ and vanishes on the outer boundary ∂B_{2R_0}. We bring up the inequality

$$\int_{\Omega^C \cap B_{2R_0}} |\xi|^p dx \leq c_{p,n}(\Omega) \int_{\Omega^C \cap B_{2R_0}} |\nabla \xi|^p dx. \tag{5.6}$$

Here the constant $c_{p,n}(\Omega)$ depends on $2R_0$ too, but we don't have to write it out explicitly because $2R_0$ in turn depends on Ω.

We only provide a sketch of a proof of (5.6). First, we mention a similar inequality on a rectangular box for a smooth function that vanishes on one side. Such an inequality can be easily proved by expressing the function in terms of an integral of its derivatives and then applying the Hölder inequality. Next, we cover the closed region $\Omega^C \cap \bar{B}_{2R_0}$ with a finite number of tube-like regions each of which can be transformed nondegenerately to a rectangular box with one side of the box corresponding to a piece of the outer boundary ∂B_{2R_0}. Then the inequality (5.6) follows.

Step three: Next we recall the standard extension operator for Sobolev functions on a bounded domain. Particularly we have a linear operator E such that for any smooth function $\eta(x)$ on $\Omega^C \cap \bar{B}_{R_0}$, the function $\tilde{\eta} = E\eta$ is in $W^{1,p}(\bar{B}_{R_0})$, $\tilde{\eta} = \eta$ on $\Omega^C \cap B_{R_0}$, and

$$\int_\Omega |\nabla \tilde{\eta}|^p dx \leq c_{p,n}(\Omega) \int_{\Omega^C \cap B_{R_0}} (|\nabla \eta|^p + |\eta|^p) dx.$$

Again we refer to [EG] and [S] for a proof.

Step four: With the extension operator E in the previous step, we apply E to $C_0^\infty(R^n)$ functions so that $\tilde{\phi} = E\phi$ satisfies $\tilde{\phi} \in W_0^{1,p}(R^n)$, $\tilde{\phi} = \phi$ on Ω^C, and

$$\int_\Omega |\nabla \tilde{\phi}|^p dx \leq c_{p,n}(\Omega) \int_{\Omega^C \cap B_{R_0}} (|\nabla \phi|^p + |\phi|^p) dx.$$

To estimate the second term in the previous inequality, we choose a $C_0^\infty(R^n)$ function ψ_{R_0} as in the proof of Proposition 2.2. Recall that 1) ψ_{R_0} is one inside B_{R_0}, zero on $B_{2R_0}^C$, 2) in the annulus $B_{R_0}^C \cap B_{2R_0}$ ψ_{R_0} is between zero and one, and 3) $|\nabla \psi_{R_0}|$ is bounded by $2/R_0$. Now we have

$$\begin{aligned}
\int_{\Omega^C \cap B_{R_0}} |\phi|^p dx &\leq \int_{\Omega^C \cap B_{2R_0}} |\phi \psi_{R_0}|^p \\
&\leq c_{p,n}(\Omega) \int_{\Omega^C \cap B_{2R_0}} (|\nabla \phi \psi_{R_0}|^p + |\phi \nabla \psi_{R_0}|^p) dx \\
&\quad \text{(apply (5.6) with } \xi = \phi \psi_{R_0}) \\
&\leq c_{p,n}(\Omega) (\int_{\Omega^C} |\nabla \phi|^p dx + \int_{B_{R_0}^C \cap B_{2R_0}} |\phi|^p dx) \\
&\leq c_{p,n}(\Omega) \int_{\Omega^C} |\nabla \phi|^p dx + c_{p,n}(\Omega) (\int_{B_{R_0}^C \cap B_{2R_0}} |\phi|^{np/(n-p)} dx)^{(n-p)/n} \\
&\leq c_{p,n}(\Omega) \int_{\Omega^C} |\nabla \phi|^p dx + c_{p,n}(\Omega) (\int_{B_{R_0}^C} |\phi|^{np/(n-p)} dx)^{(n-p)/n} \\
&\leq c_{p,n}(\Omega) \int_{\Omega^C} |\nabla \phi|^p dx + c_{p,n}(\Omega) (\int_{B_{R_0}^C} |\nabla \phi|^p dx) \quad \text{(by (5.2))} \\
&\leq c_{p,n}(\Omega) \int_{\Omega^C} |\nabla \phi|^p dx.
\end{aligned}$$

In summary, we have for $\tilde{\phi} = E\phi$

$$\int_\Omega |\nabla \tilde{\phi}|^p dx \leq c_{p,n}(\Omega) \int_{\Omega^C} |\nabla \phi|^p dx. \tag{5.7}$$

Step five: Now we complete the proof of (5.2). With $\tilde{\phi}$ as defined in step three and step four, we have

$$\begin{aligned}
& (\int_{\Omega^C} |\phi|^{np/(n-p)} dx)^{(n-p)/(np)} \leq (\int_{R^n} |\tilde{\phi}|^{np/(n-p)} dx)^{(n-p)/(np)} \\
\leq & \ c_{p,n}(\Omega)((\int_{\Omega} |\nabla \tilde{\phi}|^p dx)^{1/p} + (\int_{\Omega^C} |\nabla \phi|^p dx)^{1/p}) \\
\leq & \ c_{p,n}(\Omega)(\int_{\Omega^C} |\nabla \phi|^p dx)^{1/p}.
\end{aligned}$$

The last inequality comes from (5.7). Thus we have now completed the proof of Proposition 5.1.

Q.E.D

Our next theorem is about the trace of u on the boundary $\partial\Omega$.

Theorem 5.4 *Suppose u satisfies (5.1). Then*

$$(\int_{\partial\Omega} |u - (u)_\infty|^{(n-1)p/(n-p)} dS)^{(n-p)/((n-1)p)} \leq c_{p,n}(\Omega)(\int_{\Omega^C} |\nabla u|^p dx)^{1/p}. \quad (5.8)$$

Proof: By the discussion in the proof of Theorems 5.2 and 5.3, we need only to show that for any $C_0^\infty(R^n)$ function $\phi(x)$,

$$(\int_{\partial\Omega} |\phi|^{(n-1)p/(n-p)} dS)^{(n-p)/((n-1)p)} \leq c_{p,n}(\Omega)(\int_{\Omega^C} |\nabla \phi|^p dx)^{1/p}. \quad (5.9)$$

The standard Sobolev trace theorem tells us that

$$(\int_{\partial\Omega} |\phi|^{(n-1)p/(n-p)} dS)^{(n-p)/((n-1)p)} \leq c_{p,n}(\Omega)(\int_{\Omega^C \cap B_{R_0}} (|\nabla \phi|^p + |\phi|^p) dx)^{1/p}$$

where B_{R_0} is a ball that contains the closed set $\bar{\Omega}$. Then the same argument we used in the step three through five in the proof of Proposition 5.1 gives (5.9).

6 Irrotational and incompressible fluid flows around a body in space

We apply what we developed in the previous sections to a problem of irrotational and incompressible fluid flows around a body in space.

Let Ω be a bounded domain in R^n $(n \geq 3)$ such that its boundary $\partial\Omega$ is sufficiently smooth ($C^{2,\alpha}$ for some α between zero and one is enough) and the exterior domain Ω^C is connected. Here is the problem we consider. Suppose that a body of shape Ω moves uniformly in a fluid, water or air for examples. If we move with the body, we observe that the body is still and the fluid flows around the body with a uniform velocity at infinity. A classic problem of fluid mechanics is to determine the velocity vector field of the flow, with the shape Ω of the body and the speed at infinity given.

We are interested in the irrotational, stationary flows whose velocity vector fields have a potential function and do not vary with time. By the continuity equation from conservation of mass, such a potential function is harmonic in Ω^C in case that the fluid is incompressible. At any point on the boundary $\partial\Omega$, the normal component of the velocity vector vanishes.

The problem we consider can be formulated mathematically. Let $\bar{u}(x)$ be such a potential function. For an x on $\partial\Omega$, let $\nu(x)$ denote a unit vector normal to the boundary pointing into Ω^C. Let $(s^\infty, 0, \cdots, 0)$ be the velocity of the flow at infinity. Then

$$\begin{cases} -\Delta\bar{u} &= 0 & \text{in } \Omega^C; \\ \partial\bar{u}/\partial\nu &= 0 & \text{on } \partial\Omega; \\ -\nabla\bar{u} &= (s^\infty, 0, \ldots, 0) & \text{at infinity.} \end{cases} \tag{6.1}$$

We are practically interested in the three dimensional flows only, but the mathematical treatment is the same for all higher dimensions. In case the fluid is compressible and the flow is subsonic, the Laplacian equation for $\bar{u}$ is replaced by a quasilinear elliptic partial differential equation of second order.

Much is known about (6.1) and the corresponding problem for subsonic flows. For examples, if Ω is a sphere, then the solution has a simple formula; if Ω is an ellipsoid, then the solution can be obtained by the method of separation of variables using the triply orthogonal ellipsoidal coordinate system (cf. Lamb [L]). A general existence theorem is in the book of Kellogg [K] by formulating the problem into an integral equation. In [DO] Dong and Ou proved an existence theorem for subsonic flows by formulating the problem into a variational problem. This variational method also works for the problem (6.1) above (cf. [O]). The general uniqueness theorem for subsonic flows in space of dimension $n \geq 3$ was established in the paper of Serrin and Weinberger [SW], improving on earlier results of Finn-Gilbarg and Meyers-Serrin.

Here we present another proof of the existence of a solution to (6.1) using our new knowledge of the space $U^{1,2}(\Omega^C)$ defined in the previous section. This proof has an easy extension to subsonic flows.

Let $\bar{u}(x) = u(x) + s^\infty x_1$. We consider the following variational problem

$$\min_{u \in U^{1,2}(\Omega^C)} \int_{\Omega^C} \frac{1}{2}|\nabla u|^2 dx - s^\infty \int_{\partial\Omega} \nu_1 u dS. \tag{6.2}$$

We will prove that this minimization problem has a solution and the solution is unique up to an added constant. For the moment, let's assume $u(x)$ is a solution of the variational problem. Then for any $\phi \in C_0^\infty(R^n)$,

$$\begin{aligned} 0 &= \int_{\Omega^C} \nabla u \cdot \nabla \phi dx - s^\infty \int_{\partial\Omega} \nu_1 \phi dS \\ &= -\int_{\Omega^C} \Delta u \phi dx - \int_{\partial\Omega} (\frac{\partial u}{\partial \nu} + s^\infty \nu_1) \phi dS. \end{aligned}$$

Thus u satisfies in the distributional sense $-\Delta u = 0$ in Ω^c and $\partial u/\partial \nu = -s^\infty \nu_1$ on $\partial\Omega$. By the standard elliptic theory, u is smooth up to the boundary and the same equations are satisfied in the classical sense. Moreover, as x approaches infinity, $|\nabla u|$ approaches zero because each component of ∇u is a harmonic function on Ω^C and $\int_{\Omega^C} |\nabla u|^2 dx$ is bounded by a constant depending on Ω and s^∞. Indeed, for a large x, applying the mean value theorem to a component of ∇u on the ball centered at x and with radius $|x|/2$, one can easily prove that

$$|\nabla u(x)| \leq \frac{c}{(1 + |x|)^{n/2}}.$$

Then the function $\bar{u} = u + s^\infty x_1$ satisfies the equations in (6.1).

To prove the existence of a minimizer for (6.2), we notice that because $\int_{\partial\Omega} \nu_1 dS = 0$ by the divergence theorem, the functional is invariant when a constant is added to or subtracted from $u(x)$. Let $u = w + (u)_\infty$. Then w is in the subset of $U^{1,2}(\Omega^C)$ defined as follows:

$$W_0 = \{w \in U^{1,2}(\Omega^C) \mid (w)_\infty = 0\}. \tag{6.3}$$

It follows from our theorems in the previous sections that W_0 is a complete Hilbert space with norm

$$||w|| = (|\int_{\Omega^C} |\nabla w|^2 dx)^{1/2}.$$

Moreover, the trace of w on $\partial\Omega$ satisfies

$$(\int_{\partial\Omega} |w|^{2(n-1)/(n-2)} dS)^{(n-2)/(2n-2)} \leq c_{2,n}(\Omega)||w||.$$

Now the minimization problem (6.2) becomes

$$\min_{w\in W_0} \int_{\Omega^C} \frac{1}{2}|\nabla w|^2 dx - s^\infty \int_{\partial\Omega} \nu_1 w dS. \tag{6.4}$$

Apparently, the functional is of form $||w||^2/2 + f(w)$ for a continuous linear functional $f(w)$ on W_0. It follows from the standard Hilbert method that a unique minimizer exists for (6.4). For the problem (6.2) a minimizer exists and is unique up to an added constant.

We would like to mention that the Hilbert space W_0 was established in the paper of Dong and Ou [DO], using a weighted Sobolev inequality that can be traced back to a Hardy inequality. Our current proof, based largely on the inequality (5.2), is more direct. As in the earlier works, note that the variational method we have presented does not work if the dimension of the space is two. For a two dimensional irrotational and incompressible flow, complex analysis is a more effective tool, and we refer the reader to many textbooks on this subject.

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Guozhen Lu
Department of Mathematics
Wayne State University
Detroit, MI 48202
USA
Email: gzlu@math.wayne.edu

Biao Ou
Department of Mathematics
University of Toledo
Toledo, OH 43606
USA
Email: bou@math.utoledo.edu